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Distributional convolutors for wavelet transform

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Abstract In the present paper, the concept of convolutors on distribution spaces has been extended to establish some results of the wavelet transform. The space of convolutors is characterized in terms of its wavelet transform with example. Further, Calderón-type reproducing formula is derived in distributional sense as an application of the same.

Keywords Fourier transform · wavelet transform · Distribution spaces · Convolution operators

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1 Introduction

By applying translation and dilation to the function $\psi \in L^2(\mathbb{R})$, the wavelet $\psi_{b,a}(t)$ is obtained by

$$\psi_{b,a}(t) = |a|^{-\varrho}\psi\left(\frac{t-b}{a}\right), \quad t, b \in \mathbb{R}, \quad a \in \mathbb{R}_0, \quad \varrho > 0. \tag{1}$$

If $\varrho = \frac{1}{2}$, then we can construct a unitary operator which maps ψ to $\psi_{b,a}$ on $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, the wavelet transform $W_{\psi_a}(b)$ with respect to the basic wavelet $\psi_{b,a}(t) \in L^2(\mathbb{R})$ is defined by

$$W_{\psi_a}(b) = \int_{\mathbb{R}} f(t)\overline{\psi_{b,a}}(t)dt. \tag{2}$$

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Hence, we can write

$$W_{\psi_a}(b) = (f * \Theta_{0,a})(b), \tag{3}$$

where $\Theta(x) = \overline{\psi(-x)}$, provided the integral exists. If $f \in L^p(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$ then by [7]:

$$f * \Theta_{a,0}(b) \in L^r(\mathbb{R}), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

and $*$ denotes the classical convolution defined as

$$(f * g)(y) = \langle f(x-y), g(x) \rangle, \ y \in \mathbb{R}.$$

Equation (3) describes the wavelet transform $W_{\psi_a}(b)$ in terms of the operation convolution.
The relation between Fourier and wavelet transform is given as follows:

$$W_{\psi_a}(b) = \frac{1}{2\pi}|a|^{-\rho} \int\limits_{\mathbb{R}} e^{ib\omega} \hat{f}(\omega)\overline{\hat{\psi}(a\omega)} d\omega. \tag{4}$$

Further, if the admissibility condition holds for $\psi \in L^2(\mathbb{R})$, i.e.,

$$C_\psi = \int\limits_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|} dw < \infty, \tag{5}$$

then the following inversion formula for the wavelet transform $W(b,a)$ with $\rho = \frac{1}{2}$, holds at every point x of continuity of $f(x)$[2]:

$$\frac{1}{C_\psi} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}_0} \frac{1}{|a|^{\frac{1}{2}}} W_{\psi_a}(b) \psi\left(\frac{x-b}{a}\right) \frac{dbda}{a^2} = f(x). \tag{6}$$

The notion of classical functions have been generalized by the distributions. Through distributions, we can differentiate the non-differentiable functions in classical sense [16]. To analyze the continuity results of classical functions, the wavelet transform has been extended to distribution spaces [7,10,11]. Wavelet transform behaves to be a continuous linear mapping on such spaces, which was given by many authors including Pathak *et al.* [13]. Also, Pathak extended the continuous wavelet transform to certain distributions and their continuity results were discussed with boundedness results in some generalized spaces such as Sobolev space, Besove space and Lizorkin space [8]. Wavelets which are approximately of exponential decay were investigated by Dziubański and Hernández [3]. With these wavelets Paley-Wiener-Schwartz type theorem for wavelet transform using distribution was developed by Pathak *et al* [12]. The continuous wavelet transfrom has been extended to Schwartz distributions and an inversion formula derived by interpreting the convergence of distributions in weak sense by Pandey [6].

A convolutor is an operator which preserves the convolution operation. Most of the generalized function spaces can be regarded as spaces of convolutors by a proper selection of the test function space. Generalized functions are treated as objects which convolute with test functions and satisfy some conditions. Any operations which can be studied regarding convolution have extensions to the space of convolutors. Thus convolutors are operators on a suitable testing function space which satisfy $T(\phi * \psi) = T(\phi) * \psi$. Therefore, the corresponding space of distributions can be characterized by convolutors. Also, the operations which are defined in respect of convolutions can be extended on the space of convolutors. Because of its structure preserving property, convolutors have been extended to a number of generalized functions. Recently, Pathak [14] investigated the convolutors on the space of ultradistributions for Fourier transform. Mikusiński *et al.* [5] investigated the foundation of convolutors, which allows us to define the wavelet transform for a wide class of spaces of generalized functions (distributions) more simply. Since the wavelet transform is itself a convolution with the mother wavelet, it can be extended to convolutors. Further, the results for test function spaces can be extended onto distributions through wavelet transform. Which allows us to define convolutors for distributions in terms of wavelet transform.

Note that the structure of the convolutor varies from space to space. Hence to construct convolutors for the distributional space, we need a suitable test function space. The algebraic structure of these test functions should be similar to that of a convolution algebra. Also, the convergence in the corresponding test function spaces makes the convolutors continuous. Thus, the convolutors act on a test function like an extension of the convolution operation. Therefore, we can use the notion of convolution to specify the effect of convolutors [1].

Now, we recollect the definitions of the suitable test function space and the corresponding space of convolutors [1]. Let $\mathcal{G}$ be the space which contains all regular functions ϕ satisfying

$$\gamma_k(\phi) = \sup_{|Imz| \leq k} e^{k|\mathrm{Re}z|} |\phi(z)| < \infty, \forall\ k \in \mathbb{N} \tag{7}$$

where $\{\gamma_k\}_{k \in \mathbb{N}}$ is the family of seminorms. Let $\mathcal{M}_{\mathcal{G}}$ denotes the space of all regular functions ϕ such that, there exists $m \in \mathbb{N}$ for every $k \in \mathbb{N}$ satisfying

$$\sup_{|Imz| \leq k} e^{-m|\mathrm{Re}z|} |\phi(z)| < \infty.$$

Then each multiplier defined on the space will satisfy the condition given by (7). Thus the space of multipliers of $\mathcal{G}$ will coincide with $\mathcal{M}_{\mathcal{G}}$ [14]. A convolutor for the space $\mathcal{G}'$ is the generalized function $T \in \mathcal{G}'$ such that for every $\phi \in \mathcal{G}$, $T * \phi \in \mathcal{G}$.
Let $m \in \mathbb{Z}$. $O_{\mathcal{G},m,c}$ is the space constituted by the entire functions ϕ such that

$$\gamma_k^m(\phi) = \sup_{|Imz| \leq k} e^{m|\mathrm{Re}z|} |\phi(z)| < \infty, \quad k \in \mathbb{N}. \tag{8}$$

Now we denote the space $O_{\mathcal{G},c} = \cup_{m \in \mathbb{Z}} O_{\mathcal{G},m,c}$, i.e., topologized with the inductive limit topology of the spaces $\cup O_{\mathcal{G},m,c},\ m \in \mathbb{Z}$. Here, the dual space $O'_{\mathcal{G},c}$ is same as the space of convolutors for $\mathcal{G}'$ [1].

In the next section, we shall investigate the wavelet transform on the space of convolutors. Inversion formula for the corresponding distributional wavelet transform is also obtained. In the last section, we established Calderón's formula in the space $\mathcal{G}'$. Further, an example for distributional convolutors are also provided.

2 Wavelet transform on the space of convolutors

In this section, we characterize the space of convolutors in terms of its wavelet transform. Since convolutors are associated with convolution, to incorporate wavelet transform on the convolutors we require wavelet convolution.
Along with each integral transform there is a suitable translation, leading to the construction of the convolution. Assume that $D(x,y,z)$ is a basic generalized function satisfying

$$\phi(t,x)\phi(t,y) = \int_{-\infty}^{\infty} D(x,y,z)\phi(t,z)dz. \tag{9}$$

$D(x,y,z)$ can be treated as a measure and in some cases as a distribution. In order to define the wavelet convolution, assume the basic function is such that

$$\begin{aligned} W_{\phi_a}[D(x,y,z)](b) &= \int_{-\infty}^{\infty} D(x,y,z)\overline{\phi_{b,a}}(x)dx \\ &= \overline{\psi_{b,a}}(z)\overline{\theta_{b,a}}(y). \end{aligned}$$

Its translation is given by

$$\begin{aligned} (\tau_x f)(y) &= \int_{-\infty}^{\infty} D(x,y,z)f(z)dz \\ &= {C_\phi}^{-1}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} \overline{\psi_{b,a}}(z)\overline{\theta_{b,a}}(y)\phi_{b,a}(x)f(z)|a|^{\frac{1}{2}}dbdadz, \end{aligned}$$

where θ is the distribution taken from the same distribution space as f. Thus, the corresponding wavelet convolution is given by [7]

$$\begin{aligned} (f\#g)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} D(x,y,z)f(z)g(y)dzdy \\ &= {C_\phi}^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} \overline{\psi_{b,a}}(z)\overline{\theta_{b,a}}(y)\phi_{b,a}(x)f(z)g(y)|a|^{\frac{1}{2}}dbdadzdy. \end{aligned}$$

The wavelet transform of a convolutor $T \in O'_{\mathcal{G},c}$ is given by

$$(W_{\psi_a}T)(b) = (T\#\psi_a)(b), \quad b \in \mathbb{C}, \ a \in \mathbb{R}_+,$$

where $\psi_a(x) = |a|^{\frac{-1}{2}}\psi\left(\frac{-x}{a}\right)$.

Now we give a structure formula for the space of convolutors. The result in the theorem is a characterization of the space of convolutors in terms of the wavelet transform.

Theorem 1 *$T \in \mathcal{G}'$ is a convolutor for $\mathcal{G}'$ if and only if $\forall\ k \in \mathbb{N}$ there exists a non-negative d, $n \in \mathbb{N}$ such that $T = (\tau_{d_i} + \tau_{-d_i})^n f$, for a regular function f on the set $\{z \in \mathbb{C} : -k < Imz < k\}$.*

Proof Assume that $\forall\ k \in \mathbb{N}_0$ there exists a non-negative d, $n \in \mathbb{N}$ such that $T = (\tau_{d_i} + \tau_{-d_i})^n f$, for a regular function f on the set $\{z \in \mathbb{C} : -k < Imz < k\}$. Then we have $T \in O'_{\mathcal{G},c}$. Also, by using Lemma 2.3 of [1], we have

$$\langle T, \phi\rangle = \int\limits_{C_l} e^{-m|\mathrm{Re}z|}\phi(z)d\mu(z), \quad \phi \in \mathcal{G}, \tag{10}$$

where

$$C_l = \{z \in \mathbb{C} : |\mathrm{Im}z| \leq l\}.$$

Consider

$$\begin{aligned}\langle (W_\psi T)(b,a), \phi\rangle &= \langle W_{\psi_a} T(b), \phi\rangle \\ &= \langle T(b), (W_{\psi_a}\phi)(-b)\rangle \\ &= \int e^{-m|\mathrm{Re}b|}(W_{\psi_a}\phi)(-b)d\mu(b).\end{aligned}$$

Thus, by the definition of wavelet transform

$$\begin{aligned}\langle (W_{\psi_a} T)(b), \phi\rangle &= \int\limits_{C_l} e^{-m|\mathrm{Re}b|}\int \phi(x)\psi_a(-b-x)dxd\mu(b) \\ &= \int \phi(x)\int e^{-m|\mathrm{Re}b|}\psi_a(-b-x)d\mu(b)dx.\end{aligned}$$

Therefore,

$$(W_{\psi_a} T)(x) = \int e^{-m|\mathrm{Re}x|}\psi_a(-b-x)d\mu(b), \quad |\mathrm{Im}b| \leq k,$$

and

$$|(W_{\psi_a} T)(x)| \leq |\mu|\, M\ e^{l|\mathrm{Re}b|},$$

where $M = \sup|\psi_a(-b-x)|$. Thus, we obtain

$$\sup e^{l|\mathrm{Re}b|}e^{-pa}\,|(W_{\psi_a} T)(x)| < \infty.$$

Hence,

$$(W_{\psi_a} T)(b) \in \mathcal{M}_{\mathcal{G}}(\mathbb{C} \times \mathbb{R}_+).$$

Consider the continuous linear mapping that maps $T \to T\#\phi$ on $\mathcal{G}$ [4]. Since $(W_{\psi_a} T)(b) = (T\#\psi_a)(b), \psi_a \in \mathcal{G}$, then T is a convolutor of $\mathcal{G}'$.

Conversely, assume that T is a convolutor of $\mathcal{G}'$. Then $(W_{\psi_a} T)(b)$ is a multiplier of $\mathcal{G}$. Consider the function defined by $h_{\lambda,n}(z) = (e^{z\lambda} + e^{-z\lambda})^n$, $z \in \mathbb{C}$, with the following properties:

(i) $h_{\lambda,n}(z) = 0 \Leftrightarrow z = \frac{\pi i}{2\lambda}(2k+1),\ k \in \mathbb{Z}$.

(ii) $|h_{\lambda,n}(z)| \leq Ce^{n\lambda|\mathrm{Re}z|},\ z \in \mathbb{C}$.

(iii) $|h_{\lambda,n}(z)| \geq Ce^{n\lambda|\mathrm{Re}z|},\ |\mathrm{Im}z| \leq \frac{\pi}{4\lambda}$.

By using (4), we obtain

$$e^{-\sigma b}(W_{\psi_a}T)(b) = (W_{\psi_a}(\tau_{-i\sigma}T))(b).$$

Hence,

$$\begin{aligned} h_{\gamma,n}(b)(W_{\psi_a}T)(b) &= \left(e^{-\gamma b} + e^{\gamma b}\right)^n (W_{\psi_a}T)(b) \\ &= W_{\psi_a}((\tau_{-i\gamma} + \tau_{i\gamma})^n T)(b). \end{aligned}$$

Now, define

$$G_{n,\gamma}(b,a) = \frac{(W_{\psi_a}T)(b)}{h_{\gamma,n}(b)},\ \ |\mathrm{Im}b| \leq \frac{\pi}{2\gamma}.$$

Thus,

$$\begin{aligned} |G_{n,\gamma}(b,a)| &= \left|\frac{(W_{\psi_a}T)(b)}{h_{\gamma,n}(b)}\right| \\ &\leq \frac{|\mu|\ M\ e^{l|\mathrm{Re}b|}}{Ce^{n\gamma|\mathrm{Re}b|}} \\ &\leq M' e^{(l-n\gamma)|\mathrm{Re}b|},\quad l,n \in \mathbb{N}, \end{aligned}$$

where $\frac{|\mu|M}{C} \leq M'$.
Therefore, for $\phi \in \mathcal{G}$, we have

$$\begin{aligned} \langle T, \phi\rangle &= C_\psi \left\langle (W_{\psi_a}T)(b), (W_{\psi_a}\phi)(b)\right\rangle \quad \text{(by Parseval relation)} \\ &= C_\psi \int\int G_{n,\gamma}(b,a) h_{\gamma,n}(b)(W_{\psi_a}\phi)(b)dbda \\ &= C_\psi \int\int G_{n,\gamma}(b,a) W_{\psi_a}((\tau_{-i\gamma} + \tau_{i\gamma})^n\phi)(b)dbda \\ &= C_\psi \left\langle G_{n,\gamma}(b,a), W_{\psi_a}((\tau_{-i\gamma} + \tau_{i\gamma})^n\phi)(b)\right\rangle \\ &= \left\langle (\tau_{-i\gamma} + \tau_{i\gamma})^n g, \phi\right\rangle. \end{aligned}$$

This completes the proof of the theorem.

Now we establish inversion formula for the distributional wavelet transform on the space of convolutors.

Theorem 2 *Assume that* $(W_{\psi a}T)(b)$ *be the wavelet transform of* $T \in O'_{\mathcal{G},m,c}$. *Then*

$$T(x) = \lim_{\substack{r\to\infty \\ n\to\infty}} \frac{1}{C_\psi} \int_0^r \int_{-n}^n (W_{\psi_a}T)(b)\psi_{b,a}(x)\frac{dbda}{a^2}.$$

Proof We consider

$$H_{r,n}(x) = \frac{1}{C_\psi} \int_0^r \int_{-n}^n (W_{\psi_a} T)(b) \psi_{b,a}(x) \frac{dbda}{a^2}.$$

Then

$$\begin{aligned}
\langle H_{r,n}, \phi \rangle &= \int_{-\infty}^{\infty} H_{r,n}(x)\phi(x)dx \\
&= \frac{1}{C_\psi} \int_{-\infty}^{\infty} \phi(x) \int_0^r \int_{-n}^n (W_{\psi_a} T)(b) \psi_{b,a}(x) \frac{dbda}{a^2} dx \\
&= \frac{1}{C_\psi} \int_{-\infty}^{\infty} \phi(x) \int_0^r \int_{-n}^n \psi_{b,a}(x) \int_{-\infty}^{\infty} e^{m|\mathrm{Re} w|} \psi_a(-w-x) d\mu(w) \frac{dbda}{a^2} dx \\
&= \frac{1}{C_\psi} \int_{-\infty}^{\infty} e^{m|\mathrm{Re} w|} \left(\int_0^r \int_{-n}^n (W_{\psi_a} \phi)(b) \psi_a(-w-x) \frac{dbda}{a^2} \right) d\mu(w) \\
&= \frac{1}{C_\psi} e^{m|\mathrm{Re} w|} \left(\int_0^\infty \int_{-\infty}^\infty (W_{\psi_a} \phi)(b) \psi_a(-w-x) \frac{dbda}{a^2} \right) d\mu(w) \\
&- \frac{1}{C_\psi} e^{m|\mathrm{Re} w|} \left(\int_{a>r} \int_{|b|>n} (W_{\psi_a} \phi)(b) \psi_a(-w-x) \frac{dbda}{a^2} \right) d\mu(w) \\
&= \int e^{m|\mathrm{Re} w|} \phi(w) d\mu(w) \\
&- \frac{1}{C_\psi} \int e^{m|\mathrm{Re} w|} \left(\int_{a>r} \int_{|b|>n} (W_{\psi_a} \phi)(b) \psi_a(-w-x) \frac{dbda}{a^2} \right) d\mu(w) \\
&= I_1 - I_2.
\end{aligned}$$

As $r \to \infty$ and $n \to \infty$, I_2 converges to zero. Further, by [1, Lemma 2.3]

$$\langle T, \phi \rangle = \int e^{m|\mathrm{Re} w|} \phi(w) d\mu(w) = I_1.$$

Therefore,

$$\lim_{r\to\infty, n\to\infty} H_{r,n}(x) = T(x).$$

This completes the proof of the theorem.

Corollary 1 *Let $m \in \mathbb{Z}^-$, $m < 0$ and $T \in O'_{\mathcal{G},m,c}$. If $(W_{\psi_a}T)(b) = 0$ for $b \in \mathbb{C}$ and $a \in \mathbb{R}$, then $T = 0$.*

Example 1. Consider the Mexican hat wavelet defined as the second derivative of Gaussian function [9] by

$$\psi(t) = (1 - t^2)e^{(\frac{-t^2}{2})} = \frac{d^2}{dt^2}e^{(\frac{-t^2}{2})}.$$

Then, we have

$$\begin{aligned}\langle (W_{\psi_a}T)(b), \phi\rangle &= \langle W_{\psi_a}T(b), \phi\rangle \\ &= \langle T(b), (W_{\psi_a}\phi)(-b)\rangle \\ &= \int_{\mathbb{R}} e^{-m|\mathrm{Re}b|}(W_{\psi_a}\phi)(-b)d\mu(b) \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} e^{-m|\mathrm{Re}b|} \int_{\mathbb{R}} \phi(x) D_x^2 e^{(\frac{(-b-x)^2}{2a^2})} dx d\mu(b) \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} \phi(x) \int_{\mathbb{R}} e^{-m|\mathrm{Re}b|} D_x^2 e^{(\frac{(-b-x)^2}{2a^2})} dx d\mu(b).\end{aligned}$$

Thus, we get

$$\begin{aligned}(W_{\psi_a}T)(x) &= a^{\frac{3}{2}} \int_{\mathbb{R}} e^{-m|\mathrm{Re}b|} D_x^2 e^{(\frac{(-b-x)^2}{2a^2})} d\mu(b) \\ &= \int_{\mathbb{R}} e^{-m|\mathrm{Re}b|}\psi_a(-b-x)d\mu(b).\end{aligned}$$

Now, by using Theorem 1, we obtain

$$\begin{aligned}e^{-\sigma b}W_{\psi_a}T(b) &= \frac{1}{2\pi}|a|^{-\rho}\int_{\mathbb{R}} e^{-\sigma b}e^{ib\omega}\hat{T}(\omega)\overline{\hat{\psi}(a\omega)}d\omega \\ &= \frac{1}{2\pi}|a|^{-\rho}\int_{\mathbb{R}} e^{i(i\sigma+\omega)b}\hat{T}(\omega)\overline{\hat{\psi}(a\omega)}d\omega \\ &= \frac{1}{2\pi}|a|^{-\rho}\int_{\mathbb{R}} e^{i\xi b}\hat{T}(\xi - i\sigma)\overline{\hat{\psi}(a(\xi - i\sigma))}d\xi \\ &= W_{\psi_a}(\tau_{-i\sigma}T)(b),\end{aligned}$$

which is the corresponding structure formula.

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3 Calderón's formula

In this section, we obtain Calderóns reproducing formula on the space of convolutors by using the following theorem [10].

Theorem 3 *Suppose* $0 < \varepsilon < \delta < \infty$*, and* ψ *be a real valued radial satisfies* $\int\limits_0^\infty [\hat{\psi}(t\xi)]^2 \frac{dt}{t} = 1$*. If*

$$f_{\varepsilon,\delta}(x) = \int\limits_{\varepsilon}^{\delta} (f * \psi_a * \psi_a)(x) \frac{da}{a},$$

then $\| f - f_{\varepsilon,\delta} \|_2 \to 0$ *as* $\varepsilon \to 0$ *and* $\delta \to \infty$*.*

Now we use the structure formula given in condition (iii) of Theorem 1 for the space $\mathcal{G}'$ to derive the following result.

Theorem 4 *Let* $\psi \in \mathcal{G}$ *satisfies the conditions of Theorem 3. Then the following Calderón's formula holds:*

$$T(x) = \int\limits_0^\infty (T * \psi_a * \psi_a)(x) \frac{da}{a}, \ \forall\, T \in \mathcal{G}'.$$

Proof Let $T \in \mathcal{G}'$. Then by Theorem 1,

$T = (\tau_{d_i} + \tau_{-d_i})^n f$, for an analytic function f on the strip $\{z \in \mathbb{C} : |Imz| < k\}$.

Also, let $\phi \in \mathcal{G}$. Hence,

$$\begin{aligned}
\int\limits_{\varepsilon}^{R} \left\langle (T * \psi_a * \psi_a) \frac{da}{a}, \phi \right\rangle &= \int\limits_{\varepsilon}^{R} \left\langle ((\tau_{d_i} + \tau_{-d_i})^n f * \psi_a * \psi_a) \frac{da}{a}, \phi \right\rangle \\
&= \left\langle \int\limits_{\varepsilon}^{R} (f * \psi_a * \psi_a) \frac{da}{a}, (\tau_{d_i} + \tau_{-d_i})^n \phi \right\rangle . \qquad (11)
\end{aligned}$$

$$\begin{aligned}
\left\langle \lim_{\varepsilon \to 0, \delta \to \infty} \int\limits_{\varepsilon}^{R} (f * \psi_a * \psi_a) \frac{da}{a}, (\tau_{d_i} + \tau_{-d_i})^n \phi \right\rangle &= \langle f, (\tau_{d_i} + \tau_{-d_i})^n \phi \rangle \\
&= \langle (\tau_{d_i} + \tau_{-d_i})^n f, \phi \rangle \\
&= \langle T, \phi \rangle . \qquad (12)
\end{aligned}$$

Then from (11) and (12), we obtain

$$\int\limits_0^\infty \left\langle (T * \psi_a * \psi_a) \frac{da}{a}, \phi \right\rangle = \langle T, \phi \rangle .$$

This completes the proof of the theorem.

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