Existence and regularity of spacelike hypersurfaces for mean curvature equation in the static spacetime*

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Abstract

Consider the following mean curvature equation in the static spacetime

$$\operatorname{div}\left(\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right)+\frac{\nabla u\nabla f(x)}{\sqrt{1-f^2(x)|\nabla u|^2}}=\lambda NH$$

with Dirichlet boundary condition on a bounded domain. We investigate the existence and uniqueness of classical solution. By variational method, we also establish the multiplicity of strong solutions. Moreover, according to the behavior of H near 0, we obtain the global structure of positive solutions for this problem. The symmetry of positive solutions is also investigated.

Keywords: Bifurcation; Mean curvature operator; Static spacetime; Variational method; Positive solutions; Uniqueness; Regularity; Symmetry

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Contents

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1 Introduction and main results

Let $\Omega \subseteq \mathbb{R}^N$ with $N \geq 1$ be a domain and f be a smooth positive function on $\overline{\Omega}$. Denote by $\mathcal{M}$ the $N+1$-dimensional product manifold $I \times \Omega$ endowed the Lorentzian metric

$$g = -f^2(x)dt^2 + dx^2.$$

From Lemma 12.37 of [22], we know that $\mathcal{M}$ is *static* relative to ∂_t/f.

For any $u \in C^2(\Omega)$, let $M = \{(x,u) : x \in \Omega, u \in C^2(\Omega)\}$. A spacetime M is static relative to an observer field U provided U is irrotational and there is a smooth positive function such that fU is a Killing vector field. Then $(M, g) := \mathscr{M}$ is an N-dimensional hypersurface in $\mathcal{M}$ at time t which can be represented by the graph of $t = u$. $\mathscr{M}$ is called *spacelike* if $f|\nabla u| < 1$ in Ω (see [16]). While, $\mathscr{M}$ is called *weakly spacelike* if $f|\nabla u| \leq 1$ a.e. in Ω. Given mean curvature H for spacelike graph $\mathscr{M}$, we shall derive the following mean curvature equation in Section 2

$$\operatorname{div}\left(\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) + \frac{\nabla u \nabla f(x)}{\sqrt{1-f^2(x)|\nabla u|^2}} = NH. \tag{1.1}$$

When $f \equiv 1$, $H \equiv 0$ and $\Omega = \mathbb{R}^N$, Calabi [5] proved that equation (1.1) has only linear entire solutions for $N \leq 4$. Cheng and Yao [7] improved Calabi's result for all N. When $f \equiv 1$ and H is a positive constant, some celebrated results for equation (1.1) were obtained by Treibergs [26]. If $f \equiv 1$, Ω is a bounded domain and H is a bounded function defined on $\Omega \times \mathbb{R}$, Bartnik and Simon [1] proved that the equation (1.1) with Dirichlet boundary condition has a spacelike solution. By critical point theory or topological degree, the authors of [4, 8] studied the nonexistence, existence and multiplicity of positive solutions in the case of $f \equiv 1$ and Ω being a bounded domain. When $f \equiv 1$ and $\Omega = B_R = B_R(0) := \{x \in \mathbb{R}^N : |x| < R\}$ with $R > 0$, the authors of [2, 3] obtained some existence results for positive radial solutions of equation (1.1) with $u = 0$ on $\partial\Omega$. When $f \equiv 1$, in [9], we studied the nonexistence, existence and multiplicity of positive radial solutions of equation (1.1) with $u = 0$ on $\partial\Omega$ and $NH = -\lambda f(x, s)$ on the unit ball via bifurcation method, which had been extended to the general domain in [12, 13]. Recently, we studied equation (1.1) on a ball.

The *aim* of this paper is to investigate the existence, regularity, uniqueness, symmetry and multiplicity of spacelike solutions for equation (1.1) on general bounded domain.

By equation (1.1) we have that

$$\begin{aligned} -\operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) &= -\operatorname{div}\left(f(x)\cdot\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) \\ &= -f(x)\operatorname{div}\left(\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) - f(x)\frac{\nabla u \nabla f(x)}{\sqrt{1-f^2(x)|\nabla u|^2}} \\ &= -Nf(x)H. \end{aligned}$$

When Ω is bounded, we consider the following 0-Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) = -\lambda N f(x) H(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where λ is a nonnegative parameter which can represent the strength of mean curvature function. Let d denote the diameter of Ω. For any $x \in \Omega$, clearly, we can choose fixed $y \in \partial\Omega$ such that $|x-y| \leq d/2$. For any solution $u \in C^1\left(\overline{\Omega}\right)$, since the graph associated to u is spacelike, we have that

$$|u(x)| = |u(x)-u(y)| = \left|f(\xi)\nabla u(\xi)\frac{(x-y)}{f(\xi)}\right| \leq \left\|f(\xi)\nabla u(\xi)\frac{(x-y)}{f(\xi)}\right\|_\infty \leq d/\left(2f_0\right) := \delta$$

for some $\xi \in \Omega$, where $\|\cdot\|_\infty$ denotes the usual sup-norm on $\overline{\Omega}$ and $f_0 = \min_{\overline{\Omega}} f(x)$. It follows that the image of u lies in $[-\delta, \delta] = I_\delta$. Thus, we assume that H is a real function defined on $\Omega \times I_\delta$.

For convenience, u is called (classical) solution if it belongs to $C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ and satisfies problem (1.2). We first have the following *existence and uniqueness* of classical solution.

Theorem 1.1. *Assume that Ω has $C^{2,\alpha}$ boundary $\partial\Omega$ for some $\alpha \in (0,1)$. If $H \in C^{0,\alpha}\left(\overline{\Omega} \times I_\delta\right)$, problem (1.2) with $\lambda = 1$ has at least one spacetime solution $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$ such that $\max_{\overline{\Omega}}\left(f(x)|\nabla u|\right) \leq 1-\theta$ for some positive constant θ, which only depends on N, f, Ω and $\sup_{\overline{\Omega}\times I_\delta} f(x)|H(x,t)|$. Moreover, the solution is unique if $H(x,t)$ is increasing with respect to t.*

When $f(x) \equiv 1$, Theorem 1.1 is just the Theorem 3.6 of [1] with $\varphi \equiv 0$ on $\partial\Omega$. As we have pointed out in [13] that the proof of [1, Proposition 1.1] contains a gap. In [13], we have shown that the uniqueness is right when H is independent on t. Here, we further show that the uniqueness is also holding when H is increasing with respect to t. So, we complement the arguments of [1, Proposition 1.1 and Theorem 3.6] even in the case of $f(x) \equiv 1$.

The **natural question** is whether there exist multiple solutions of problem (1.2). We will use variational method to give a confirmed answer for this question. To show this, we state the following assumption on H:

(H1) $H : \Omega \times [0, \delta] \to \mathbb{R}$ satisfies the Carathéodory condition and the L^∞-growth condition

$$H(x,t) \leq h(x) \text{ for a.e. } x \in \Omega, \forall t \in [0,\delta] \tag{1.3}$$

for some function $h \in L^\infty(\Omega)$.

Following the terminology of [8], a function $u \in W^{2,p}(\Omega)$ for some $p > N$ with $\|f(x)\nabla u\|_\infty < 1$ satisfying problem (1.2) a.e. in Ω is called a *strong* (spacelike) solution. Then, we have the following *multiplicity* of strong spacelike solutions.

Theorem 1.2. *Besides (H1), assume that Ω has C^2 boundary $\partial\Omega$ and $H(x,t) < 0$*

for a.e. $x \in \Omega$ and $\forall t \in (0,R)$ with any fixed $R \in (0,\delta)$ such that

$$\lim_{t \to 0^+} \frac{H(x,t)}{t} = 0 \text{ uniformly with a.e. } x \in \Omega. \tag{1.4}$$

Then, there exists $\lambda_* > 0$ such that problem (1.2) has at least two nontrivial nonnegative strong spacelike solutions for any $\lambda > \lambda_*$.

Furthermore, if Ω has $C^{2,\alpha}$ boundary and $H \in C^{0,\alpha}\left(\overline{\Omega} \times [0,\delta]\right)$, following Theorem 1.1, the nontrivial nonnegative strong spacelike solutions obtained in Theorem 1.2 are also belonging to $C^{2,\alpha}\left(\overline{\Omega}\right)$.

Now, **two natural questions are in order:**

*Q*1. (*Global structure of solutions*) If we patch those solutions obtained in Theorem 1.2 together, what does it look like?

*Q*2. (*Positive solutions*) Whether those solutions obtained in Theorem 1.2 are positive?

We will use bifurcation method to answer *Q*1. While, *Q*2 is a directly corollary of the strong maximum principle (see Lemma 4.1).

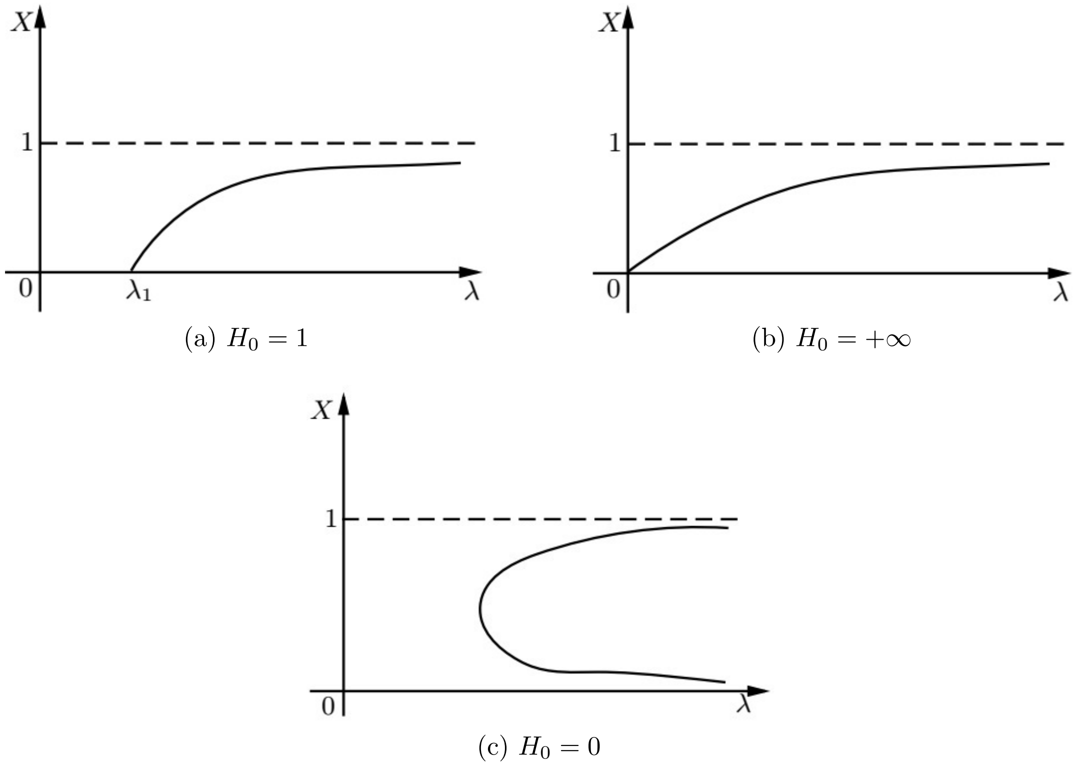

Figure 1: Bifurcation diagrams of Theorem 1.3.

Let λ_1 be the first eigenvalue of

$$\begin{cases} -\operatorname{div}\left(f^2(x)\nabla u\right) = \lambda f(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

It is well known that λ_1 is simple, isolated and the unique principal eigenvalue.

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Let

$$X = \left\{ u \in C^1\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega \right\}$$

with the norm $\|u\| := \|f(x)\nabla u\|_\infty$. It is not difficult to verify that $\|u\|_\infty \leq \left(d \|\nabla u\|_\infty\right)/2$ and $\|u\|/f_M \leq \|\nabla u\|_\infty \leq \|u\|/f_0$, where $f_M = \max_{\overline{\Omega}} f(x)$. It follows that the norm $\|u\|$ is equivalent to the usual norm $\|u\|_\infty + \|\nabla u\|_\infty$. Let $P = \{u \in X : u > 0 \text{ on } \Omega\}$. From now on, following [24], we add the point ∞ to our space $\mathbb{R} \times X$.

Then, we have the following theorem, which is also one of the *main results* of this paper.

Theorem 1.3. *Assume that Ω has $C^{2,\alpha}$ boundary, $H \in C^{0,\alpha}\left(\overline{\Omega} \times [0, \delta]\right)$ such that $H(x, t) < 0$ for any $x \in \overline{\Omega}$ and $t \in (0, \delta]$, and there exists $H_0 \in [0, +\infty]$ such that*

$$\lim_{t \to 0^+} \frac{NH(x,t)}{t} = -H_0$$

uniformly for $x \in \Omega$. Then,

(a) *if $H_0 = 1$, there is an unbounded component $\mathscr{C}$ of the set of nontrivial solutions of problem (1.2) bifurcating from $(\lambda_1, 0)$ such that $\mathscr{C} \subseteq (\mathbb{R} \times P) \cup \{(\lambda_1, 0)\}$, $(\lambda_1, +\infty) \subseteq pr_{\mathbb{R}}(\mathscr{C})$, $\|u_\lambda\| < 1$ and $\lim_{\lambda \to +\infty} \|u_\lambda\| = 1$ for $(\lambda, u_\lambda) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$, where $pr_{\mathbb{R}}(\mathscr{C})$ denotes the projection of $\mathscr{C}$ on $\mathbb{R}$,*

(b) *if $H_0 = +\infty$, there is an unbounded component $\mathscr{C}$ of the set of nontrivial solutions of problem (1.2) emanating from $(0, 0)$ such that $\mathscr{C} \subseteq (\mathbb{R} \times P) \cup \{(0, 0)\}$, joins to $(+\infty, 1)$ and $\|u_\lambda\| < 1$ for $(\lambda, u_\lambda) \in \mathscr{C} \setminus \{(0, 0)\}$,*

(c) *if $H_0 = 0$, there is an unbounded component $\mathscr{C}$ of the set of nontrivial solutions of problem (1.2) such that $\mathscr{C} \subseteq \mathbb{R} \times P$, joins $(+\infty, 1)$ to $(+\infty, 0)$ and $\|u_\lambda\| < 1$ for any $(\lambda, u_\lambda) \in \mathscr{C}$ with $\lambda < +\infty$.*

Figure 1 illustrates the global bifurcation branches of Theorem 1.3. The existence or multiplicity of positive solutions of problem (1.2) can be easily derived from these diagrams. Again using Theorem 1.1, we see that these positive solutions are also belonging to $C^{2,\alpha}\left(\overline{\Omega}\right)$.

Finally, we consider equation (1.1) with more general Dirichlet boundary condition

$$\begin{cases} \operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) = \lambda N f(x) H(x, u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\varphi : \partial\Omega \to \mathbb{R}$ is a given function whose values will be taken in the sense of (1.1) of [1].

Assume that $H : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the corresponding assumptions of Theorem 1.1, φ is bounded and has an extension $\overline{\varphi} \in C^{2,\alpha}\left(\overline{\Omega}\right)$ $\left(\overline{\varphi} \in C^1\left(\overline{\Omega}\right)\right)$ satisfying $f\left|\nabla\overline{\varphi}\right| \leq 1 - \theta_0$ in $\overline{\Omega}$, for some $\theta_0 > 0$. Repeating the argument with obvious changes, we can see that the conclusion of Theorem 1.1 is also valid for problem (1.6). In this case, θ is also dependent on θ_0.

By Lemma 3.3 (clearly, which is valid for problem (1.6)), if $H(x, t)$ is increasing with respect to t, classical solution is also the ground state solution of problem (1.6). Conversely, the natural question is whether the ground state solution is also the classical solution of problem (1.6)? To answer it, we need the following "anti-peeling" theorem, which extends the corresponding result of [1, Theorem 3.2].

Theorem 1.4. *Let u_0 be the ground state solution of problem (1.6) with a.e. bounded H. If there is a line segment $\overline{x_0x_1} \subset\subset \Omega$ such that*

$$f\left(x^s\right)\left|\nabla u_0\left(x^s\right)\right| = 1, \forall s \in [0,1], \tag{1.7}$$

where $x^s = x_0 + s\left(x_1 - x_0\right)$, then this equality holds for all $s \in \mathbb{R}$ such that $x^s \in \Omega$ and $\overline{x_0x^s} \subset\subset \Omega$.

On the basis of Theorem 1.4, we have the following regularity.

Theorem 1.5. *Suppose that φ is bounded and has an weakly spacelike extension $\psi : \Omega \to \mathbb{R}$ with $\psi = \varphi$ on $\partial\Omega$. Define the singular set*

$$K = \left\{\overline{xy} : x, y \in \partial\Omega, x \neq y, \overline{xy} \subset \Omega \text{ and } |\widetilde{\varphi}(x) - \widetilde{\varphi}(y)| = |x-y|\right\},$$

where $\widetilde{\varphi}$ is determined by $f\nabla\varphi = \nabla\widetilde{\varphi}$. Let H be a given function, measurable on $\Omega \times \mathbb{R}$ and continuous in the $\mathbb{R}$-component, with $\sup_{\Omega\times\mathbb{R}} |H| \leq \Lambda < +\infty$ and $H \in C^{0,\alpha}\left((\Omega \setminus K) \times \mathbb{R}\right)$. Then any ground state solution u of problem (1.6) is strictly spacelike on $\Omega \setminus K$ and satisfies the first equation of problem (1.6) on $\Omega \setminus K$. Furthermore, $f|\nabla u| \equiv 1$ on $\overline{xy}$, where $\overline{xy} \in K$.

The rest of this paper is arranged as follows. In Section 2, we derive the equation (1.1). In Section 3, we study the uniqueness of solution and present the proof of Theorem 1.1. The proof of Theorem 1.2 will be given in Section 4. Section 5 is mainly concerns the proof of Theorem 1.3. In Section 6, we give the proofs of Theorems 1.4–1.5. In the last Section, we show a result concerning the radial symmetry of positive solutions when Ω is the unit ball.

2 Formulation of equation (1.1)

We denote Minkowski space by $\mathbb{L}^{N+1} := \left\{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\right\}$, with the flat metric $\sum_{i=1}^{N} dx_i^2 - dt^2$. Let e_i, $i = 1, \ldots, N$, denote the natural basis of $\mathbb{R}^N$. Choose e_{N+1} such that

$$\langle e_{N+1}, e_i\rangle = \begin{cases} -1 & \text{if } i = N+1, \\ 0 & \text{if } i \in \{1, \ldots, N\}, \end{cases}$$

where $\langle\cdot,\cdot\rangle$ denotes the flat metric of $\mathbb{L}^{N+1}$. Then, we see that

$$e_1, \ldots, e_N, fe_{N+1}$$

are the natural basis of $\mathcal{M}$. Let

$$g^* = -dt^2 + \frac{dx^2}{f^2}.$$

Clearly, $g^* = g/f^2$ is a conformal metric of g. Correspondingly, we use $\mathcal{M}^*$ to denote the $N+1$-dimensional product manifold $I \times \Omega$ with the new Lorentzian metric g^*. Clearly,

$$e_1/f, \ldots, e_N/f, e_{N+1}$$

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are the natural basis of $\mathcal{M}^*$. From now on, we use a superscript * the geometric quantities to indicate which are related to the metric g^*.

Assume that $u \in C^2(\Omega)$ and let $M = \{(x, u(x)) : x \in \Omega\}$. Then we have coordinate tangent vectors $X_i = e_i + f u_i e_{N+1}$ and $X_i^* = e_i/f + u_i e_{N+1}$, where $u_i = \nabla_i u = \partial u/\partial x_i$, $i = 1, \ldots, N$. Then, the induced metric on M under the metric g is

$$g_{ij} = \langle X_i, X_j \rangle = \delta_{ij} - f^2 u_i u_j, \ i, j \in \{1, \ldots, N\},$$

where $\delta_{ij} = 1$ (0) if $j = i$ ($j \neq i$). Similarly, we also have that

$$g_{ij}^* = \left\langle X_i^*, X_j^* \right\rangle = \frac{1}{f^2} \langle X_i, X_j \rangle = \frac{1}{f^2} g_{ij}.$$

It follows that the inverse matrix of g_{ij}^* is $f^2 g^{ij} := \left(g_{ij}^*\right)^{-1}$, where (g^{ij}) is the inverse matrix of (g_{ij}).

Let υ be the upward normal to M under the metric g, normalized by $\langle \upsilon, \upsilon \rangle = -1$,

$$\upsilon = \frac{(f^2 \nabla u, 1)}{f\sqrt{1 - f^2 |\nabla u|^2}}.$$

Similarly, we choose

$$\upsilon^* = \frac{(f^2 \nabla u, 1)}{\sqrt{1 - f^2 |\nabla u|^2}}$$

be the upward normal to M under the metric g^*. Obviously, one has that $\upsilon^* = f\upsilon$.

It is well known that the second fundamental form can be calculated by

$$A_{ij} = \left\langle X_i, \overline{\nabla}_{X_j} \upsilon \right\rangle, \ A_{ij}^* = \left\langle X_i^*, \overline{\nabla}^*_{X_j^*} \upsilon^* \right\rangle,$$

where $\overline{\nabla}$ and $\overline{\nabla}^*$ are the Levi-Civita connection under g and g^*, respectively. For any point $p := (x, u(x)) \in M$, we have that $p = \sum_{i=1}^N x_i e_i / f(x) + u e_{N+1} = (x/f, u) := (x^*, u(x))$ under the metric g^*. Then, we have that

$$\nabla_i^* u = \nabla_i u \frac{\partial x_i}{\partial x_i^*} = \nabla_i u(x) f(x),$$

where $\nabla_i u(x) = \partial u(x)/\partial x_i$, $\nabla_i^* u(x) = \partial u(x)/\partial x_i^*$. It follows that $\overline{\nabla}^* = f\overline{\nabla}$, $\nabla^* = f\nabla$ and $\mathrm{div}^* = f\mathrm{div}$, where ∇^* means the g^*-gradient and div^* represents the divergence operator corresponding to g^*. Then, using Proposition 7.35 of [22], we have that

$$\begin{aligned} A_{ij}^* &= \left\langle X_i^*, \overline{\nabla}^*_{X_j^*} \upsilon^* \right\rangle = \left\langle X_i, \overline{\nabla}_{X_j^*} (f\upsilon) \right\rangle = f \left\langle X_i, \overline{\nabla}_{X_j^*} \upsilon \right\rangle + \left\langle X_i, \upsilon \overline{\nabla}_{X_j^*} f \right\rangle \\ &= f \left\langle X_i, \overline{\nabla}_{X_j^*} \upsilon \right\rangle = f \left\langle X_i, \overline{\nabla}_{X_j/f} \upsilon \right\rangle = \left\langle X_i, \overline{\nabla}_{X_j} \upsilon \right\rangle \\ &= A_{ij}. \end{aligned}$$

The mean curvature of M associated to g is

$$H = \frac{\sum_{i,j=1}^N g^{ij} A_{ij}}{N}.$$

Then, related to g^*, we have that

$$NH^* = \sum_{i,j=1}^{N} \left(g^*_{ij}\right)^{-1} A^*_{ij} = f^2 \sum_{i,j=1}^{N} g^{ij} A_{ij} = N f^2 H. \tag{2.1}$$

On the other hand, we have known that

$$NH^* = \operatorname{div}^* \left(\frac{\nabla^* u}{\sqrt{1 - |\nabla^* u|^{*2}}} \right),$$

where $|\nabla^* u|^{*2} = g^*\left(\nabla^* u, \nabla^* u\right)$. Noting the fact of $1 - |\nabla^* u|^{*2} = 1/f^2 - |\nabla u|^2$, by some elementary calculations, we obtain that

$$NH^* = \operatorname{div}^* \left(\frac{\nabla^* u}{\sqrt{1 - |\nabla^* u|^{*2}}} \right) = f \operatorname{div} \left(\frac{f^2(x) \nabla u}{\sqrt{1 - f^2(x) |\nabla u|^2}} \right).$$

Therefore, in view of the relation (2.1), we reach that

$$\operatorname{div} \left(\frac{f^2(x) \nabla u}{\sqrt{1 - f^2(x) |\nabla u|^2}} \right) = N f(x) H,$$

which is equivalent to (1.1).

Another way to derive (1.1) is to obtain it as the Euler-Lagrange equation of a variational problem. The metric g can be rewritten by

$$ds^2 = \psi(x) \left(-dt^2 + g_{ij}(x) dx_i dx_j\right), \ i, j \in \{1, \ldots, N\},$$

where $\psi(x) = f^2(x)$ and

$$g_{ij}(t) = \begin{cases} \frac{1}{f^2(x)} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $g^{ij} = \left(g_{ij}\right)^{-1}$. Consider the functional

$$J(\eta) = \int_\Omega \left(1 - |\nabla \eta|^2\right)^{1/2} \widetilde{g}^{1/2}(x) \psi^{n/2}(x)\, dx + N \int_\Omega \left(\int_0^\eta H(x,t) \psi^{1/2}(x)\, dt \right) \widetilde{g}^{1/2}(x)\, dx$$

in

$$\left\{ \eta \in H^{1,\infty}(\Omega) : \eta = 0 \text{ on } \partial\Omega \text{ and } |\nabla \eta| \leq 1 \text{ a.e. in } \Omega \right\},$$

where

$$|\nabla \eta|^2 = \sum_{i,j=1}^{N} g^{ij}(x) \nabla_i \eta \nabla_j \eta$$

and

$$\widetilde{g} = \det\left(g_{ij}\right) = \frac{1}{f^{2N}(x)}.$$

Next we use the summation convention that repeats indices from 1 to N. The corresponding Euler-Lagrange equation for a spacelike solution u looks like

$$\frac{1}{\widetilde{g}^{1/2}} \frac{\partial}{\partial x^i} \left(\widetilde{g}^{1/2} \upsilon g^{ij} D_j u \right) - \frac{\upsilon}{2} \frac{\partial g^{ij}}{\partial t} D_i u D_j u + \frac{g^{ij}}{2\upsilon} \frac{\partial g_{ij}}{\partial t} + \frac{N}{2\upsilon} \frac{\partial}{\partial t} \ln \psi + \frac{N}{2} \upsilon D_i \ln \psi g^{ij} D_j u = N \sqrt{\psi} H, \tag{2.2}$$

where $D_i u = \partial u/\partial x^i$ and

$$\upsilon = \left(1 - \sum_{i,j=1}^{N} g^{ij}(x) D_i u \cdot D_j u\right)^{-1/2} = \left(1 - f^2(x)|\nabla u|^2\right)^{-1/2}.$$

From (2.2), we have that

$$\begin{aligned} NfH &= f^N \operatorname{div}\left(\frac{1}{f^N}\frac{f^2\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right) + \frac{Nf\nabla f \nabla u}{\sqrt{1-f^2|\nabla u|^2}} \\ &= \operatorname{div}\left(\frac{f^2\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right) - \frac{Nf\nabla f\nabla u}{\sqrt{1-f^2|\nabla u|^2}} + \frac{Nf\nabla f\nabla u}{\sqrt{1-f^2|\nabla u|^2}} \\ &= \operatorname{div}\left(\frac{f^2\nabla u}{\sqrt{1-f^2|\nabla u|^2}}\right). \end{aligned}$$

We would like to point out that equation (2.2) has been derived in [17]. While, there lost the term $\psi^{1/2}$ in functional J. The last g^{ij} on the left of (2.2) is also lost in (1.24) of [17]. So, we give the detailed derivation process here. Clearly, this small gap does not affect other arguments and results of [17]. In particular, the global gradient estimates [17, Theorem 2.1 and Theorem 4.1] is valid to problem (1.2).

Define the connected Lorentz ball by

$$K_R(\xi) = \left\{x \in \Omega : \overline{x\xi} \subset \Omega, l(x,\xi) < R\right\} \subset \Omega, \tag{2.3}$$

where $\overline{x\xi}$ denotes the line segment joining x and ξ, $l(x,\xi) = \sqrt{|x-\xi|^2 - |u(x) - u(\xi)|^2}$. By Lemma 2.1 of [1] and the global gradient estimates [17, Theorem 2.1 and Theorem 4.1], we can easily arrive the following estimate.

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded nonempty domain. Assume $\sup_{\Omega\times I_\delta} |H(x,t)| \leq \Lambda < +\infty$. Let $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ be any nontrivial spacelike solution of problem (1.2) with $\lambda = 1$. Let $\xi \in \Omega$ and $R > 0$ be such that $K_{2R}(\xi) \subset\subset \Omega$. Then there are positive constants $\alpha < 1/N$ and C depending only on N, such that*

$$Ce^{\left(N\Lambda + \frac{f^0\gamma^*}{\sqrt{1-(\gamma^*)^2}}\right)^2 R^2+1} \upsilon^\alpha(\xi) \geq R^{-N}\int_{K_R(\xi)} \upsilon^{\alpha+1}\,dx + R^{2-N}\int_{K_R(\xi)} \sum \left(f_j u_i + f u_{ij}\right)^2\,dx,$$

where $\gamma^* = 1 - \theta$ and $\upsilon = \sqrt{1 - f^2(x)|\nabla u|^2}$.

3 Uniqueness

In this section, we always assume that $\lambda = 1$. Recall that $C^{0,1}(\Omega)$ is the class of locally Lipschitz functions on Ω. Let

$$\mathscr{S} = \left\{w \in C^{0,1}(\Omega) : w = 0 \text{ on } \partial\Omega \text{ and } f|\nabla w| \leq 1 \text{ a.e. in } \Omega\right\}.$$

Define the energy functional $I : \mathscr{S} \to \mathbb{R}$ by

$$I(u) = \int_\Omega \left(1 - \sqrt{1 - f^2(x)|\nabla u|^2}\right)\,dx + N\int_\Omega \left(f(x)\int_0^u H(x,t)\,dt\right)\,dx.$$

For any $w \in \mathscr{S}$, as in the Introduction, we can see that $|w(x)| \leq \delta$ for a.e. $x \in \Omega$. *In this section, we always assume that $H(x,t) : \Omega \times I_\delta \to \mathbb{R}$ satisfies the L^∞-growth condition (1.3).* So, I is uniformly bounded on $\mathscr{S}$. The equicontinuity of $\mathscr{S}$ gives a uniformly convergent minimizing sequence $u_n \rightrightarrows u \in \mathscr{S}$ as $n \to +\infty$. Since $\sqrt{1-f^2(x)p^2}$ is concave with respect to p, a semi-continuity theorem of [21, Theorem 1.8.1] shows that

$$I(u) \leq \liminf_{n\to+\infty} I\left(u_n\right).$$

It follows that u is the *ground state* (least energy) solution of problem (1.2). Set

$$A(u) := \int_\Omega \sqrt{1-f^2(x)|\nabla u|^2}\, dx - N\int_\Omega \left(f(x)\int_0^u H(x,t)\, dt\right) dx.$$

Clearly, u is the ground state solution of problem (1.2) if and only if it is a maximizer of A on $\mathscr{S}$.

Moreover, we have the following *uniqueness of the ground state solution*.

Lemma 3.1. *The ground state solution of problem (1.2) is unique if $H(x,s) \equiv H(x) \in L^\infty(\Omega)$, which is denoted by $\Psi(H)$.*

Proof. Suppose u, w are two ground state solutions of problem (1.2). By an argument similar to that of [1, Proposition 1.1], we can obtain that

$$\begin{aligned}\int_\Omega \left(\sqrt{1-f^2\left|\nabla u_t\right|^2} - u_t N f H(x)\right) dx &= (1-t)\int_\Omega \left(\sqrt{1-f^2\left|\nabla u\right|^2} - NufH(x)\right) dx \\ &\quad + t\int_\Omega \left(\sqrt{1-f^2\left|\nabla w\right|^2} - NwfH(x)\right) dx,\end{aligned}$$

where $u_t = u + t(w-u)$. It follows that

$$\int_\Omega \sqrt{1-f^2(x)\left|\nabla u_t\right|^2}\, dx = (1-t)\int_\Omega \sqrt{1-f^2(x)\left|\nabla u\right|^2}\, dx + t\int_\Omega \sqrt{1-f^2(x)\left|\nabla w\right|^2}\, dx.$$

Then, the concavity of $\sqrt{1-f^2(x)p^2}$ and $u = w$ on $\partial\Omega$ imply that $u = w$ in Ω. ∎

Furthermore, by Lemma 3.1 and an argument similar to that of [1, Lemma 1.2] with obvious changes, we have the following comparison principle for the ground state solution.

Lemma 3.2. *Assume that u_i, $i = 1, 2$, is the ground state solution of problem (1.2) with $H_i \in L^\infty(\Omega)$ and $H_1(x) \leq H_2(x)$ for a.e. $x \in \Omega$. Then, $u_2 \leq u_1$ in Ω.*

In addition, we have the following uniqueness for monotonous curvature function.

Proposition 3.1. *The ground state solution of problem (1.2) in $\mathscr{S}$ is unique if $H(x,t)$ is increasing with respect to t.*

Proof. Let u, w be any two ground state solutions of problem (1.2) in $\mathscr{S}$. By Lemmas 3.1–3.2, we have that

$$0 \leq (u-w)^2 = (\Psi(H(x,u)) - \Psi(H(x,w)))(u-w) \leq 0.$$

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It follows that $u = w$ in Ω. ■

If H is increasing with respect to t, the following lemma roughly says that the classical solution is also the ground state solution of problem (1.2).

Lemma 3.3. *If $\partial\Omega \in C^2$, $H(x,t)$ is increasing with respect to t, any spacelike solution $u \in C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$ is also the ground state solution of problem (1.2).*

Proof. For any $v \in \mathscr{S}$, by the concavity of $\sqrt{1-f^2(x)p^2}$ with respect to p, we have that

$$\int_\Omega \sqrt{1-f^2(x)|\nabla v|^2}\,dx - \int_\Omega \sqrt{1-f^2(x)|\nabla u|^2}\,dx \le \int_\Omega \frac{f^2(x)\nabla u\nabla(u-v)}{\sqrt{1-f^2(x)|\nabla u|^2}}\,dx.$$

Multiplying problem (1.2) by $v-u$ and integrating over Ω, we obtain that

$$\int_\Omega \operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right)(v-u)\,dx = N\int_\Omega f(x)H(x,u)(v-u)\,dx.$$

In view of [17, Theorem 2.1 and Theorem 4.1], using integration by parts, we have that

$$\int_\Omega \frac{f^2(x)\nabla u(\nabla u-\nabla v)}{\sqrt{1-f^2(x)|\nabla u|^2}}\,dx = N\int_\Omega f(x)H(x,u)(v-u)\,dx.$$

Since $H(x,t)$ is increasing with respect to t, we obtain that

$$\begin{aligned}
\int_\Omega \left(\int_0^v H(x,t)\,dt - \int_0^u H(x,t)\,dt\right)dx &= \int_\Omega\int_u^v H(x,t)\,dt\,dx \\
&= \int_{u\le v}\int_u^v H(x,t)\,dt\,dx \\
&\quad + \int_{u>v}\int_u^v H(x,t)\,dt\,dx \\
&\ge \int_\Omega H(x,u)(v-u)\,dx.
\end{aligned}$$

Therefore, we have that

$$\int_\Omega \sqrt{1-f^2|\nabla v|^2}\,dx - \int_\Omega \sqrt{1-f^2|\nabla u|^2}\,dx \le N\int_\Omega f(x)\left(\int_0^v H(x,t)\,dt - \int_0^u H(x,t)\,dt\right)dx,$$

that is to say

$$\begin{aligned}
\int_\Omega \sqrt{1-f^2(x)|\nabla v|^2}\,dx - N\int_\Omega f(x)\int_0^v H(x,t)\,dt\,dx &\le \int_\Omega \sqrt{1-f^2(x)|\nabla u|^2}\,dx \\
&\quad -N\int_\Omega f(x)\int_0^u H(x,t)\,dt\,dx.
\end{aligned}$$

It follows that u is a maximizer of A on $\mathscr{S}$. So, u is the ground state solution of problem (1.2). ■

Using Lemma 3.3 and reasoning as that of [1, Lemma 1.3], we can obtain the following result, which roughly says that the limit of the classical solutions is the ground state solution of problem (1.2).

Lemma 3.4. *Suppose that $\partial\Omega \in C^2$ and there is a sequence $\{u_k\}_1^\infty$ in $C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$ of spacelike functions with mean curvatures H_k, H_k is measurable on Ω and $\sup_\Omega |H_k| \leq \Lambda < +\infty$, such that $\{u_k\}$ converges uniformly and $\{H_k\}_1^\infty$ converges weakly,*

$$\begin{aligned} &u_k \rightrightarrows u \text{ in } C^0\left(\overline{\Omega}\right), \\ &H_k \rightharpoonup H \text{ in } L^2(\Omega). \end{aligned}$$

Then u is weakly spacelike and is the ground state solution of problem (1.2) with mean curvature H.

We end this section by presenting the proof of Theorem 1.1, which is a modification of the argument given in [1, Theorem 3.6].

Proof of Theorem 1.1. For any $\sigma \in [0, 1]$, we consider the following problem

$$\begin{cases} \operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) = N\sigma f(x)H(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

For every solution $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$ of problem (3.1), from [17, Theorem 2.1 and Theorem 4.1], we get that there exists a positive constant θ depending on N, f, $\sup_{\overline{\Omega}\times I_\delta} f(x)|H(x,t)|$ and Ω such that $\max_{\overline{\Omega}}(f(x)|\nabla u|) \leq 1-\theta$. Applying Theorem 13.7 of [19], we get a priori estimate for $\|u\|_{C^{1,\beta}(\overline{\Omega})}$ with some $\beta \in (0, 1)$. Finally, by Theorem 11.4 of [19], Proposition 3.1 and Lemma 3.3, we obtain the desired conclusions. ∎

4 Existence of solutions via variational method

Define

$$K_0 = \left\{u \in W^{1,\infty}(\Omega) : \|f\nabla u\|_\infty \leq 1, u = 0 \text{ on } \partial\Omega\right\}.$$

Let $\Phi : C\left(\overline{\Omega}\right) :\longrightarrow (-\infty, +\infty]$ be defined by

$$\Phi(u) = \begin{cases} \int_\Omega \left(1 - \sqrt{1-f^2(x)|\nabla u|^2}\right) dx & \text{if } u \in K_0, \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, Φ is convex. Moreover, as that of [4, Lemma 4], we can show that Φ is lower semicontinuous.

In this section, we assume that H satisfies the Carathéodory conditions on $\Omega \times I_\delta$ and the L^∞-growth condition (1.3). So, for any $u \in C\left(\overline{\Omega}\right)$, the Nemytskii operator $\mathcal{N}_H(u) := H(x, u(x))$ is continuous and maps the bounded sets in $C\left(\overline{\Omega}\right)$ into the bounded sets in $L^1(\Omega)$. Clearly, $\mathcal{N}_H(u) \in L^\infty(\Omega)$ for any $u \in C\left(\overline{\Omega}\right)$. Define the functional

$$\mathcal{H}(u) = N\int_\Omega \left(f(x)\int_0^u H(x,t)\,dt\right) dx$$

on $C\left(\overline{\Omega}\right)$. Clearly, $\mathcal{H}$ is C^1 and its derivative is given by

$$\langle \mathcal{H}'(u), v\rangle = N\int_\Omega f(x)\mathcal{N}_H v\,dx$$

for all $u, v \in C\left(\overline{\Omega}\right)$. So, $I = \Phi + \mathcal{H}$ has the structure required by Szulkin's critical point theory [25]. Following the definition of [25], $u \in K_0$ is critical point of I if it satisfies the following variational inequality

$$\Phi(v) - \Phi(u) + \langle \mathcal{H}'(u), v-u\rangle \geq 0 \text{ for all } v \in K_0.$$

According to [25], I is said to satisfy the (PS)-condition if $\{u_n\}$ is a sequence contained in K_0 such that $I\left(u_n\right) \to c \in \mathbb{R}$ and

$$\Phi(v) - \Phi\left(u_n\right) + \left\langle \mathcal{H}'\left(u_n\right), v-u_n\right\rangle \geq -\varepsilon_n \left\|v-u_n\right\|_\infty, \ \forall v \in K_0,$$

where $\varepsilon_n \to 0^+$ as $n \to +\infty$, then $\{u_n\}$ possesses a convergent subsequence. By Lemma 2 of [4] and the positivity of f, we see that K_0 is compact in $C\left(\overline{\Omega}\right)$. Thus, I satisfies the (PS)-condition.

To prove Theorem 1.2, we first prove the following preliminary result.

Lemma 4.1. *Assume that Ω has C^2 boundary $\partial\Omega$ and $h \in L^\infty(\Omega)$. Then, the problem*

$$\begin{cases} -\mathrm{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) = h & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

has a unique solution $u \in W^{2,p}(\Omega)$ for some $p > N$ and there exists a positive constant $\theta = \theta\left(f, \|h\|_\infty, \Omega\right)$ such that $\max_{\overline{\Omega}}\left(f(x)|\nabla u|\right) \leq 1-\theta$. Moreover, if $h \geq 0$ in Ω, then $u \geq 0$ in Ω and u cannot achieve a minimum in Ω unless it is the trivial solution.

Proof. The existence can be obtained from Theorem 5.1 [17]. The uniqueness can be deduced easily from Lemmas 4.1 and 4.3. Finally, if $h \geq 0$ in Ω, by virtue of Theorem 9.1 of [19], we know that $u \geq 0$ in Ω. Further, using Theorem 9.6 of [19], we have that u cannot achieve a minimum in Ω unless it is a constant. ∎

If Ω has C^2 boundary $\partial\Omega$ and $h \in L^\infty(\Omega)$, reasoning as that of Lemma 3.3, we can show that the solution obtained in Lemma 4.1 is the unique ground state solution of problem (4.1) in K_0. Conversely, if Ω has C^2 boundary $\partial\Omega$ and u is a critical point of I, then it is also ground state solution of the following problem

$$\begin{cases} -\mathrm{div}\left(\frac{f^2(x)\nabla w}{\sqrt{1-f^2(x)|\nabla w|^2}}\right) = -Nf\mathcal{N}_H(u) & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

So, u is a strong spacelike solution of problem (1.2) with $\lambda = 1$.

The first existence result of this section is the following proposition.

Proposition 4.1. *Assume that Ω has C^2 boundary $\partial\Omega$. Then, problem (1.2) with $\lambda = 1$ has a strong spacelike solution, which is also the ground state solution.*

Proof. Since H satisfies the L^∞-growth condition, we can easily derive that I is bounded from below on $C\left(\overline{\Omega}\right)$. Noting that I satisfies the (PS)-condition, by Theorem 1.7 of [25], we obtain a critical point $u_0 \in K_0$ of I such that

$$I\left(u_0\right) = \inf_{u \in C\left(\overline{\Omega}\right)} I(u).$$

So, u_0 is a strong spacelike solution of problem (1.2) with $\lambda = 1$ and it is also the ground state solution. ■

Obviously, the energy functional associated to problem (1.2) is

$$I_\lambda(u) = \Phi(u) + \lambda N \int_\Omega \left(f(x) \int_0^u H(x,t)\,dt \right) dx$$

on $C\left(\overline{\Omega}\right)$. Clearly, one has $I_\lambda(0) = 0$. Then, we have the following existence result.

Proposition 4.2. *Besides the conditions of Proposition 4.1, we also assume that there exists $R \in (0, \delta)$ such that $H(x,t) < 0$ for a.e. $x \in \Omega$ and $\forall t \in (0, R)$. Then, there exists $\lambda_* > 0$ such that problem (1.2) has at least one nontrivial strong spacelike solution for any $\lambda > \lambda_*$ which is a minimizer of I_λ.*

Proof. Let $x_0 \in \Omega$ and $r \in (0, R)$ such that $\overline{B}_r\left(x_0\right) \subset \Omega$. Like [4, Theorem 2], choose the bump function

$$\eta(x) = \begin{cases} e^{\frac{r^2}{|x-x_0|^2 - r^2}}, & x \in B_r\left(x_0\right), \\ 0, & x \in \Omega \setminus B_r\left(x_0\right) \end{cases}$$

and define

$$\eta_0(x) = \min\left\{r, \|\nabla \eta\|^{-1} f_M^{-1}\right\} \eta(x).$$

It is not difficult to verify that $\eta_0 \in K_0$ and $0 \le \eta_0(x) < R$ in Ω. So, we have that

$$N \int_\Omega \left(f(x) \int_0^{\eta_0} H(x,t)\,dt \right) dx < 0.$$

Taking

$$\lambda_* = \frac{-\Phi\left(\eta_0\right)}{N \int_\Omega \left(f(x) \int_0^{\eta_0} H(x,t)\,dt \right) dx},$$

then for $\lambda > \lambda_*$, we have that $I_\lambda\left(\eta_0\right) < 0$. By Proposition 4.1, we get the desired conclusion. ■

On the basis of Proposition 4.2, we can present the proof of Theorem 1.2 via the Mountain Pass Theorem.

Proof of Theorem 1.2. We first extend H continuously to the whole I_δ by taking $H = 0$ on $\Omega \times [-\delta, 0]$. And for simplicity, we'll still use H to denote the extended function. For any fixed $\lambda > \lambda_*$, by Proposition 4.2, I_λ has a nontrivial minimizer $e_\lambda \in K_0$ such that $I_\lambda\left(e_\lambda\right) < 0$. To obtain the second critical point of I_λ via the Mountain Pass

Theorem [25, Theorem 3.2], it is sufficient to show that there exist two positive constants α and $\rho < \|e_\lambda\|_\infty$ such that

$$I_\lambda(u) \geq \alpha \text{ for all } u \in K_0 \text{ with } \|u\|_\infty = \rho. \tag{4.2}$$

For any $u \in K_0$, by the elementary inequality

$$1 - \sqrt{1-p^2} \geq \frac{p^2}{2}$$

and the Poncaré inequality, we obtain that

$$\Phi(u) \geq \frac{\lambda_1 N}{2} \int_\Omega f(x)|u|^2 \, dx.$$

Since (1.4) holds, there exists $\sigma > 0$ such that

$$H(x,t) \geq -\frac{\lambda_1 N}{2\lambda}|t| \text{ for a.e. } x \in \Omega \text{ and } \forall t \in [-\sigma, \sigma].$$

It follows that

$$\lambda N \int_\Omega \left(f(x) \int_0^u H(x,t)\, dt \right) dx \geq -\frac{\lambda_1 N}{4} \int_\Omega f(x)|u|^2 \, dx.$$

So, we obtain that

$$I_\lambda(u) \geq \frac{\lambda_1 N}{4} \int_\Omega f(x)|u|^2 \, dx$$

for any $u \in K_0$ with $\|u\|_\infty \in [-\sigma, \sigma]$. Letting $\rho \in (0, \min\{\sigma, \|e_\lambda\|_\infty\})$, by an argument similar to that of [4, Theorem 3], we can show that

$$0 < \inf_{u \in K_0, \|u\|_\infty = \rho} \int_\Omega f(x)|u|^2 \, dx := \gamma,$$

which implies (4.2) with $\alpha = (\lambda_1 N \gamma)/4$. Therefore, using the Mountain Pass Theorem, we obtain a critical point $u_\lambda \in C\left(\overline{\Omega}\right)$ of I_λ such that $\alpha \leq I_\lambda(u_\lambda) < +\infty$. It follows that $u_\lambda \in K_0 \setminus \{e_\lambda\}$ is the nontrivial solution of problem (1.2). Finally, we show that e_λ and u_λ are nonnegative. In fact, for any strong spacelike solution u, setting $u^- = \min\{0, u\}$, multiplying the first equation of problem (1.2) by u^- and integrating over Ω, we find that

$$\int_\Omega \frac{f^2(x)|\nabla u^-|^2}{\sqrt{1 - f^2(x)|\nabla u|^2}} \, dx = 0.$$

Hence $u^- \equiv 0$ and u is a nonnegative solution. ■

Example 4.1. We consider the case of $H(x,t) = -t^p$ for any $t \in [0, \delta]$. If $p \in (0,1]$, then by Proposition 4.2 there exists at least one nontrivial nonnegative strong spacelike solution for any $\lambda > \lambda_*$. If $p > 1$, according to Theorem 1.2, problem (1.2) possesses at least two nontrivial nonnegative strong spacelike solutions for any $\lambda > \lambda_*$.

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5 Bifurcation

For any $t \in (0,1]$, we consider the following auxiliary problem

$$\begin{cases} -\operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-tf^2(x)|\nabla u|^2}}\right) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

for a given $g \in C^\alpha\left(\overline{\Omega}\right)$ with some $\alpha \in (0,1)$. Setting $v = \sqrt{t}u$, problem (5.1) is equivalent to

$$\begin{cases} -\operatorname{div}\left(\frac{f^2(x)\nabla v}{\sqrt{1-f^2(x)|\nabla v|^2}}\right) = \sqrt{t}g(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

By Theorem 1.1, we know that problem (5.2) has a unique spacelike solution $v \in C^{2,\alpha}(\Omega)$ which is denoted by $\Psi(\sqrt{t}g)$. So, $u = \Psi(\sqrt{t}g)/\sqrt{t}$ is the unique solution of problem (5.1). We also consider the following auxiliary problem

$$\begin{cases} -\operatorname{div}\left(f^2(x)\nabla u\right) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

By Theorem 8.34 of [19], we know that problem (5.3) has a unique solution $u \in C^{1,\alpha}\left(\overline{\Omega}\right)$ for some constant $\alpha \in (0,1)$, which is denoted by $\Phi(g)$. Clearly, $\Phi : C^\alpha\left(\overline{\Omega}\right) \longrightarrow C^{1,\alpha}\left(\overline{\Omega}\right)$ is continuous and linear. So, $\Phi : C^\alpha\left(\overline{\Omega}\right) \longrightarrow C^1\left(\overline{\Omega}\right)$ is completely continuous and linear. Define

$$G(t,g) = \begin{cases} \frac{\Psi(\sqrt{t}g)}{\sqrt{t}} & \text{if } t \in (0,1], \\ \Phi(g) & \text{if } t = 0. \end{cases}$$

Then, we have that:

Lemma 5.1. $G : [0,1] \times C^\alpha\left(\overline{\Omega}\right) \longrightarrow X$ *is completely continuous.*

Proof. We first show the *continuity* of G. For any $g_n, g \in C^\alpha\left(\overline{\Omega}\right)$ and $t_n, t \in [0,1]$ with $g_n \to g$ in $C^\alpha\left(\overline{\Omega}\right)$ and $t_n \to t$ in $[0,1]$ as $n \to +\infty$, it is sufficient to show that $u_n := G(t_n, g_n) \to u := G(t,g)$ in X.

If $t > 0$, without loss of generality, we assume that $t_n > t/2$ for any $n \in \mathbb{N}$. By Theorem 1.1, $u_n\sqrt{t_n} := v_n, u\sqrt{t} := v \in C^{2,\alpha}\left(\overline{\Omega}\right)$ and $\|v_n\| \le 1 - \theta < 1$ for any $n \in \mathbb{N}$ and some positive constant θ which is independent on n. Theorem 13.7 of [19] gives an a priori estimate for $\|v_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)}$ for some $\alpha \in (0,1)$. So, there exist $w \in C^1\left(\overline{\Omega}\right)$ and a subsequence v_{n_k} such that $v_{n_k} \to w$ in $C^1\left(\overline{\Omega}\right)$ as $k \to +\infty$. From Lemma 3.4, we have that w is the maximum point of

$$A(w) = \int_\Omega \left(\sqrt{1 - f^2(x)|\nabla w|^2} - \sqrt{t}g(x)w\right) dx$$

in $\mathscr{S}$. Further, Lemma 3.1 implies that w is also the unique maximum point of A. From Lemma 3.3, we get that $w = v$. It follows that $u_{n_k} \to u$ in X as $k \to +\infty$. Furthermore, we *claim* that $u_n \to u$ in X as $n \to +\infty$. Indeed, if there exist a subsequence u_{m_k} of u_n and $\varepsilon_0 > 0$ such that $\|u_{m_k} - u\| \ge \varepsilon_0$ for any $k \in \mathbb{N}$, similar to the discussion above, we can obtain that u_{m_k} contains a further subsequence $u_{m_{k_j}}$ such that $u_{m_{k_j}} \to u$ in X as $j \to +\infty$, which is a contradiction.

If $t = 0$ and there exists a subsequence t_{n_i} of t_n such that $t_{n_i} = 0$, then $u_{n_i} = G(t_{n_i}, g_{n_i}) = \Phi(g_{n_i}) \to \Phi(g) = u$ in X as $i \to +\infty$. So, next we assume that $t = 0$ and $t_n > 0$ for any $n \in \mathbb{N}$. From Theorem 1.1, we know that problem (5.2) has only the trivial solution when $t = 0$. Reasoning as above, we can show that $v_n \to 0$ in X as $n \to +\infty$.

Note that u_n satisfies

$$\begin{cases} -\sum_{i,j=1}^N a^{ij} u_{ij} - \sum_{i=1}^N b^i u_i = g_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where

$$a^{ij} = \delta_{ij} \frac{f^2(x)}{\sqrt{1 - f^2(x)\left|\nabla v_n\right|^2}} + \frac{f^4 \nabla_i v_n \nabla_j v_n}{\left(1 - f^2(x)\left|\nabla v_n\right|^2\right)^{3/2}}, \; b^i = \frac{2 f f_i - f^3 f_i \left|\nabla v_n\right|^2}{\left(1 - f^2(x)\left|\nabla v_n\right|^2\right)^{3/2}}.$$

Since $\|v_n\| \leq 1 - \theta < 1$, the above problem is a priori uniformly elliptic. Thus, Theorem 3.7 of [19] implies an a priori estimate for $\|u_n\|_{C^0(\overline{\Omega})}$. Further, by Theorem 6.6 of [19], we have that $\|u_n\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$ for some positive constant C independent of n. So, up to a subsequence, there exists $w \in C^2\left(\overline{\Omega}\right)$ such that $u_n \to w$ in $C^2\left(\overline{\Omega}\right)$ as $n \to +\infty$. Letting $n \to +\infty$ in (5.4), we obtain that

$$\begin{cases} -\operatorname{div}\left(f^2(x)\nabla w\right) = g(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that $w = \Phi(g) = G(0, g) = u$. Then, similarly to the case of $t > 0$, we obtain that $u_n \to u$ in X as $n \to +\infty$.

Next, we show the *compactness* of G. For any $(t_n, g_n) \in [0,1] \times C^\alpha\left(\overline{\Omega}\right)$ where g_n is bounded in $C^\alpha\left(\overline{\Omega}\right)$ for any $n \in \mathbb{N}$, it is enough to show that $\{G(t_n, g_n)\}$ possesses a convergent subsequence. Without loss of generality, we assume that $t_n \to t_0 \in [0,1]$. Clearly, $G(t, \cdot)$ is compact for any $t \in [0,1]$. So, $\{G(t_1, g_n)\}$ has a convergent subsequence. Hence, there exists a subsequence $\left\{g_n^{(1)}\right\}$ of $\{g_n\}$ such that the diameter of $\left\{G\left(t_1, g_n^{(1)}\right)\right\}$ less than 1. Similarly, there exists $\left\{g_n^{(2)}\right\} \subseteq \left\{g_n^{(1)}\right\}$ such that the diameter of $\left\{G\left(t_2, g_n^{(2)}\right)\right\}$ less than 1/2. In general, there exists $\left\{g_n^{(k)}\right\} \subseteq \left\{g_n^{(k-1)}\right\}$ such that the diameter of $\left\{G\left(t_k, g_n^{(k)}\right)\right\}$ less than $1/k$, $k \geq 3$.

We *claim* that for any $\varepsilon > 0$ and $g \in C^\alpha\left(\overline{\Omega}\right)$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|G(t, g) - G(t_0, g)\| < \varepsilon/3$ when $|t - t_0| < \delta$ with any $t \in [0,1]$. Suppose, by contradiction, that there exist $\varepsilon_0 > 0$, $g_0 \in C^\alpha\left(\overline{\Omega}\right)$ such that for any $n \in \mathbb{N}$, existing $t_n' \in [0,1]$ with $|t_n' - t_0| < 1/n$ such that

$$\|G(t_n', g_0) - G(t_0, g_0)\| \geq \varepsilon_0. \tag{5.5}$$

Up to a subsequence, we have that $t_n' \to t_0 \in [0,1]$ as $n \to +\infty$. Letting $n \to +\infty$ in (5.5) and noting the continuity of G, we obtain that

$$0 = \lim_{n \to +\infty} \|G(t_n', g_0) - G(t_0, g_0)\| \geq \varepsilon_0,$$

which is a contradiction.

Now we show that $\left\{G\left(t_n, g_n^{(n)}\right)\right\}$ is convergent. Obviously, there exists an $N_0 > 3/\varepsilon$ such that $|t_n - t_0| < \delta$ for any $n > N_0$. Hence, when $m > n > N_0$, we have that

$$\begin{aligned} \left\|G\left(t_m, g_m^{(m)}\right) - G\left(t_n, g_n^{(n)}\right)\right\| &< \left\|G\left(t_m, g_m^{(m)}\right) - G\left(t_0, g_m^{(m)}\right)\right\| \\ &\quad + \left\|G\left(t_0, g_m^{(m)}\right) - G\left(t_n, g_m^{(m)}\right)\right\| \\ &\quad + \left\|G\left(t_n, g_m^{(m)}\right) - G\left(t_n, g_n^{(n)}\right)\right\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{1}{n} < \varepsilon. \end{aligned}$$

So, $\left\{G\left(t_n, g_n^{(n)}\right)\right\}$ is the Cauchy sequence. Consequently, $G\left(t_n, g_n^{(n)}\right) \to u_0$ for some $u_0 \in X$.

Finally, we show that $G\left(t_n^{(n)}, g_n^{(n)}\right) \to u_0$. Obviously, there exists an $N_1 > 0$ such that $|t_n - t_0| < \delta$, $\left|t_n^{(n)} - t_0\right| < \delta$ and $\left\|G\left(t_n, g_n^{(n)}\right) - u_0\right\| < \varepsilon/3$ for any $n > N_1$. So, when $n > N_1$, we have that

$$\begin{aligned} \left\|G\left(t_n^{(n)}, g_n^{(n)}\right) - u_0\right\| &< \left\|G\left(t_n^{(n)}, g_n^{(n)}\right) - G\left(t_0, g_n^{(n)}\right)\right\| \\ &\quad + \left\|G\left(t_0, g_n^{(n)}\right) - G\left(t_n, g_n^{(n)}\right)\right\| \\ &\quad + \left\|G\left(t_n, g_n^{(n)}\right) - u_0\right\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

Therefore, we obtain that $G\left(t_n^{(n)}, g_n^{(n)}\right) \to u_0$ in X. ∎

For any fixed λ, we consider the following problem

$$\begin{cases} -\mathrm{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) = \lambda f(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

Obviously, problem (5.6) is equivalent to the operator equation $u = \Psi(\lambda u) := \Psi_\lambda(u)$. By virtue of Lemma 5.1, we can obtain the following topological degree jumping result.

Lemma 5.2. *For any $r > 0$, we have that*

$$\deg\left(I - \Psi_\lambda, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1), \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta) \end{cases}$$

for some $\delta > 0$, where $B_r(0) = \{w \in X : \|w\| < r\}$.

Proof. Choose δ small enough such that there is no any eigenvalue of problem (1.5) in $(\lambda_1, \lambda_1 + \delta)$. We *claim* that the Leray-Schauder degree $\deg\left(I - G(t, \lambda\cdot), B_r(0), 0\right)$ is well defined for any $\lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}$ and $t \in [0, 1]$. The claim is obvious for $t = 0$. So, it is enough to show that $u = G(t, \lambda u)$ has no solution with $\|u\| = r$ for r sufficiently small and any $t \in (0, 1]$. Otherwise, there exists a sequence $\{u_n\}$ such that $u_n = \Psi_\lambda\left(\sqrt{t} f u_n\right)/\sqrt{t}$ and $\|u_n\| \to 0$ as $n \to +\infty$. Let $w_n := u_n/\|u_n\|$, then by an argument similar to that of Lemma 5.1, we can show that for some convenient subsequence

$w_n \to w$ as $n \to +\infty$ and w verifies problem (1.5) with $\|w\| = 1$. This implies that λ is an eigenvalue of problem (1.5), which is a contradiction.

By the invariance of the degree under homotopies and Lemma 5.1, we obtain that

$$\begin{aligned} \deg\left(I - \Psi_\lambda, B_r(0), 0\right) &= \deg\left(I - G(1, \lambda\cdot), B_r(0), 0\right) \\ &= \deg\left(I - G(0, \lambda\cdot), B_r(0), 0\right) = \deg\left(I - \lambda\Phi, B_r(0), 0\right). \end{aligned}$$

By Theorem 8.10 of [15], we have that

$$\deg\left(I - \lambda\Phi, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1), \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta), \end{cases}$$

which implies the desired conclusion. ■

Now we present the proof of Theorem 1.3.

Proof of Theorem 1.3. (a) Let $\xi : \Omega \times [0, \delta] \to \mathbb{R}$ be such that

$$-NH(x, s) = s + \xi(x, s)$$

with

$$\lim_{s \to 0^+} \frac{\xi(x, s)}{s} = 0$$

uniformly for $x \in \Omega$. Then, problem (1.2) is equivalent to

$$\begin{cases} -\operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1 - f^2(x)|\nabla u|^2}}\right) = \lambda f(x)u + \lambda f(x)\xi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Define

$$F(\lambda, u) = \lambda f(x)u + \lambda f(x)\xi(x, u) + \operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1 - f^2(x)|\nabla u|^2}}\right)$$

for any $(\lambda, u) \in \mathbb{R} \times X$. Then, by some simple calculations, we have that

$$F_u(\lambda, 0)v = \lim_{t \to 0} \frac{F(\lambda, tv)}{t} = \lambda f(x)v + \operatorname{div}(f^2(x)\nabla v).$$

It follows that if $(\mu, 0)$ is a bifurcation point of problem (5.7), then μ is an eigenvalue of problem (1.5).

Consider the following problem

$$\begin{cases} -\operatorname{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1 - f^2(x)|\nabla u|^2}}\right) = \lambda f(x)u + \lambda s f(x)\xi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.8}$$

for any $s \in [0, 1]$. Clearly, problem (5.8) is equivalent to

$$u = \Psi\left(\lambda f(x)u + \lambda s f(x)\xi(x, u)\right) := F_\lambda(s, u).$$

In view of Lemma 5.1, $F_\lambda : [0, 1] \times X \longrightarrow X$ is completely continuous. In particular, $H_\lambda := F_\lambda(1, \cdot) : X \to X$ is completely continuous.

Let

$$\widetilde{\xi}(x,w)=\max_{0\le s\le w}|\xi(x,s)| \text{ for any } x\in\Omega.$$

Then, $\widetilde{\xi}$ is nondecreasing with respect to w and

$$\lim_{w\to 0^+}\frac{\widetilde{\xi}(x,w)}{w}=0. \tag{5.9}$$

It follows from (5.9) that

$$\left|\frac{\xi(x,u)}{\|u\|}\right|\le\frac{\widetilde{\xi}(x,u)}{\|u\|}\le\frac{\widetilde{\xi}\left(x,\|u\|_\infty\right)}{\|u\|}\le\delta\frac{\widetilde{\xi}(x,\delta\|u\|)}{\delta\|u\|}\to 0 \text{ as } \|u\|\to 0 \tag{5.10}$$

uniformly in $x\in\Omega$.

By (5.10) and an argument similar to that of Lemma 5.2 with obvious changes, we can show that the Leray-Schauder degree $\deg\left(I-F_\lambda(s,\cdot),B_r(0),0\right)$ is well defined for $\lambda\in(0,\lambda_1+\delta)\setminus\{\lambda_1\}$. By the invariance of the degree under homotopies, we have that

$$\begin{aligned}\deg\left(I-H_\lambda,B_r(0),0\right) &= \deg\left(I-F_\lambda(1,\cdot),B_r(0),0\right)=\deg\left(I-F_\lambda(0,\cdot),B_r(0),0\right)\\ &= \deg\left(I-\Psi_\lambda,B_r(0),0\right).\end{aligned}$$

From Lemma 5.2, we obtain that

$$\deg\left(I-H_\lambda,B_r(0),0\right)=\begin{cases}1 & \text{if } \lambda\in(0,\lambda_1)\,,\\ -1 & \text{if } \lambda\in(\lambda_1,\lambda_1+\delta)\,.\end{cases}$$

By the global bifurcation theorem of [23], there exists a continuum $\mathscr{C}$ of nontrivial solution of problem (1.2) bifurcating from $(\lambda_1,0)$ which is either unbounded or $\mathscr{C}\cap(\mathbb{R}\setminus\{\lambda_1\}\times\{0\})\neq\emptyset$. Since $u\equiv 0$ is the only solution of problem (1.2) for $\lambda=0$ and 0 is not an eigenvalue of problem (1.5), so $\mathscr{C}\cap(\{0\}\times X)=\emptyset$. By Lemma 4.1, we have $u\ge 0$ in Ω for any $(\lambda,u)\in\mathscr{C}$.

We *claim* that $\mathscr{C}\cap(\mathbb{R}\setminus\{\lambda_1\}\times\{0\})=\emptyset$. Otherwise, there exists a nontrivial solution sequence $(\lambda_n,u_n)\in\mathscr{C}\setminus\{(\lambda_1,0)\}$ such that $\lambda_n\to\mu$ and $u_n\to 0$ as $n\to+\infty$. Letting $w_n:=u_n/\|u_n\|$, by (5.10) and reasoning as that of Lemma 5.1, we can show that $w_n\to w$ as $n\to+\infty$ and w verifies problem (1.5) with $\|w\|=1$. It follows that $\mu=\lambda_1$, which is a contradiction. So, $\mathscr{C}$ is unbounded. Moreover, using Lemma 4.1 again, we know that $u>0$ in Ω for any $(\lambda,u)\in\mathscr{C}\setminus\{(\lambda_1,0)\}$. The fact of $\|u\|<1$ for any fixed $(\lambda,u)\in\mathscr{C}$ implies that the projection of $\mathscr{C}$ on $\mathbb{R}_+$ is unbounded.

Finally, we show the *asymptotic behavior* of u_λ as $\lambda\to+\infty$ for $(\lambda,u_\lambda)\in\mathscr{C}\setminus\{(\lambda_1,0)\}$. Otherwise, there exist a constant $\delta>0$ and $(\lambda_n,u_n)\in\mathscr{C}\setminus\{(\lambda_1,0)\}$ with $\lambda_n\to+\infty$ as $n\to+\infty$ such that $\|u_n\|^2\le 1-\delta^2$ for any $n\in\mathbb{N}$.

Our assumptions on H imply that there exists a positive positive $\rho>0$ such that

$$\frac{-NH(x,u_n(x))}{u_n(x)}\ge\rho$$

for any $x\in\Omega$ and $n\in\mathbb{N}$. Let φ_1 be a positive eigenfunction associated to λ_1. Multiplying the first equation of problem (1.2) by φ_1, we obtain after integrations by parts that

$$\begin{aligned}\frac{\lambda_1}{\delta}\int_\Omega f(x)u_n\varphi_1\,dx &= \frac{1}{\delta}\int_\Omega f^2(x)\nabla u_n\nabla\varphi_1\,dx\ge\int_\Omega\frac{f^2(x)\nabla u_n\nabla\varphi_1}{\sqrt{1-f^2(x)\left|\nabla u_n\right|^2}}\,dx\\ &= \lambda_n\int_\Omega f(x)\frac{-NH\left(x,u_n\right)}{u_n}u_n\varphi_1\,dx\ge\lambda_n\rho\int_\Omega f(x)u_n\varphi_1\,dx.\end{aligned}$$

It follows that $\lambda_n \leq \lambda_1/(\delta\rho)$, which is a contradiction.

(b) For any $n \in \mathbb{N}$, define

$$H^n(x,s) = \begin{cases} -ns, & s \in \left[0, \frac{1}{n}\right], \\ n\left(H\left(x, \frac{2}{n}\right)+1\right)\left(s-\frac{1}{n}\right)-1, & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ H(x,s), & s \in \left[\frac{2}{n}, +\infty\right). \end{cases}$$

Clearly, we see that $\lim_{n\to+\infty} H^n(x,s) = H(x,s)$ and $H_0^n = n$. Consider the following approximation problem

$$\begin{cases} -\text{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) = -\lambda N f(x) H^n(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

By the conclusion of (a), there exists a sequence unbounded continua $\mathscr{C}_n$ of the set of nontrivial solutions of problem (5.11) emanating from $(\lambda_1/n, 0)$ and joining to $(+\infty, 1)$ such that

$$\mathscr{C}_n \subseteq ((\mathbb{R}_+ \times P) \cup \{(\lambda_1/n, 0)\}).$$

Taking $z^* = (0,0)$, clearly, one has that $z^* \in \liminf_{n\to+\infty} \mathscr{C}_n$. The compactness of Ψ implies that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right) \cap \mathbb{B}_R$ is pre-compact, where $\mathbb{B}_R = \{z \in \mathbb{R} \times X : \|z\| < R\}$ for any $R > 0$. By Theorem 2.1 of [10], $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$ is unbounded and connected such that $z^* \in \mathscr{C}$ and $(+\infty, 1) \in \mathscr{C}$.

For any $(\lambda, u) \in \mathscr{C}$, the definition of superior limit (see [27]) shows that there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}_n$ such that $(\lambda_n, u_n) \to (\lambda, u)$ as $n \to +\infty$. Clearly, one has that

$$u_n = \Psi\left(\lambda_n N f(x) H^n\left(x, u_n\right)\right).$$

Letting $n \to +\infty$, in view of Lemma 5.1, we get that

$$u = \Psi\left(-\lambda N f(x) H\left(x, u\right)\right),$$

which shows that u is a solution of problem (1.2). Obviously, u is nonnegative for any $(\lambda, u) \in \mathscr{C}$ because $u_n \geq 0$ in Ω. We *claim* that $\mathscr{C} \cap ((0,+\infty) \times \{0\}) = \emptyset$. Suppose, by contradiction, that there exists $\mu > 0$ such that $(\mu, 0) \in \mathscr{C}$. There exists N_0 such that $\mu > \lambda_1/n$ for any $n > N_0$, which implies that $(\mu, 0) \notin \mathscr{C}_n$ for any $n > N_0$. We then have that $(\mu, 0) \notin \mathscr{C}$, which is impossible. Therefore, by virtue of Lemma 4.1, we have that $u > 0$ in Ω for any $(\lambda, u) \in \mathscr{C} \setminus \{(0,0)\}$.

(c) For any $n \in \mathbb{N}$, define

$$H_n(x,s) = \begin{cases} -\frac{1}{n}s, & s \in \left[0, \frac{1}{n}\right], \\ \left(H\left(x, \frac{2}{n}\right)+\frac{1}{n^2}\right)n\left(s-\frac{1}{n}\right)-\frac{1}{n^2}, & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ H(x,s), & s \in \left[\frac{2}{n}, +\infty\right). \end{cases}$$

Next, we consider the following problem

$$\begin{cases} -\text{div}\left(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\right) = -\lambda N f(x) H_n(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.12}$$

It is easy to see that $\lim_{n\to+\infty} H_n(x,s) = H(x,s)$ and

$$\lim_{s\to 0^+} \frac{H_n(x,s)}{s} = \frac{1}{n} \text{ uniformly for } x \in \Omega.$$

By the conclusion of (a), there exists a sequence unbounded continua $\mathscr{C}_n$ of positive solutions set of problem (5.12) in $\mathbb{R}_+ \times X$ emanating from $(\lambda_1 n, 0)$ for any $n \in \mathbb{N}$ and joining to $(+\infty, 1) := z_*$.

Taking $z^* = (+\infty, 0)$, clearly, we have that $z^* \in \liminf_{n\to+\infty} \mathscr{C}_n$ with $\|z^*\|_{\mathbb{R}\times X} = +\infty$. Let

$$S = \{(+\infty, u) : 0 < \|u\| < 1\}.$$

For fixed $n \in \mathbb{N}$, we *claim* that $\mathscr{C}_n \cap S = \emptyset$. Suppose, by contradiction, that there exists a sequence $(\lambda_m, u_m) \in \mathscr{C}_n$ such that $(\lambda_m, u_m) \to (+\infty, u_*) \in S$ with $\|u_*\| \in (0, 1)$. Then, as that of (a), we obtain that $\lambda_m \le c_n$ for some positive constant c_n, which is a contradiction. It follows that $\left(\cup_{n=1}^{+\infty} \mathscr{C}_n\right) \cap S = \cup_{n=1}^{+\infty} (\mathscr{C}_n \cap S) = \emptyset$. Letting $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$, since $\mathscr{C} \subseteq \cup_{n=1}^{+\infty} \mathscr{C}_n$, we have that $\mathscr{C} \cap S = \emptyset$. And then we get that $\mathscr{C} \cap \{\infty\} = \{z_*, z^*\}$.

Now, we show that $\mathscr{C} \setminus \{\infty\} \neq \emptyset$. It is enough to show that the projection of $\mathscr{C}$ on $\mathbb{R}$ is nonempty. From the argument of (a), we have known that $\mathscr{C}_n$ has unbounded projection on $\mathbb{R}$ for any fixed $n \in \mathbb{N}$. By Proposition 2 of [11], for any fixed $\sigma > 0$ there exists an $N_1 > 0$ such that for every $n > N_1$, $\mathscr{C}_n \subset V_\sigma(\mathscr{C})$, where $V_\sigma(\mathscr{C})$ denotes the σ-neighborhood of $\mathscr{C}$ in $\mathbb{R} \times X$. It follows that

$$(\lambda_1 n, +\infty) \subseteq \mathrm{pr}_{\mathbb{R}}(\mathscr{C}_n) \subseteq \mathrm{pr}_{\mathbb{R}}(V_\sigma(\mathscr{C})).$$

So, we have that $(n\lambda_1 + \sigma, +\infty) \subseteq \mathrm{pr}_{\mathbb{R}}(\mathscr{C})$, which implies $\mathscr{C} \setminus \{\infty\} \neq \emptyset$. Using Lemma 3.1 of [13], we obtain that $\mathscr{C}$ is connected. By an argument similar to that of (b), we can show that $\mathscr{C} \cap ([0, +\infty) \times \{0\}) = \emptyset$ and u is a positive solution of problem (1.2) for any $(\lambda, u) \in \mathscr{C}$. ■

Finally, we show a result concerning the nonexistence of positive solution.

Theorem 5.1. *Assume that* $\sup_{\Omega\times I_\delta} |H(x,t)| \le \Lambda < +\infty$ *and there exists a positive constant* ϱ *such that*

$$\frac{-NH(x,s)}{s} \le \varrho$$

for any $s \in I_\delta \setminus \{0\}$ *and a.e.* $x \in \Omega$. *Then there exists* $\varrho_* > 0$ *such that problem (1.2) has no any positive classical solution for* $\lambda \in (0, \varrho_*)$.

Proof. Assume that u is a positive classical solution of problem (1.2) with some $\lambda > 0$. Multiplying the first equation of problem (1.2) by u, in view of [17, Theorem 2.1 and Theorem 4.1], by integrating by parts we obtain that

$$\begin{aligned}\int_\Omega f^2(x)\,|\nabla u|^2\,dx \;&\le\; \int_\Omega \frac{f^2(x)\,|\nabla u|^2}{\sqrt{1-f^2(x)|\nabla u|^2}}\,dx = \lambda \int_\Omega f(x)\frac{-NH(x,u)}{u}u^2\,dx \\ &\le\; \lambda\varrho \int_\Omega f(x)u^2\,dx \le \frac{\lambda\varrho}{\lambda_1}\int_\Omega f^2(x)\,|\nabla u|^2\,dx,\end{aligned}$$

which follows that $\lambda \ge \lambda_1/\varrho$. ■

6 Proofs of Theorems 1.4–1.5

Now, we present the proofs of Theorems 1.4–1.5. Here, we borrowed some ideas of [1, Theorem 4.1].

Proof of Theorem 1.4. By Lemma 3.4, there exists a sequence $\{u_k\}_1^\infty$ in $C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$ of spacelike functions with mean curvatures H_k, H_k is measurable on $\Omega \times \mathbb{R}$ and $\sup_{\Omega\times\mathbb{R}} |H_k| \leq \Lambda < +\infty$, such that $\{u_k\}$ converges uniformly and $\{H_k\}_1^\infty$ converges weakly,

$$u_k \rightrightarrows u_0 \text{ in } C^0\left(\overline{\Omega}\right),$$

$$H_k\left(x, u_k(x)\right) \rightharpoonup H(x, u(x)) \text{ in } L^2\left(\Omega\right).$$

Hence, the desired conclusion can be obtained from [17, Theorem 3.1] immediately. ■

Proof of Theorem 1.5. By mollification, we can construct $C^{2,\alpha}$ approximants Ω_k, ψ_k such that

$$\Omega_k \subset \Omega, \ \operatorname{dist}\left(\Omega_k, \partial\Omega\right) \leq 1/k,$$

$$\sup_{\overline{\Omega}_k} |\nabla \psi_k| \leq 1 - \theta_k \text{ for some } \theta_k > 0,$$

$$\sup_{\overline{\Omega}_k} |\psi - \psi_k| \leq 1/k.$$

Meantime, choose $H_k \in C^{0,\alpha}\left(\overline{\Omega} \times \mathbb{R}\right)$ such that $H_k \to H$ in $L^2\left(\Omega \times [-a, a]\right)$ for every $a > 0$. By Theorem 1.1, we have a strictly spacelike solution $u^{(k)} \in C^{2,\alpha}\left(\overline{\Omega}_k\right)$ to the problem on Ω_k with H_k and ψ_k, where (k) denotes a superscript. Passing to a subsequence, we find a weakly spacelike $w \in C^{0,1}\left(\overline{\Omega}\right)$ such that $u^{(k)} \rightrightarrows w$ and $\sup_{\Omega_k}\left|u^{(k)} - w\right| \leq 1/k$. We could rearrange w such that it identically equal to u.

Let $f\left(x\right)\left|\nabla u\left(x\right)\right| = |\nabla v(x)|$ and $f\left(x\right)\left|\nabla u^{(k)}\left(x\right)\right| = |\nabla v^{(k)}(x)|$. Then, we can see that

$$v^{(k)} \rightrightarrows v \text{ in } C^{0,1}\left(\overline{\Omega}\right).$$

Let $l(x, y)$ be the Lorentz distance function with respect to v,

$$l(x, y) = \sqrt{|x - y|^2 - (v(x) - v(y))^2}, \ x, y \in \Omega.$$

And let $K_R(x)$ be the projected Lorentz ball defined by (2.3). The corresponding objects with respect to $v^{(k)}$ will be denoted by $l^{(k)}$ and $K_R^{(k)}$.

For any fixed $x_0 \in \Omega$, if there exists $y \in \partial\Omega$ such that $l\left(x_0, y\right) = 0$ and $\overline{x_0 y} \subset \Omega$, noting that l is increasing on outward rays from x_0 (see [1, page 138]), Theorem 1.4 shows that this segment extends to the boundary $\partial\Omega$. Another point of intersection is denoted by x. So, we have that $f|\nabla u| \equiv 1$ on $\overline{xy}$, which implies that $\overline{xy} \in K$ and $x_0 \in K$. Therefore, if $x_0 \notin K$, one has that $l\left(x_0, \partial\Omega\right) > 0$. Hence, there is $R > 0$ such that $K_{4R}\left(x_0\right) \subset\subset \Omega$. From the argument of [1, Theorem 4.1], we have known that there is $r \in (0, R/4)$ and k_1 such that for $|x - x_0| < r$ and $k > k_1$, $K_{R/r}\left(x_0\right) \subset K_{R/2}^k(x) \subset K_R\left(x_0\right)$ and $K_{2R}^k(x) \subset\subset \Omega$. Applying Corollary 2.1 to $v^{(k)}$, using these inclusions and noting that $B_r\left(x_0\right) \subset K_r\left(x_0\right)$, we obtain that

$$\int_{K_{R/2}^{(k)}(x)} \sum_{i,j=1}^{N} \left(v_{ij}^{(k)}\right)^2 dx \leq c\left(N, \Lambda, R, f^0, \Omega\right)\left(v^{(k)}(x)\right)^\alpha, \ x \in B_r = B_r\left(x_0\right), \tag{6.1}$$

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$$\int_{B_r(x_0)} \sum_{i,j=1}^{N} \left(v_{ij}^{(k)}\right)^2 dx \leq c\left(N, \Lambda, R, f^0, \Omega\right), \tag{6.2}$$

$$\|v^{(k)}\|_{L^2(B_r)} \leq \|v^{(k)}\|_{L^2\left(K_{R/4}(x_0)\right)} \leq c\left(N, \Lambda, R, f^0, \Omega\right)\left(v^{(k)}(x)\right)^{\alpha}, \; x \in B_r. \tag{6.3}$$

The estimate (6.2) shows that $\{v^{(k)}\}$ is bounded in the space $W^{2,2}(B_r)$. So, by Rellich's theorem and the weak compactness of bounded sets in $W^{2,2}$, there is a subsequence converging strongly in $W^{1,2}(B_r)$ and weakly in $W^{2,2}(B_r)$. Up to a subsequence, we obtain that

$$v^{(k)} \to v \text{ in } W^{1,2}(B_r), \tag{6.4}$$

$$v^{(k)} \rightharpoonup v \text{ in } W^{2,2}(B_r). \tag{6.5}$$

Obviously, (6.4) implies that $\|v^{(k)}\|_{L^2(B_r)} \to \|v\|_{L^2(B_r)}$. If $\|v\|_{L^2(B_r)} = 0$, (6.1) and (6.5) show that

$$\|v_{ij}\|_{L^2(B_r)} \leq \liminf_{k\to\infty} \|v_{ij}^{(k)}\|_{L^2(B_r)} \leq c \liminf_{k\to\infty}\left(\inf_{B_r}\left(v^{(k)}\right)^{\alpha}\right) = 0.$$

It follows that ∇v is constant in B_r. Since $|\nabla v| = 1$ a.e. in B_r, $v|_{B_r}$ is linear with slope 1. Using Theorem 1.4 as before, we have that $l(x_0, \partial\Omega) = 0$, which is a contradiction. Thus, $\|v\|_{L^2(B_r)} > 0$, and then (6.3) indicates that $v^{(k)}(x) \geq c > 0$ for all $x \in B_r$ and $k \geq k_1$, for some constant c. Hence, one has that

$$|\nabla v|^2 \leq 1 - c^2 < 1, \; \forall x \in B_r.$$

Applying [19, Theorem 8.24] to ∇v on B_r, we have that ∇v is Hölder continuous in a smooth neighbourhood $\Omega_0 \subset B_r$ of x_0. So, one has that $v \in C^{1,\beta}\left(\overline{\Omega}_0\right)$ for some $\beta > 0$. By [19, Theorem 11.4] and the arbitrary of x_0, for some $\alpha \in (0,1)$, we have that $u \in C^{2,\alpha}(\Omega \setminus K)$ is strictly spacelike on $\Omega \setminus K$ and satisfies the first equation of problem (1.6) on $\Omega \setminus K$. ■

Proceed exactly as in Theorem 1.5, we can show the following corollary.

Corollary 6.1. *Suppose that $\varphi : \partial\Omega \to \mathbb{R}$ is bounded. Let $H \in C^{0,\alpha}(\Omega \times \mathbb{R})$ with $\sup_{\Omega\times\mathbb{R}} |H| \leq \Lambda < +\infty$. Then there is a strictly spacelike $u \in C^{2,\alpha}(\Omega)$ satisfying (1.6) if and only if there is a spacelike function $\psi : \Omega \to \mathbb{R}$ with $\psi = \varphi$ on $\partial\Omega$.*

7 Radial symmetry of positive solutions

The aim of this section is to provide sufficient conditions on the prescription function to ensure that any eventual positive solution of problem (1.2) must be radially symmetric when Ω is the unit ball B. More precisely, we shall use the moving plane method (see [6, 18]) to show the following result.

Theorem 7.1. *Assume that f is radially symmetric and decreasing on $(0,1)$. Also* *assume that H is radially symmetric, increasing on $(0,1)$ with respect to the first variable and satisfies the following uniformly Lipschitz condition in the second variable*

$$|H(x,p)-H(x,q)|\leq C_0|p-q|$$

for any $x\in B$ and some constant C_0. Then, any positive solution u of problem (1.2) with $\nabla f\cdot\nabla u\geq 0$ in B is radially symmetric and monotone decreasing about the origin.

Proof. For convenience, we assume $N\lambda=1$. Let u be any positive solution of problem (1.2). By [17, Theorem 2.1 and Theorem 4.1], we know that $f\,|\nabla u|\leq\theta<1$ on $\overline{B}$. We define the truncated function

$$\varphi(t)=\begin{cases}\frac{1}{\sqrt{1-t}} & \text{if } t\in[0,\theta^2]\,,\\ \alpha(t) & \text{if } t\in(\theta^2,1)\,,\\ c & \text{if } t\geq 1,\end{cases}$$

where the function α and the constant c make $\varphi\in C^1(\mathbb{R}_+)$ increasing and convex, where $\mathbb{R}_+=[0,+\infty)$.

Then, consider the following problem

$$\begin{cases}-\frac{1}{f(x)}\operatorname{div}\left(\varphi\left(f^2|\nabla u|^2\right)f^2\nabla u\right)=-H(x,u) & \text{in } B,\\ u=0 & \text{on } \partial B.\end{cases}\tag{7.1}$$

Set

$$F\left(x,\nabla u,D^2u\right):=\operatorname{div}\left(\varphi\left(f^2|\nabla u|^2\right)f^2\nabla u\right)\frac{1}{f(x)}.$$

It is not difficult to show that

$$F\left(x,\nabla u,D^2u\right)=\sum_{i,j=1}^{N}f^{ij}(x)u_{ij}+2f^2\varphi'\left(f^2|\nabla u|^2\right)|\nabla u|^2\nabla f\nabla u+2\varphi\left(f^2|\nabla u|^2\right)\nabla f\nabla u,$$

where

$$f^{ij}(x)=\varphi\left(f^2|\nabla u|^2\right)f\delta_{ij}+2f^3\varphi'\left(f^2|\nabla u|^2\right)u_iu_j.$$

Observe that both φ and φ' are bounded on $\mathbb{R}_+$. It is easy to see that there exist two constants m and M such that

$$0<m|\xi|^2\leq\sum_{i,j=1}^{N}f^{ij}(x)\xi_i\xi_j\leq M|\xi|^2\tag{7.2}$$

for all $\xi\in\mathbb{R}^N\setminus\{0\}$. Hence, F is uniformly elliptic.

Without loss of generality, we choose a direction to be the x_1-direction and let

$$T_\lambda=\left\{x\in\mathbb{R}^N:x_1=\lambda\right\},\ \Sigma_\lambda=\{x\in B:x_1<\lambda\}\,.$$

For any fixed $z\in\mathbb{R}^{N-1}$ with $(0,z)\in B$, define

$$l_z=\{(x_1,z):x_1\in(-1,1)\}\,.$$

Clearly, $l_z\subset B$. Then, the reflection of $x=(x_1,z)\in l_z\cap\Sigma_\lambda$ about T_λ is

$$x^\lambda=(2\lambda-x_1,z)\,.$$

Let $u^\lambda(x)=u\left(x^\lambda\right)$ and $w^\lambda(x)=u^\lambda(x)-u(x)$. Then, it is easy to see that u^λ also satisfies problem (7.1). Note that $u_iu_j=u_1^2\geq 0$ on $l_z\cap\Sigma_\lambda$. Further, due to the monotone of f and $\nabla f\cdot\nabla u\geq 0$ in B, we have that

$$
\begin{aligned}
H\left(x^\lambda,u^\lambda\right)-H(x,u) &= -\left(F\left(x,\nabla u,D^2u\right)-F\left(x^\lambda,\nabla u^\lambda,D^2u^\lambda\right)\right)\\
&\geq -\left(F\left(x,\nabla u,D^2u\right)-F\left(x,\nabla u^\lambda,D^2u^\lambda\right)\right)\\
&= -\int_0^1\frac{d}{d\tau}F\left(x,\nabla u,D^2\left(\tau u+(1-\tau)u^\lambda\right)\right)\,d\tau\\
&= \sum_{i,j=1}^N\left(\int_0^1 F_{z_{ij}}\,d\tau\right)w_{ij}^\lambda\\
&= \sum_{i,j=1}^N f^{ij}(x)w_{ij}^\lambda,
\end{aligned}
$$

where $z_{ij}=\tau u_{ij}+(1-\tau)u_{ij}^\lambda$. It follows from the monotone of H with respect to x that

$$
-\sum_{i,j=1}^N f^{ij}(x)w_{ij}^\lambda+H\left(x,u^\lambda\right)-H(x,u)\geq 0.
$$

Let

$$
c(x)=c(x,\lambda)=\begin{cases}\frac{H\left(x,u^\lambda\right)-H(x,u)}{w^\lambda} & \text{if } w^\lambda\neq 0,\\ 0 & \text{if } w^\lambda=0.\end{cases}
$$

The Lipschitz condition implies that $|c(x)|\leq C_0$. Apparently, for λ close to -1, we have that

$$
w^\lambda(x)\geq 0,\ x\in\partial\left(l_z\cap\Sigma_\lambda\right).
$$

By Corollary of [18], in view of (7.2), we obtain that

$$
w^\lambda(x)\geq 0,\ x\in l_z\cap\Sigma_\lambda
$$

for λ near -1. By the arbitrary of z, we reach that

$$
w^\lambda(x)\geq 0,\ x\in\Sigma_\lambda
$$

Define

$$
\overline{\lambda}=\sup\left\{\lambda: w^\lambda(x)\geq 0,\forall x\in\Sigma_\lambda\right\}.
$$

We claim that $\overline{\lambda}\geq 0$. Otherwise, the reflection of $\partial\Sigma_{\overline{\lambda}}\cap\partial B$ falls inside B. It follows that

$$
w^{\overline{\lambda}}(x)\geq 0,\ \not\equiv 0\ \text{ on }\ \partial\Sigma_{\overline{\lambda}}.
$$

By the strong maximum principle [20, Theorem 2.7], we have that

$$
w^{\overline{\lambda}}(x)>0\ \text{ in }\ \Sigma_{\overline{\lambda}}. \tag{7.3}
$$

Let d be the maximum width of the narrow regions so that we can employ Corollary of [18]. Set $\delta =: \min\left\{-\overline{\lambda}, d/2\right\}$. We investigate function $w^{\overline{\lambda}+\delta}(x)$ on the narrow region (see Figure 2)

$$\Omega_\delta = \Sigma_{\overline{\lambda}+\delta} \cap \left\{x : x_1 > \overline{\lambda} - \frac{d}{2}\right\}.$$

We have shown that

$$-\sum_{i,j=1}^{N} f^{ij}(x) w_{ij}^{\overline{\lambda}+\delta} + c\left(x, \overline{\lambda}+\delta\right) w^{\overline{\lambda}+\delta} \geq 0 \text{ in } l_z \cap \Omega_\delta.$$

Furthermore, we claim that

$$w^{\overline{\lambda}+\delta} \geq 0 \quad \text{on } \partial\Omega_\delta.$$

In particular, this claim follows that

$$w^{\overline{\lambda}+\delta} \geq 0 \quad \text{on } \partial\left(l_z \cap \Omega_\delta\right).$$

First, because $\overline{\lambda}+\delta \leq 0$, the reflection of the two curved parts of $\partial\Omega_\delta$ falls inside $\overline{B}$. Hence, we have $w^{\overline{\lambda}+\delta} \geq 0$ on $\partial\Omega_\delta \cap \partial B$. Obviously, we have $w^{\overline{\lambda}+\delta} = 0$ on the flat part of $\partial\Omega_\delta$ where $\lambda_1 = \overline{\lambda}+\delta$. So, it suffices to show that $w^{\overline{\lambda}+\delta} \geq 0$ on the flat part of $\partial\Omega_\delta$ where $\lambda_1 = \overline{\lambda} - \delta/2$. while, inequality (7.3) indicates that there exists a positive constant c such that

$$w^{\overline{\lambda}}(x) \geq c, \ x \in \Sigma_{\overline{\lambda}-d/2}.$$

By the continuity of w^{λ} with respect to λ, we have that

$$w^{\overline{\lambda}+\delta}(x) \geq 0, \ x \in \Sigma_{\overline{\lambda}-d/2}$$

for sufficiently small δ.

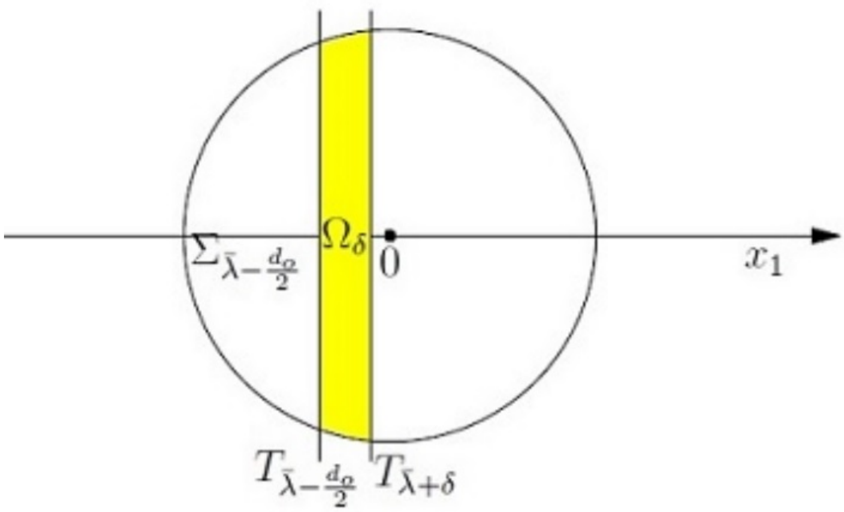

Figure 2: Narrow region Ω_δ.

Now we can apply the narrow region principle of [18, Corollary] to derive that

$$w^{\overline{\lambda}+\delta}(x) \geq 0, \;\; x \in l_z \cap \Omega_\delta.$$

By the arbitrary of z, we have that

$$w^{\overline{\lambda}+\delta}(x) \geq 0, \;\; x \in \Omega_\delta.$$

Thus,

$$w^{\overline{\lambda}+\delta}(x) \geq 0, \;\; x \in \Sigma_{\overline{\lambda}+\delta},$$

which contradicts the definition of $\overline{\lambda}$. Therefore, we verify that $\overline{\lambda} \geq 0$. It follows that

$$u\left(x_1, x'\right) \leq u\left(-x_1, x'\right), \;\; \forall x_1 \leq 0. \tag{7.4}$$

Similarly, moving the plane from near $x_1 = 1$ to the left, we can derive

$$u\left(-x_1, x'\right) \geq u\left(x_1, x'\right), \;\; \forall x_1 \geq 0.$$

It follows that

$$u\left(x_1, x'\right) \geq u\left(-x_1, x'\right), \;\; \forall x_1 \leq 0. \tag{7.5}$$

Combining (7.4) and (7.5), we obtain that u is symmetric about the plane T_0. The arbitrariness of the x_1-direction leads to the radial symmetry of u about the origin. The monotonicity comes directly from the above argument. ■

We would like to point out that the conclusion of Theorem 7.1 cannot be deduced from Theorem 2.1′ or Corollary 1 of [18] because F does not satisfy the condition (c) or (c_2') when $f(x)$ is not a constant. Here we overcome this difficulty by introducing the line l_z.

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