Bounding the A_α-spectral radius of k-connected irregular graphs*

Hongzhang Chen[a], Jianxi Li[a]†, Peng Huang[b]

[a]School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian, 363000, P.R. China

[b]School of Sciences, Nantong University, Nantong, Jiangsu, China.

Abstract: Let G be a simple graph of order n. For $\alpha \in [0,1]$, the A_α-matrix of G is defined as $A_\alpha = \alpha D(G) + (1-\alpha)A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The largest eigenvalues of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$, is called the A_α spectral radius of G. In this paper, we give an upper bound on $\rho_\alpha(G)$ for k-connected irregular graphs. Moreover, we also derive an upper bound on $\rho_\alpha(G)$ when G is a subgraph of a k-connected regular graph. Our results improve or extend the existing results, respectively.

Keywords: A_α spectral radius, irregular graph, k-connected.

AMS 2010 MSC: 05C50.

1 Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $v_i \in V(G)$, let $d_G(v_i)$ and $N_G(v_i)$ (or d_i and $N(v_i)$ for short) be the degree and the set of neighbors of v_i, respectively. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$ (or δ and Δ for short). We say G is *regular* if $\delta = \Delta$, and also call G a Δ-regular graph. The vertex-connectivity and edge-connectivity of G are denoted by $\kappa(G)$ and $\kappa'(G)$, respectively. A graph G is *k-connected* if $\kappa(G) \geq k$ and G is *h-edge-connected* if $\kappa'(G) \geq h$. Other undefined notation are found in [1].

Let $A(G)$ and $D(G) = \{d_1, \ldots, d_n\}$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. For every $\alpha \in [0,1]$, the matrix $A_\alpha(G)$ of G is defined by Nikiforov in [7] as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G).$$

In particular,

$$A_0(G) = A(G), \quad A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G), \quad A_1(G) = D(G).$$

Note that $A_\alpha(G)$ is a real symmetric matrix. Then the eigenvalues of $A_\alpha(G)$ (also called the A_α-eigenvalues of G) are real. The largest eigenvalue of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$ is called the

*Partially supported by NSFC (No. 12171089), NSF of Fujian (No. 2021J02048), NSF of Jiangsu (No. 06190028).

†Email addresses: m[illegible]chern@gmail.com (H.-Z. Chen); ptjxli@hotmail.com (J. Li); phuang@ntu.edu.cn (P. Huang).

A_α*-spectral radius* of G. Moreover, note that $A_\alpha(G)$ is irreducible if and only if G is connected for $\alpha \in [0,1)$. Therefore when G is a connected graph, the Perron-Frobenius theorem implies that the multiplicity of $\rho_\alpha(G)$ is one and there exists a positive unit eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, which is called the *Perron vector* of $A_\alpha(G)$. The Perron vector satisfies $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2} = 1$ and the *eigenvalue equation* $A_\alpha(G)\mathbf{x} = \rho_\alpha(G)\mathbf{x}$, which is

$$\rho_\alpha(G)x_i = \alpha d_i x_i + (1-\alpha) \sum_{v_j \in N(v_i)} x_j. \tag{1}$$

In addition, we have

$$\rho_\alpha(G) = \mathbf{x}^T A_\alpha(G)\mathbf{x} = \sum_{v_i v_j \in E(G)} (\alpha x_i^2 + 2(1-\alpha)x_i x_j + \alpha x_j^2). \tag{2}$$

The A_α-spectral radius of a graph contains lots of information about the graph. Many studies on this topic have been conducted, see [3, 5, 7, 8]. In particular, establishing upper and lower bounds for $\rho_\alpha(G)$ is of great interest. For any graph G of order n with m edges, it is known that $\frac{2m}{n} \le \rho_\alpha(G) \le \Delta$ and the equality holds if and only if G is regular [7]. It is natural to ask how small $\rho_\alpha(G) - \frac{2m}{n}$ and $\Delta - \rho_\alpha(G)$ can be when G is irregular. In [4], Ji *et al.* presented the upper and lower bounds for $\rho_\alpha(G) - \frac{2m}{n}$, which extends Nikiforov's bounds on $\rho_0(G) - \frac{2m}{n}$ [6]. Recently, motivated by the results in [2, 9, 10], Guo and Zhou [3] presented several lower bounds of $\Delta - \rho_\alpha(G)$ for an irregular graph G with fixed diameter. They also derived an upper bound for $\rho_\alpha(G)$ in terms of its maximum degree when G is a k-connected irregular graph. In this paper, we further study the relationships between $\rho_\alpha(G)$ and $\Delta(G)$ for k-connected irregular graphs, and present an upper bound for $\rho_\alpha(G)$, which improves the Guo and Zhou's bound. Moreover, we also derive an upper bound on $\rho_\alpha(G)$ when G is a subgraph of a k-connected regular graph, which extends the result of Huang, Shiu and Sun [11] on the signless Laplacian spectral radius.

2 Lemmas and main results

In order to prove our results, the following lemmas are needed.

Lemma 2.1 ([10]) *If $a, b > 0$, then $a(x-y)^2 + by^2 \ge \frac{abx^2}{a+b}$ with equality if and only if $y = \frac{ax}{a+b}$.*

Lemma 2.2 ([7]) *If G is connected and H is a proper subgraph of G, then for $\alpha \in [0,1)$, $\rho_\alpha(H) < \rho_\alpha(G)$.*

For k-connected irregular graphs, Guo and Zhou [3] presented the following upper bound for $\rho_\alpha(G)$ in terms of its maximum degree.

Theorem 2.3 ([3]) *Let G be a k-connected irregular graph of order n with m edges and maximum degree Δ. Then for $\alpha \in [0,1)$,*

$$\rho_\alpha(G) < \Delta - \frac{(1-\alpha)(n\Delta - 2m)k^2}{(n\Delta - 2m)[n^2 - (\Delta - k + 2)(n-k)] + (1-\alpha)nk^2}. \tag{3}$$

We slightly improve this upper bound by proving the following.

Theorem 2.4 *Let G be a k-connected irregular graph of order n with m edges, maximum degree Δ and minimum degree δ. Then for $\alpha \in [0,1)$, we have*

$$\rho_\alpha(G) < \Delta - \frac{(1-\alpha)(n\Delta - 2m)k^2}{(n\Delta - 2m)[(n - 2 + 2k - \Delta)(n - \delta - 1) + k^2] + (1-\alpha)nk^2}. \tag{4}$$

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$. Let w and v be the two vertices of G with $x_w = \max\{x_i | 1 \le i \le n\}$ and $x_v = \min\{x_i | 1 \le i \le n\}$. Then $x_w > \frac{1}{\sqrt{n}} > x_v > 0$ as $\sum_{i=1}^n x_i^2 = 1$ and G is irregular.

If $d_w \le \Delta - 1$, then we have

$$\begin{aligned}\rho_\alpha(G)x_w &= \alpha d_w x_w + (1-\alpha)\sum_{j\in N(w)} x_j \\ &\le \alpha d_w x_w + (1-\alpha)d_w x_w = d_w x_w \le (\Delta - 1)x_w,\end{aligned}$$

which leads to

$$\Delta - \rho_\alpha(G) \ge 1 > \frac{(1-\alpha)(n\Delta - 2m)k^2}{(n\Delta - 2m)[(n-2+2k-\Delta)(n-\delta-1)+k^2] + (1-\alpha)nk^2},$$

as desired. In what follows, we consider the case $d_w = \Delta$. From (2) we can get

$$\begin{aligned}\Delta - \rho_\alpha(G) &= \Delta\sum_{i=1}^n x_i^2 - \mathbf{x}^T A_\alpha \mathbf{x} \\ &= \sum_{i=1}^n (\Delta - d_i)x_i^2 + (1-\alpha)\sum_{v_iv_j\in E(G)} (x_i - x_j)^2 \\ &> (n\Delta - 2m)x_v^2 + (1-\alpha)\sum_{v_iv_j\in E(G)} (x_i - x_j)^2. \end{aligned} \tag{5}$$

Recall that G is k-connected. Then by Menger's Theorem, we may choose k vertex-disjoint paths between w and v such that the sum of the lengths of these k paths is as small as possible, denoted by $P_1, P_2, \ldots, P_k$. Obviously, each of these paths contains only one vertex in $N(w)$, *i.e.* $\bigcup_{t=1}^k V(P_t)$ contains exactly k vertices in $N(w)$. Otherwise, we can find another k paths that have a smaller sum of lengths. Then there exists at least $\Delta - k$ vertices v_j, where $j \in \{1, 2, \ldots, \Delta - k\}$, such that $v_j \in N(w)$, but $v_j \notin \bigcup_{t=1}^k V(P_t)$. Thus $\sum_{t=1}^k (|V(P_t)| - 2) + 2 + \Delta - k \le n$, which is

$$\sum_{t=1}^k (|V(P_t)| - 1) \le n - \Delta + 2k - 2.$$

Moreover, by Cauchy-Schwarz inequality and Arithmetic-Harmonic Mean inequality, we have

$$\begin{aligned}\sum_{v_iv_j\in E(G)} (x_i - x_j)^2 &\ge \sum_{t=1}^k \sum_{v_iv_j\in E(P_t)} (x_i - x_j)^2 \\ &\ge \sum_{t=1}^k \frac{1}{|V(P_t)| - 1}\left(\sum_{v_iv_j\in E(P_t)} (x_i - x_j)\right)^2 \\ &= (x_w - x_v)^2 \sum_{t=1}^k \frac{1}{|V(P_t)| - 1} \\ &\ge (x_w - x_v)^2 \frac{k^2}{\sum\limits_{t=1}^k (|V(P_t)| - 1)} \\ &\ge \frac{k^2}{n - \Delta + 2k - 2}(x_w - x_v)^2. \end{aligned} \tag{6}$$

Therefore, combining (5) and (6), and using Lemma 2.1, we have

$$\begin{aligned}\Delta - \rho_\alpha(G) &> (n\Delta - 2m)x_v^2 + (1-\alpha)\sum_{v_iv_j\in E(G)}(x_i - x_j)^2 \\ &\geq (n\Delta - 2m)x_v^2 + (1-\alpha)\frac{k^2}{n-\Delta+2k-2}(x_w - x_v)^2 \\ &\geq \frac{(1-\alpha)(n\Delta-2m)k^2}{(n\Delta-2m)(n-\Delta+2k-2)+(1-\alpha)k^2}x_w^2. \end{aligned} \tag{7}$$

For convenience, let

$$\beta = \frac{(1-\alpha)(n\Delta-2m)k^2}{(n\Delta-2m)[(n-2+2k-\Delta)(n-\delta-1)+k^2]+(1-\alpha)nk^2}.$$

If $x_v^2 \geq \frac{\beta}{n\Delta-2m}$, then by (7), we have $\Delta - \rho_\alpha(G) > (n\Delta-2m)x_v^2 \geq \beta$, as desired. So we may assume that $x_v^2 < \frac{\beta}{n\Delta-2m}$. Note that there are at least δ vertices in $N(v)$ since $d_v \geq \delta$. We now choose δ vertices in $N(v)$, labelled by $v_1, v_2, \ldots, v_\delta$. If $\sum_{t=1}^{\delta} x_{v_t}^2 > \frac{n\Delta-2m+(1-\alpha)\delta}{(1-\alpha)(n\Delta-2m)}\beta$, then by (5) and Lemma 2.1, we obtain

$$\begin{aligned}\Delta - \rho_\alpha(G) &> (n\Delta-2m)x_v^2 + (1-\alpha)\sum_{t=1}^{\delta}(x_{v_t} - x_v)^2 \\ &= \sum_{t=1}^{\delta}\left(\frac{n\Delta-2m}{\delta}x_v^2 + (1-\alpha)(x_{v_t}-x_v)^2\right) \\ &\geq \sum_{t=1}^{\delta}\frac{(1-\alpha)(n\Delta-2m)}{n\Delta-2m+(1-\alpha)\delta}x_{v_t}^2 \\ &= \frac{(1-\alpha)(n\Delta-2m)}{n\Delta-2m+(1-\alpha)\delta}\sum_{t=1}^{\delta}x_{v_t}^2 > \beta. \end{aligned}$$

We now assume that $\sum_{t=1}^{\delta} x_{v_t}^2 \leq \frac{n\Delta-2m+(1-\alpha)\delta}{(1-\alpha)(n\Delta-2m)}\beta$. Recall that $x_v^2 < \frac{\beta}{n\Delta-2m}$, $\sum_{i=1}^{n} x_i^2 = 1$ and $\delta \leq n-2$ since G is irregular. Then we have

$$\begin{aligned} x_w^2 &\geq \frac{1 - x_v^2 - \sum\limits_{i=1}^{\delta} x_{v_t}^2}{n-\delta-1} \\ &> \frac{1}{n-\delta-1}\left[1 - \frac{\beta}{n\Delta-2m} - \frac{n\Delta-2m+(1-\alpha)\delta}{(1-\alpha)(n\Delta-2m)}\beta\right] \\ &= \frac{1}{n-\delta-1}\left[1 - \frac{n\Delta-2m+(1-\alpha)(\delta+1)}{(1-\alpha)(n\Delta-2m)}\beta\right]. \end{aligned} \tag{8}$$

Combining (7) and (8), we then have

$$\begin{aligned}\Delta - \rho_\alpha(G) &> \frac{(1-\alpha)(n\Delta-2m)k^2}{(n\Delta-2m)(n-\Delta+2k-2)+(1-\alpha)k^2} \\ &\quad \times \frac{1}{n-\delta-1}\left[1 - \frac{n\Delta-2m+(1-\alpha)(\delta+1)}{(1-\alpha)(n\Delta-2m)}\beta\right] \\ &= \frac{(1-\alpha)(n\Delta-2m)k^2}{(n\Delta-2m)[(n-2+2k-\Delta)(n-\delta-1)+k^2]+(1-\alpha)nk^2} = \beta. \end{aligned}$$

This completes the proof. ■

Remark 2.1 The upper bound in Theorem 2.4 is always better than that in Theorem 2.3. This is shown by the following fact

$$\begin{aligned}&[n^2-(\Delta-k+2)(n-k)]-[(n-2+2k-\Delta)(n-\delta-1)+k^2]\\=&\Delta k-kn-2k^2+n+4k+n\delta-2\delta-2+2k\delta-\Delta\delta-\Delta\\=&(\delta-k)(n+2k-2-\Delta)+n+2k-2-\Delta\\=&(\delta-k+1)(n+2k-2-\Delta)>0.\end{aligned}$$

Remark 2.2 The result in Theorem 2.4 can be extended to the case of h-edge-connected irregular graphs. Note that there are at least h edge-disjoint paths in an h-edge-connected graph. Applying an similar argument as that of Theorem 2.4, we can obtain the following edge-connected version of Theorem 2.4:

$$\rho_\alpha<\Delta-\frac{(1-\alpha)(n\Delta-2m)h^2}{(n\Delta-2m)[(m+h-\Delta)(n-\delta-1)+h^2]+(1-\alpha)nh^2}. \tag{9}$$

However, in some cases, (9) is not better than (4). For example, if G satisfies $k=h$ and $n-2+k<m$, where $\kappa(G)=k$ and $\kappa'(G)=h$, then (4) is always better than (9).

Moreover, note that $2m\le(n-1)\Delta+\delta$ when G is a k-connected irregular graph. Then we have the following result immediately.

Corollary 2.5 *Let G be a k-connected irregular graph of order n with m edges, maximum degree Δ and minimum degree δ. For $\alpha\in[0,1)$, then*

$$\rho_\alpha(G)<\Delta-\frac{(1-\alpha)(\Delta-\delta)k^2}{(\Delta-\delta)[(n-2+2k-\Delta)(n-\delta-1)+k^2]+(1-\alpha)nk^2}.$$

Huang, Shiu and Sun [11] shown that if H is a proper subgraph of a k-connected Δ-regular graph G, then

$$\rho_{1/2}(H)<\Delta-\frac{(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4)+(n+1)(k-1)^2}. \tag{10}$$

We now extend this result to A_α-spectral radius by proving the following.

Theorem 2.6 *Let H be a proper subgraph of a k-connected Δ-regular graph G of order n. If $k\ge2$, $n\ge3$ and $G\ne K_n$, then for $\alpha\in[0,1)$,*

$$\rho_\alpha(H)<\Delta-\frac{(1-\alpha)(k-1)^2}{(n-\Delta-1)(n-\Delta+2k-4)+((1-\alpha)n+\alpha)(k-1)^2}. \tag{11}$$

Proof. By Lemma 2.2, we known that the A_α-spectral radius will not increase when deleting an edge. So we may assume that $H=G-e$ for some $e=uv\in E(G)$. Clearly, $d_H(u)=d_H(v)=\Delta-1$. Obviously, H is connected when $k\ge2$. Let $\mathbf{x}=(x_1,x_2,\dots,x_n)^T$ be the Perron vector of $A_\alpha(H)$ corresponding to $\rho_\alpha(H)$.

Let w be the vertex of G such that $x_w=\max\{x_i|1\le i\le n\}$. We claim that $u\ne w$ and $v\ne w$. Indeed, if $u=w$, then by the eigenvalue equation (1), we have

$$\begin{aligned}\rho_\alpha(H)x_w&=\alpha d_H(w)x_w+(1-\alpha)\sum_{j\in N_H(w)}x_j\\&\le\alpha d_H(w)x_w+(1-\alpha)d_H(w)x_w\\&=d_H(w)x_w=(\Delta-1)x_w\end{aligned}$$

As a result, $\Delta - \rho_\alpha(H) \ge 1$. Moreover, using the fact that

$$\rho_\alpha(H) > \frac{2m}{n} = \frac{n\Delta - 2}{n} = \Delta - \frac{2}{n} \ge \Delta - \frac{2}{3} > \Delta - 1,$$

a contradiction. Therefore, $u \ne w$ and $v \ne w$.

Note that, by (2),

$$\begin{aligned}\Delta - \rho_\alpha(H) &= \Delta \sum_{i=1}^{n} x_i^2 - \mathbf{x}^T A_\alpha(H)\mathbf{x} \\ &= \sum_{i=1}^{n}(\Delta - d_i)x_i^2 + (1-\alpha)\sum_{v_iv_j \in E(H)}(x_i - x_j)^2 \\ &= x_u^2 + x_v^2 + (1-\alpha)\sum_{v_iv_j \in E(H)}(x_i - x_j)^2. \end{aligned} \tag{12}$$

Since G is k-connected, then $\kappa(H) \ge k-1$. Thus by Menger's Theorem, there are at least $k-1$ vertex-disjoint paths between w and v. Similar to the proof of (6), we can get

$$\sum_{t=1}^{k-1}(|V(P_t)| - 1) \le n - \Delta + 2k - 4$$

and

$$\sum_{ij \in E(H)}(x_i - x_j)^2 \ge \frac{(k-1)^2}{n - \Delta + 2k - 4}(x_w - x_u)^2. \tag{13}$$

Therefore, from (12), (13) and Lemma 2.1, we have

$$\begin{aligned}\Delta - \rho_\alpha(H) &> x_u^2 + (1-\alpha)\sum_{v_iv_j \in E(H)}(x_i - x_j)^2 \\ &\ge x_u^2 + (1-\alpha)\frac{(k-1)^2}{n-\Delta+2k-4}(x_w - x_u)^2 \\ &\ge \frac{(1-\alpha)(k-1)^2}{n - \Delta + 2k - 4 + (1-\alpha)(k-1)^2}x_w^2. \end{aligned} \tag{14}$$

Let

$$\mu = \frac{(1-\alpha)(k-1)^2}{(n-\Delta-1)(n-\Delta+2k-4) + ((1-\alpha)n+\alpha)(k-1)^2}.$$

If $x_u^2 + x_v^2 \ge \mu$, then by (12), $\Delta - \rho_\alpha(H) > x_u^2 + x_v^2 \ge \mu$, as desired.

Let $N_H(u) = \{u_1, u_2, \ldots, u_{\Delta-1}\}$ as $d_H(u) = \Delta - 1$. If $\sum_{t=1}^{\Delta-1} x_{u_t}^2 \ge \frac{(1-\alpha)(\Delta-1)+1}{1-\alpha}\mu$, then

$$\begin{aligned}\Delta - \rho_\alpha(G) &> x_u^2 + (1-\alpha)\sum_{t=1}^{\Delta-1}(x_{u_t} - x_u)^2 \\ &= \sum_{t=1}^{\Delta-1}\left(\frac{1}{\Delta-1}x_u^2 + (1-\alpha)(x_{u_t} - x_u)^2\right) \\ &\ge \sum_{t=1}^{\Delta-1}\frac{(1-\alpha)}{(1-\alpha)(\Delta-1)+1}x_{u_t}^2 \ge \mu, \end{aligned}$$

as desired. It remains to consider the case that $x_u^2 + x_v^2 < \mu$ and $\sum_{t=1}^{\Delta-1} x_{u_t}^2 < \frac{(1-\alpha)(\Delta-1)+1}{1-\alpha}\mu$. Recall that $\sum_{i=1}^{n} x_i^2 = 1$. Then we have

$$\begin{aligned} x_w^2 &\geq \frac{1 - x_u^2 - x_v^2 - \sum\limits_{i=1}^{\Delta-1} x_{u_t}^2}{n - \Delta - 1} \\ &> \frac{1}{n-\Delta-1}\left(1 - \mu - \frac{(1-\alpha)(\Delta-1)+1}{1-\alpha}\mu\right) \\ &= \frac{1}{n-\Delta-1}\left(1 - \frac{(1-\alpha)\Delta+1}{1-\alpha}\mu\right). \end{aligned} \tag{15}$$

As a consequence, combining (14) and (15), we can obtain

$$\begin{aligned} \Delta - \rho_\alpha(G) &> \frac{(1-\alpha)(k-1)^2}{n-\Delta+2k-4+(1-\alpha)(k-1)^2} x_w^2 \\ &> \frac{(1-\alpha)(k-1)^2}{n-\Delta+2k-4+(1-\alpha)(k-1)^2} \cdot \frac{1}{n-\Delta-1}\left(1 - \frac{(1-\alpha)\Delta+1}{1-\alpha}\mu\right) = \mu, \end{aligned}$$

as desired. The proof is completed. ■

Remark 2.3 Recall that $A_{1/2}(G) = \frac{1}{2}Q(G)$. Then by Theorem 2.6, we have

$$\rho_{1/2}(H) < \Delta - \frac{(k-1)^2}{2(n-\Delta-1)(n-\Delta+2k-4)+(n+1)(k-1)^2}. \tag{16}$$

Obviously, (16) is always better than (10) when $G \neq K_n$.

Acknowledgments: The authors would like to thank the anonymous referees for their constructive corrections and valuable comments on this paper, which have considerably improved the presentation of this paper.

References

[1] J. Bondy, U. Murty. *Graph theory with applications*, London, Macmillan, 1976.

[2] S.M. Cioabă, D.A. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, *J. Comb. Theory. Ser. B*, **97**(2007), 483–486.

[3] H. Guo, B. Zhou, On the α-spectral radius of graphs, *Appl. Anal. Discr. Math.*, **14**(2020), 431–458.

[4] Y. Ji, J. Wang, M. Brunetti, N. Bian, A_α-spectral radius and measures of graph irregularity, *Linear Multilinear Algebra*, 2021, https://doi.org/10.1080/03081087.2021.1910617.

[5] H. Lin, X. Liu, J. Xue, Graphs determined by their A_α-spectra, *Discr. Math.*, **342**(2019), 441–450.

[6] V. Nikiforov, Eigenvalues and degree deviation in graphs, *Linear Algebra Appl.*, **414**(2006), 347–360.

[7] V. Nikiforov, Merging the A- and Q-spectral theories, *Appl. Anal. Discr. Math.*, **11**(2016), 81–107.

[8] V. Nikiforov, G. Pastén, O. Rojo, R. Soto, On the A_α-spectra of trees, *Linear Algebra Appl.*, **520**(2017), 286–305.

[9] W. Ning, H. Li, M. Lu, On the signless Laplacian spectral radius of irregular graphs, *Linear Algebra Appl.*, **438**(2013), 2280–2288.

[10] L. Shi, The spectral radius of irregular graphs, *Linear Algebra Appl.*, **431**(2009), 189–196.

[11] W.C. Shiu, P. Huang, P.K. Sun, The signless Laplacian spectral radius of k-connected irregular graphs, *Linear Multilinear Algebra*, **65**(2017), 830–839.