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MONOTONICITY OF RESISTANCE DISTANCE IN LINEAR 2-TREES

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ABSTRACT. Recall that a linear 2-tree, sometimes called a 2-path, is a 2-tree with exactly two vertices of degree two. In this article we will address two open questions regarding resistance distance in linear 2–trees. The first question is: given an arbitrary linear 2-tree does resistance distance between two vertices u, v increase as $|v - u|$ increases? We answer this question in the affirmative. As a corollary to this result, we show that the maximal resistance distance in a linear 2-tree occurs between the vertices of degree 2 (the extremal vertices). The second question concerns the optimal location of bends in a linear 2-tree. We show that for a linear 2-tree with a single bend, the location of the bend that minimizes the maximal resistance distance (i.e., the resistance distance between the degree 2-vertices) is as close as possible to a degree 2 vertex. We show empirically and provide a conjecture that for a linear 2-tree with an arbitrary number of bends the configuration that will result in the smallest maximal resistance distance is to place the bends consecutively and as close as possible to one of the degree two vertices.

1. Introduction

Resistance distance, also referred to as effective resistance, is a graph metric that has gained popularity in a wide variety of fields due to its ability to quantify structural properties of a graph. The application of resistance distance to graph theory originated in the analysis of the structure of compounds in chemistry [12], but has since been applied to fields as diverse as spectral sparsification and fast linear system solving [18], Kemeny's constant [16], distributed control [4], combinatorial matrix theory [3, 20] and spectral graph theory [1, 7, 8, 18].

We recall that given a graph G, we may determine the resistance distance between two points on the graph by assuming that the graph G represents an electrical circuit with resistances on each edge. The resistance on a weighted edge is the reciprocal of its edge weight. Given any two nodes i and j assume that one unit of current flows into node i and one unit of current flows out of node j. The potential difference $v_i - v_j$ between nodes i and j needed to maintain this current is the *resistance distance* between i and j.

One natural way of determining the resistance distance in a graph is to perform equivalent electrical circuit transformations, such as the familiar parallel and series rule to analyze the resistance distance between two vertices in the graph (for an explanation of such rules see [19] and for a worked example see [11]). A significant number of mathematical techniques to determine resistance distance in a graph have also been developed. These include:

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- counting the number of spanning tress and spanning 2-forests that separate a given pair of vertices [1];
- recursion techniques where the edge weight of a single edge in the graph is changed [20];
- resistance distance sum rules [14] and similar local sum rules [8] which rely on symmetries to create a solvable system of equations;
- generalized inverses of the combinatorial Laplacian [2, 12];
- as a solution to an optimization problem [7], which relies on Thompson's principle to recast the problem as minimizing the energy of a set of springs;
- considering the graph as a simplex in a higher dimensional space where the resistance distances of the graph are equivalent to the Euclidean distances [9];
- considering the commute time and escape probability of random walks [10];
- determining the eigenvalues and eigenvectors of the Laplacian matrix [13] and the normalized Laplacian matrix [8].

In addition, fast numerical techniques have been developed for approximating the resistance distance in graphs [18, 15].

This paper addresses two questions regarding the monotonicity of the resistance distance in linear 2–trees, extending the results in [5] and [6]. The first question is: Given a linear 2-tree with the nodes ordered consecutively, (e.g., as in Figure 2) does resistance distance monotonically increase as $|u-v|$ increases? It turns out that $r(u,v)$, the resistance distance between nodes u and v, depends not only on the locations of u and v, but also of the bends in the 2-tree. (A bent linear 2-tree is formally defined in Definition 2.8, and a tree with a single bend can be see in the right panel of Figure 1.) Even so, we are able to prove the following important result:

Theorem A (Theorem 3.4)**.** *Given a linear 2-tree G with n vertices, $r_G(u,v) < r_G(u,v+1)$ for any $u<v$.*

An important strategy in the proof of the preceding theorem is to separate the bends into three (possibly empty) groups, those that occur between u and v, and those that occur on either side. An important corollary to this result shows that for a linear 2-tree the maximal resistance distance occurs between the vertices of degree 2 (the extremal vertices).

The second question addresses the placement of the bends that minimizes the maximal resistance distance. We show the following result for a linear 2-tree with a single bend

Theorem B (Theorem 3.8)**.** *Given a bent linear 2-tree G_k with n vertices and one bend, the location k of the bend that minimizes the maximal effective resistance is $k=4$ (and also $n-2$ by symmetry). In this case*

$$r_{G_4}(1,n) = \frac{n-1}{5} + \frac{4F_{n-1}}{5L_{n-1}} - \frac{F_{n-5}(F_n+F_{n-4})}{F_{2n-2}},$$

where F_p is the pth Fibonacci number and L_q is the qth Lucas number (see Definition 2.4).

We also conjecture that for a linear 2-tree with an arbitrary number of bends, the configuration that minimizes the maximal resistance distance is the one that places the bends consecutively at the ends of the linear 2-tree. (See Conjecture 3.9.)

The structure of the paper is as follows. In Section 2 we formally define 2-trees and linear 2-trees, and provide some preliminary results on resistance distance for this family of graphs. Next, in Section 3 we show that resistance distance is monotonic (Theorem 3.4) and show that the maximal resistance distance occurs between the vertices of degree 2. Finally, we show that the configuration that minimizes the maximal resistance distance in a linear 2-tree with a single bend is the configuration where the bend is located at vertex 4 or $n-2$ (Theorem 3.8). We conclude Section 3 with a collection of open conjectures and questions. We note that many of the lemmas and theorems in this paper, in addition to determining resistance distances in linear two trees, yield relationships between Fibonacci numbers. When a direct proof of an equality-type relationship can be provided in a few lines, we provide that proof in the text. When such a proof would require many lines, or even pages, we refer the reader to an algorithmic verification technique for Fibonacci identities [17].

2. Preliminary Results on Resistance Distance in Linear 2-trees

First, we recall the relationship between resistance distance and spanning 2-forests, demonstrated in the following theorem [1, Th. 4 and (5)].

Theorem 2.1. *Given a graph G, the resistance distance between vertices u and v is given by*

$$r_G(u,v) = \frac{\mathscr{F}_G(u,v)}{T(G)},$$

where $\mathscr{F}_G(u,v)$ is the number of spanning 2-forests of G that separate u and v, $T(G)$ is the number of spanning trees of G, and w is any vertex of G.

Several of the results that follow in this section were proved by taking advantage of combinatorial methods for enumerating spanning trees and spanning forests in simple graphs. In this paper, we will use Theorem 2.1 as a way to present various mathematical statements more compactly.

Here we consider the infinite class of graphs termed *linear 2-trees*, also known as 2-paths, which we now define.

Definition 2.2. *A 2-tree is defined inductively as follows*

(1) *K_3 is a 2-tree;*
(2) *if G is a 2-tree, the graph obtained by inserting a vertex adjacent to the two vertices of an edge of G is a 2-tree.*

An alternative and more compact definition of a 2-tree is: G is a 2-tree on n vertices if G is chordal, has $2n-3$ edges, and K_4 is not a subgraph of G. (Recall that a chord of a cycle is an edge whose endpoints lie on the cycle, but is not itself an edge in the cycle; a graph is called chordal if all of its cycles of length ≥ 4 have a chord.)

Definition 2.3. *A linear 2-tree (or 2-path) is a 2-tree in which exactly two vertices have degree 2.*

See Figures 1, 2, and 3 for examples of 2-trees.

In [5], Barrett and the authors of this paper used network transformations to determine the resistance distance and number of spanning 2-forests separating two vertices in a linear 2-tree with n vertices. Before stating the results we recall the recursive definitions of both the Fibonacci and Lucas numbers.

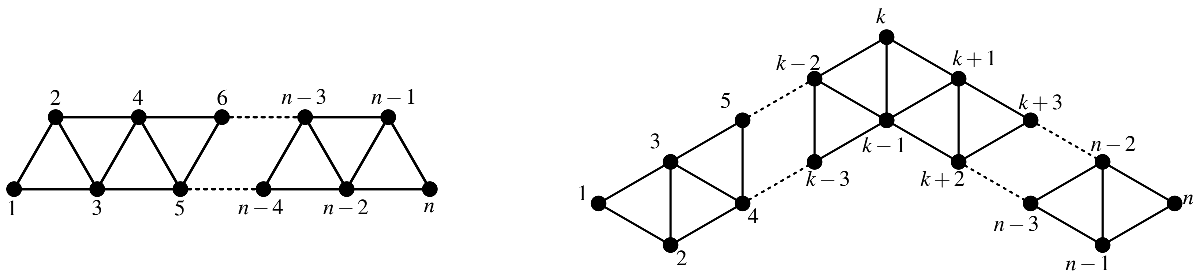


FIGURE 1. On the left, a straight linear 2-tree with n vertices. On the right, a linear 2-tree with n vertices and single bend at vertex k.

Definition 2.4 (Fibonacci and Lucas Numbers)**.** *Define $F_0 = 0$ and $F_1 = 1$, the nth Fibonacci number is defined recursively as*

$$F_n = F_{n-1} + F_{n-2}.$$

Similarly, define $L_0 = 2$ and $L_1 = 1$, the nth Lucas number is defined recursively as

$$L_n = L_{n-1} + L_{n-2}.$$

With these definitions at hand we can thus state the main results of [5].

Theorem 2.5. [5, Th. 20] *Let S_n be the straight linear 2-tree on n vertices labeled as in the graph on the left in Figure 1. Then for any two vertices u and v of S_n with $u < v$,*

$$r_{S_n}(u,v) = \frac{\sum_{i=1}^{v-u}(F_iF_{i+2u-2} - F_{i-1}F_{i+2u-3})F_{2n-2i-2u+1}}{F_{2n-2}}, \tag{1}$$

where F_p is the pth Fibonacci number.

It is natural to ask how resistance distance changes when one of u or v is exchanged for an adjacent node. The answer is stated below.

Corollary 2.6. *Under the assumptions of Theorem 2.5 the following equality holds*

$$r_{S_n}(u,v+1) - r_{S_n}(u,v) = (F_v^2 - F_{v-1}^2 + 2(-1)^{v-u}F_{u-1}^2)F_{2n-2v-1}/F_{2n-2}.$$

Proof. Recall that the number of spanning trees in S_n is F_{2n-2} [5]. Then, (1) gives

$$\begin{aligned}
&\mathscr{F}_{S_n}(u,v+1) - \mathscr{F}_{S_n}(u,v)\\
&\quad= \sum_{i=1}^{v+1-u}(F_iF_{i+2u-2} - F_{i-1}F_{i+2u-3})F_{2n-2i-2u+1} - \sum_{i=1}^{v-u}(F_iF_{i+2u-2} - F_{i-1}F_{i+2u-3})F_{2n-2i-2u+1}\\
&\quad= (F_{v+1-u}F_{v+u-1} - F_{v-u}F_{v+u-2})F_{2n-2v-1}.
\end{aligned}$$

Catalan's identity yields $F_{v+1-u}F_{v+u-1} = F_v^2 - (-1)^{v+u-1}F_{u-1}^2$ and $F_{v-u}F_{v+u-2} = F_{v-1}^2 - (-1)^{v+u}F_{u-1}^2$. Thus,

$$\begin{aligned}
\mathscr{F}_{S_n}(u,v+1) - \mathscr{F}_{S_n}(u,v) &= (F_v^2 - (-1)^{v-u+1}F_{u-1}^2 - F_{v-1}^2 + (-1)^{v-u}F_{u-1}^2)F_{2n-2v-1}\\
&= (F_v^2 - F_{v-1}^2 + 2(-1)^{v-u}F_{u-1}^2)F_{2n-2v-1}.
\end{aligned}$$

□

In [6] the authors generalized the formulas for a straight linear 2-tree to a linear 2-tree with any number of bends. As we also consider linear 2-trees with bends in this paper, we formalize the definition as follows:

Definition 2.7. *We define the graph G_n with $V(G_n) = V(S_n)$ and $E(G_n) = (E(S_n) \cup \{k-1, k+2\}) \setminus (\{k, k+2\})$ to be the bent linear 2-tree with a single bend at vertex k. See the graph on the right in Figure 1.*

In essence, performing a bend operation on a straight linear 2-tree at vertex k results in vertex $k-1$ having degree 5, vertex k having degree 3 and all other vertices having the same degrees as before. We generalize the idea of a linear 2-tree with a single bend to a linear 2-tree with two or more bends recursively as follows.

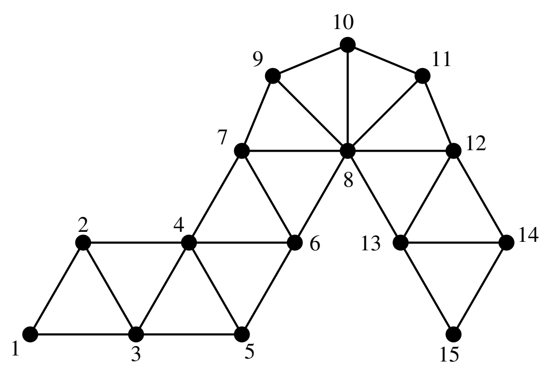


FIGURE 2. A linear 2-tree on 15 vertices with a single bend at vertex 5 and three consecutive bends at vertices 9, 10, and 11. For a complete definition of a bend in a linear 2-tree see Definition 2.8.

Definition 2.8. *We define the bent linear 2-tree G_n with n vertices and p bends located at nodes $k_1, k_2, \ldots, k_p$, with $k_1 < k_2 < \cdots < k_{p-1} < k_p$, iteratively as follows: Let G_n^1 be the bent linear 2-tree with a single bend located at k_1. For $i = 2$ to p perform a bend operation as follows:*

1. *If $k_i > k_{i-1} + 1$, bend the tree as in Definition 2.7, replacing S_n with G_n^{i-1}.*
2. *If $k_i = k_{i-1} + 1$, iterate backward through the k_j locations until $k_i - k_j \neq i - j$. Define G_n^i with $V(G_n^i) = V(G_n^{i-1})$ and $E(G_n) = (E(G_n^{i-1}) \cup \{k_{j+1} - 1, k_i + 2\}) \setminus (\{k_i, k_i + 2\})$. See Figure 2.*

The following is the main result from [6] and is the primary tool used in the following section.

Theorem 2.9. [6, Th. 3.1] *Given a bent linear 2-tree with n vertices and $p = p_1 + p_2 + p_3$ single bends located at nodes $k_1, k_2, \ldots, k_p$ with $k_1 < k_2 < \cdots < k_{p-1} < k_p$, the number of spanning 2-forests* *separating nodes u and v where* $k_{p_1} < u \le k_{p_1+1}$ *and* $k_{p_1+p_2} < v \le k_{p_1+p_2+1}$ *is given by*

(2)

$$\mathscr{F}_G(u,v) = \mathscr{F}_{S_n}(u,v) - \sum_{j=p_1+1}^{p_1+p_2}\left[F_{k_j-3}F_{k_j} - 2\sum_{i=p_1+1}^{j-1}[(-1)^{k_j-k_i+1+j-i}F_{k_i}F_{k_i-3}] + 2(-1)^{j+u+k_j}F_{u-1}^2\right]\times \left[F_{n-k_j+2}F_{n-k_j-1} + 2(-1)^{v-k_j}F_{n-v}^2\right],$$

and the resistance distance between nodes u and v is given by

(3)

$$r_G(u,v) = r_{S_n}(u,v) - \sum_{j=p_1+1}^{p_1+p_2}\left[F_{k_j-3}F_{k_j} - 2\sum_{i=p_1+1}^{j-1}[(-1)^{k_j-k_i+1+j-i}F_{k_i}F_{k_i-3}] + 2(-1)^{j+u+k_j}F_{u-1}^2\right]\times \left[F_{n-k_j+2}F_{n-k_j-1} + 2(-1)^{v-k_j}F_{n-v}^2\right]/F_{2n-2}.$$

As was done in Corollary 2.6, we consider how resistance distance changes if u or v is exchanged for an adjacent vertex. This time, to simplify the statement, we give the difference in terms of spanning 2-forests which separate the appropriate vertices.

Corollary 2.10. *Under the hypotheses of Theorem 2.9, assume further that* $v < k_{p_1+p_2+1}$ *if* $p_3 > 0$. *Then* $\mathscr{F}_G(u,v+1) - \mathscr{F}_G(u,v)$ *is equivalent to:*

$$F_{2n-2v-1}\left(F_v^2 - F_{v-1}^2 + 2(-1)^{v+u}F_{u-1}^2\left(1 + 2\sum_{j=p_1+1}^{p_1+p_2}(-1)^j\right) + 2\sum_{i=p_1+1}^{p_1+p_2}(-1)^{v+p_1+p_2+k_i+i}F_{k_i-3}F_{k_i}\right).$$

Proof. This relationship can be verified through algorithmic techniques for Fibonacci numbers, see [17]. □

3. Monotonicity of resistance distance in linear 2-trees

In this section we consider two open questions regarding the monotonicity of resistance distance in bent linear 2-trees.

Question 3.1. *Given an arbitrary linear 2-tree, labeled as in Figure 2, does resistance distance between vertices u and v increase as* $|v-u|$ *increases?*

This question is answered in the affirmative in Section 3.1; as a corollary we find that the resistance distance is maximized between the extremal vertices (i.e., the vertices with degree 2).

The second question addressed is:

Question 3.2. *Given a linear 2–tree with n vertices and p bends, where should the bends be placed so that the maximal resistance distance is minimized?*

We answer this question for the special case when $p = 1$ and provide empirical evidence for the case where the p bends are consecutive, and for the general case.

3.1. ***Resistance distance for a fixed linear 2-tree.*** With an aim toward answering Question 3.1, we first restrict to the case where all bends occur *between* the vertices u and v.

Theorem 3.3. *Given a linear 2-tree G with n vertices, if there are p bends located at nodes $k_1, k_2, \dots, k_p$ with $k_1 < k_2 < \cdots < k_{p-1} < k_p$, then $r_G(u,v) < r_G(u,v+1)$, for $1 \le u < k_1 < \cdots k_p < v < n$.*

Proof. We consider just the numerators, that is $\mathscr{F}_G(u,v)$ and $\mathscr{F}_G(u,v+1)$, since the denominators are the same for both $r(u,v)$ and $r(u,v+1)$. We will demonstrate that $\mathscr{F}_G(u,v+1) - \mathscr{F}_G(u,v) > 0$. Applying (2) and Corollary 2.6 together with Corollary 2.10 in the case where $p_1 = 0$ and $p_2 = p$ yields

$$\mathscr{F}_G(u,v+1) - \mathscr{F}_G(u,v) = F_{2n-2v-1}\left(F_v^2 - F_{v-1}^2 + 2(-1)^{v+u}F_{u-1}^2\left(1 + 2\sum_{j=1}^{p}(-1)^j\right) + 2\sum_{i=1}^{p}(-1)^{v+p+k_i+i}F_{k_i-3}F_{k_i}\right).$$

Considering the most negative possible scenario, we find

$$\mathscr{F}_G(u,v+1) - \mathscr{F}_G(u,v) \ge F_{2n-2v-1}\left(F_v^2 - F_{v-1}^2 - 2F_{u-1}^2 - 2\sum_{j=1}^{p}F_{k_j-3}F_{k_j}\right).$$

Note that $F_{2n-2v-1} > 0$ (since $n > v$), so we just need to show that

$$F_v^2 - F_{v-1}^2 - 2\sum_{i=1}^{p}F_{k_i}F_{k_i-3} - 2F_{u-1}^2 > 0.$$

The most extreme case (i.e., the most possible bends) is that with bends located at: $u+1, u+2, u+3, \dots, v-2, v-1$. In this extreme case the above equation becomes

$$\begin{aligned}
F_v^2 - F_{v-1}^2 - 2\sum_{i=u+1}^{v-2}F_iF_{i-3} - 2F_{u-1}^2 &= F_v^2 - F_{v-1}^2 - 2\sum_{i=u+1}^{v-2}(F_{i-1}^2 - F_{i-2}^2) - 2F_{u-1}^2 \\
&= F_v^2 - F_{v-1}^2 - 2(F_{v-3}^2 - F_{u-1}^2) - 2F_{u-1}^2 \\
&= F_v^2 - F_{v-1}^2 - 2F_{v-2}^2 \\
&= F_{2v-2} - F_{2v-3} = F_{2v-4} > 0.
\end{aligned}$$

□

We now consider the general case, where an arbitrary number of bends can be placed at arbitrary locations.

Theorem 3.4. *Given a linear 2-tree G with n vertices, $r_G(u,v) < r_G(u,v+1)$ for any $u < v$.*

Proof. Here we must consider the case that in a given linear 2-tree we have $p = p_1 + p_2 + p_3$ total bends with $p_1, p_2, p_3 \in \mathbb{N}_{\ge 0}$, such that p_1 bends occur to the left of u, p_2 bends occur between u and v, and p_3 bends occur to the right of v. The bends are located at nodes $k_1, k_2, \dots, k_p$ with $k_1 < k_2 < \cdots < k_{p-1} < k_p$ where $k_{p_1} \le u < k_{p_1+1} < \cdots k_{p_1+p_2} < v < n$.

As before, we consider just the numerators, that is $\mathscr{F}_G(u,v)$ and $\mathscr{F}_G(u,v+1)$, since the denominators are the same for both $r_G(u,v)$ and $r_G(u,v+1)$. For the case that $v \ne k_{p_1+p_2+1}$, applying the prior theorem and Corollary 3.3 of [6] gives the result.

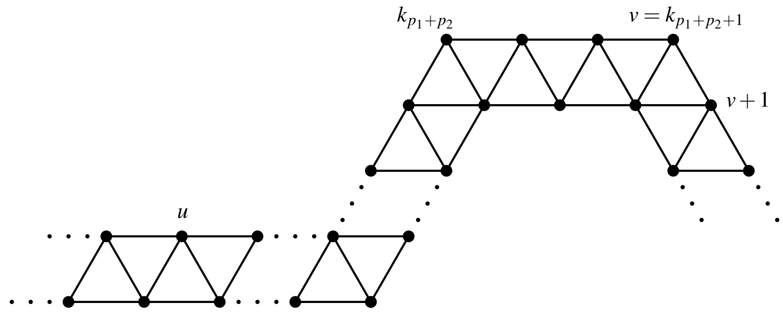


FIGURE 3. An example graph showing the dilemma we face traveling from v to $v+1$, through a bend.

In the case that $v = k_{p_1+p_2+1}$, $v+1$ no longer satisfies the hypotheses of Theorem 2.6, that is, moving from v to $v+1$ forces us to consider the p_1+p_2+1st bend. In a straight linear 2-tree, increasing $|u-v|$ increases effective resistance. However, each time we add a bend between vertices, we expect the resistance to decrease. Thus, to show that $r(u,v+1) > r(u,v)$ we must show that the increase in resistance due to the move from v to $v+1$ outweighs the decrease in resistance due to the additional bend (see Figure 3). We now consider the case where $v = k_{p_1+p_2+1}$.

In this case, we compute $\mathscr{F}_G(u,v+1)$ using (2) to be:

$$\mathscr{F}_{S_n}(u,v+1) - \sum_{j=p_1+1}^{p_1+p_2+1}\left[F_{k_j-3}F_{k_j} - 2\sum_{i=p_1+1}^{j-1}[(-1)^{k_j-k_i+1+j-i}F_{k_i}F_{k_i-3}] + 2(-1)^{j+u+k_j}F_{u-1}^2\right]\times \left[F_{n-k_j+2}F_{n-k_j-1} + 2(-1)^{v+1-k_j}F_{n-v-1}^2\right].$$

Without loss of generality, we assume that $v > \lfloor (n+1)/2 \rfloor$ (if not, we can reorder the vertices in reverse so that now $u < \lfloor (n+1)/2 \rfloor$). Applying Corollary 2.10, we find that $\mathscr{F}_G(u,v+1) - \mathscr{F}_G(u,v)$ is equivalent to

$$F_{2n-2v-1}\left(F_v^2 - F_{v-1}^2 + 2(-1)^{v+u}F_{u-1}^2\left(1 + 2\sum_{j=p_1+1}^{p_1+p_2}(-1)^j\right) + 2\sum_{i=p_1+1}^{p_1+p_2}(-1)^{v+p_1+p_2+k_i+i}F_{k_i-3}F_{k_i}\right)$$
$$-\left(F_{v-3}F_v - 2\sum_{i=p_1+1}^{p_1+p_2}[(-1)^{v+k_i+i+p_1+p_2}F_{k_i}F_{k_i-3}] - 2(-1)^{p_1+p_2+u+v}F_{u-1}^2\right)\left(F_{n-v+2}F_{n-v-1} - 2F_{n-v-1}^2\right). \tag{4}$$

Set

$$Q_G(u,v) := \mathscr{F}_G(u,v+1) - \mathscr{F}_G(u,v).$$

We will now demonstrate that $Q_G(u,v)$ is nonnegative for all $v > \lfloor (n+1)/2 \rfloor$.

$$Q_G(u,v) = F_{2n-2v-1}\left(F_v^2 - F_{v-1}^2 + 2(-1)^{v+u}F_{u-1}^2\left(1+2\sum_{j=p_1+1}^{p_1+p_2}(-1)^j\right) + 2\sum_{i=p_1+1}^{p_1+p_2}(-1)^{v+p_1+p_2+k_i+i}F_{k_i-3}F_{k_i}\right)$$
$$-\left(F_{v-3}F_v - 2\sum_{i=p_1+1}^{p_1+p_2}[(-1)^{v+k_i+i+p_1+p_2}F_{k_i}F_{k_i-3}] - 2(-1)^{p_1+p_2+u+v}F_{u-1}^2\right)\left(F_{n-v+2}F_{n-v-1} - 2F_{n-v-1}^2\right).$$

Or, equivalently,

$$Q_G(u,v) = 2(-1)^{v+u}F_{u-1}^2\left(F_{2n-2v-1}\left(1+2\sum_{j=p_1+1}^{p_1+p_2}(-1)^j\right) + (-1)^{p_1+p_2}F_{2n-2v-2}\right)$$
$$+2(-1)^{v+p_1+p_2}(F_{2n-2v})\sum_{i=p_1+1}^{p_1+p_2}(-1)^{k_i+i}F_{k_i-3}F_{k_i} + F_{2n-2v-1}(F_v^2 - F_{v-1}^2) - F_{2n-2v-2}(F_{v-3}F_v).$$

It is not difficult to check that

$$(5)\quad F_{2n-2v-1}\left(1+2\sum_{j=p_1+1}^{p_1+p_2}(-1)^j\right) + (-1)^{p_1+p_2}F_{2n-2v-2} = \begin{cases} L_{2n-2v} & \text{if } p_1 \text{ is odd}, p_2 \text{ is odd}, \\ F_{2n-2v-3} & \text{if } p_1 \text{ is odd}, p_2 \text{ is even}, \\ -F_{2n-2v} & \text{if } p_1 \text{ is even}, p_2 \text{ is odd}, \\ F_{2n-2v} & \text{if } p_1 \text{ is even}, p_2 \text{ is even}. \end{cases}$$

Recall that $v \geq u + p_2 + 1$.

Case 1. $v = u + p_2 + 1$. In this case $k_{p_1+i} = u + i$ for $i = 0, \ldots, p_2 + 1$.

Using (5), it is easy to see that

$$2(-1)^{v+u}F_{u-1}^2\left(F_{2n-2v-1}\left(1+2(-1)^{p_1+1}\sum_{j=p_1+1}^{q-1}(-1)^j\right) - (-1)^q F_{2n-2v-2}\right) \geq -2F_{u-1}^2 F_{2n-2v}$$

and thus

$$\begin{aligned} Q(u,v) &\geq -2F_{u-1}^2F_{2n-2v} - 2F_{2n-2v}\sum_{i=1}^{p_2}F_{u+i-3}F_{u+i} + F_{2n-2v-1}(F_v^2 - F_{v-1}^2) - F_{2n-2v-2}F_{v-3}F_v \\ &= F_{4v-2n-2} > 0, \end{aligned}$$

since, by assumption, $v > \lfloor (n+1)/2 \rfloor$, and thus $4v - 2n - 2 \geq 0$.

Case 2. Set $v = u + p_2 + a$ for some $a > 1$.

Case 2a We start by assuming $u + v$ is even and assume the worst case scenario, that p_1 is even and p_2

is odd, and we use (5) to obtain

$$Q(u,v) = -2F_{u-1}^2 F_{2n-2v} + 2(-1)^v F_{2n-2v} \sum_{i=p_1+1}^{p_1+p_2} (-1)^{k_i+i} F_{k_i-3} F_{k_i} + F_{2n-2v-1}(F_v^2 - F_{v-1}^2) - F_{2n-2v-2}(F_{v-3}F_v).$$

Further, the worst case scenario for the summed term (i.e., the bend placement which makes $\sum_{i=p_1+1}^{p_2+p_1} F_{k_i-3}F_{k_i}$ as large and negative as possible) is for $k_{p_1+i} = v - p_2 + i - 1$ for $i = 1, \ldots, p_2$. In this case, we have

$$\sum_{i=p_1+1}^{p_1+p_2} F_{k_i-3}F_{k_i} = \sum_{i=1}^{p_2} F_{v-p_2+i-4}F_{v-p_2+i-1} = \sum_{i=1}^{p_2} (F_{v-p_2+i-2}^2 - F_{v-p_2+i-3}) = F_{v-2}^2 - F_{v-p_2-2}^2.$$

So,

$$\begin{aligned} Q(u,v) &\geq -2F_{u-1}^2 F_{2n-2v} - 2F_{2n-2v}(F_{v-2}^2 - F_{u+a-2}^2) + F_{2n-2v-1}(F_v^2 - F_{v-1}^2) - F_{2n-2v-2}(F_{v-1}^2 - F_{v-2}^2) \\ &= F_{4v-2n-2} + 2F_{2n-2v}(F_{u+a-2}^2 - 2F_{u-1}^2). \end{aligned}$$

Since, by assumption, $v > \lfloor (n+1)/2 \rfloor$, and thus $4v - 2n - 2 \geq 0$ we are done.

Case 2b. We now assume $u+v$ is odd and also assume the worst case scenario, that both p_1 and p_2 are odd, and we use (5) to obtain

$$Q(u,v) = -2F_{u-1}^2 L_{2n-2v} + 2(-1)^v F_{2n-2v} \sum_{i=p_1+1}^{p_1+p_2} (-1)^{k_i+i} F_{k_i-3} F_{k_i} + F_{2n-2v-1}(F_v^2 - F_{v-1}^2) - F_{2n-2v-2}(F_{v-3}F_v).$$

Further, the worst case scenario for the summed term (i.e., the bend placement which makes $\sum_{i=p_1+1}^{p_2+p_1} F_{k_i-3}F_{k_i}$ as large and negative as possible) is for $k_{p_1+i} = v - p_2 + i - 1$ for $i = 1, \ldots, p_2$. In this case, we have

$$\sum_{i=p_1+1}^{p_1+p_2} F_{k_i-3}F_{k_i} = \sum_{i=1}^{p_2} F_{v-p_2+i-4}F_{v-p_2+i-1} = \sum_{i=1}^{p_2} (F_{v-p_2+i-2}^2 - F_{v-p_2+i-3}) = F_{v-2}^2 - F_{v-p_2-2}^2.$$

So,

$$\begin{aligned} Q(u,v) &\geq -F_{u-1}^2 L_{2n-2v} - 2F_{2n-2v}(F_{v-2}^2 - F_{u+a-2}^2) + F_{2n-2v-1}(F_v^2 - F_{v-1}^2) - F_{2n-2v-2}(F_{v-1}^2 - F_{v-2}^2) \\ &= -2F_{u-1}^2 L_{2n-2v} + F_{4v-2n-2} + 2F_{2n-2v}F_{u+a-2}^2. \end{aligned}$$

Here, we note that since $u+v$ is odd by assumption, so is a. Thus, we have

$$\begin{aligned} Q(u,v) &\geq -2F_{u-1}^2 L_{2n-2v} + F_{4v-2n-2} + 2F_{2n-2v}F_{u+a-2}^2 \\ &= F_{4v-2n-2} + 2F_{2n-2v}((F_{u-1}F_{u+a-1} + F_{u-2}F_{u+a})F_{a-1}) - 4F_{2n-2v-1}F_{u-1}^2 \\ &\geq F_{4v-2n-2} + 2F_{2n-2v}(2F_{u-1}^2) - 4F_{2n-2v-1}F_{u-1}^2 \geq 0, \end{aligned}$$

since $F_{a-1} > 2$.

□

Corollary 3.5. *Given a linear 2-tree G with n vertices and p bends $r_G(1,n) > r_G(i,j)$ for any $\{i,j\} \neq \{1,n\}$.*

3.2. *Resistance distance between fixed vertices on a linear 2-tree with fixed diameter.* The goal of this subsection is to show that placing a bend at the location $k = 4$ (or, by symmetry, $k = n-2$) minimizes the effective resistance between the end vertices in the bent linear 2-tree. We also provide empirical evidence that in a linear 2-tree with p bends and n vertices, the bends should be placed, consecutively, at either end of G in order to minimize $r_G(1,n)$. Our main result requires two lemmas giving new Fibonacci identities which we first provide.

Lemma 3.6. *For $k = 3,4,\dots,n-2$,*

$$\sum_{j=3}^{k} [(-1)^j F_{n-2j+1}(F_n + F_{j-2}F_{n-j-1})] = -F_{k-2}F_{k+1}F_{n-k-2}F_{n+1-k}. \tag{6}$$

Proof. It is easy to verify that (6) holds for $k = 3$. For arbitrary k the equality can be shown through algorithmic techniques as shown in [17]. □

Lemma 3.7. *Given $n \geq 8$, let $g(j) = F_{n-2j+1}(F_n + F_{j-2}F_{n-j-1})$, where F_p is the pth Fibonacci number. If n is even then*

$$\begin{cases} g(j) > g(j+1) & \text{for } 3 \leq j < n/2, \\ g(j) = g(j+1) & \text{if } j = n/2, \text{ and} \\ -g(j) > -g(j+1) & \text{for } n/2 < j \leq n-3. \end{cases}$$

If n is odd then $g(j) > g(j+1)$ for all j.

Proof. Algorithmic techniques for Fibonacci numbers ([17]) can be used to verify that

$$g(j) - g(j+1) = F_{n-2j}(F_{j+1}F_{n-j-2} + F_jF_{n-j-1} + F_{j-2}F_{n-j-1} + F_jF_{n-j}).$$

Observe that for $3 \leq j \leq n-3$ we have $F_{j+1}F_{n-j-2} + F_jF_{n-j-1} + F_{j-2}F_{n-j-1} + F_jF_{n-j} > 0$. If n is even then $F_{n-2j} > 0$ if $j < n/2$, $F_{n-2j} = 0$ if $j = n/2$ and $F_{n-2j} < 0$ if $n/2 < j < n-3$. If n is odd $F_{n-2j} > 0$ for all j such that $3 \leq j \leq n-3$. Hence we have shown the claim. □

We now state and prove our main result for this section.

Theorem 3.8. *Given a bent linear 2-tree G_k with n vertices and one bend, the location k of the bend that minimizes $r_{G_k}(1,n)$ is $k = 4$ (and also $n-2$ by symmetry). In this case*

$$r_{G_k}(1,n) = \frac{n-1}{5} + \frac{4F_{n-1}}{5L_{n-1}} - \frac{F_{n-5}(F_n + F_{n-4})}{F_{2n-2}}, \tag{7}$$

where F_p is the pth Fibonacci number and L_q is the qth Lucas number.

Proof. Due to symmetry we will only consider bends locations k with $4 \leq k \leq \lfloor (n+2)/2 \rfloor + 1$.

By Theorem 2.9 and Lemma 3.6 the formula for the resistance distance between node 1 and node n in a bent linear 2-tree with n vertices and one bend located at vertex $k \in \{4,5,\dots,n-2\}$ is given by

$$r_{G_k}(1,n) = \frac{n-1}{5} + \frac{4F_{n-1}}{5L_{2n-2}} + \frac{\sum_{j=3}^{k-1}\left[(-1)^j F_{n-2j+1}(F_n + F_{j-2}F_{n-j-1})\right]}{F_{2n-2}}. \tag{8}$$

We consider the final term in the sum, that is

$$\frac{\sum_{j=3}^{k-1}\left[(-1)^j F_{n-2j+1}(F_n + F_{j-2}F_{n-j-1})\right]}{F_{2n-2}},$$

and observe that the denominator is constant for a fixed n. Moreover, the numerator is an alternating sum where the first term in the sum is negative and the absolute value of each term in the sum is equal to $g(k)$ where g is defined as in Lemma 3.7.

From this same lemma we know that $g(j) > g(j+1)$ for $3 \le j \le \lfloor n/2 \rfloor$. Hence $r_{G_4}(1,n) < r_{G_\ell}(1,n)$ for integers ℓ such that $5 \le k \le \lfloor n/2 \rfloor$.

□

This result invites several observations and conjectures. The first observation can be seen by considering Theorem 2.9, and noting that the addition of bends always results in a decrease in the resistance distance between the extremal points in the graph, and that the resistance distance either decreases or remains the same between other pairs of vertices.

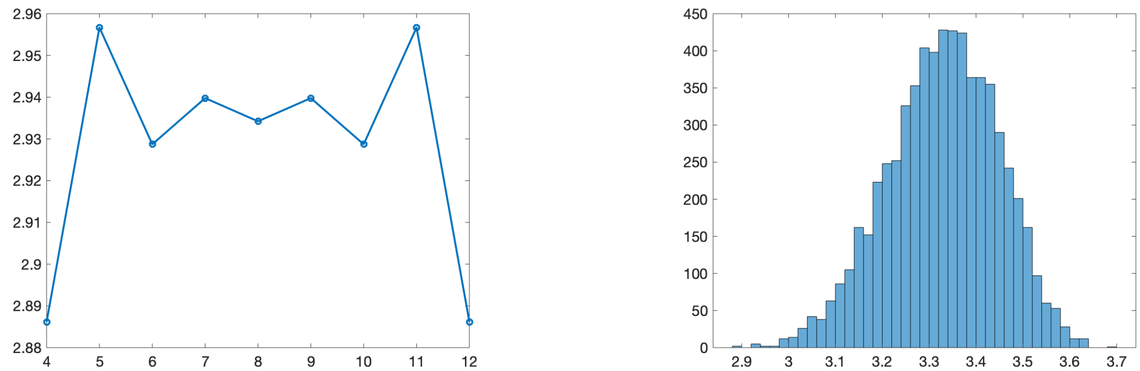
This observation motivates Question 3.2. A preliminary step toward answering this question is to first assume that the p bends are placed consecutively (i.e., at nodes $k_{i+1}, k_{i+2}, \dots, k_{i+p}$). In Figure 4a we consider this question for the case of the linear 2–tree with 20 nodes and 7 bends. As can be seen, clustering the bends at the two ends of the linear 2–tree results in the lowest maximal resistance distance. We also observe that translating the locations of all bends by one results in the same oscillatory behavior seen for the placement of a single bend in the linear 2–tree (see Equation 8, for example).

Next, we consider the question of where p bends can be placed in a linear 2–tree to minimize the resistance distance between the extremal vertices, without the "clustering" constraint we imposed in the previous paragraph. For a linear 2–tree with n nodes and p bends there are $\binom{n-5}{p}$ choices of node locations. In Figure 4b we display a histogram of the resistance distance between the extremal points for a linear 2-tree with 20 nodes and 7 bends using all 6,435 bend location choices. The bin on the far left, i.e., the bin corresponding to the lowest resistance distance has two entries, corresponding to placing p consecutive nodes at the two ends of the linear 2–tree. Empirically this holds true for every value of n and every value of p which inspires the following conjecture.

Conjecture 3.9. *Given a bent linear 2-tree with n vertices and p bends, the location of the bends that minimizes the maximal effective resistance between the end vertices is* $k_1 = 4$, $k_2 = 5, \dots, k_p = p+3$. *In this case*

$$r_G(1,n) = \frac{n-1}{5} + \frac{4F_{n-1}^2}{5F_{2n-2}} - \frac{\sum_{j=1}^{p}\left[F_{8+2j-5} - 2\right]F_{n-j-6}F_{n-j-3}}{F_{2n-2}}, \tag{9}$$

where F_p is the pth Fibonacci number.



(A) Resistance distance for a linear 2-tree on 20 nodes with 7 consecutive bends between the extremal vertices. The x–axis show the location of the first bend. Note that the graph is symmetric and that the resistance distance between the extremal vertices oscillates as we increase the index of the starting node.

(B) A histogram of resistance distance values between the extremal vertices of a linear 2–tree with 20 nodes and 7 bends. We note that the left most bin contains 2 entries corresponding to placing all of the bends consecutively at one or the other end of the linear 2–tree.

FIGURE 4. A comparison of resistance distance in linear 2–trees as the bend location is varied.

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