ON COARSE DIRECTED LIMITS OF METRIC SPACES

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ABSTRACT. It was shown in [1] that every coarse space is induced by a family of metrics defined on the underlying set of that coarse space. Motivated by this, we study the relation between prescribed properties of a coarse space and the corresponding system of metrics on the underlying set. In order to do this, we introduce the notion of coarse directed limits of metric spaces, which is, in general, *different* from metric space directed limits. We will show that every coarse space having asymptotic dimension dominated by an integer n (respectively, having FWCDC, being locally finite or being uniformly locally finite with property A) is a coarse directed limit of metric spaces which have asymptotic dimensions dominated by n (respectively, have FWCDC, are locally finite or are uniformly locally finite with property A). Moreover, the converse of this statement is true. More generally, we will also introduce and study coarse directed limits of general coarse spaces.

1. INTRODUCTION AND PRELIMINARY

Coarse structures were first introduced by Higson, Pedersen and Roe in [11] to study large scale geometric properties of metric spaces. In fact, the study of large scale geometric properties for metric spaces derives from Gromov's seminal work concerning finitely generated groups equipped with word-length metrics (see [8]), which has now become an important topic. An early famous theorem due to Gromov in [7] is that a finitely generated group has polynomial growth (which is a large scale property) if and only if it has a nilpotent subgroup of finite index. Latter, Yu proved in [19] that the Novikov Conjecture holds for finitely generated groups of finite asymptotic dimension (which is also a large scale property), which was then improved in [20] for finitely generated groups coarsely embeddable into Hilbert spaces (and this is again a large scale property). More generally, if a discrete metric space with bounded geometry has property A (as defined in [20]), then it admits a coarse embedding into a Hilbert space, which in turn implies the coarse Baum–Connes conjecture to be true for this space.

To abstractly quantify important features of metric spaces in a large scale perspective, Higson, Pedersen and Roe introduced in [11] the notion of coarse structure in an axiomatic approach, which was later refined by Roe in [15]. Many large scale geometric properties for metric spaces can be defined and discussed parallelly for coarse spaces. For example, it has been proved that property C implies property A for coarse spaces in [2], which extends its counterpart in the metric space setting (see [4]). These extensive researches lead to a great deal of work on general coarse structures. A good reference for coarse spaces is [15].

It was shown in [1] that every coarse structure on a set is induced by a system of metrics defined on the same set. This can be rephrased as saying that every coarse space is a coarse directed limit of a coarse inductive system of metric spaces, in the sense of Definition 1.6(d) below. It is natural to ask whether there is any relation between properties of a coarse space and the corresponding properties of those metric spaces in the coarse inductive system. This is the main concern of this article.

In particular, we obtain the following result in Corollary 2.20 and Proposition 3.2. Actually, by Proposition 3.2, the backward implications of parts (a) to (d) as well as part (e) hold for coarse inductive systems of coarse spaces (instead of just metric spaces).

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Theorem 1.1. *Let $(X, \mathcal{E})$ be a coarse space.*
(a) The asymptotic dimension of $(X, \mathcal{E})$ is dominated by a non-negative integer n if and only if $(X, \mathcal{E})$ is the coarse directed limit of a coarse inductive system of metric spaces whose asymptotic dimensions are dominated by n.
(b) $(X, \mathcal{E})$ has FWCDC if and only if $(X, \mathcal{E})$ is the coarse directed limit of a coarse inductive system of metric spaces having FWCDC.
(c) $(X, \mathcal{E})$ is uniformly locally finite (respectively, locally finite) if and only if $(X, \mathcal{E})$ is the coarse directed limit of a coarse inductive system of uniformly locally finite (respectively, locally finite) metric spaces.
(d) $(X, \mathcal{E})$ is a uniformly locally finite coarse space having property A if and only if $(X, \mathcal{E})$ is the coarse directed limit of a coarse inductive system of uniformly locally finite metric spaces having property A.
(e) If $(X, \mathcal{E})$ is the coarse directed limit of a coarse inductive system, over a countably-upward directed index set, of metric spaces having asymptotic property C, then $(X, \mathcal{E})$ has coarse property C.
(f) If $(X, \mathcal{E})$ has bounded geometry, then $(X, \mathcal{E})$ is the coarse directed limit of a coarse inductive system of metric spaces having bounded geometry.

We warn the reader that even in the very special case when the connection maps in a coarse inductive system are metric preserving (which is not assumed in Definition 1.6(d) nor in Theorem 1.1 above), the "coarse directed limit" is NOT the same as the "metric space directed limit" in general (see Remark 1.7(e)).

Note also that Theorem 1.1(d) in the above was also obtained independently in Proposition 3.2 of [3].

As shown in [4] (respectively, [2]), asymptotic property C (respectively, coarse property C) implies property A for uniformly locally finite metric spaces (respectively, uniformly locally finite coarse spaces). However, it is not known whether FWCDC implies property A. One possible application of Theorem 1.1 is the following.

Suppose that every uniformly locally finite metric space with FWCDC has property A.
Then every uniformly locally finite coarse space with FWCDC has property A.

On the other hand, the general form of Theorem 1.1, as obtained in Proposition 3.2, allows one to use the coarse directed limit construction to get some coarse spaces satisfying certain desired properties.

In the following, let us recall some basic materials on coarse spaces. For a set X, the collection of all subsets of X will be denoted by $\mathcal{P}(X)$ and

$$\Delta_X := \{(x, x) \in X \times X : x \in X\}.$$

For $A, B \subseteq X \times X$, we define the *inverse* $A^{-1} := \{(y, x) \in X \times X : (x, y) \in A\}$ and the *product*

$$A \circ B := \{(x, z) \in X \times X : \text{there exists } y \in X \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\}.$$

Let us also set $A[x] := \{y \in X : (x, y) \in A\}$.

A *coarse structure* $\mathcal{E}$ on a set X is a subcollection of $\mathcal{P}(X \times X)$ such that $\Delta_X \in \mathcal{E}$ and that $\mathcal{E}$ is closed under the formation of subsets, inverses, products and finite unions. In this case, $(X, \mathcal{E})$ is called a *coarse space* and elements in $\mathcal{E}$ are called *controlled sets*. Moreover, a subcollection $\mathcal{B} \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if every element of $\mathcal{E}$ is contained in an element of $\mathcal{B}$. If $\mathcal{F}$ is another coarse structure on X, then we say that $\mathcal{F}$ is *finer than* $\mathcal{E}$ if $\mathcal{F} \subseteq \mathcal{E}$.

For any subcollection $\mathcal{C} \subseteq \mathcal{P}(X \times X)$, the intersection of all coarse structures on X containing $\mathcal{C}$ is called the *coarse structure generated by* $\mathcal{C}$. A coarse structure is *countably generated* if it is generated by a countable collection (could be finite).

A coarse space $(X, \mathcal{E})$ is said to be *locally finite* if for each $E \in \mathcal{E}$ and $x \in X$, the set $E[x]$ is finite. Furthermore, $(X, \mathcal{E})$ is said to be *uniformly locally finite* if for each $E \in \mathcal{E}$, there exists $n \in \mathbb{N}$ such that the cardinality of $E[x]$ is dominated by n, for every $x \in X$.

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Remark 1.2. Note that our definition of uniform local finiteness follows that of metric spaces. In [15], this kind of coarse spaces are called "uniformly discrete". However, since this name conflicts with the corresponding notion in metric spaces (a metric $\mu : X \times X \to \mathbb{R}_+$ is *uniformly discrete* if there is $\kappa > 0$ such that $\mu(x, y) \geq \kappa$ for every distinct elements $x, y \in X$), we call it "uniformly locally finite" instead.

Definition 1.3. *For a coarse space $(X, \mathcal{E})$, a family $\{S_i\}_{i\in\Lambda}$ of subsets of X satisfying*

$$\bigcup_{i\in\Lambda} S_i \times S_i \in \mathcal{E}$$

is said to be $\mathcal{E}$-uniformly bounded.

Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces. A map $f : X \to Y$ is said to be

- *bornologous* if $(f \times f)(\mathcal{E}) \subseteq \mathcal{F}$;
- *effectively proper* if $(f \times f)^{-1}(\mathcal{F}) \subseteq \mathcal{E}$;
- *rough* if it is both bornologous and effectively proper.

Two maps $f, f' : X \to Y$ are said to be *close* if $\{(f(x), f'(x)) : x \in X\} \in \mathcal{F}$. Two coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are said to be *coarsely equivalent* if there are rough maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is close to the identity map id_X on X and that $f \circ g$ is close to the identity map id_Y on Y.

Denote $\mathbb{R}_+^\infty := \mathbb{R}_+ \cup \{+\infty\}$. If $\mu : X \times X \to \mathbb{R}_+^\infty$ is a function satisfying $\mu(x, x) = 0$, $\mu(x, y) = \mu(y, x)$ and $\mu(x, z) \leq \mu(x, y) + \mu(y, z)$ $(x, y, z \in X)$, then it is called a *pseudo-metric* on X. A pseudo-metric μ is a *metric* if $\mu(x, y) = 0$ implies $x = y$.

Suppose that μ is a pseudo-metric on X. We set

$$\mathbf{D}_r^\mu := \{(x, y) \in X \times X : \mu(x, y) \leq r\} \qquad (r \in \mathbb{R}_+).$$

If $\mathcal{M}$ is a collection of pseudo-metrics on X, the coarse structure $\mathcal{E}_\mathcal{M}$ generated by $\{\mathbf{D}_r^\mu : r \in \mathbb{R}_+; \mu \in \mathcal{M}\}$ is called the *coarse structure induced by* $\mathcal{M}$. In this case, we also say that $\mathcal{M}$ *induces* $\mathcal{E}_\mathcal{M}$. For a single pseudo-metric μ, we will denote $\mathcal{E}_\mu := \mathcal{E}_{\{\mu\}}$, and call it the *coarse structure induced by* μ. If a coarse structure $\mathcal{E}$ is induced by a single pseudo-metric μ, then $\mathcal{E}$ (and also $(X, \mathcal{E})$) is said to be *metrizable*. We say that two metric spaces (X, μ) and (Y, ν) are *coarsely equivalent* if $(X, \mathcal{E}_\mu)$ is coarsely equivalent to $(Y, \mathcal{E}_\nu)$.

Remark 1.4. (a) The usual definitions for pseudo-metrics and metrics take values in $\mathbb{R}_+$ instead of $\mathbb{R}_+^\infty$. Note that μ takes values in $\mathbb{R}_+$ if and only if $(X, \mathcal{E}_\mu)$ is *coarsely connected*, in the sense that $\bigcup \mathcal{E}_\mu = X \times X$. Since we do not assume our coarse spaces to be coarsely connected, we will consider metrics taking values in $\mathbb{R}_+^\infty$.
(b) For a pseudo-metric $\mu : X \times X \to \mathbb{R}_+^\infty$, there is a metric $\nu : X \times X \to \mathbb{R}_+^\infty$ such that $\mathcal{E}_\mu = \mathcal{E}_\nu$. Hence, we will only consider metrics on X.

Let (X, μ) be a metric space. A family $\{S_i\}_{i\in\Lambda}$ of subsets of X is said to be *μ-uniformly bounded* if it is $\mathcal{E}_\mu$-uniformly bounded in the sense of Definition 1.3; in other words, there exists $r \in \mathbb{R}_+$ with $\bigcup_{i\in\Lambda} S_i \times S_i \subseteq \mathbf{D}_r^\mu$; or equivalently,

$$\sup_{i\in\Lambda} \sup\{\mu(x, y) : x, y \in S_i\} < \infty.$$

On the other hand, for $A, B \in \mathcal{P}(X) \setminus \{\emptyset\}$, we set

$$\mu(A, B) := \inf\{\mu(x, y) : x \in A; y \in B\}.$$

The following result is crucial to our discussion.

Proposition 1.5. *([15, Theorem 2.55]) A coarse structure $\mathcal{E}$ on a set X is countably generated if and only if it is metrizable.*

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Definition 1.6. *(a) When a set $\mathfrak{I}$ is equipped with a reflexive, transitive and anti-symmetric relation $\leq$, we call $\mathfrak{I}$ a* partially ordered set*. A partially ordered set $\mathfrak{I}$ is called an* upward directed set *if for every $i, j \in \mathfrak{I}$, there exists $k \in \mathfrak{I}$ satisfying $i \leq k$ and $j \leq k$. Moreover, we say that $\mathfrak{I}$ is a* countably-upward directed set *if for every $i_1, i_2, \cdots \in \mathfrak{I}$, there exists i_∞ with $i_k \leq i_\infty$ for all $k \in \mathbb{N}$.*

(b) A coarse inductive system of coarse spaces *is a family $\{(X_i, \mathcal{E}_i)\}_{i\in\mathfrak{I}}$ of coarse spaces indexed by an upward directed set $\mathfrak{I}$, equipped with an injective bornologous map $\phi_{j,i} : X_i \to X_j$ for every $i, j \in \mathfrak{I}$ with $i \leq j$ such that $\phi_{k,i} = \phi_{k,j} \circ \phi_{j,i}$ for $i \leq j \leq k$ in $\mathfrak{I}$ and that $\phi_{i,i}$ is the identity map on X_i ($i \in \mathfrak{I}$).*

(c) Let $\big\{\big((X_i, \mathcal{E}_i), \phi_{j,i}\big)\big\}_{i\leq j\in\mathfrak{I}}$ be a coarse inductive system of coarse spaces. Denote by $\big(\lim_i X_i, \{\phi_i\}_{i\in\mathfrak{I}}\big)$ the "set directed limit" of this system. We define the directed limit coarse structure*, $\lim_i \mathcal{E}_i$, on $\lim_i X_i$ to be the coarse structure generated by $\bigcup_{i\in\mathfrak{I}}(\phi_i \times \phi_i)(\mathcal{E}_i)$. In this case, $\big((\lim_i X_i, \lim_i \mathcal{E}_i), \{\phi_i\}_{i\in\mathfrak{I}}\big)$ is called the* coarse directed limit *of the system.*

(d) A coarse inductive system of metric spaces *is a coarse inductive system of coarse spaces of the form $\big\{\big((X_i, \mathcal{E}_{\mu_i}), \phi_{j,i}\big)\big\}_{i\leq j\in\mathfrak{I}}$, where μ_i is a metric on X_i ($i \in \mathfrak{I}$). The corresponding coarse directed limit is called the* coarse directed limit *of that coarse inductive system of metric spaces.*

It is well-known that the set directed limit always exists (this can be shown by considering the set of equivalence classes of the disjoint union $\bigsqcup_{i\in\mathfrak{I}} X_i$ under an equivalence relation induced by $\{\phi_{j,i}\}_{i\leq j\in\mathfrak{I}}$). Thus, the coarse directed limit exists by its construction.

Remark 1.7. (a) Clearly, all the maps ϕ_i in Definition 1.6 are bornologous. One can identify the coarse directed limit via the following "universal property": if $(Y, \mathcal{F})$ is a coarse space and for each $i \in \mathfrak{I}$, there exists a bornologous map $f_i : X_i \to Y$ satisfying $f_j \circ \phi_{j,i} = f_i$ for every $i \leq j$ in $\mathfrak{I}$, then there is a unique bornologous map $f : \lim_i X_i \to Y$ such that $f \circ \phi_i = f_i$ ($i \in \mathfrak{I}$).

(b) One can also consider coarse inductive systems with the connection maps $\phi_{j,i}$ being non-injective. The corresponding coarse directed limits also exist and the corresponding fact as in part (a) above holds for such coarse directed limits as well. Nevertheless, since we will only consider coarse inductive systems with injective connection maps, we restrict our attention to such systems so that we do not need to put the description "with injective connection maps" here and there.

(c) Since we assume that all $\phi_{j,i}$ are injective, the maps $\phi_i : X_i \to \lim_i X_i$ are all injective, and we have

$$\lim{}_i\mathcal{E}_i = \big\{F_1 \cup F_2 : F_1 \in (\phi_j \times \phi_j)(\mathcal{E}_j) \text{ and } F_2 \subseteq \Delta_{\lim_i X_i} \text{ for some } j \in \mathfrak{I}\big\}.$$

Without the injectivity assumption on $\phi_{j,i}$, the collection on the right-hand side above may not be closed under products. In that case, a base for $\lim_i\mathcal{E}_i$ is the collection consisting of finite products of sets of the form $(\phi_i \times \phi_i)(E) \cup \Delta_{\lim_i X_i}$ (see [13, Lemma 2.5]).

(d) Suppose that $\big\{\big((Y_i, \mathcal{F}_i), \psi_{j,i}\big)\big\}_{i\leq j\in\mathfrak{I}}$ is another coarse inductive system over the same index set $\mathfrak{I}$ such that for each $i \in \mathfrak{I}$, there exist rough maps $f_i : X_i \to Y_i$ and $g_i : Y_i \to X_i$ inducing the coarse equivalence between X_i and Y_i and respecting the connection maps $\phi_{j,i}$ and $\psi_{j,i}$. Then one obtains two maps $f : \lim_i X_i \to \lim_i Y_i$ and $g : \lim_i Y_i \to \lim_i X_i$. As in Proposition 3.6, these two maps are bornologous. However, compositions of these two maps may not be close to the respective identity maps (since such a property involves an infinite union of a family of controlled sets over $\mathfrak{I}$). In fact, Example 3.8 is an explicit example for which $\lim_i X_i$ is not coarsely equivalent to $\lim_i Y_i$.

(e) In the case of a coarse inductive system of metric spaces with the connection maps $\phi_{j,i}$ being metric preserving, the coarse directed limit may not be the same as the metric space directed limit (see Example 3.3(b)). The reason is that every controlled set for the coarse directed limit contains a very huge "diagonal part" (see part (c) above). Nevertheless, the coarse directed limit coincides with the metric space directed limit in some very special case (see Example 1.8 below).

The coarse directed limits appeared in some disguised forms in the literature. The following is one of them.

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Example 1.8. *For $k \leq l \in \mathbb{N}$, let $\phi_{l,k} : \mathbb{Z}^k \to \mathbb{Z}^l$ be the map sending $(n_1, ..., n_k)$ to $(n_1, \dots, n_k, 0, \dots, 0)$. One may identify $\lim_k \mathbb{Z}^k$ with the set*

$$\big\{(x_k)_{k\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} : x_k = 0 \text{ for all but a finite number of } k\big\}.$$

We recall the following metric defined on $\lim_k \mathbb{Z}^k$ from [18] (which was first considered in [9] and [14]):

$$\mathbf{d}(x,y) := \sum\nolimits_{k=1}^{\infty} k|x_k - y_k|, \quad \text{where } x = (x_k)_{k\in\mathbb{N}}, y = (y_k)_{k\in\mathbb{N}} \in \lim_k \mathbb{Z}^k.$$

If $\mathcal{E}_k$ is the coarse structure on $\mathbb{Z}^k$ induced by the Euclidean metric, then $\lim_k \mathcal{E}_k = \mathcal{E}_{\mathbf{d}}$.

In fact, it follows from Remark 1.7(a) that $\lim_k \mathcal{E}_k \subseteq \mathcal{E}_{\mathbf{d}}$. Conversely, consider $n \in \mathbb{N}$. If $(x,y) \in \mathbb{Z}^{\mathbb{N}}$ satisfying $\mathbf{d}(x,y) \leq n$, then $x_k = y_k$ for $k \geq n+1$. This shows that $\{(x,y) \in \mathbb{Z}^{\mathbb{N}} : \mathbf{d}(x,y) \leq n\} \subseteq (\phi_n \times \phi_n)(E) \cup \Delta_{\mathbb{Z}^{\mathbb{N}}}$, for some $E \in \mathcal{E}_n$. Thus, Remark 1.7(c) implies that $\mathcal{E}_{\mathbf{d}} \subseteq \lim_k \mathcal{E}_k$.

Moreover, it is clear that $\mathcal{E}_k$ coincides with $\mathcal{E}_{\mathbf{d}|_{\mathbb{Z}^k\times\mathbb{Z}^k}}$. Hence, the coarse directed limit and the metric space directed limit of the inductive system $\big\{\big((\mathbb{Z}^k, \mathcal{E}_{\mathbf{d}|_{\mathbb{Z}^k\times\mathbb{Z}^k}}), \phi_{l,k}\big)\big\}_{k\leq l\in\mathbb{N}}$ coincide.

The referee has kindly informed us of a related work, namely [10]. Let us end this introduction by comparing the results in this paper and some results found in [10].

More precisely, another abstract framework to study large scale geometric properties of metric spaces was introduced in [5], under the name "large scale structure". According to [5, Proposition 2.4] and [5, Proposition 2.5], coarse structures and large scale structures are basically the same. Indeed, given a set X, for every coarse structure $\mathcal{E}$ on X, one can associate with it a large scale structure $\mathcal{L}_{\mathcal{E}}$ that consists of $\mathcal{E}$-uniformly bounded families $\mathcal{B}$ of subsets of X. Conversely, for a large scale structure $\mathcal{L}$ defined on X, one can associate with it a coarse structure $\mathcal{E}_{\mathcal{L}}$ consisting of subsets $E \subseteq X \times X$ satisfying $E \subseteq \bigcup_{B\in\mathcal{B}} B \times B$ for some $\mathcal{B} \in \mathcal{L}$.

Furthermore, the notion of *asymptotic filtered colimit* was introduced in [10]. The notion of coarse directed limit in Definition 1.6 can be deemed as an extension of asymptotic filtered colimit. More precisely, the requirement in [10, Definition 9] that for $r, s \in S$, there exists $t \in S$ with $X_r \cup X_s \subseteq X_t$ tells us that S is an upward directed set under the ordering induced by inclusions of subsets of X. Furthermore, the requirement of the restrictions of the large scale structures of X_r and X_s on $X_r \cap X_s$ being the same (see [10, Definition 9]), means that the connection maps $\phi_{j,i}$ are rough maps, instead of bornologous maps in Definition 1.6.

In this respect, some statements in Proposition 3.2 in the current paper (which are extensions of the backward implication of some statements in Theorem 1.1) can be regarded as generalizations of some results in [10]. On the other hand, the forward implication in some statements in Theorem 1.1 above were also considered in [10, Proposition 2.8] and [10, Corollary 2.9]. Note that since we only assume the connection maps in the inductive system to be bornologous maps instead of rough maps, we can consider the directed limit of a family of metrics defined on the same set (see Corollary 2.20), which is not available in the context of [10].

2. Coarse spaces as coarse directed limits of metric spaces

As said in the Introduction, it was proved in [1] that coarse spaces are induced by systems of metric spaces. In fact, one has a refined statement that every coarse structure $\mathcal{E}$ on a set X is induced by a "coarsely increasing family" of metrics defined on X. Let us begin with some words about the latter.

Definition 2.1. *(a) Suppose that μ_1 and μ_2 are two metrics defined on the same set X. We denote $\mu_2 \preceq \mu_1$ if μ_2 is* coarser *than μ_1, in the sense that $\mathcal{E}_{\mu_1} \subseteq \mathcal{E}_{\mu_2}$. We say that μ_1 and μ_2 are* coarsely equivalent *if $\mathcal{E}_{\mu_1} = \mathcal{E}_{\mu_2}$.*

(b) A collection $\mathcal{M}$ of metrics defined on X is said to be

- quasi-directed *if for $\mu_1, \mu_2 \in \mathcal{M}$, one can find $\mu_3 \in \mathcal{M}$ such that $\mu_3 \preceq \mu_1$ and $\mu_3 \preceq \mu_2$;*
- countably-quasi-directed *if for $\mu_1, \mu_2, \dots \in \mathcal{M}$, there is $\nu \in \mathcal{M}$ with $\nu \preceq \mu_k$ for every $k \in \mathbb{N}$;*

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- effective *if for distinct elements* $\mu, \nu \in \mathcal{M}$, *one has* $\mathcal{E}_\mu \neq \mathcal{E}_\nu$.

Remark 2.2. (a) Notice that if two metrics μ_1 and μ_2 satisfy $\mu_2(x,y) \leq \mu_1(x,y)$ $(x,y \in X)$, then one has $\mu_2 \preceq \mu_1$. This is the reason why we use the notation $\preceq$.

(b) Let $\mathcal{M}$ be a family of metrics on X. One can define the opposite "pre-ordering" (i.e., a reflexive and transitive relation, which is not necessarily anti-symmetric) on $\mathcal{M}$ by declaring that $\mu \leq_{\mathrm{c}} \nu$ when $\nu \preceq \mu$. In this case, $\mathcal{M}$ is a partially ordered set under $\leq_{\mathrm{c}}$ (i.e., $\leq_{\mathrm{c}}$ is anti-symmetric) if and only if $\mathcal{M}$ is effective.

(c) Suppose that $\mathcal{M}$ is a family of metrics on X. Let us pick one element from each coarse equivalence class in $\mathcal{M}$ and form a set $\mathcal{M}_0$. Obviously, $\mathcal{M}_0$ is effective and $\mathcal{E}_{\mathcal{M}} = \mathcal{E}_{\mathcal{M}_0}$. Moreover, $\mathcal{M}_0$ is an upward directed set (respectively, countably-upward directed set) under $\leq_{\mathrm{c}}$ if and only if $\mathcal{M}$ is quasi-directed (respectively, countably-quasi-directed).

(d) Suppose that $\mathcal{M}$ is an effective quasi-directed family of metrics inducing $\mathcal{E}$. Then $(X,\mathcal{E})$ is the coarse directed limit of the coarse inductive system $\big\{\big((X,\mathcal{E}_\mu), \mathrm{id}_{\nu,\mu}\big)\big\}_{\nu \preceq \mu \in \mathcal{M}}$, where $\mathrm{id}_{\nu,\mu} : X \to X$ is the identity map.

It is trivial that a family consisting of only one metric is countably-quasi-directed. Moreover, the following is the example that we are interested in.

Example 2.3. *Let* $(X,\mathcal{E})$ *be a coarse space and*

$$\mathcal{M}^{\mathcal{E}} := \{\mu : \mu \text{ is a metric on } X \text{ with } \mathcal{E}_\mu \subseteq \mathcal{E}\}.$$

Then $\mathcal{M}^{\mathcal{E}}$ *is a countably-quasi-directed family of metrics. For every* $E \in \mathcal{E}$*, we know from Proposition 1.5 that the coarse structure generated by* E *coincides with* $\mathcal{E}_\mu$ *for some* $\mu \in \mathcal{M}^{\mathcal{E}}$*. This shows that* $\mathcal{M}^{\mathcal{E}}$ *induces* $\mathcal{E}$*.*

The following proposition is the motivation of this study. Part (b) of it is an alternative form of [1, Proposition 4.10]. By Remark 2.2(d), one can rephrase it as saying that every coarse space is the coarse directed limit of a coarse inductive system of metric spaces over a countably-upward directed set.

Proposition 2.4. *Let* X *be a set.*

(a) If $\mathcal{M}$ *is a quasi-directed family of metrics on* X*, then* $\{\mathbf{D}^\mu_k : k \in \mathbb{N}; \mu \in \mathcal{M}\}$ *is a base for* $\mathcal{E}_{\mathcal{M}}$*.*

(b) If $\mathcal{E}$ *is a coarse structure on* X*, then there is an effective countably-quasi-directed family* $\mathcal{M}_0$ *of metrics on* X *inducing* $\mathcal{E}$*.*

Proof. (a) From the definition, $\mathcal{E}_{\mathcal{M}}$ is the finest coarse structure on X containing $\{\mathbf{D}^\mu_r : r \in \mathbb{N}; \mu \in \mathcal{M}\}$. Thus, it suffices to show that $\{\mathbf{D}^\mu_r : r \in \mathbb{N}; \mu \in \mathcal{M}\}$ is closed under formations of inverses, products, and unions. In fact, it is obvious that $(\mathbf{D}^\mu_r)^{-1} = \mathbf{D}^\mu_r$. Moreover, since $\mathcal{M}$ is quasi-directed, given $\mu_1, \mu_2 \in \mathcal{M}$ and $r_1, r_2 \in \mathbb{N}$, one can find $\nu \in \mathcal{M}$ and $k_1, k_2 \in \mathbb{N}$ such that $\mathbf{D}^{\mu_1}_{r_1} \subseteq \mathbf{D}^\nu_{k_1}$ and $\mathbf{D}^{\mu_2}_{r_2} \subseteq \mathbf{D}^\nu_{k_2}$. Thus, $\mathbf{D}^{\mu_1}_{r_1} \circ \mathbf{D}^{\mu_2}_{r_2}$ and $\mathbf{D}^{\mu_1}_{r_1} \cup \mathbf{D}^{\mu_2}_{r_2}$ are both contained in $\mathbf{D}^\nu_{k_1+k_2}$.

(b) Let $\mathcal{M}^{\mathcal{E}}$ be the countably-quasi-directed family as in Example 2.3. Let us take one element from every coarse equivalence class of elements in $\mathcal{M}^{\mathcal{E}}$ and form a set $\mathcal{M}_0$. By parts (b) and (c) of Remark 2.2, we know that $\mathcal{M}_0$ is an effective countably-quasi-directed family of metrics on X inducing $\mathcal{E}$. □

The following is our first result on the relation between properties of a coarse space and the corresponding properties of a family of metrics inducing that coarse space. Part (a) of this result is clear, while part (b) follows from part (a) as well as Proposition 2.4(b).

Corollary 2.5. *(a) If* $(X,\mathcal{E})$ *is a uniformly locally finite (respectively, locally finite) coarse space and* $\mu \in \mathcal{M}^{\mathcal{E}}$*, then* (X,μ) *is uniformly locally finite (respectively, locally finite).*

(b) A coarse space $(X,\mathcal{E})$ *is uniformly locally finite (respectively, locally finite) if and only if there exists an effective quasi-directed family* $\mathcal{N}$ *of metrics on* X *inducing* $\mathcal{E}$ *such that* (X,μ) *is a uniformly locally finite (respectively, locally finite) metric space for every* $\mu \in \mathcal{N}$*.*

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Let us recall some other properties of coarse spaces that we will study (see [2,6,15]).

Definition 2.6. *Let* $(X, \mathcal{E})$ *be a coarse space and*

$$\hat{\mathcal{E}} := \{E \in \mathcal{E} : \Delta_X \subseteq E = E^{-1}\}. \tag{2.1}$$

Then $(X, \mathcal{E})$ *is said to have*

- asymptotic dimension dominated by a non-negative integer d, *denoted by* $\operatorname{asdim}(X, \mathcal{E}) \leq d$, *if for any* $E \in \hat{\mathcal{E}}$, *there is a partition* $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,...,d\}}$ *of* X *such that for each* $k \in \{0, ..., d\}$,
  (1) $\bigcup_{j\in\Lambda_k} X_{k,j} \times X_{k,j} \in \mathcal{E}$*;*
  (2) $E \cap X_{k,i} \times X_{k,j} = \emptyset$ *when* $i \neq j$ *in* Λ_k*;*
- finite weak coarse decomposition complexity (FWCDC) *if for every* $E \in \hat{\mathcal{E}}$, *there exist* $d \in \mathbb{Z}_+$ *and a partition* $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d\}}$ *of* X *satisfying Conditions (1) and (2) for* $k \in \{0, \dots, d\}$*;*
- coarse property C *if for each increasing sequence* $\{E_k\}_{k\in\mathbb{N}}$ *in* $\hat{\mathcal{E}}$, *there exist* $n \in \mathbb{N}$ *and a partition* $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{1,\dots,n\}}$ *of* X *such that for every* $k \in \{1, \dots, n\}$, *Condition (1) above holds and one has* $E_k \cap X_{k,i} \times X_{k,j} = \emptyset$ *when* $i \neq j$ *in* Λ_k.

Moreover, we say that $(X, \mathcal{E})$ *has* finite asymptotic dimension *if it has asymptotic dimension dominated by some* $d \in \mathbb{Z}_+$.

Note that the notion of asymptotic dimension dominated by d is a direct extension of the corresponding property for metric spaces.

One may regard $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,...,d\}}$ as a "colored partition" of X (with $(d+1)$-colors), and express Definition 2.6 in terms of "coloring". However, we will not give the details here.

Remark 2.7. (a) In Definition 2.6, we may replace the word "partition" by "covering" without changing the meanings in all the three parts. In fact, let $\{Y_{k,j}\}_{j\in\Lambda_k;k\in\{0,...,d\}}$ be a covering of X, and $E_0, \dots, E_d \in \hat{\mathcal{E}}$ satisfy:

- $\{Y_{k,j}\}_{j\in\Lambda_k;k\in\{0,...,d\}}$ is $\mathcal{E}$-uniformly bounded;
- $E_k \cap Y_{k,i} \times Y_{k,j} = \emptyset$ when $i \neq j$ in Λ_k, for every $k \in \{0, \dots, d\}$.

Fix $l \in \{0, \dots, d\}$. Since $\Delta_X \subseteq E_l$, we know that the collection $\{Y_{l,j}\}_{j\in\Lambda_l}$ is disjoint. Let us define inductively, $Z_{0,j} := Y_{0,j}$ ($j \in \Lambda_0$), and for $l < d$,

$$Z_{l+1,i} := Y_{l+1,i} \setminus \bigcup_{k\leq l} \bigcup_{j\in\Lambda_k} Y_{k,j} \qquad (i \in \Lambda_{l+1}).$$

Then $\{Z_{k,j}\}_{j\in\Lambda_k;k\in\{0,...,d\}}$ becomes a partition of X, after we remove those empty subsets from this collection. Moreover, one has

- $\{Z_{k,j}\}_{j\in\Lambda_k;k\in\{0,...,d\}}$ is $\mathcal{E}$-uniformly bounded;
- $E_k \cap Z_{k,i} \times Z_{k,j} = \emptyset$ when $i \neq j$ in Λ_k, for every $k \in \{0, \dots, d\}$.

(b) Coarse property C stands between FWCDC and finite asymptotic dimension.

(c) The original definition in [2, Definition 4.1] for finite weak coarse decomposition complexity involves a decomposition game with multiple rounds. However, it was proved in [2, Proposition 4.9], that [2, Definition 4.1] is equivalent to the one in Definition 2.6 above.

(d) Let $\{F_i\}_{i\in\mathbb{N}}$ be a sequence in $\hat{\mathcal{E}}$. If we set $E_k := \bigcup_{i=1}^k F_i$, then $\{E_k\}_{k\in\mathbb{N}}$ is an increasing sequence in $\hat{\mathcal{E}}$ with $F_i \subseteq E_i$ ($i \in \mathbb{N}$). Therefore, it makes no difference if we drop the assumption of $\{E_k\}_{k\in\mathbb{N}}$ being increasing in the definition of coarse property C.

(e) We recall from [4] that a metric space (X, μ) is said to have *asymptotic property C* if for any sequence $r_1 \leq r_2 \leq r_3 \leq \dots$ in $\mathbb{R}_+$, there is a finite sequence $\{\mathcal{U}_k\}_{k\in\{1,...,n\}}$ of μ-uniformly bounded families of open subsets such that $\bigcup_{k=1}^n \mathcal{U}_k$ is a covering of X and $\mu(V, W) > r_l$ when $V \neq W$ in $\mathcal{U}_l$ for $l \in \{1, \dots, n\}$.

Notice that one may remove the assumption of subsets in $\bigcup_{k=1}^n \mathcal{U}_k$ being open. In fact, suppose that $\bigcup_{k=1}^n \mathcal{U}_k$ is a collection of non-open subsets satisfying the above. If we replace a set $V \in \bigcup_{k=1}^n \mathcal{U}_k$ by $\bigcup_{x\in V} O(x, \epsilon)$, where $O(x, \epsilon)$ is the open ball with center x and radius ϵ, then the resulting collection is also μ-uniformly bounded and the distance between two subsets will be decreased by at most 2ϵ.

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This, together with part (a) above, tells us that a metric space (X, μ) has asymptotic property C if and only if the coarse space $(X, \mathcal{E}_\mu)$ has coarse property C.

Theorem 2.8. *Let $(X, \mathcal{E})$ be a coarse space. The following statements are equivalent.*

W1). $(X, \mathcal{E})$ has FWCDC.

W2). For any countably-quasi-directed family $\mathcal{M}$ of metrics on X inducing $\mathcal{E}$ and any $\mu \in \mathcal{M}$, there exists $\nu \in \mathcal{M}$ with $\nu \preceq \mu$ such that for each $s \in \mathbb{N}$, one can find $d \in \mathbb{Z}_+$ and a ν-uniformly bounded partition $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d\}}$ of X with $\mu(X_{k,i}, X_{k,j}) > s$ for $k \in \{0, \dots, d\}$ and $i \neq j$ in Λ_k.

W3). There exists an effective quasi-directed family $\mathcal{N}$ of metrics on X inducing $\mathcal{E}$ such that (X, ν) has FWCDC, for every $\nu \in \mathcal{N}$.

Proof. $(W1) \Rightarrow (W2)$. Let us first show that $(W1)$ implies the following statement.

W2'). For any quasi-directed family $\mathcal{M}$ of metrics inducing $\mathcal{E}$, any $\mu \in \mathcal{M}$ and any $n \in \mathbb{N}$, there exist $\tilde{\mu} \in \mathcal{M}$, $d \in \mathbb{Z}_+$ and a $\tilde{\mu}$-uniformly bounded partition $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d\}}$ of X with $\mu(X_{k,i}, X_{k,j}) > n$ for $k \in \{0, \dots, d\}$ and $i \neq j$ in Λ_k.

Indeed, if we set $E := \mathbf{D}^\mu_n$, then by the definition, there exist $d \in \mathbb{Z}_+$ and a partition $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d\}}$ of X satisfying Conditions (1) and (2) in Definition 2.6 for E. Condition (2) implies $\mu(X_{k,i}, X_{k,j}) > n$. Moreover, by Condition (1), we can find $\tilde{\mu} \in \mathcal{M}$ and $r \in \mathbb{N}$ with $\bigcup_{k=0}^d \bigcup_{j\in\Lambda_k} X_{k,j} \times X_{k,j} \subseteq \mathbf{D}^{\tilde{\mu}}_r$, which means that $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d\}}$ is $\tilde{\mu}$-uniformly bounded.

Now, we assume that Statement (W2') holds. Fix $\mu \in \mathcal{M}$. For $n \in \mathbb{N}$, it follows from Statement (W2') that one can find $\mu_n \in \mathcal{M}$, $d_n \in \mathbb{Z}_+$ and a partition $\{X^{(n)}_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d_n\}}$ of X such that for each $k \in \{0, \dots, d_n\}$, one has

$$\sup_{j\in\Lambda_k} \sup\big\{\mu_n(x, y) : x, y \in X^{(n)}_{k,j}\big\} < \infty \quad \text{and} \quad \mu\big(X^{(n)}_{k,j}, X^{(n)}_{k,i}\big) > n \text{ when } j \neq i \text{ in } \Lambda_k.$$

Since $\mathcal{M}$ is countably-quasi-directed, there is $\nu \in \mathcal{M}$ with $\nu \preceq \mu$ and $\nu \preceq \mu_n$ $(n \in \mathbb{N})$. Thus, for each $s \in \mathbb{N}$, the partition $\{X^{(s)}_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d_s\}}$ of X as in the above is ν-uniformly bounded.

$(W2) \Rightarrow (W3)$. Let $\mathcal{M}_0$ be the effective countably-quasi-directed family as in the proof of Proposition 2.4(b). Pick any $\mu \in \mathcal{M}_0$. Set $\mu_1 := \mu$. By Statement (W2), there is $\mu_2 \in \mathcal{M}_0$ such that $\mu_2 \preceq \mu_1$ and that for any $s \in \mathbb{N}$, one can find $d_{1,s} \in \mathbb{Z}_+$ and a μ_2-uniformly bounded partition $\{X^{1,s}_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d_{1,s}\}}$ of X with

$$\mu_1\big(X^{1,s}_{k,j}, X^{1,s}_{k,i}\big) > s \quad \text{when } j \neq i \in \Lambda_k \text{ and } k \in \{0, \dots, d_{1,s}\}.$$

Inductively, for $m \in \mathbb{N}$, there exists $\mu_{m+1} \in \mathcal{M}_0$ such that $\mu_{m+1} \preceq \mu_m$ and that for each $s \in \mathbb{N}$, one can find $d_{m,s} \in \mathbb{Z}_+$ and a μ_{m+1}-uniformly bounded partition $\{X^{m,s}_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d_{m,s}\}}$ of X with $\mu_m\big(X^{m,s}_{k,j}, X^{m,s}_{k,i}\big) > s$ when $j \neq i$ in Λ_k and $k \in \{0, \dots, d_{m,s}\}$. The coarse structure $\mathcal{F}_\infty$ induced by $\{\mu_1, \mu_2, \dots\}$ is the one generated by $\{\mathbf{D}^{\mu_m}_n : m, n \in \mathbb{N}\}$, and hence by Proposition 1.5 and the construction of $\mathcal{M}_0$, there exists $\bar{\mu} \in \mathcal{M}_0$ with $\mathcal{E}_{\bar{\mu}} = \mathcal{F}_\infty$.

Consider $r \in \mathbb{N}$. We know from Proposition 2.4(a) that there exist $m, n \in \mathbb{N}$ with $\mathbf{D}^{\bar{\mu}}_r \subseteq \mathbf{D}^{\mu_m}_n$. Let $d_{m,n} \in \mathbb{Z}_+$ and $\{X^{m,n}_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d_{m,n}\}}$ be as in the above; i.e., $\{X^{m,n}_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d_{m,n}\}}$ is a μ_{m+1}-uniformly bounded partition of X and

$$\mu_m\big(X^{m,n}_{k,j}, X^{m,n}_{k,i}\big) > n \quad \text{when } j \neq i \text{ in } \Lambda_k \text{ and } k \in \{0, \dots, d_{m,n}\}.$$

From this, we see that $\bar{\mu}\big(X^{m,n}_{k,j}, X^{m,n}_{k,i}\big) > r$ when $j \neq i$ in Λ_k as well as $k \in \{0, \dots, d_{m,n}\}$, and that $\{X^{m,n}_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d_{m,n}\}}$ is a $\bar{\mu}$-uniformly bounded partition of X. In other words, $(X, \bar{\mu})$ has FWCDC.

Moreover, since $\bar{\mu} \preceq \mu_1 = \mu$, we know that $\mathcal{E}_\mu \subseteq \mathcal{E}_{\bar{\mu}} \subseteq \mathcal{E}$. This, together with the fact that $\bigcup_{\mu\in\mathcal{M}_0} \mathcal{E}_\mu$ generates $\mathcal{E}$, implies that the collection

$$\mathcal{N} := \{\nu \in \mathcal{M}_0 : (X, \nu) \text{ has FWCDC}\}$$

induces $\mathcal{E}$. Furthermore, as $\mathcal{M}_0$ is effective, so is $\mathcal{N}$. For a sequence $\nu_1, \nu_2, \dots \in \mathcal{N} \subseteq \mathcal{M}_0$, there exists $\nu \in \mathcal{M}_0$ with $\nu \preceq \nu_k$ $(k \in \mathbb{N})$. The above construction then gives $\bar{\nu} \in \mathcal{N}$ with $\bar{\nu} \preceq \nu$ and we see that $\mathcal{N}$ is also countably-quasi-directed.

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$(W3) \Rightarrow (W1)$. Consider any $E \in \hat{\mathcal{E}}$. One can find $\nu \in \mathcal{N}$ and $r \in \mathbb{N}$ with $E \subseteq \mathbf{D}_r^\nu$ (by Proposition 2.4(a)). As (X, ν) has FWCDC, there exist $d \in \mathbb{Z}_+$ and a ν-uniformly bounded partition $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d\}}$ of X such that $\nu(X_{k,j}, X_{k,i}) > r$ if $k \in \{0, \dots, d\}$ and $j \neq i$ in Λ_k. It is clear that Conditions (1) and (2) of Definition 2.6 will then be satisfied for the given controlled set E. □

Using the same argument as that for Theorem 2.8, but with d, d_n and $d_{m,s}$ etc. in the proof being replaced by a fixed integer $N \in \mathbb{Z}_+$, we have the following result.

Theorem 2.9. *Let $(X, \mathcal{E})$ be a coarse space and $N \in \mathbb{Z}_+$. The following statements are equivalent.*

F1). $\mathrm{asdim}(X, \mathcal{E}) \leq N$.

F2). For any countably-quasi-directed family $\mathcal{M}$ of metrics on X inducing $\mathcal{E}$ and any $\mu \in \mathcal{M}$, there is $\nu \in \mathcal{M}$ with $\nu \preceq \mu$ such that for each $s \in \mathbb{N}$, one can find a ν-uniformly bounded partition $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,N\}}$ of X satisfying $\mu(X_{k,j}, X_{k,i}) > s$ when $k \in \{0, \dots, N\}$ and $j \neq i$ in Λ_k.

F3). There exists an effective quasi-directed family $\mathcal{N}$ of metrics on X inducing $\mathcal{E}$ such that $\mathrm{asdim}(X, \mu) \leq N$*, for every $\mu \in \mathcal{N}$.*

Unfortunately, it seems that the argument for Theorem 2.8 does not work for coarse property C. We only have a partial result here.

Proposition 2.10. *Let $(X, \mathcal{E})$ be a coarse space. Suppose that there is a countably-quasi-directed family $\mathcal{M}$ of metrics inducing $\mathcal{E}$ such that (X, μ) has asymptotic property C, for each $\mu \in \mathcal{M}$. Then $(X, \mathcal{E})$ has coarse property C.*

Proof. Consider $\{E_k\}_{k\in\mathbb{N}}$ to be a sequence in $\hat{\mathcal{E}}$. For each $k \in \mathbb{N}$, there exists $\nu_k \in \mathcal{M}$ such that $E_k \in \mathcal{E}_{\nu_k}$. Since $\mathcal{M}$ is countably-quasi-directed, one can find $\nu \in \mathcal{M}$ with $\nu \preceq \nu_k$ for all $k \in \mathbb{N}$. The relation $E_k \in \mathcal{E}_\nu$ then produces $s_k \in \mathbb{N}$ with $E_k \subseteq \mathbf{D}^\nu_{s_k}$ $(k \in \mathbb{N})$. Without loss of generality, we may assume that the sequence $\{s_k\}_{k\in\mathbb{N}}$ is increasing. As (X, ν) has asymptotic property C, Remark 2.7(e) gives $n \in \mathbb{N}$ and a ν-uniformly bounded partition $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{1,\dots,n\}}$ of X such that

$$\nu(X_{k,j}, X_{k,i}) > s_k \quad \text{for} \quad k \in \{1, \dots, n\} \quad \text{and} \quad j \neq i \quad \text{in} \quad \Lambda_k.$$

Thus, $\bigcup_{k=1}^n \bigcup_{j\in\Lambda_k} X_{k,j} \times X_{k,j} \in \mathcal{E}$ and for $k \in \{1, \dots, n\}$, one has $E_k \cap X_{k,j} \times X_{k,i} = \emptyset$ if $j \neq i$ in Λ_k. It follows that $(X, \mathcal{E})$ has coarse property C. □

Unlike the third statements in both Theorems 2.8 and 2.9, in the hypothesis of Proposition 2.10, we need a "countably-quasi-directed family", instead of simply a "quasi-directed family" of metrics, satisfying asymptotic property C. This requirement means that it is not possible to apply Proposition 2.10 to coarse directed limits over the index set $\mathbb{N}$ (see e.g., the paragraph preceding Proposition 3.9). Because of this, we ask the following question, for which we do not have an answer. Observe that if this question have a positive answer, then we can use it to reproduce Yamauchi's result in [18, Theorem 2.1] that the space in Example 1.8 has asymptotic property C.

Question 2.11. *Let $(X, \mathcal{E})$ be a coarse space. If there is a quasi-directed family $\mathcal{M}$ of metrics inducing $\mathcal{E}$ with (X, μ) having asymptotic property C, for each $\mu \in \mathcal{M}$, does $(X, \mathcal{E})$ have coarse property C?*

On the other hand, adapting the argument of Theorem 2.8, we also have the corresponding statement for Property A. Let us first recall from [16] its meaning here.

Definition 2.12. *A uniformly locally finite coarse space $(X, \mathcal{E})$ is said to have* property A *if for each $\epsilon > 0$ and $E \in \mathcal{E}$, there is a map $a : X \to S_{\ell^1(X)}$, where*

$$S_{\ell^1(X)} := \{f \in \ell^1(X) : \|f\|_1 = 1\},$$

such that $\{(x, y) \in X \times X : a_x(y) \neq 0\} \in \mathcal{E}$ and that $\|a_x - a_y\|_1 < \epsilon$ for every $(x, y) \in E$; here $a_x := a(x) \in S_{\ell^1(X)}$.

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Remark 2.13. According to the proof of [16, Theorem 3.1] as well as [17, Theorem 1.2.4], a uniformly locally finite metric space (X, μ) has property A (which was defined in a different way from the above) if and only if the uniformly locally finite coarse space $(X, \mathcal{E}_\mu)$ has property A in the above sense.

Theorem 2.14. *Let $(X, \mathcal{E})$ be a uniformly locally finite coarse space. The following are equivalent.*

A1). $(X, \mathcal{E})$ has property A.

A2). For any countably-quasi-directed family $\mathcal{M}$ of metrics on X inducing $\mathcal{E}$ and any $\mu \in \mathcal{M}$, there is $\nu \in \mathcal{M}$ with $\nu \preceq \mu$ such that for $k, r \in \mathbb{N}$, one can find $a : X \to S_{\ell^1(X)}$ satisfying $\sup_{x,y\in X} \sup\{\nu(x, y) : a_x(y) \neq 0\} < \infty$ and $\|a_x - a_y\|_1 < 1/k$ when $\mu(x, y) \leq r$.

A3). There exists an effective quasi-directed family $\mathcal{N}$ of metrics on X inducing $\mathcal{E}$ such that (X, μ) is uniformly locally finite and has property A, for each $\mu \in \mathcal{N}$.

Proof. $(A1) \Rightarrow (A2)$. It is obvious that Statement (A1) implies the following statement.

A2'). For any quasi-directed family $\mathcal{M}$ of metrics on X inducing $\mathcal{E}$, any $\mu \in \mathcal{M}$ and any $k, r \in \mathbb{N}$, there exist $a : X \to S_{\ell^1(X)}$ and $\nu \in \mathcal{M}$ with $\nu \preceq \mu$ such that $\sup_{x,y\in X} \sup\big\{\nu(x, y) : a_x(y) \neq 0\big\} < \infty$ and $\|a_x - a_y\|_1 < 1/k$ when $(x, y) \in \mathbf{D}^\mu_r$.

Assume that Statement (A2') holds. Consider $\mu \in \mathcal{M}$ and $k, r \in \mathbb{N}$. There is a map $a^{k,r} : X \to S_{\ell^1(X)}$ as well as $\nu_{k,r} \in \mathcal{M}$ with $\nu_{k,r} \preceq \mu$ such that

$$\sup_{x,y\in X} \sup\big\{\nu_{k,r}(x, y) : a^{k,r}_x(y) \neq 0\big\} < \infty \quad \text{and} \quad \big\|a^{k,r}_x - a^{k,r}_y\big\|_1 < 1/k \quad \text{when} \quad \mu(x, y) \leq r.$$

As $\mathcal{M}$ is countably-quasi-directed, one can find $\nu \in \mathcal{M}$ with $\nu \preceq \mu$ and $\nu \preceq \nu_{k,r}$ for every $k, r \in \mathbb{N}$. This ν will then satisfy the requirement in Statement (A2).

$(A2) \Rightarrow (A3)$. Let $\mathcal{M}_0$ be the effective countably-quasi-directed family as in the proof of Proposition 2.4. Then $\mathcal{M}_0$ is a family of uniformly locally finite metrics on X (see Corollary 2.5(a)). Fix an arbitrary $\mu \in \mathcal{M}_0$ and set $\mu_1 := \mu$. Statement (A2) gives $\mu_2 \in \mathcal{M}_0$ with $\mu_2 \preceq \mu_1$ such that for $k, r \in \mathbb{N}$, one can find $a^{1,k,r} : X \to S_{\ell^1(X)}$ with

$$\sup_{x,y\in X} \sup\big\{\mu_2(x, y) : a^{1,k,r}_x(y) \neq 0\big\} < \infty \quad \text{and} \quad \big\|a^{1,k,r}_x - a^{1,k,r}_y\big\|_1 < 1/k \quad \text{when} \quad \mu_1(x, y) \leq r.$$

Inductively, for every $n \in \mathbb{N}$, we can find $\mu_{n+1} \in \mathcal{M}_0$ with $\mu_{n+1} \preceq \mu_n$ satisfying: for $k, r \in \mathbb{N}$, there exists $a^{n,k,r} : X \to S_{\ell^1(X)}$ with $\sup_{x,y\in X} \sup\big\{\mu_{n+1}(x, y) : a^{n,k,r}_x(y) \neq 0\big\} < \infty$ and $\big\|a^{n,k,r}_x - a^{n,k,r}_y\big\|_1 < 1/k$ when $\mu_n(x, y) \leq r$. By Proposition 1.5 and the construction of $\mathcal{M}_0$, the coarse structure $\mathcal{F}_\infty$ induced by $\{\mu_n : n \in \mathbb{N}\}$ coincides with $\mathcal{E}_{\bar\mu}$ for some $\bar\mu \in \mathcal{M}_0$.

Now, for $s, k \in \mathbb{N}$, one can find $r, n \in \mathbb{N}$ with $\mathbf{D}^{\bar\mu}_s \subseteq \mathbf{D}^{\mu_n}_r$ (see Proposition 2.4(a)). The above tells us that there is a map $a^{n,k,r} : X \to S_{\ell^1(X)}$ satisfying

$$\sup_{x,y\in X} \sup\big\{\mu_{n+1}(x, y) : a^{n,k,r}_x(y) \neq 0\big\} < \infty \quad \text{and} \quad \big\|a^{n,k,r}_x - a^{n,k,r}_y\big\|_1 < 1/k \quad \text{when} \quad \mu_n(x, y) \leq r.$$

From this, we see that $\sup_{x,y\in X} \sup\big\{\bar\mu(x, y) : a^{n,k,r}_x(y) \neq 0\big\} < \infty$ and $\|a^{n,k,r}_x - a^{n,k,r}_y\|_1 < 1/k$ when $\bar\mu(x, y) \leq s$. This means that $(X, \mathcal{E}_{\bar\mu})$ has property A. Finally, as in the proof of $(W2) \Rightarrow (W3)$ in Theorem 2.8, we know that

$$\mathcal{N} := \{\mu \in \mathcal{M}_0 : (X, \mu) \text{ is uniformly locally finite and has property A}\}$$

is an effective countably-quasi-directed family of metrics on X inducing $\mathcal{E}$.

$(A3) \Rightarrow (A1)$. This follows more or less from the definition. □

The last property that we want to consider in this section is the notion of bounded geometry, as defined in [15], which is an extension of the corresponding notion for metric spaces. Let us recall that the proof of [12, Theorem 5(b)] as well as [12, Proposition 9] produce an alternative description for coarse spaces with bounded geometry. For simplicity, we will use this alternative form as our definition.

Definition 2.15. *Let $(X,\mathcal{E})$ be a coarse space. Then $(X,\mathcal{E})$ is said to have* bounded geometry *if there exists $E_0 \in \hat{\mathcal{E}}$ (see (2.1)) satisfying: for each $F \in \hat{\mathcal{E}}$, one can find $n \in \mathbb{N}$ such that every set of the form $F[x]$ (for some $x \in X$) is contained in $E_0[y_1] \cup \cdots \cup E_0[y_n]$ for some $y_1, \ldots, y_n \in X$. In this case, the set E_0 as in the above is called a* gauge *for $(X,\mathcal{E})$.*

Remark 2.16. (a) Note that a coarse space $(X,\mathcal{E})$ is uniformly locally finite if and only if Δ_X is a gauge for $(X,\mathcal{E})$. This tells us that every uniformly locally finite coarse space has bounded geometry.

(b) From the argument in [15, Remark 3.26], if the coarse structure of a metric space has bounded geometry, then this metric space is coarsely equivalent to a uniformly locally finite metric space. In fact, the same is true for coarse space having bounded geometry.

(c) Suppose that $(X,\mathcal{E})$ is a coarse space and $\mathcal{M}$ is a quasi-directed family of metrics on X inducing $\mathcal{E}$. It follows from the definitions that $(X,\mathcal{E})$ has bounded geometry if and only if there exist $\mu_0 \in \mathcal{M}$ and $r_0 \in \mathbb{N}$ satisfying: whenever $\mu \in \mathcal{M}$ and $s \in \mathbb{N}$, one can find $n \in \mathbb{N}$ such that for any $x \in X$, there are $y_1, \ldots, y_n \in X$ with $\mathbf{D}^{\mu}_{s}[x] \subseteq \bigcup_{k=1}^{n} \mathbf{D}^{\mu_0}_{r_0}[y_k]$.

Proposition 2.17. *Let $(X,\mathcal{E})$ be a coarse space and $\mathcal{M}$ be a quasi-directed family of metrics on X inducing $\mathcal{E}$. If $(X,\mathcal{E})$ has bounded geometry, then one can find $\mu_0 \in \mathcal{M}$ such that for every $\mu \in \mathcal{M}$ with $\mu \preceq \mu_0$, the coarse space $(X,\mathcal{E}_\mu)$ has bounded geometry.*

Proof. Let μ_0 and r_0 satisfy the property in Remark 2.16(c). Consider $\mu \in \mathcal{M}$ with $\mu \preceq \mu_0$. Then there exists $r_1 \in \mathbb{N}$ such that $\mathbf{D}^{\mu_0}_{r_0} \subseteq \mathbf{D}^{\mu}_{r_1}$. For any $s \in \mathbb{N}$ and $x \in X$, one can find $y_1, ..., y_n \in X$ satisfying $\mathbf{D}^{\mu}_{s}[x] \subseteq \bigcup_{k=1}^{n} \mathbf{D}^{\mu_0}_{r_0}[y_k] \subseteq \bigcup_{k=1}^{n} \mathbf{D}^{\mu}_{r_1}[y_k]$. It follows that $(X,\mathcal{E}_\mu)$ has bounded geometry. □

The converse of Proposition 2.17 is unfortunately not true in general, as can be seen in Example 2.19 below. Let us do some preparation for it.

Let $(A_n, \mathbf{d}_n)$ be a metric space for each $n \in \mathbb{N}$. Set X to be the disjoint union $\bigsqcup_{n\in\mathbb{N}} A_n$. We will identify A_n with the corresponding subset of X. For each $n \in \mathbb{N}$, define a metric μ_n on X by

$$\mu_n(x,y) := \begin{cases} \mathbf{d}_k(x,y) & \text{when } x, y \in A_k \text{ for some } k \leq n; \\ 0 & \text{when } x = y; \\ \infty & \text{otherwise.} \end{cases} \tag{2.2}$$

Notice that $\mathbf{D}^{\mu_n}_{r} \subseteq \bigcup_{k\in\mathbb{N}} A_k \times A_k$ $(r \in \mathbb{N})$. We denote by $\mathcal{E}$ the coarse structure on X induced by $\{\mu_n\}_{n\in\mathbb{N}}$, and call it the *coarse disjoint union coarse structure.*

Proposition 2.18. *Let $\{(A_k, \mathbf{d}_k)\}_{k\in\mathbb{N}}$ be a sequence of metric spaces. Let $X := \bigsqcup_{n\in\mathbb{N}} A_n$ and $\mathcal{E}$ be the coarse disjoint union coarse structure on X. Suppose that $n \in \mathbb{N}$ and μ_n is the metric on X as in* (2.2)*.*

(a) If $(A_k, \mathcal{E}_{\mathbf{d}_k})$ has bounded geometry for $k = 1, \ldots, n$, then $(X, \mathcal{E}_{\mu_n})$ has bounded geometry.

(b) $(X,\mathcal{E})$ has bounded geometry if and only if all $\{(A_k, \mathcal{E}_{\mathbf{d}_k})\}_{k\in\mathbb{N}}$ have bounded geometry and all but a finite number of $\{(A_k, \mathcal{E}_{\mathbf{d}_k})\}_{k\in\mathbb{N}}$ are uniformly locally finite.

Proof. (a) Let E_k be a gauge for $(A_k, \mathcal{E}_{\mathbf{d}_k})$, for $k \in \mathbb{N}$. We put

$$F_n := \bigcup_{k=1}^{n} E_k \cup \Delta_X \in \hat{\mathcal{E}}_{\mu_n}.$$

Consider $E \in \hat{\mathcal{E}}_{\mu_n}$. There exists $r \in \mathbb{N}$ such that $E \subseteq \mathbf{D}^{\mu_n}_{r} \subseteq \bigcup_{k=1}^{n} \mathbf{D}^{\mathbf{d}_k}_{r} \cup \Delta_X$. For any $k \in \{1, \ldots, n\}$, one can find $m_k \in \mathbb{N}$ such that when $x \in A_k$, one has

$$E[x] \subseteq \mathbf{D}^{\mathbf{d}_k}_{r}[x] \subseteq E_k[x_1] \cup \cdots \cup E_k[x_{m_k}], \quad \text{for some } x_1, \ldots, x_{m_k} \in A_k.$$

Set $m := \max\{m_1, \ldots, m_n\}$. Pick any $x \in X$. If $x \notin \bigcup_{k=1}^{n} A_k$, then $E[x] = \{x\} \subseteq F_n[x]$. If $x \in A_k$ for some $k \leq n$, and $x_1, ..., x_{m_k}$ are the elements as in the above, then

$$E[x] \subseteq E_k[x_1] \cup \cdots \cup E_k[x_{m_k}] \subseteq F_n[x_1] \cup \cdots \cup F_n[x_m]$$

as required (we may set $x_l := x_{m_k}$ for $m_k \leq l \leq m$).

(b) We first show that if $(X, \mathcal{E})$ has bounded geometry, then all the coarse spaces $(A_k, \mathcal{E}_{\mathbf{d}_k})$ are of bounded geometry and $(A_k, \mathbf{d}_k)$ is uniformly locally finite for large enough k. In fact, since $(X, \mathcal{E})$ has bounded geometry, it is easy to see that $(A_k, \mathcal{E}_{\mathbf{d}_k})$ has bounded geometry. Moreover, suppose that E_0 is a gauge for $(X, \mathcal{E})$. Since $\mathcal{E}$ is induced by the quasi-directed family $\{\mu_k\}_{k\in\mathbb{N}}$ of metrics, Proposition 2.4(a) gives $n, r \in \mathbb{N}$ such that $E_0 \subseteq \mathbf{D}_r^{\mu_n}$. Fix any $k > n$. Consider $F \in \hat{\mathcal{E}}_{\mathbf{d}_k} \subseteq \mathcal{E}$. There is $N \in \mathbb{N}$ such that for every $x \in A_k \subseteq X$,

$$F[x] \subseteq E_0[y_1] \cup \cdots \cup E_0[y_N], \quad \text{for some } y_1, \ldots, y_N \in X.$$

Since $F[x] \subseteq A_k$, we may assume that $y_1, \ldots, y_N \in A_k$ (because $\mathbf{D}_r^{\mu_n}[y] \cap A_k = \emptyset$ when $y \notin A_k$). Furthermore, as $E_0 \subseteq \mathbf{D}_r^{\mu_n}$ and $k > n$, one has $E_0[y_i] \subseteq \{y_i\} = \Delta_{A_k}[y_i]$ for $i = 1, \ldots, N$. This means that Δ_{A_k} is a gauge for $(A_k, \mathcal{E}_{\mathbf{d}_k})$, and hence $(A_k, \mathbf{d}_k)$ is uniformly locally finite.

Conversely, assume that all $\{(A_k, \mathcal{E}_{\mathbf{d}_k})\}_{k\in\mathbb{N}}$ have bounded geometry and there is $N \in \mathbb{N}$ such that $(A_k, \mathbf{d}_k)$ is uniformly locally finite for $k > N$. Let E_k be a gauge for $(A_k, \mathcal{E}_{\mathbf{d}_k})$ when $k \in \{1, \ldots, N\}$ and set $E_k := \Delta_{A_k}$ for every $k > N$. As in the argument for part (a) above, $F_n := \bigsqcup_{k=1}^n E_k \cup \Delta_X$ is a gauge for $(X, \mathcal{E}_{\mu_n})$ ($n \in \mathbb{N}$). Since $E_k = \Delta_{A_k}$ for $k > N$, one has $F_n \subseteq F_N$ for every $n \in \mathbb{N}$. Pick any $E \in \hat{\mathcal{E}}$. There exist $n \in \mathbb{N}$ and $s \in \mathbb{N}$ such that $E \subseteq \mathbf{D}_s^{\mu_n}$. Since F_n is a gauge for $(X, \mathcal{E}_{\mu_n})$, there exists $m \in \mathbb{N}$ such that for every $x \in X$, one can find $y_1, \ldots, y_m \in X$ with

$$E[x] \subseteq F_n[y_1] \cup \cdots \cup F_n[y_m] \subseteq F_N[y_1] \cup \cdots \cup F_N[y_m].$$

This shows that F_N is a gauge for $(X, \mathcal{E})$. □

Example 2.19. *Let us consider the case when all $(A_n, \mathbf{d}_n)$ are the Euclidean metric space $(\mathbb{R}, \mathbf{d})$. Then $X := \bigsqcup_{n\in\mathbb{N}} \mathbb{R}$ does not have bounded geometry, when it is equipped with the coarse disjoint union coarse structure, because $(\mathbb{R}, \mathbf{d})$ is not uniformly locally finite (see Proposition 2.18(b)). However, as in Proposition 2.18(a), the quasi-directed family $\{\mu_n\}_{n\in\mathbb{N}}$ of metrics induces $\mathcal{E}$ and every $(X, \mathcal{E}_{\mu_n})$ has bounded geometry.*

By Remark 2.2, the following result is a direct application of Corollary 2.5, Theorems 2.8, 2.9 and 2.14 as well as Proposition 2.17.

Corollary 2.20. *Each coarse space having asymptotic dimension dominated by an integer n (respectively, having FWCDC, having bounded geometry, being locally finite, being uniformly locally finite or being uniformly locally finite with property A) is a coarse directed limit of a coarse inductive system of metric spaces, each of which has asymptotic dimension dominated by n (respectively, has FWCDC, has bounded geometry, is locally finite, is uniformly locally finite or is uniformly locally finite with property A).*

In Proposition 3.2 in the next section, we will see that a more general form of the converse of Corollary 2.20 holds, except for the part concerning bounded geometry.

3. General coarse directed limits of coarse spaces

In this section, we consider the coarse directed limits of coarse inductive systems of general coarse spaces (see Definition 1.6). Let us recall from Remark 1.7(c) that if $\big\{\big((X_i, \mathcal{E}_i), \phi_{j,i}\big)\big\}_{i\leq j\in\mathfrak{I}}$ is a coarse inductive system of coarse spaces, then

$$\lim_i \mathcal{E}_i = \bigcup\nolimits_{i\in\mathfrak{I}} \big\{(\phi_i \times \phi_i)(E) \cup F : E \in \mathcal{E}_i \text{ and } F \subseteq \Delta_{\lim_i X_i}\big\}.$$

Thus, in order to study $\lim_i \mathcal{E}_i$, we need to have a look at the coarse structure generated by $\big\{(\phi_i \times \phi_i)(E) \cup \Delta_{\lim_i X_i} : E \in \mathcal{E}_i\big\}$ for a fixed $i \in \mathfrak{I}$. We will do it in the following lemma.

Lemma 3.1. *Let $(X,\mathcal{E})$ be a coarse space and $\phi : X \to Y$ be an injection. Let $\mathcal{E}^\phi$ be the coarse structure on Y generated by $\{(\phi\times\phi)(E)\cup\Delta_Y : E\in\mathcal{E}\}$.*

(a) If $(X,\mathcal{E})$ is uniformly locally finite (respectively, locally finite), then so is $(Y,\mathcal{E}^\phi)$.

(b) If $\operatorname{asdim}(X,\mathcal{E})\le N$ for some $N\in\mathbb{Z}_+$ (respectively, $(X,\mathcal{E})$ has coarse property C or FWCDC), then $\operatorname{asdim}(Y,\mathcal{E}^\phi)\le N$ (respectively, $(Y,\mathcal{E}^\phi)$ has coarse property C or FWCDC).

(c) If $(X,\mathcal{E})$ is a uniformly locally finite coarse space having property A, then $(Y,\mathcal{E}^\phi)$ has property A.

Proof. The injectivity of ϕ implies that $\mathcal{E}^\phi = \{F_1\cup F_2 : F_1\in(\phi\times\phi)(\mathcal{E}) \text{ and } F_2\subseteq\Delta_Y\}$.

(a) We will consider the case when $(X,\mathcal{E})$ is uniformly locally finite (the argument for the locally finite case is the same). Consider $F\in\mathcal{E}^\phi$. There is $E\in\mathcal{E}$ with $F\subseteq(\phi\times\phi)(E)\cup\Delta_Y$. As $(X,\mathcal{E})$ is uniformly locally finite, one can find $n\in\mathbb{N}$ such that for each $x\in X$, the number of elements in $E[x]$ is dominated by n. Pick any $y\in Y$. If $y\notin\phi(X)$, then $\big((\phi\times\phi)(E)\cup\Delta_Y\big)[y]$ has exactly one element. If $y=\phi(x)$ for some $x\in X$, then $\big((\phi\times\phi)(E)\cup\Delta_Y\big)[y]\subseteq\phi(E[x])$, which has at most n elements.

(b) Suppose that $E\in\hat{\mathcal{E}}$ (see (2.1)) and $\{X_j\}_{j\in\Lambda}$ is a family of subsets of X such that $\bigcup_{j\in\Lambda}X_j\times X_j\in\mathcal{E}$ and $E\cap X_j\times X_k=\emptyset$ whenever $j\neq k$ in Λ. We consider the disjoint union

$$\Lambda^\phi := \Lambda\sqcup(Y\setminus\phi(X)).$$

For $j\in\Lambda$, we set $X^\phi_j := \phi(X_j)$. For $y\in\Lambda^\phi\setminus\Lambda$, we set $X^\phi_y := \{y\}$. It is not hard to check that $\bigcup_{j\in\Lambda^\phi}X^\phi_j\times X^\phi_j\in\mathcal{E}^\phi$, and that $\big((\phi\times\phi)(E)\cup\Delta_Y\big)\cap X^\phi_j\times X^\phi_k=\emptyset$ when $j\neq k$ in Λ^ϕ.

Consider$\{F_k\}_{k\in\mathbb{N}\cup\{0\}}$ to be an increasing sequnce in $\hat{\mathcal{E}}^\phi$. For each $k\in\mathbb{N}\cup\{0\}$, one can find $E_k\in\hat{\mathcal{E}}$ with $F_k\subseteq(\phi\times\phi)(E_k)\cup\Delta_Y$ such that $\{E_k\}_{k\in\mathbb{N}\cup\{0\}}$ is again increasing. Suppose that $d\in\mathbb{N}$ and $\{X_{k,j}\}_{j\in\Lambda_k;k\in\{0,\dots,d\}}$ is a partition of X satisfying $\bigcup_{j\in\Lambda_k}X_{k,j}\times X_{k,j}\in\mathcal{E}$ and $E_k\cap X_{k,i}\times X_{k,j}=\emptyset$ when $i\neq j$ in Λ_k and $k\in\{0,\dots,d\}$. Then

$$\{X^\phi_{0,j} : j\in\Lambda^\phi_0\}\cup\{\phi(X_{k,j})\}_{j\in\Lambda_k;k\in\{1,\dots,d\}}$$

is a partition of Y satisfying the corresponding condition for $\mathcal{E}^\phi$ and the one for $F_0,\dots,F_d$.

(c) Suppose that $(X,\mathcal{E})$ is a uniformly locally finite coarse space having property A. Then by part (a) above, $(Y,\mathcal{E}^\phi)$ is a uniformly locally finite coarse space. Consider $F\in\mathcal{E}^\phi$ and $\epsilon>0$. There exists $E\in\mathcal{E}$ such that $F\subseteq(\phi\times\phi)(E)\cup\Delta_Y$. Since $(X,\mathcal{E})$ has property A, one can find $a : X\to S_{\ell^1(X)}$ such that

$$\big\{(x,x')\in X\times X : a_x(x')\neq 0\big\}\in\mathcal{E} \quad\text{and}\quad \|a_x-a_{x'}\|_1<\epsilon \text{ when } (x,x')\in E.$$

Consider $y,y'\in Y$. If $y=\phi(x)$ for some $x\in X$, then we set

$$b_y(y') = \begin{cases} a_x(x') & \text{if } y'=\phi(x') \text{ for some } x'\in X,\\ 0 & \text{otherwise.}\end{cases}$$

If $y\notin\phi(X)$, then we set

$$b_y(y') = \begin{cases} 1 & \text{if } y'=y,\\ 0 & \text{if } y'\neq y.\end{cases}$$

As ϕ is injective and $a_x\in S_{\ell^1(X)}$ $(x\in X)$, we know that b_y is well-defined and belongs to $S_{\ell^1(Y)}$, for each $y\in Y$. We also have

$$\big\{(y,y')\in Y\times Y : b_y(y')\neq 0\big\}\subseteq(\phi\times\phi)\big(\big\{(x,x')\in X\times X : a_x(x')\neq 0\big\}\big)\cup\Delta_Y\in\mathcal{E}^\phi.$$

Moreover, if $(y,y')\in F$ with $y\neq y'$, then there exists $(x,x')\in E$ such that $y=\phi(x)$ and $y'=\phi(x')$, which implies $\|b_y-b_{y'}\|_1=\|a_x-a_{x'}\|_1<\epsilon$. Therefore, $(Y,\mathcal{E}^\phi)$ has property A. □

Let us now go back to the coarse inductive system $\big\{\big((X_i,\mathcal{E}_i),\phi_{j,i}\big)\big\}_{i\le j\in\mathfrak{I}}$ of coarse spaces. By its definition, $\lim_i\mathcal{E}_i$ is the finest coarse structure coarser than all the coarse structures $\mathcal{E}_i^{\phi_i}$ on $\lim_i X_i$. Therefore, using the previous lemma as well as the arguments for the corresponding results in Theorems 2.8, 2.9 and 2.14 as well as Corollary 2.5 and Proposition 2.10, we obtain the following.

Proposition 3.2. *Let $\{((X_i,\mathcal{E}_i),\phi_{j,i})\}_{i\leq j\in\mathfrak{I}}$ be a coarse inductive system of coarse spaces, and let $(X,\mathcal{E})$ be its coarse directed limit.*

(a) If there is $N\in\mathbb{Z}_+$ such that $\operatorname{asdim}(X_i,\mathcal{E}_i)\leq N$ for all $i\in\mathfrak{I}$, then $\operatorname{asdim}(X,\mathcal{E})\leq N$.

(b) If $(X_i,\mathcal{E}_i)$ has FWCDC for each $i\in\mathfrak{I}$, then $(X,\mathcal{E})$ has FWCDC.

(c) If $(X_i,\mathcal{E}_i)$ is uniformly locally finite (respectively, locally finite) for each $i\in\mathfrak{I}$, then $(X,\mathcal{E})$ is uniformly locally finite (respectively, locally finite).

(d) If $(X_i,\mathcal{E}_i)$ is a uniformly locally finite coarse space having property A for each $i\in\mathfrak{I}$, then $(X,\mathcal{E})$ has property A.

(e) Suppose that $\mathfrak{I}$ is a countably-upward directed set. If $(X_i,\mathcal{E}_i)$ has coarse property C for each $i\in\mathfrak{I}$, then $(X,\mathcal{E})$ has coarse property C.

Example 3.3. *(a) Let X be a set and $\mathfrak{F}$ be the collection of all non-empty finite subsets of X. Then X can be identified with the set directed limit $\lim_{Y\in\mathfrak{F}}Y$ canonically. For every $Y\in\mathfrak{F}$, we equip Y with the coarse structure $\mathcal{E}^0_Y:=\mathcal{P}(Y\times Y)$. Then a subset $E\subseteq X\times X$ belongs to $\lim_{Y\in\mathfrak{F}}\mathcal{E}^0_Y$ if and only if $E\setminus\Delta_X$ is a finite set. Since each $Y\in\mathfrak{F}$ has only finitely many elements, we know that $(Y,\mathcal{E}^0_Y)$ is uniformly locally finite and $\operatorname{asdim}(Y,\mathcal{E}^0_Y)=0$. By Proposition 3.2, $(X,\lim_Y\mathcal{E}^0_Y)$ is uniformly locally finite and $\operatorname{asdim}(X,\lim_Y\mathcal{E}^0_Y)=0$. However, when X is not countable, $\lim_{Y\in\mathfrak{F}}\mathcal{E}^0_Y$ is not countably generated and hence is not metrizable.*

(b) Let X be a set and $\mathbf{d}:X\times X\to\mathbb{R}_+$ be a metric. Fix an element $x_0\in X$. For $n\in\mathbb{N}$, we set

$$\mathbf{D}_n(x_0):=\{x\in X:\mathbf{d}(x,x_0)\leq n\},$$

and equip $\mathbf{D}_n(x_0)$ with the coarse structure $\mathcal{E}_n$ induced by $\mathbf{d}|_{\mathbf{D}_n(x_0)\times\mathbf{D}_n(x_0)}$; i.e., $\mathcal{E}_n=\mathcal{P}(\mathbf{D}_n(x_0)\times\mathbf{D}_n(x_0))$. Then X can be identified with the set directed limit $\lim_n\mathbf{D}_n(x_0)$. By Proposition 3.2(a), we have $\operatorname{asdim}(X,\lim_n\mathcal{E}_n)=0$, although it could happen that $\operatorname{asdim}(X,\mathbf{d})\neq 0$ (e.g., if $\mathbf{d}$ is the Euclidean metric on $X=\mathbb{N}$).

It is not hard to see that $E\subseteq X\times X$ belongs to $\lim_n\mathcal{E}_n$ if and only if there is a bounded subset $B\subseteq X$ such that $E\setminus\Delta_X\subseteq B\times B$ (note that the metric $\mathbf{d}$ only takes finite values), and $\lim_n\mathcal{E}_n$ does not depend on the choice of x_0.

From this, we know that $\lim_n\mathcal{E}_n=\mathcal{E}_\mathbf{d}$ if and only if for every $m\in\mathbb{N}$, there is $n\in\mathbb{N}$ satisfying

$$\{(x,y)\in X\times X:0<\mathbf{d}(x,y)\leq m\}\subseteq\mathbf{D}_n(x_0)\times\mathbf{D}_n(x_0).$$

In particular, $\lim_n\mathcal{E}_n\neq\mathcal{E}_\mathbf{d}$ when $\mathbf{d}$ is the Euclidean metric on $X:=\mathbb{N}$. However, if $\mathbf{d}$ is the metric on $X:=\mathbb{N}$ satisfying

$$\mathbf{d}(k,l):=\max\{k,l\}\qquad(k,l\in\mathbb{N};k\neq l),$$

then $\lim_n\mathcal{E}_n=\mathcal{E}_\mathbf{d}$ on $\mathbb{N}$ (choose $x_0=1$). Another example for $\lim_n\mathcal{E}_n=\mathcal{E}_\mathbf{d}$ is the one in Example 1.8.

Via Example 2.19, we know that the corresponding statement as Proposition 3.2 for bounded geometry does not hold; i.e., bounded geometry does not pass to coarse directed limit. In order to have it in more precise term, let us give the following remark concerning coarse disjoint unions and coarse directed limits.

Remark 3.4. Let $\{(A_k,\mathbf{d}_k)\}_{k\in\mathbb{N}}$ be a sequence of metric spaces. For $n\in\mathbb{N}$, we define a metric $\bar{\mathbf{d}}_n$ on the finite disjoint union $\bigsqcup_{k=1}^n A_k$ by

$$\bar{\mathbf{d}}_n(x,y):=\begin{cases}\mathbf{d}_k(x,y) & \text{when } x,y\in A_k \text{ for some } k\leq n;\\ \infty & \textit{otherwise}.\end{cases}$$

(a) Let $\varphi_{m,n}:\bigsqcup_{k=1}^n A_k\to\bigsqcup_{k=1}^m A_k$ be the canonical map for $m\geq n$. Clearly,

$$\{((\bigsqcup_{k=1}^n A_k,\mathcal{E}_{\bar{\mathbf{d}}_n}),\varphi_{m,n})\}_{n\leq m\in\mathbb{N}}$$

is a coarse inductive system of coarse spaces. The set directed limit of this system is the disjoint union $\bigsqcup_{k=1}^\infty A_k$. Consider $\varphi_n:\bigsqcup_{k=1}^n A_k\to\bigsqcup_{k=1}^\infty A_k$ to be the inclusion map. Then the coarse structure $\mathcal{E}^{\varphi_n}_{\bar{\mathbf{d}}_n}$

as in Lemma 3.1 coincides with the coarse structure on $\bigsqcup_{k=1}^{\infty} A_k$ induced by the metric μ_n as defined in (2.2). Thus, $\lim_n \mathcal{E}_{\bar{\mathbf{d}}_n}$ coincides with the coarse disjoint union coarse structure as considered in the paragraph preceeding Proposition 2.18.

(b) For a fixed $n \in \mathbb{N}$, it follows from the proof of Proposition 2.18(a) that if $(A_1, \mathcal{E}_{\mathbf{d}_1}), \ldots, (A_n, \mathcal{E}_{\mathbf{d}_n})$ have bounded geometry and E_k is a gauge for $(A_k, \mathcal{E}_{\mathbf{d}_k})$ $(k = 1, \ldots, n)$, then $\bigsqcup_{k=1}^{n} E_k$ is a gauge for $\big(\bigsqcup_{k=1}^{n} A_k, \mathcal{E}_{\bar{\mathbf{d}}_n}\big)$.

Example 3.5. *Suppose that $(A_k, \mathbf{d}_k)$ is the Euclidean metric space $(\mathbb{R}, \mathbf{d})$ for each $k \in \mathbb{N}$. Then Remark 3.4 and Example 2.19 tell us that $\big(\bigsqcup_{k=1}^{\infty} A_k, \lim_n \mathcal{E}_{\bar{\mathbf{d}}_n}\big)$ does not have bounded geometry, while all $\big(\bigsqcup_{k=1}^{n} A_k, \mathcal{E}_{\bar{\mathbf{d}}_n}\big)$ have bounded geometry.*

The following result is direct consequence of the definitions.

Proposition 3.6. *Let $\big\{\big((X_i, \mathcal{E}_i), \phi_{j,i}\big)\big\}_{i \le j \in \mathfrak{I}}$ and $\big\{\big((Y_i, \mathcal{F}_i), \psi_{j,i}\big)\big\}_{i \le j \in \mathfrak{I}}$ be two coarse inductive systems of coarse spaces. Suppose that for each $i \in \mathfrak{I}$, there is a bornologous map $f_i : X_i \to Y_i$ satisfying $f_j \circ \phi_{j,i} = \psi_{j,i} \circ f_i$ when $i \le j$ in $\mathfrak{I}$. Then there exists a unique bornologous map $f : \lim_i X_i \to \lim_i Y_i$ such that $f \circ \phi_i = \psi_i \circ f_i$ $(i \in \mathfrak{I})$.*

Note that the map f in the above is effectively proper if and only if

$$\bigcup\nolimits_{j \ge i} (\phi_j \times \phi_j)(f_j \times f_j)^{-1}\big((\psi_{j,i} \times \psi_{j,i})(F) \cup \Delta_{Y_j}\big) \in \mathcal{E} \qquad (F \in \mathcal{F}_i, i \in \mathfrak{I}).$$

Hence, in general, even if all f_i are effectively proper, the map f may not be effectively proper (a description for the case when such a map f is effective proper will be given in [13]). Nevertheless, it is not hard to see that the following is true.

Proposition 3.7. *Suppose that $\{(A_n, \mathbf{d}_n)\}_{n \in \mathbb{N}}$ and $\{(B_n, \mathbf{d}'_n)\}_{n \in \mathbb{N}}$ be two sequences of metric spaces with $\big(X, \mathcal{E}\big)$ and $\big(Y, \mathcal{F}\big)$ being their respective coarse disjoint unions. If $f_n : A_n \to B_n$ is an injective rough map for each $n \in \mathbb{N}$, then the resulting map $\bigsqcup_{n \in \mathbb{N}} f_n : X \to Y$ is a rough map.*

However, the following example tells us that even if $(A_n, \mathbf{d}_n)$ is coarsely equivalent to $(B_n, \mathbf{d}'_n)$ for every $n \in \mathbb{N}$, it is possible that $\big(\bigsqcup_{k=1}^{\infty} A_k, \lim_n \mathcal{E}_{\bar{\mathbf{d}}_n}\big)$ is not coarsely equivalent to $\big(\bigsqcup_{k=1}^{\infty} B_k, \lim_n \mathcal{E}_{\bar{\mathbf{d}}'_n}\big)$.

Example 3.8. *For every $k \in \mathbb{N}$, we let $(A_k, \mathbf{d}_k)$ be the Euclidean metric space $(\mathbb{R}, \mathbf{d})$, and $(B_k, \mathbf{d}'_k)$ be the subspace $(\mathbb{Z}, \mathbf{d})$. It is not hard to check that $\big(\bigsqcup_{k=1}^{n} A_k, \mathcal{E}_{\bar{\mathbf{d}}_n}\big)$ (see Remark 3.4) is coarsely equivalent to $\big(\bigsqcup_{k=1}^{n} B_k, \mathcal{E}_{\bar{\mathbf{d}}'_n}\big)$. Since $\big(\bigsqcup_{k=1}^{n} B_k, \mathcal{E}_{\bar{\mathbf{d}}'_n}\big)$ is uniformly locally finite for each $n \in \mathbb{N}$, we know from Proposition 3.2(c) that $\big(\bigsqcup_{k=1}^{\infty} B_k, \lim_n \mathcal{E}_{\bar{\mathbf{d}}'_n}\big)$ is uniformly locally finite, and hence it has bounded geometry. However, by Example 3.5, the coarse space $\big(\bigsqcup_{k=1}^{\infty} A_k, \lim_n \mathcal{E}_{\bar{\mathbf{d}}_n}\big)$ does not have bounded geometry, and hence cannot be coarsely equivalent to $\big(\bigsqcup_{k=1}^{\infty} B_k, \lim_n \mathcal{E}_{\bar{\mathbf{d}}'_n}\big)$.*

In the case when $\mathfrak{I}$ is countable and all $(X_i, \mathcal{E}_i)$ are metrizable, the coarse directed limit $(X, \mathcal{E})$ is also metrizable (see Proposition 1.5). Thus, Proposition 3.2 gives a way to construct metric spaces having some prescribed properties.

Now, suppose that $\mathfrak{I} = \mathbb{N}$ and there is a metric μ_k on each X_k with $\mathcal{E}_k = \mathcal{E}_{\mu_k}$ such that the injection $\phi_{k+1,k} : X_k \to X_{k+1}$ is metric decreasing. Then one can even obtain an explicit metric on $\lim_k X_k$ that induces $\lim_k \mathcal{E}_k$. In fact, let us put

$$F_n := \big\{\big(\phi_n(x), \phi_n(y)\big) : x, y \in X_n; \mu_n(x, y) \le 2^n\big\} \cup \Delta_{\lim_k X_k} \qquad (n \in \mathbb{N}). \tag{3.1}$$

It is easy to see that $F_n \circ F_n \subseteq F_{n+1}$, and $\lim_k \mathcal{E}_k$ is generated by $\{F_n\}_{n \in \mathbb{N}}$. This, together with the proof of [15, Theorem 2.55], tells us that $\lim_k \mathcal{E}_k$ is induced by the metric μ on $\lim_k X_k$ defined as

$$\mu(x, y) := \inf\{n \in \mathbb{N} : x = \phi_n(x_n) \text{ and } y = \phi_n(y_n) \text{ for some } x_n, y_n \in X_n \text{ with } \mu_n(x_n, y_n) \le 2^n\}$$

when $x \neq y$ in $\lim_k X_k$.

In the following, we will have a closer look at the very special situation when there exists a metric μ_k on each X_k such that all $\phi_{k+1,k}$ are metric preserving. Equivalently, there is a metric $\mathbf{d}$ defined on the set directed limit $\lim_k X_k$ such that the coarse structure $\mathcal{E}_k$ on each X_k is induced by $\mathbf{d}|_{X_k\times X_k}$.

As we have seen in Example 3.3(b), the directed limit coarse structure $\lim_k \mathcal{E}_k$ is in general NOT the coarse structure on $\lim_k X_k$ induced by the metric $\mathbf{d}$. Nevertheless, there are certain relations between them. In particular, since the properties of asymptotic dimension dominated by N, FWCDC, (uniform) local finiteness and property A all pass to subspaces, we know from Proposition 3.2 that

> *if* $\mathrm{asdim}(\lim_k X_k, \mathbf{d}) \leq N$*, or* $(\lim_k X_k, \mathbf{d})$ *has FWCDC, or* $(\lim_k X_k, \mathbf{d})$ *is (uniformly) locally finite or is uniformly locally finite with property A, then so is* $(\lim_k X_k, \lim_k \mathcal{E}_k)$.

However, a similar consideration as the above does not work for coarse property C, because we need to assume countably-upward directedness for the index set in Proposition 3.2(e), which does not hold for $\mathbb{N}$. Nevertheless, the following proposition tells us that the conclusion similar to the statement displayed above still holds. Note that this result is not an obvious fact, since there is no guarantee that a coarse structure that is finer than a coarse structure with coarse property C will have coarse property C.

Proposition 3.9. *Let* $(X, \mathbf{d})$ *be a metric space and* $\{X_k\}_{k\in\mathbb{N}}$ *be an increasing sequence of subspaces of* X *with* $X = \bigcup_{k\in\mathbb{N}} X_k$*. Suppose that* $\mathcal{E}_k$ *is the coarse structure on* X_k *induced by* $\mu_k := \mathbf{d}|_{X_k\times X_k}$*. If* $(X, \mathbf{d})$ *has asymptotic property C, then* $(X, \lim_k \mathcal{E}_k)$ *has coarse property C.*

Proof. Set $\mathcal{E} := \lim_k \mathcal{E}_k$ and let $E_1 \subseteq E_2 \subseteq \dots$ be an increasing sequence of elements in $\hat{\mathcal{E}}$ (see (2.1)). As said in the above, the collection $\{F_n\}_{n\in\mathbb{N}}$ as defined in (3.1) is a base for $\mathcal{E}$. Thus, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ with $n_k \geq n_{k-1}$ and

$$E_k \subseteq F_{n_k} = \big\{(x, y) : x, y \in X_{n_k}; \mathbf{d}(x, y) \leq 2^{n_k}\big\} \cup \Delta_X.$$

Since $(X, \mathbf{d})$ has asymptotic property C, for the sequence $2^{n_1} \leq 2^{n_2} \leq \dots$, one obtains, via Remark 2.7(e), an integer $m \in \mathbb{N}$ and a partition $\{U_{k,j}\}_{j\in\Lambda_k;k\in\{1,\dots,m\}}$ of X such that

$$\sup\nolimits_{j\in\Lambda_k;k\in\{1,\dots,m\}} \sup\{\mathbf{d}(x, y) : x, y \in U_{k,j}\} < \infty \tag{3.2}$$

and $\mathbf{d}(U_{k,j}, U_{k,i}) > 2^{n_k}$ when $k \in \{1, \dots, m\}$ and $j \neq i$ in Λ_k. Consider $k \in \{1, \dots, m\}$. We set

$$\bar{\Lambda}_k := \Lambda_k \sqcup \Big(\bigcup\nolimits_{j\in\Lambda_k} U_{k,j} \setminus X_{n_k}\Big),$$

where $\sqcup$ means the disjoint union. For $j \in \Lambda_k$, we put $\bar{U}_{k,j} := U_{k,j} \cap X_{n_k}$. For $x \in \bar{\Lambda}_k \setminus \Lambda_k$, we set $\bar{U}_{k,x} := \{x\}$. Then $\bigcup_{i\in\bar{\Lambda}_k} \bar{U}_{k,i} = \bigcup_{j\in\Lambda_k} U_{k,j}$. Moreover, Relation (3.2) and $\bar{U}_{k,j} \subseteq X_{n_m}$ when $k = 1, \dots, m$ and $j \in \Lambda_k$ imply that

$$\bigcup\nolimits_{k=1}^m \bigcup\nolimits_{j\in\bar{\Lambda}_k} \bar{U}_{k,j} \times \bar{U}_{k,j} \subseteq F_n \quad \text{for some } n \geq n_m.$$

Now, assume that $k \in \{1, \dots, m\}$ and $i, j \in \bar{\Lambda}_k$ with $i \neq j$. When $i \notin \Lambda_k$ or $j \notin \Lambda_k$, we know from the definitions that $F_{n_k} \cap \bar{U}_{k,j} \times \bar{U}_{k,i} = \emptyset$. When $i, j \in \Lambda_k$, one has $\bar{U}_{k,j} \subseteq U_{k,j}$ and $\bar{U}_{k,i} \subseteq U_{k,i}$, which implies that

$$\mathbf{d}(\bar{U}_{k,j}, \bar{U}_{k,i}) \geq \mathbf{d}(U_{k,j}, U_{k,i}) > 2^{n_k},$$

and hence $F_{n_k} \cap \bar{U}_{k,j} \times \bar{U}_{k,i} = \emptyset$. Therefore, $(X, \mathcal{E})$ has coarse property C. □

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References

[1] K. Austin and Ž. Virk, Coarse metric approximation, Top. and its Appl. **202** (2016), 194-204.

[2] G. Bell, D. Moran and A. Nagórko, Coarse property C and decomposition complexity, Top. and its Appl. **227** (2017), 30-50.

[3] B.M. Braga, I. Farah and A. Vignati, General uniform Roe algebra rigidity, Ann. Inst. Fourier (Grenoble) **72** (2022), no. 1, 301–337.

[4] A.N. Dranishnikov, Asymptotic topology, Russ. Math. Surv. **55**(2000), 1085-1129.

[5] J. Dydak and C.S. Hoffland, An alternative definition of coarse structures, Top. and its Appl. **155** (2008), 1013-1021.

[6] B. Grave, Asymptotic dimension of coarse spaces, New York J. Math. **12** (2006), 249-256.

[7] M. Gromov, Groups of polynomial growth and expanding maps, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53-73.

[8] M. Gromov, *Asymptotic invariants of infinite groups*, Cambridge University Press, 1993.

[9] E. Guentner, R. Tessera and G. Yu, Discrete groups with finite decomposition complexity, Groups Geom. Dyn. **7** (2013), 377–402.

[10] L. Higginbotham and K. Sinclair, Asymptotic filtered colimits, Top. and its Appl. **270** (2020), 106944.

[11] N. Higson, E.K. Pedersen and J. Roe, C^*-algebras and controlled topology, K-Theory **11** (1997), 209-239.

[12] C.K. Ng, Coarse metric and uniform metric, Top. and its Appl. **260** (2019), 1-12.

[13] C.K. Ng and R. Tian, Coarse directed limits of metric spaces admitting coarse embeddings into Hilbert spaces, in preparation.

[14] P.W. Nowak and G. Yu, *Large scale geometry*, EMS Textbooks in Mathematics, Eur. Math. Soc. (2012).

[15] J. Roe, *Lectures on Coarse Geometry*, Univ. Lecture Ser. **31**, Amer. Math. Soc. (2003).

[16] H. Sako, Property A for coarse spaces, Top. and its Appl. **300** (2021)107751.

[17] R. Willett, Some notes on property A, *Limits of Graphs in Group Theory and Computer Science* (2006), 191-281.

[18] T. Yamauchi, Asymptotic property C of the countable direct sum of the integers, Top. and its Appl. **184** (2015), 50-53.

[19] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension, Ann. Math. **147** (1998), 325-355.

[20] G. Yu, The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Inventiones Math. **139** (2000), 201–240.

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