THE BSE-PROPERTY FOR VECTOR-VALUED L^p-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a separable Banach algebra, G be a locally compact Hausdorff group and $1 < p < \infty$. In this paper, we first provide a necessary and sufficient condition, for which $L^p(G,\mathcal{A})$ is a Banach algebra, under convolution product. Then we characterize the character space of $L^p(G,\mathcal{A})$, in the case where $\mathcal{A}$ is commutative and G is abelian. Moreover, we investigate the BSE-property for $L^p(G,\mathcal{A})$ and prove that $L^p(G,\mathcal{A})$ is a BSE-algebra if and only if $\mathcal{A}$ is a BSE-algebra and G is finite. Finally, we study the BSE-norm property for $L^p(G,\mathcal{A})$ and show that if $L^p(G,\mathcal{A})$ is a BSE-norm algebra then $\mathcal{A}$ is so. We prove the converse of this statement for the case where G is finite and $\mathcal{A}$ is unital.

1. Introduction and preliminaries

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a without order commutative Banach algebra, in the sense that $a\mathcal{A} = \{0\}$ implies $a = 0$ $(a \in \mathcal{A})$. We denote by $\Delta(\mathcal{A})$, the space consisting of all nonzero multiplicative linear functionals on $\mathcal{A}$, called the character space of $\mathcal{A}$. Throughout the paper, we assume that $\Delta(\mathcal{A})$ is nonempty. It shoulde be noted that $\Delta(\mathcal{A})$, equipped with the weak*topology, inherited from $\mathcal{A}^*$, is a locally compact Hausdorff space. We denote by $C_b(\Delta(\mathcal{A}))$ the space consisting of all bounded and continuous functions on $\Delta(\mathcal{A})$. A bounded net $\{e_\alpha\}_{\alpha\in I}$ in $\mathcal{A}$, is called a bounded Δ-weak approximate identity for $\mathcal{A}$ if $\lim_\alpha \varphi(ae_\alpha) = \varphi(a)$, for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$; see [18]. Following [21], a bounded linear operator T on $\mathcal{A}$, satisfying $T(ab) = aT(b)$, $(a, b \in \mathcal{A})$, is called a multiplier. The set of all multipliers on $\mathcal{A}$ is denoted by $M(\mathcal{A})$, which is a unital commutative Banach algebra, called the multiplier algebra of $\mathcal{A}$. By [21, Theorem 1.2.2], for any $T \in M(\mathcal{A})$, there exists a unique function $\widehat{T} \in C_b(\Delta(\mathcal{A}))$ such that

$$\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\widehat{a}(\varphi),$$

for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. A bounded and continuous function σ on $\Delta(\mathcal{A})$ is called a BSE-function if there exists a constant $\beta > 0$ such that following

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the inequality holds:

$$\left|\sum_{k=1}^{n} \alpha_k \sigma(\varphi_k)\right| \leq \beta \left\|\sum_{k=1}^{n} \alpha_k \varphi_k\right\|_{\mathcal{A}^*}, \tag{1.1}$$

for any finite number of $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and the same number of $\varphi_1, ..., \varphi_n \in \Delta(\mathcal{A})$. The BSE-norm of σ is the infimum of all β, which satisfying (1.1) and will be denoted by $\|\sigma\|_{BSE}$. By [27], $C_{BSE}(\Delta(\mathcal{A}))$, the space consisting of all BSE-functions on $\Delta(\mathcal{A})$, is a commutative and semisimple Banach algebra, equipped with the norm $\|\cdot\|_{BSE}$ and pointwise product. Then $\mathcal{A}$ is called a BSE-algebra (or has the BSE-property) $\widehat{M(\mathcal{A})} = C_{BSE}(\Delta(\mathcal{A}))$, where $\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}$. By [27, Corollary 5], $\widehat{M(\mathcal{A})} \subseteq C_{BSE}(\Delta(\mathcal{A}))$, if and only if $\mathcal{A}$ has a bounded Δ-weak approximate identity. It follows that all BSE-algebras possesses a bounded Δ-weak approximate identity. The Gelfand mapping $\Gamma_{\mathcal{A}} : \mathcal{A} \to C_b(\Delta(\mathcal{A}))$ is defined by $a \mapsto \widehat{a}$, such that $\widehat{a}$ is the Gelfand transform of a. The Banach algebra $\mathcal{A}$ is called semisimple if $ker(\Gamma_{\mathcal{A}}) = \{0\}$. It should be noted that we always have $\widehat{\mathcal{A}} \subseteq C_{BSE}(\Delta(\mathcal{A}))$ and for any $a \in \mathcal{A}$,

$$\|\widehat{a}\|_\infty \leq \|\widehat{a}\|_{BSE} \leq \|a\|_{\mathcal{A}},$$

where $\widehat{\mathcal{A}}$ is the range of Gelfand mapping of $\mathcal{A}$. Let $\mathcal{M}(\mathcal{A})$ is the normed algebra, consisting of all $\Phi \in C_b(\Delta(\mathcal{A}))$ such that $\Phi \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$. If $\mathcal{A}$ is semisimple then $\widehat{M(\mathcal{A})} = \mathcal{M}(\mathcal{A})$ [21, p. 30]. Consequently, a semisimple and commutative Banach algebra $\mathcal{A}$ is a BSE-algebra if $C_{BSE}(\Delta(\mathcal{A})) = \mathcal{M}(\mathcal{A})$.

BSE-algebras and BSE functions were introduced and investigated by Takahashi and Hatori [27], and then by some other authors, for various kinds of Banach algebras. The acronym "BSE" stands for Bochner-Schoenberg-Eberlein and refers to a famous theorem, for the additive group of real numbers, proved by Bochner and Schoenberg [5, 25]. Then the result was generalized by Eberlein [11], for an abelian locally compact group G, which is indicating the BSE-property of the group algebra $L^1(G)$ [22]. This result, has led Takahashi and Hatori [27] to introduce the BSE-property for any commutative and without order Banach algebra $\mathcal{A}$. The interested reader is referred to [1], [2], [8], [12], [13], [14], [15], [19], [20], [28], and [29].

In this paper, we investigate the BSE-property for $L^p(G, \mathcal{A})$. In fact, we first provide a necessary and sufficient condition, under which $L^p(G, \mathcal{A})$ is a Banach algebra, with convolution product. In fact, we generalize L^p-conjecture [24], for the vector-valued case. We also characterize the character space of $L^p(G, \mathcal{A})$, in the case where $\mathcal{A}$ is commutative and G is abelian. Moreover, we prove that $L^p(G, \mathcal{A})$ is a BSE-algebra if and only if $\mathcal{A}$ is a BSE-algebra and G is finite. Finally, we study the BSE-norm property for $L^p(G, \mathcal{A})$ and show that if $L^p(G, \mathcal{A})$ is a BSE-norm algebra then $\mathcal{A}$ is so. We prove the converse of this statement for the case where G is finite and $\mathcal{A}$ is unital.

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2. Some basic results

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra, G be a locally compact Hausdorff group with a left Haar measure λ and $1 < p < \infty$. A function $f : G \longrightarrow \mathcal{A}$ is called strongly measurable, if f is Borel measurable and also $f(G)$ is separable in $\mathcal{A}$. Thus in the case where $\mathcal{A}$ is separable, the concept of measurability and strong measurability are equivalent. Most of the properties of integral theory in the complex version, are also valid for the vector-valued case. We refer to [10], as a complete survey in this issue.

Throughout the paper, $\mathcal{A}$ is a Banach algebra, G is a locally compact Hausdorff group with a left Haar measure λ and $1 \le p < \infty$. In the case where $1 < p < \infty$, we assume in addition that $\mathcal{A}$ is separable. For any Borel measurable function $f : G \to \mathcal{A}$, let

$$\|f\|_{p,\mathcal{A}} = \left(\int_G \|f(x)\|_{\mathcal{A}}^p d\lambda(x)\right)^{\frac{1}{p}}.$$

Then $L^p(G, \mathcal{A})$ is the Banach space, consisting of all Borel measurable functions $f : G \longrightarrow \mathcal{A}$ such that $\|f\|_{p,\mathcal{A}} < \infty$. For each $f \in L^p(G, \mathcal{A})$, define $\overline{f}(x) = \|f(x)\|_{\mathcal{A}}$, for all $x \in G$. Then $f \in L^p(G, \mathcal{A})$ if and only if $\overline{f} \in L^p(G)$. In this case, $\|\overline{f}\|_p = \|f\|_{p,\mathcal{A}}$. Recall that for measurable vector-valued functions $f, g : G \longrightarrow \mathcal{A}$, the convolution multiplication

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\lambda(y),$$

is defined at each point $x \in G$ for which this makes sense.

In this section, we study some basic results about $L^p(G, \mathcal{A})$. We give a necessary and sufficient condition for that $L^p(G, \mathcal{A})$ is a Banach algebra under convolution product. Note that by a famous conjecture, proved by Saeki [24], $L^p(G)$ is closed under convolution product if and only if G is compact.

Theorem 2.1. *Let $\mathcal{A}$ be a separable Banach algebra, G be a locally compact Hausdorff group and $1 < p < \infty$. Then $L^p(G, \mathcal{A})$ is a Banach algebra under convolution product if and only if G is compact.*

Proof. Let $L^p(G, \mathcal{A})$ be a Banach algebra. To prove that G is compact, by [24] it is sufficient to show that $L^p(G)$ is closed under convolution. For any $f, g \in L^p(G)$, define $\widetilde{f}(x) = f(x)a$ and $\widetilde{g}(x) = g(x)a$, where $a \in \mathcal{A}$ with

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$\|a\| = 1$. Thus

$$\begin{aligned}\left\|\widetilde{f} * \widetilde{g}\right\|_{p,\mathcal{A}}^p &= \int_G \left\|(\widetilde{f} * \widetilde{g})(x)\right\|_{\mathcal{A}}^p d\lambda(x) \\ &= \int_G \left\|\int_G \widetilde{f}(y)\widetilde{g}(y^{-1}x)d\lambda(y)\right\|_{\mathcal{A}}^p d\lambda(x) \\ &= \int_G \left\|\int_G f(y)g(y^{-1}x)a^2 d\lambda(y)\right\|_{\mathcal{A}}^p d\lambda(x) \\ &= \int_G |(f*g)(x)|^p \left\|a^2\right\|_{\mathcal{A}}^p d\lambda(x) \\ &= \|a^2\|_{\mathcal{A}}^p \, \|f*g\|_p^p.\end{aligned}$$

Consequently,

$$\|\widetilde{f} * \widetilde{g}\|_{p,\mathcal{A}} = \|a^2\|_{\mathcal{A}} \, \|f*g\|_p. \tag{2.1}$$

By the hypothesis, we have

$$\|\widetilde{f} * \widetilde{g}\|_{p,\mathcal{A}} \leq \|\widetilde{f}\|_{p,\mathcal{A}} \, \|\widetilde{g}\|_{p,\mathcal{A}}.$$

Thus by the equality (2.1) we obtain

$$\|a^2\|_{\mathcal{A}} \, \|f*g\|_p \leq \|a\|_{\mathcal{A}} \, \|f\|_p \, \|a\|_{\mathcal{A}} \, \|g\|_p = \|f\|_p \, \|g\|_p$$

and so

$$\|f*g\|_p \leq \tfrac{1}{\|a^2\|_{\mathcal{A}}} \|f\|_p \, \|g\|_p.$$

It follows that $L^p(G)$ is closed under convolution, which implies that G is compact.

Conversely, suppose that G is compact. Thus $L^p(G)$ is a Banach algebra under convolution product. For all $f, g \in L^p(G, \mathcal{A})$, we have $\overline{f}, \overline{g} \in L^p(G)$ and

$$\|\overline{f} * \overline{g}\|_p^p = \int_G \left|\int_G \|f(y)\|_A \, \|g(y^{-1}x)\|_{\mathcal{A}} d\lambda(y)\right|^p d\lambda(x).$$

Moreover, by the properties of Bochner integrals, we obtain

$$\begin{aligned}\|f*g\|_{p,\mathcal{A}}^p &= \int_G \left\|\int_G f(y)g(y^{-1}x)d\lambda(y)\right\|_{\mathcal{A}}^p d\lambda(x) \\ &\leq \int_G \left(\int_G \|f(y)\|_{\mathcal{A}} \, \|g(y^{-1}x)\|_{\mathcal{A}} d\lambda(y)\right)^p d\lambda(x) \\ &= \|\overline{f} * \overline{g}\|_p^p.\end{aligned}$$

Consequently,

$$\|f*g\|_{p,\mathcal{A}} \leq \|\overline{f} * \overline{g}\|_p \leq \|\overline{f}\|_p \, \|\overline{g}\|_p = \|f\|_{p,\mathcal{A}} \|g\|_{p,\mathcal{A}}.$$

Therefore $L^p(G, \mathcal{A})$ is a Banach algebra, under convolution product. □

The following corollary is obtained by Theorem 2.1 and [24], immediately.

Corollary 2.2. *Let $\mathcal{A}$ be a separable Banach algebra, G be a locally compact Hausdorff group and $1 < p < \infty$. Then the following statements are equivalent.*

(i) *$L^p(G, \mathcal{A})$ is a Banach algebra, under convolution product.*
(ii) *$L^p(G)$ is a Banach algebra, under convolution product.*
(iii) *G is compact.*

Remark 2.3. Let $\mathcal{A}$ be a separable Banach algebra, G be a compact Hausdorff group and $1 \leq p < \infty$. By the Holder's inequality, we have $L^p(G) \subseteq L^1(G)$ and

$$\|f\|_1 = \int_G |f(x)| d\lambda(x) \leq \left(\int_G |f(x)|^p d\lambda(x) \right)^{1/p} = \|f\|_p,$$

for all $f \in L^p(G)$. Now suppose that $\mathcal{A}$ is a separable Banach algebra and $f \in L^p(G, \mathcal{A})$. Then

$$\|f\|_{1,\mathcal{A}} = \int_G ||f(x)|| d\lambda(x) \leq \left(\int_G ||f(x)||^p d\lambda(x) \right)^{1/p} = \|f\|_{p,\mathcal{A}}.$$

It follows that $f \in L^1(G, \mathcal{A})$ and $\|f\|_{1,\mathcal{A}} \leq \|f\|_{p,\mathcal{A}}$. In fact, we obtain $L^p(G, \mathcal{A}) \subseteq L^1(G, \mathcal{A})$.

It is known that $C_{00}(G)$, the space consisting of all complex-valued continuous functions on G with compact support, is dense in $L^p(G)$ $(1 \leq p < \infty)$. Now let

$$C_{00}(G, \mathcal{A}) = \{f : G \longrightarrow \mathcal{A}, \ \text{f is continuous with compact support}\}.$$

We denote it by $C(G, \mathcal{A})$, whenever G is compact. In the next result, we investigate the same statement for the vector-valued case. First, we introduce some notations, which we require for convenience. For any $f \in L^p(G)$ and $a \in \mathcal{A}$, we denote by $f \otimes a$ the function, defined as

$$f \otimes a(x) = f(x)a \qquad (x \in G).$$

Then $f \in L^p(G, \mathcal{A})$ and

$$\|f \otimes a\|_{p,\mathcal{A}} = \|f\|_p \|a\|_{\mathcal{A}}.$$

It should be noted that if $f \in C(G)$, then $f \otimes a$ belongs to $C(G, \mathcal{A})$. Let $C(G) \otimes \mathcal{A}$ be the vector space, generating by all $f \otimes a$, where $f \in C(G)$ and $a \in \mathcal{A}$. It is easily verified that if G is compact, then

$$C(G) \otimes \mathcal{A} \subseteq C(G, \mathcal{A}) \subseteq L^p(G, \mathcal{A}) \subseteq L^1(G, \mathcal{A}). \tag{2.2}$$

Proposition 2.4. *Let $\mathcal{A}$ be a separable Banach algebra and G be a compact Hausdorff group. Then $C(G) \otimes \mathcal{A}$ is dense in $L^1(G, \mathcal{A})$.*

Proof. Take $f \in L^1(G, \mathcal{A})$. By [20, Proposition 1.5.4.], there exist the sequences $\{f_n\}_n$ in $L^1(G)$ and also $\{a_n\}_n$ in $\mathcal{A}$ such that $f = \sum_{n=1}^{\infty} f_n \otimes a_n$

and $\sum_{n=1}^{\infty}\|f_n\|_1\|a_n\|_{\mathcal{A}} < \infty$. Since $C(G)$ is dense $L^1(G)$, for each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $g_n \in C(G)$ such that

$$\|g_n - f_n\|_1 < \frac{\varepsilon}{2^{n+1}(\|a_n\|_{\mathcal{A}} + 1)}.$$

Define $g = \sum_{n=1}^{\infty} g_n \otimes a_n$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \|g_n\|_1 \|a_n\|_{\mathcal{A}} &\leq \sum_{n=1}^{\infty} \|f_n - g_n\|_1 \|a_n\|_{\mathcal{A}} + \sum_{n=1}^{\infty} \|f_n\|_1 \|a_n\|_{\mathcal{A}} \\
&\leq \sum_{n=1}^{\infty} \frac{\varepsilon \|a_n\|_{\mathcal{A}}}{2^{n+1}(\|a_n\|_{\mathcal{A}} + 1)} + \sum_{n=1}^{\infty} \|f_n\|_1 \|a_n\|_{\mathcal{A}} \\
&\leq \varepsilon + \sum_{n=1}^{\infty} \|f_n\|_1 \|a_n\|_{\mathcal{A}} \\
&< \infty,
\end{aligned}$$

which implies that $g \in L^1(G, \mathcal{A})$. Moreover,

$$\begin{aligned}
\|g - f\|_{1,\mathcal{A}} &= \left\| \sum_{n=1}^{\infty} (g_n - f_n) \otimes a_n \right\|_{1,\mathcal{A}} \\
&\leq \sum_{n=1}^{\infty} \|g_n - f_n\|_1 \|a_n\|_{\mathcal{A}} \\
&\leq \sum_{n=1}^{\infty} \frac{\varepsilon \|a_n\|_{\mathcal{A}}}{2^{n+1}(\|a_n\|_{\mathcal{A}} + 1)} \\
&< \varepsilon.
\end{aligned}$$

Now let $h_n = \sum_{k=1}^{n} g_k \otimes a_k$ $(n \in \mathbb{N})$. Then $\{h_n\}_n \subseteq C(G) \otimes \mathcal{A}$ and

$$\lim_n \|h_n - g\|_{1,\mathcal{A}} = 0.$$

It follows that there exists $n \in \mathbb{N}$ such that $\|h_n - g\|_{1,\mathcal{A}} < \varepsilon$. Hence, we obtain

$$\|f - h_n\|_{1,\mathcal{A}} \leq \|f - g\|_{1,\mathcal{A}} + \|g - h_n\|_{1,\mathcal{A}} < 2\varepsilon.$$

This completes the proof. □

Proposition 2.5. *Let $\mathcal{A}$ be a separable commutative Banach algebra, G be an abelian compact Hausdorff group and $1 < p < \infty$. Then $L^p(G, \mathcal{A})$ is a dense ideal in $L^1(G, \mathcal{A})$.*

Proof. Proposition 2.4 together with the inclusions (2.2) imply that $L^p(G, \mathcal{A})$ is dense in $L^1(G, \mathcal{A})$. We show that $L^p(G, \mathcal{A})$ is an ideal in $L^1(G, \mathcal{A})$. To this end, let $f \in L^1(G, \mathcal{A})$ and $g \in L^p(G, \mathcal{A})$. We indicate that $f * g \in L^p(G, \mathcal{A})$. Since $\overline{f} \in L^1(G)$ and $\overline{g} \in L^p(G)$, together with the fact that $L^p(G)$ is an ideal in $L^1(G)$, we obtain $\overline{f} * \overline{g} \in L^p(G)$. Moreover,

$$\|f * g\|_{p,\mathcal{A}} \leq \|\overline{f} * \overline{g}\|_p < \infty.$$

It follows that $f * g \in L^p(G, \mathcal{A})$, as claimed. □

The next result is obtained from [4, Theorem 2.3]. We refer to [4], for the general definition of abstract Segal algebras.

Corollary 2.6. *Let $\mathcal{A}$ be a separable commutative Banach algebra, G be an abelian compact Hausdorff group and $1 \leq p < \infty$. Then $L^p(G, \mathcal{A})$ is an abstract Segal algebra in $L^1(G, \mathcal{A})$.*

Remark 2.7. Let $\mathcal{A}$ be a separable commutative Banach algebra with a bounded approximate identity, G be an abelian compact Hausdorff group and $1 < p < \infty$. Then by [7, Theorem 2.9.21] and [20, Proposition 1.5.4], $L^1(G, \mathcal{A})$ possesses a bounded approximate identity. Now Cohen factorization theorem implies that $L^1(G, \mathcal{A}) * L^p(G, \mathcal{A}) = L^p(G, \mathcal{A})$.

Let $\mathcal{A}$ be a commutative Banach algebra and G be an abelian locally compact Hausdorff group with dual group $\widehat{G}$, introduced in [22]. By [20, Proposition 1.5.4], $L^1(G, \mathcal{A})$ is isomorphic with the projective tensor product $L^1(G)$ and $\mathcal{A}$. Moreover, by [20, Theorem 2.11.2], the character space of $L^1(G, \mathcal{A})$ is homeomorphic with $\widehat{G} \times \Delta(\mathcal{A})$, such that for all $\chi \in \widehat{G}$ and $\varphi \in \Delta(\mathcal{A})$,

$$\chi \otimes \varphi : L^1(G)\widehat{\otimes}\mathcal{A} \longrightarrow \mathbb{C},$$

defined as,

$$(\chi \otimes \varphi)(f \otimes a) = \chi(f)\varphi(a) \quad (f \in L^1(G), a \in \mathcal{A}).$$

Moreover, for all $f \in L^1(G, \mathcal{A})$, we have

$$(\chi \otimes \varphi)(f) = \varphi\left(\int_G \overline{\chi(x)} f(x) d\lambda(x)\right) = \int_G \overline{\chi(x)} \varphi(f(x)) d\lambda(x).$$

Now Corollary 2.6 together with [3, Lemma 2.2] yield the next result.

Proposition 2.8. *Let $\mathcal{A}$ be a separable commutative Banach algebra, G be an abelian compact Hausdorff group and $1 < p < \infty$. Then*

$$\Delta(L^p(G, \mathcal{A})) = \{(\chi \otimes \varphi)|_{L^p(G,\mathcal{A})} : \ \chi \in \widehat{G}, \varphi \in \Delta(\mathcal{A})\}.$$

3. $L^p(G, \mathcal{A})$ as a BSE-algebra

Let G be an abelian compact Hausdorff group and $1 < p < \infty$. Then $L^p(G)$ is a commutative Banach algebra, which is an ideal in its second dual. Now by [19, Theorem 3.1], the following assertions are equivalent.

(i) $L^p(G)$ is a BSE-algebra.
(ii) $L^p(G)$ possesses a bounded $\Delta-$weak approximate identity.
(iii) $L^p(G)$ admits a bounded approximate identity.

Since $L^p(G)$ is reflexive, the above equivalent conditions imply that $L^p(G)$ is unital and so G is finite.

In this section, we investigate the BSE-property for $L^p(G, \mathcal{A})$. We commence with the following proposition.

Proposition 3.1. *Let $\mathcal{A}$ be a separable commutative Banach algebra, G be an abelian compact Hausdorff group and $1 < p < \infty$. Then $L^p(G, \mathcal{A})$ is semisimple if and only if $\mathcal{A}$ is semisimple.*

Proof. Let $L^p(G,\mathcal{A})$ be semisimple. Then $\Delta(L^p(G,\mathcal{A})) = \widehat{G} \times \Delta(\mathcal{A})$ separates the points of $L^p(G,\mathcal{A})$. Take $a, b \in \mathcal{A}$ with $a \neq b$. Then

$$1_G \otimes a \neq 1_G \otimes b,$$

where 1_G is the constant function 1 on G. By the assumption, there exist $\chi \in \widehat{G}$ and $\varphi \in \Delta(\mathcal{A})$ such that

$$(\chi \otimes \varphi)(1_G \otimes a) \neq (\chi \otimes \varphi)(1 \otimes b).$$

Consequently,

$$\chi(1)\varphi(a) \neq \chi(1)\varphi(b),$$

and so $\varphi(a) \neq \varphi(b)$. Thus $\mathcal{A}$ is semisimple.

Conversely, suppose that $\mathcal{A}$ is semisimple and take $f, g \in L^p(G,\mathcal{A})$ with $f \neq g$. Then $\|f - g\|_{p,\mathcal{A}} \neq 0$. Thus $\lambda(\{x \in G : f(x) \neq g(x)\}) \neq 0$. It follows that $\|f - g\|_{1,\mathcal{A}} \neq 0$. By the semisimplicity of $L^1(G,\mathcal{A})$ [20, Theorem 2.11.8], there exists $\psi \in \Delta(L^1(G,\mathcal{A})) = \Delta(L^p(G,\mathcal{A}))$ such that $\psi(f) \neq \psi(g)$. It follows that $L^p(G,\mathcal{A})$ is semisimple. □

The next proposition is applied in some further results. The proof is straightforward and left to the reader.

Proposition 3.2. *Let $\mathcal{A}$ be a separable commutative unital Banach algebra with the identity element e such that $\|e\|_{\mathcal{A}} = 1$, G be an abelian compact Hausdorff group and $1 \leq p < \infty$. Then for any $\chi_1, \cdots, \chi_n \in \widehat{G}$ and $\varphi_1, \cdots, \varphi_n \in \Delta(\mathcal{A})$ and the same number of $c_1, \cdots, c_n \in \mathbb{C}$, we have*

$$\left\| \sum_{i=1}^{n} c_i \chi_i \right\|_{L^p(G)^*} \leq \left\| \sum_{i=1}^{n} c_i (\chi_i \otimes \varphi_i) \right\|_{L_p(G,\mathcal{A})^*}$$

and

$$\left\| \sum_{i=1}^{n} c_i \varphi_i \right\|_{\mathcal{A}^*} \leq \left\| \sum_{i=1}^{n} c_i (\chi_i \otimes \varphi_i) \right\|_{L_p(G,\mathcal{A})^*}$$

Proposition 3.3. *Let $\mathcal{A}$ be a separable commutative Banach algebra, G be an abelian compact Hausdorff group and $1 < p < \infty$. Then $L^p(G,\mathcal{A})$ has a bounded Δ-weak approximate identity if and only if $\mathcal{A}$ has so.*

Proof. First suppose that $\{f_\alpha\}_\alpha$ is a bounded Δ-weak approximate identity for $L^p(G,\mathcal{A})$, bounded by $M > 0$. Take $\chi_0 \in \widehat{G}$ to be fixed and for each α let

$$e_\alpha := \chi_0(f_\alpha) = \int_G f_\alpha(x)\overline{\chi_0(x)}d\lambda(x).$$

Then for each $\varphi \in \Delta(\mathcal{A})$, we have

$$\varphi(e_\alpha) = \int_G \varphi(f_\alpha(x))\overline{\chi_0(x)}d\lambda(x) = \chi_0(\varphi \circ f_\alpha) = \chi_0 \otimes \varphi(f_\alpha) \longrightarrow_\alpha 1.$$

Moreover,

$$\|e_\alpha\|_{\mathcal{A}} \leq \int_G \|f_\alpha(x)\|_{\mathcal{A}}|\overline{\chi_0(x)}|d\lambda(x) \leq M \int_G \overline{\chi_0(x)}d\lambda(x) \leq M\lambda(G).$$

Therefore $\{e_\alpha\}_\alpha$ is a bounded Δ-weak approximate identity for $\mathcal{A}$.

Conversely, let $\mathcal{A}$ has a bounded Δ-weak approximate identity, denoted by $\{e_\alpha\}_\alpha$ and $\{f_\beta\}_\beta$ be the bounded approximate identity for $L^1(G)$. Then for all $\varphi \in \Delta(\mathcal{A})$, $\varphi(e_\alpha) \longrightarrow 1$ and so for all $\chi \in \widehat{G}$, we have

$$(\chi \otimes \varphi)(f_\beta \otimes e_\alpha) \longrightarrow 1.$$

This implies that $(f_\beta \otimes e_\alpha)_{\alpha,\beta}$ is a bounded Δ-weak approximate identity for $L^p(G, \mathcal{A})$. □

We state here the main result of this section.

Theorem 3.4. *Let $\mathcal{A}$ be a semisimple commutative separable and unital Banach algebra, G be an abelian compact Hausdorff group and $1 < p < \infty$. Then $L^p(G, \mathcal{A})$ is a BSE-algebra if and only if $\mathcal{A}$ is a BSE-algebra and G is finite.*

Proof. Let $L^p(G, \mathcal{A})$ be a BSE-algebra. By [27, Corollary 5], $L^p(G, \mathcal{A})$ admits a bounded $\Delta-$weak approximate identity $\{f_\alpha\}_\alpha$, bounded by M. It follows that

$$\widehat{f_\alpha}(\chi \otimes \varphi) \longrightarrow_\alpha 1,$$

and so

$$\int_G \overline{\chi(x)}\varphi(f_\alpha(x))dx \longrightarrow_\alpha 1, \tag{3.1}$$

for all $\chi \in \widehat{G}$ and $\varphi \in \Delta(\mathcal{A})$. Now, take $\varphi_0 \in \Delta(\mathcal{A})$ to be fixed and define $g_\alpha(x) = \varphi_0(f_\alpha(x))$ $(x \in G)$. Thus

$$\begin{aligned} \left(\int |g_\alpha(x)|^p dx\right)^{1/p} &= \left(\int_G |\varphi_0(f_\alpha(x))|^p dx\right)^{1/p} \\ &\leq \|\varphi_0\| \left(\int_G \|f_\alpha(x)\|^p dx\right)^{1/p} \\ &\leq M. \end{aligned}$$

Consequently, $\{g_\alpha\}_\alpha$ is a bounded net in $L^p(G)$. Moreover, by (4.4), for all $\chi \in \widehat{G}$ we obtain

$$\widehat{g_\alpha}(\chi) = \chi(g_\alpha) = \int_G g_\alpha(x)\overline{\chi(x)}dx = \int_G \overline{\chi(x)}\varphi_0(f_\alpha(x)) \longrightarrow_\alpha 1.$$

Thus $\{g_\alpha\}_\alpha$ is a bounded $\Delta-$weak approximate identity for $L^p(G)$, which implies that G is finite. In the sequel, we show that $\mathcal{A}$ is a BSE-algebra.

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Proposition 3.3 implies that $\mathcal{A}$ has a bounded Δ-weak approximate identity. Thus [27, Corollary 5] implies that

$$\mathcal{M}(\mathcal{A}) \subseteq C_{BSE}(\Delta(\mathcal{A})). \tag{3.2}$$

Now, we prove that the reverse of the inclusion (3.2). Take $\sigma \in C_{BSE}(\Delta(\mathcal{A}))$ and define the function $\overline{\sigma} : \widehat{G} \times \Delta(A) \longrightarrow \mathbb{C}$ as $\overline{\sigma}(\chi \otimes \varphi) := \sigma(\varphi)$, for all $\chi \in \widehat{G}$ and $\varphi \in \Delta(\mathcal{A})$. We show that $\overline{\sigma} \in C_{BSE}(\Delta(L^p(G, \mathcal{A})))$. Note that since G is finite, thus $G = \widehat{G}$. Now Proposition 3.2 implies that for all complex numbers $c_1, \cdots, c_n$ and the same number of $\chi_1 \otimes \varphi_1, \cdots, \chi_n \otimes \varphi_n$ of $\widehat{G} \times \Delta(\mathcal{A})$,

$$\left|\sum_{i=1}^{n} c_i \overline{\sigma}(\chi_i \otimes \varphi_i)\right| \leq \|\sigma\|_{BSE} \left\|\sum_{i=1}^{n} c_i \varphi_i\right\|_{\mathcal{A}^*} \leq \|\sigma\|_{BSE} \left\|\sum_{i=1}^{n} c_i (\chi_i \otimes \varphi_i)\right\|_{L^p(G,\mathcal{A})^*}.$$

Consequently, $\overline{\sigma} \in C_{BSE}(\Delta(L^p(G, \mathcal{A}))) = \mathcal{M}(L^p(G, \mathcal{A}))$. Hence, for each $f \in C(G)$ and $a \in \mathcal{A}$, there exists $g \in L^p(G, \mathcal{A})$ such that $\overline{\sigma}\ (\widehat{f \otimes a}) = \widehat{g}$. It follows that

$$\sigma(\varphi)\ \chi(f)\varphi(a) = \chi(\varphi \circ g). \qquad (\chi \in \widehat{G}, \varphi \in \Delta(\mathcal{A}))$$

Taking $\chi \in \widehat{G}$ and $f \in C(G)$ with $\chi(f) = 1$, we obtain

$$\sigma(\varphi)\widehat{a}(\varphi) = \chi(\varphi \circ g) = \varphi\left(\int_G g(x)\overline{\chi(x)} d\lambda(x)\right) := \varphi(b),$$

where $b := \int_G g(x)\overline{\chi(x)} d\lambda(x)$. Consequently, $\sigma\widehat{a} = \widehat{b}$, which implies that $\sigma \in \mathcal{M}(\mathcal{A})$. It follows that $\mathcal{A}$ is a $BSE-$algebra.

Conversely, suppose that $\mathcal{A}$ is a BSE-algebra and G is finite. Thus by Proposition 3.3, $L^p(G, \mathcal{A})$ possesses a bounded Δ-weak approximate identity and so

$$\mathcal{M}(L^p(G, \mathcal{A})) \subseteq C_{BSE}(\Delta(L^p(G, \mathcal{A}))),$$

by [27, Corollary 5]. To prove the reverse of the inclusion, suppose that $\sigma \in C_{BSE}(\Delta(L^p(G, \mathcal{A})))$ and $f \in L^p(G, \mathcal{A})$. We should find $g \in L^p(G, \mathcal{A})$ such that $\sigma\widehat{f} = \widehat{g}$. Note that since G is finite, it follows that $G = \widehat{G}$. For each $\chi \in \widehat{G}$, define

$$\sigma_\chi : \Delta(\mathcal{A}) \to \mathbb{C}$$

with

$$\sigma_\chi(\varphi) = \sigma(\chi \otimes \varphi) \qquad (\varphi \in \Delta(\mathcal{A})).$$

It is easily verified that $\sigma_\chi \in C_b(\Delta(\mathcal{A}))$. Now we show that $\sigma_\chi \in C_{BSE}(\Delta(\mathcal{A}))$. For any $c_1, ..., c_n \in \mathbb{C}$ and the same number of $\varphi_1, ..., \varphi_n$ in $\Delta(\mathcal{A})$, we easily obtain

$$\left|\sum_{i=1}^{n} c_i \sigma_\chi(\varphi_i)\right| \leq \|\sigma\|_{BSE} \left\|\sum_{i=1}^{n} c_i \varphi_i\right\|_{\mathcal{A}^*}.$$

Consequently, $\sigma_\chi \in C_{BSE}(\Delta(\mathcal{A}))$ and since $\mathcal{A}$ is a BSE algebra, $\sigma_\chi \in \mathcal{M}(\mathcal{A})$. It follows that for each $\chi \in \widehat{G}$, there exists $a_\chi \in \mathcal{A}$ such that

$$\sigma_\chi \widehat{f(\chi)} = \widehat{a_\chi}.$$

Now define the function

$$g : G = \widehat{G} \to \mathcal{A} \qquad g(\chi) := a_\chi,$$

which belongs to $L^p(G, \mathcal{A})$. Finally, for each $\chi \otimes \varphi$ in $\Delta(L^p(G, \mathcal{A}))$, we have

$$\sigma(\chi \otimes \varphi)\widehat{f}(\chi \otimes \varphi) = \sigma_\chi(\varphi)\widehat{f(\chi)}(\varphi) = \widehat{g(\chi)}(\varphi) = \widehat{g}(\chi \otimes \varphi).$$

Consequently, $\sigma\widehat{f} = \widehat{g}$. It follows that $\sigma \in \mathcal{M}(L^p(G, \mathcal{A}))$. Therefore $L^p(G, \mathcal{A})$ is a BSE algebra. □

4. $L^p(G, \mathcal{A})$ as a BSE-norm algebra

It is known that in any commutative Banach algebra $\mathcal{A}$,

$$\|\widehat{a}\|_\infty \le \|\widehat{a}\|_{BSE} \le \|a\|_{\mathcal{A}}, \tag{4.1}$$

for all $a \in \mathcal{A}$. In [9], the authors investigated a class of commutative Banach algebras which satisfies the following condition

$$\|a\|_{\mathcal{A}} \le K\|\widehat{a}\|_{BSE} \qquad (a \in \mathcal{A}),$$

for some $K > 0$, called BSE-norm algebras. They indicates that $L^p(G)$ is always a BSE-norm algebra. As the final results, we show that this result is not necessarily valid for the vector-valued case. We commence with the following elementary lemma.

Lemma 4.1. *Let $\mathcal{A}$ be a separable and semisimple Banach algebra, G be a compact Hausdorff group and $1 \le p < \infty$. Then $L^p(G, \mathcal{A})$ is unital if and only if $\mathcal{A}$ is unital and G is finite.*

Proof. Suppose that $L^p(G, \mathcal{A})$ is unital. Thus $\widehat{G} \times \Delta(\mathcal{A})$ is compact and so $\Delta(\mathcal{A})$ and $\widehat{G}$ are compact. It follows that $\mathcal{A}$ is unital and G is discrete, which implies that G finite. The converse is similar. □

Theorem 4.2. *Let $\mathcal{A}$ be a semisimple commutative separable unital Banach algebra, G be an abelian compact Hausdorff group and $1 < p < \infty$. If $L^p(G, \mathcal{A})$ is a BSE-norm algebra then $\mathcal{A}$ is so. The converse is true, whenever G is finite and $\mathcal{A}$ is unital.*

Proof. By the hypothesis, there exists $M > 0$ such that

$$\|f\|_{p,\mathcal{A}} \le M\|\widehat{f}\|_{BSE} \qquad (f \in L^p(G, \mathcal{A})).$$

Taking $a \in \mathcal{A}$, for the constant fuction f_a we have

$$\|a\|_{\mathcal{A}} = \|f_a\|_{p,\mathcal{A}} \le M\|\widehat{f_a}\|_{BSE} \qquad (f \in L^p(G, \mathcal{A})). \tag{4.2}$$

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We show that $\|\widehat{f_a}\|_{BSE} \le \|\widehat{a}\|_{BSE}$. For $\chi_1 \otimes \varphi_1, \cdots, \chi_n \otimes \varphi_n \in \Delta(L^p(G, \mathcal{A}))$ and the same number of $c_1, \cdots, c_n \in \mathbb{C}$ we have

$$\begin{aligned} \left| \sum_{i=1}^n c_i \widehat{f_a}(\chi_i \otimes \varphi_i) \right| &= \left| \sum_{i=1}^n c_i (\chi_i \otimes \varphi_i)(f_a) \right| \\ &= \left| \sum_{i=1}^n \left(c_i \int_G \overline{\chi_i(x)} d\lambda(x) \right) \varphi_i(a) \right| \\ &\le \|\widehat{a}\|_{BSE} \left\| \sum_{i=1}^n \left(c_i \int_G \overline{\chi_i(x)} d\lambda(x) \right) \varphi_i \right\|_{\mathcal{A}^*}. \end{aligned} \tag{4.3}$$

On the other hand,

$$\begin{aligned} \left\| \sum_{i=1}^n c_i (\chi_i \otimes \varphi_i) \right\|_{L^p(G,\mathcal{A})^*} &= \sup_{\|f\|_{p,\mathcal{A}} \le 1} \left| \sum_{i=1}^n c_i \int_G \overline{\chi_i(x)} \varphi_i(f(\chi_i)) d\lambda(x) \right| \\ &\ge \sup_{\|a\|_{\mathcal{A}} \le 1} \left| \sum_{i=1}^n c_i \int_G \overline{\chi_i(x)} \varphi_i(a) d\lambda(x) \right| \\ &= \left\| \sum_{i=1}^n \left(c_i \int_G \overline{\chi_i(x)} d\lambda(x) \right) \varphi_i \right\|_{\mathcal{A}^*}. \end{aligned} \tag{4.4}$$

Now (4.3) and (4.4) imply that

$$\left| \sum_{i=1}^n c_i \widehat{f_a}(\chi_i \otimes \varphi_i) \right| \le \|\widehat{a}\|_{BSE} \left\| \sum_{i=1}^n c_i (\chi_i \otimes \varphi_i) \right\|_{L^p(G,\mathcal{A})^*}.$$

It follows that

$$\|\widehat{f_a}\|_{BSE} \le \|\widehat{a}\|_{BSE}. \tag{4.5}$$

By (4.2) and (4.5) we obtain

$$\|a\|_{\mathcal{A}} \le M \|\widehat{a}\|_{BSE},$$

which implies that $\mathcal{A}$ is a BSE-norm algebra.

Conversely, suppose that G is finite and $\mathcal{A}$ is unital. By Lemma 4.1, $L^p(G, \mathcal{A})$ is unital. Moreover, Theorem 3.4 implies that $L^p(G, \mathcal{A})$ is a BSE-algebra. Thus $L^p(G, \mathcal{A})$ is a BSE-norm algebra, by [9, Theorem page 40]. □

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