Strongly $\mathcal{E}$-Gorenstein injective and flat modules*†

Zenghui Gao‡, Ying Zhong

College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, Sichuan Province, P. R. China

Abstract

Let $\mathcal{E}$ be an injectively resolving subcategory of left R-modules. The paper continues with the study of a particular case of $\mathcal{E}$-Gorenstein injective and flat modules, called strongly $\mathcal{E}$-Gorenstein injective and flat modules, respectively. We prove that a module is $\mathcal{E}$-Gorenstein injective if and only if it is a direct summand of a strongly $\mathcal{E}$-Gorenstein injective module, and every $\mathcal{E}$-Gorenstein flat module is a direct summand of a strongly $\mathcal{E}$-Gorenstein flat module. Then we show the property of being a strongly $\mathcal{E}$-Gorenstein injective (resp., flat) module can be inherited by its direct summands under certain condition. The connections between (strongly) $\mathcal{E}$-Gorenstein injective and flat modules are also discussed. Finally, we investigate FC rings in terms of strongly $\mathcal{E}$-Gorenstein injective and flat modules.

1. Introduction

Auslander and Bridger [2] generalized the notion of finitely generated modules to that of finitely generated modules of Gorenstein dimension zero over a two-sided Noetherian ring. Several decades later, Enochs et al. in [16, 18] defined Gorenstein projective, injective and flat modules for arbitrary modules over a general ring. It has been shown that Gorenstein projective, injective and flat modules share many nice properties of projective, injective and flat modules, respectively (cf.[3, 10, 17, 25]). In 2007, Bennis and Mahdou in [4] studied a particular case of Gorenstein projective, injective and flat modules, which are called strongly Gorenstein projective, injective and flat modules, respectively. They proved that a module is Gorenstein projective (resp., injective) if and only if it is a direct summand of a strongly Gorenstein projective (resp., injective) module [4, Theorem 2.7], and every Gorenstein flat module is a direct summand of a strongly Gorenstein flat module [4, Theorem 3.5]. Later, Yang and Liu [33] gave some nice characterizations of them.

On the other hand, some generalized versions of Gorenstein projective, injective and flat modules have been studied by many authors. For instance, Ding et al. introduced in [15, 27] strongly Gorenstein flat modules and Gorenstein FP-injective modules. Bravo, Gillespie and Hovey in [9] introduced the notions of Gorenstein AC-projective and AC-injective modules in terms of the so-called level modules and absolutely clean modules, and the homological

*2020 Mathematics Subject Classification: 18G25, 18G10, 16D40.

†Keywords: Injectively resolving subcategory, Strongly $\mathcal{E}$-Gorenstein flat module, Strongly $\mathcal{E}$-Gorenstein injective module, Duality pair, FC-ring

‡E-mail address: gaozenghui@cuit.edu.cn(Z. Gao), 1828323766@qq.com(Y. Zhong)

3 Nov 2022 18:24:49 PDT
220430-ZenghuiGao Version 4 - Submitted to Rocky Mountain J. Math.

behavior of them has been discussed in [8, 9, 24] etc. By replacing injective modules by absolutely clean modules in the notion of Gorenstein flat modules, the so-called Gorenstein AC-flat modules appeared in [7].

As a generalization of the notions of Gorenstein injective and flat modules, Gao et al. in [22, 23] introduced the notions of $\mathcal{E}$-Gorenstein injective and flat modules, where $\mathcal{E}$ is an injectively resolving subcategory of left R-modules. Recall that a left R-module M is said to be $\mathcal{E}$-Gorenstein injective ($\mathcal{G}_{\mathcal{E}}$-injective for short) in [22] if there exists an exact sequence of injective left R-modules

$$\mathscr{E} := \ \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Ker}(E^0 \to E^1)$ such that $\mathrm{Hom}_R(I, -)$ leaves the sequence $\mathscr{E}$ exact for any $I \in \mathcal{E}$. A right R-module M is said to be $\mathcal{E}$-Gorenstein flat ($\mathcal{G}_{\mathcal{E}}$-flat for short) in [23] if there exists an exact sequence of flat right R-modules

$$\mathscr{F} := \ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Ker}(F^0 \to F^1)$ such that $- \otimes_R I$ leaves the sequence $\mathscr{F}$ exact for any $I \in \mathcal{E}$.

It can be seen that the notion of $\mathcal{E}$-Gorenstein injective modules unifies the following notions: Gorenstein injective modules [16], Gorenstein FP-injective modules [27] and Gorenstein AC-injective modules [8], and the notion of $\mathcal{E}$-Gorenstein flat modules unifies some known modules such as Gorenstein flat modules [18] and Gorenstein AC-flat modules [7]. Also in [22, 23], the homological behavior of $\mathcal{E}$-Gorenstein injective and flat modules has been discussed. Along the same lines, it is natural to study the consequences of extending the notion of strongly Gorenstein injective and flat module to that of "strongly $\mathcal{E}$-Gorenstein injective and flat module". The purpose of this paper is to introduce and study a particular case of $\mathcal{E}$-Gorenstein injective and flat modules, called strongly $\mathcal{E}$-Gorenstein injective and flat modules, respectively. This paper is organized as follows.

In Section 2, we give some notions and notations that will be used in the paper.

In Section 3, we introduce and study strongly $\mathcal{E}$-Gorenstein injective and flat modules, many homological facts are developed and some of them generalize known results in [4, 33]. We prove that a module is $\mathcal{E}$-Gorenstein injective if and only if it is a direct summand of a strongly $\mathcal{E}$-Gorenstein injective module (Theorem 3.4), and every $\mathcal{E}$-Gorenstein flat module is a direct summand of a strongly $\mathcal{E}$-Gorenstein flat module (Theorem 3.5). Then we show the property of being a strongly $\mathcal{E}$-Gorenstein injective (resp., flat) module can be inherited by its direct summands under certain condition. The connections between the (strongly) $\mathcal{E}$-Gorenstein injective and flat modules are also discussed. Finally, we investigate FC rings in terms of strongly $\mathcal{E}$-Gorenstein injective and flat modules.

2. Preliminaries

Throughout this paper, R is an associative ring with identity and all modules are unitary. We denote by $\mathrm{Mod}\, R$ (resp., $\mathrm{Mod}\, R^{op}$) the category of all left (resp., right) R-modules and

$\mathcal{I}(R)$ the subcategory of all injective left R-modules. For an R-module M, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote the usual projective, injective and flat dimensions of M, respectively. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

In this section, we give terminology and some preliminary results.

Definition 2.1 [16, 18] (1) A left R-module M is said to be *Gorenstein injective* if there exists an exact sequence of injective left R-modules

$$\mathbf{E} = \ \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Ker}(E^0 \to E^1)$ such that $\mathrm{Hom}_R(I, -)$ leaves the sequence $\mathbf{E}$ exact whenever I is an injective left R-module. The exact sequence $\mathbf{E}$ is called a complete injective resolution. Gorenstein projective modules are defined dually.

(2) A right R-module M is said to be *Gorenstein flat* if there exists an exact sequence of flat right R-modules

$$\mathbf{F} = \ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Ker}(F^0 \to F^1)$ such that $- \otimes_R I$ leaves the sequence $\mathbf{F}$ exact whenever I is an injective left R-module. The exact sequence $\mathbf{F}$ is called a complete flat resolution.

Definition 2.2 [4] (1) A left R-module M is said to be *strongly Gorenstein injective* if there exists a complete injective resolution of the form

$$\cdots \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Ker}\, f$. Strongly Gorenstein projective modules are defined dually.

(2) A right R-module M is said to be *strongly Gorenstein flat* if there exists a complete flat resolution of the form

$$\cdots \xrightarrow{g} F \xrightarrow{g} F \xrightarrow{g} F \xrightarrow{g} \cdots$$

such that $M \cong \mathrm{Ker}\, g$.

Definition 2.3 [19] A left R-module M is said to be *strongly Gorenstein FP-injective* if there exists an exact sequence of FP-injective left R-modules

$$\cdots \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \cdots \tag{2.1}$$

with $M \cong \mathrm{Ker}\, f$ such that $\mathrm{Hom}_R(P, -)$ leaves the sequence (2.1) exact whenever P is a finitely presented left R-module with $\mathrm{pd}_R(P) < \infty$.

In what follows, $\mathcal{E}$ always denotes an injectively resolving subcategory of left R-modules.

Definition 2.4 A left R-module M is said to be *$\mathcal{E}$-Gorenstein injective* (*$\mathcal{G}_{\mathcal{E}}$-injective* for short) in [22] if there exists an exact sequence of injective left R-modules

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \tag{2.2}$$

with $M \cong \mathrm{Ker}(E^0 \to E^1)$ such that $\mathrm{Hom}_R(I,-)$ leaves the sequence (2.2) exact for any $I \in \mathcal{E}$.

A right R-module M is said to be *$\mathcal{E}$-Gorenstein flat* (*$\mathcal{G}_{\mathcal{E}}$-flat* for short) in [23] if there exists an exact sequence of flat right R-modules

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots \tag{2.3}$$

with $M \cong \mathrm{Ker}(F^0 \to F^1)$ such that $- \otimes_R I$ leaves the sequence (2.3) exact for any $I \in \mathcal{E}$.

We denote by $\mathcal{GI}_{\mathcal{E}}(R)$ (resp., $\mathcal{GF}_{\mathcal{E}}(R)$) the subcategory of the $\mathcal{E}$-Gorenstein injective left (resp., $\mathcal{E}$-Gorenstein flat right) R-modules.

Definition 2.5 [17] Let $\mathcal{C}$ be a subcategory of $\mathrm{Mod}\, R$ and M be a left R-module. A homomorphism $f : M \to C$ in $\mathrm{Mod}\, R$ with $C \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope* if $\mathrm{Hom}_R(f, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $f : M \to C$ is called a *$\mathcal{C}$-envelope* if every endomorphism $g : C \to C$ such that $gf = f$ is an isomorphism. A subcategory $\mathcal{C}$ in $\mathrm{Mod}\ R$ is called *(pre)enveloping* if every module $M \in \mathrm{Mod}\, R$ admits a $\mathcal{C}$-(pre)envelope. Dually, we have the notions of a $\mathcal{C}$-(pre)cover and a covering subcategory.

Definition 2.6 [26] Let $\mathcal{M}$ and $\mathcal{N}$ be subcategories of $\mathrm{Mod}\, R$ and $\mathrm{Mod}\, R^{op}$, respectively.

(1) The pair $(\mathcal{M}, \mathcal{N})$ is called a *duality pair* if the following conditions are satisfied.

(1.1) For a module $X \in \mathrm{Mod}\, R$, $X \in \mathcal{M}$ if and only if $X^+ \in \mathcal{N}$.

(1.2) $\mathcal{N}$ is closed under direct summands and finite direct sums.

(2) A duality pair $(\mathcal{M}, \mathcal{N})$ is called *(co)product-closed* if $\mathcal{M}$ is closed under (co)products.

(3) A duality pair $(\mathcal{M}, \mathcal{N})$ is called *perfect* if it is coproduct-closed, ${}_RR \in \mathcal{M}$ and $\mathcal{M}$ is closed under extensions.

For a subcategory $\mathcal{X}$ of $\mathrm{Mod}\, R$, we write

$$\mathcal{X}^{\perp_1} := \{A \in \mathrm{Mod}\, R \mid \mathrm{Ext}^1_R(X, A) = 0 \text{ for any } X \in \mathcal{X}\},$$

$${}^{\perp_1}\mathcal{X} := \{A \in \mathrm{Mod}\, R \mid \mathrm{Ext}^1_R(A, X) = 0 \text{ for any } X \in \mathcal{X}\}.$$

Definition 2.7 [17] Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathrm{Mod}\, R$.

(1) The pair $(\mathcal{X}, \mathcal{Y})$ is called a cotorsion pair if $\mathcal{X} = {}^{\perp_1}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp_1}$.

(2) A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called perfect if $\mathcal{X}$ is a covering subcategory and $\mathcal{Y}$ is an enveloping subcategory in $\mathrm{Mod}\, R$.

Lemma 2.8 [26, Theorem 3.1] *Let $(\mathcal{M}, \mathcal{N})$ is a duality pair. Then $\mathcal{M}$ is closed under pure submodules, pure quotients and pure extensions. Moreover, the following hold.*

(1) *If $(\mathcal{M}, \mathcal{N})$ is coproduct-closed, then $\mathcal{M}$ is a covering subcategory.*

(2) *If $(\mathcal{M}, \mathcal{N})$ is product-closed, then $\mathcal{M}$ is a preenveloping subcategory.*

(3) *If $(\mathcal{M}, \mathcal{N})$ is perfect, then $(\mathcal{M}, \mathcal{M}^{\perp_1})$ is a perfect cotorsion pair.*

3. Strongly $\mathcal{E}$-Gorenstein injective and flat modules

In this section, we introduce and study strongly $\mathcal{E}$-Gorenstein injective and flat modules, where $\mathcal{E}$ is always an injectively resolving subcategory of left R-modules. We generalize some

3 Nov 2022 18:24:49 PDT
220430-ZenghuiGao Version 4 - Submitted to Rocky Mountain J. Math.

main results in [4, 33] to strongly $\mathcal{E}$-Gorenstein injective and flat modules, and discuss the connections between (strongly) $\mathcal{E}$-Gorenstein injective and flat modules. As applications, some known results are obtained as corollaries.

Definition 3.1 A left R-module M is called *strongly $\mathcal{E}$-Gorenstein injective* (*$\mathcal{SG}_{\mathcal{E}}$-injective* for short) if there exists an exact sequence of injective left R-modules

$$\mathbf{E} = \ \cdots \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \cdots$$

with $M \cong \mathrm{Im}(f)$ such that $\mathrm{Hom}_R(I, \mathbf{E})$ is still exact for any $I \in \mathcal{E}$.

A right R-module M is called *strongly $\mathcal{E}$-Gorenstein flat* (*$\mathcal{SG}_{\mathcal{E}}$-flat* for short) if there exists an exact sequence of flat right R-modules

$$\mathbf{F} = \ \cdots \xrightarrow{g} F \xrightarrow{g} F \xrightarrow{g} F \xrightarrow{g} \cdots$$

with $M \cong \mathrm{Im}(g)$ such that $\mathbf{F} \otimes_R I$ is still exact for any $I \in \mathcal{E}$.

We denote by $\mathcal{SGI}_{\mathcal{E}}(R)$ (resp., $\mathcal{SGF}_{\mathcal{E}}(R)$) the subcategory of strongly $\mathcal{E}$-Gorenstein injective left (resp., strongly $\mathcal{E}$-Gorenstein flat right) R-modules.

Remark 3.2 (1) By definition, we have the following inclusions

$$\mathcal{I}(R) \subseteq \mathcal{SGI}_{\mathcal{E}}(R) \subseteq \mathcal{GI}_{\mathcal{E}}(R), \text{ and } \ \mathcal{F}(R) \subseteq \mathcal{SGF}_{\mathcal{E}}(R) \subseteq \mathcal{GF}_{\mathcal{E}}(R).$$

Moreover, it is known from [4, Examples 2.5 and 3.11] and [6, Corollary 3.10] that these containments can be strict.

(2) If $\mathcal{E} = \mathcal{I}(R)$, then strongly $\mathcal{E}$-Gorenstein injective (resp., flat) modules are just strongly Gorenstein injective (resp., flat) modules in [4]. By taking $\mathcal{E} = \mathcal{WI}$, where $\mathcal{WI}$ stands for the subcategory of all weak injective left R-modules [21], we have that strongly $\mathcal{E}$-Gorenstein flat modules are precisely strongly Gorenstein AC-flat modules in [7].

Proposition 3.3 (1) If $\{E_i\}_{i \in I}$ is a family of strongly $\mathcal{E}$-Gorenstein injective left R-modules, then the direct product $\prod_{i \in I} E_i$ is strongly $\mathcal{E}$-Gorenstein injective.

(2) If $\{Q_i\}_{i \in I}$ is a family of strongly $\mathcal{E}$-Gorenstein flat right R-modules, then the direct sum $\bigoplus_{i \in I} E_i$ is strongly $\mathcal{E}$-Gorenstein flat.

Proof. (1) Let $M = \prod_{i \in I} M_i$ where each M_i is a strongly $\mathcal{E}$-Gorenstein injective left R-module. Then, for each $i \in I$, there exists an exact sequence of injective left R-modules

$$\mathbb{E}_i = \ \cdots \xrightarrow{f_i} E_i \xrightarrow{f_i} E_i \xrightarrow{f_i} E_i \xrightarrow{f_i} \cdots$$

with $M_i = \mathrm{Im}\, f_i$ such that $\mathrm{Hom}_R(I, -)$ leaves the sequence $\mathbb{E}_i$ exact for any $I \in \mathcal{E}$. Put $f : \prod_{i \in I} E_i \to \prod_{i \in I} E_i$, where $f([x_i]) = [f_i(x_i)]$, $x_i \in E_i$ for all $i \in I$. Since any direct product of injective modules is also injective, we have the following exact sequence

$$\prod_{i \in I} \mathbb{E}_i = \ \cdots \xrightarrow{f} \prod_{i \in I} E_i \xrightarrow{f} \prod_{i \in I} E_i \xrightarrow{f} \prod_{i \in I} E_i \xrightarrow{f} \cdots$$

of injective left R-modules such that $M = \operatorname{Im} f$. It follows from [28, Theorem 2.6] that $\operatorname{Hom}_R(I, \prod_{i\in I} \mathbb{E}_i) \cong \prod_{i\in I} \operatorname{Hom}_R(I, \mathbb{E}_i)$ is exact for any $I \in \mathcal{E}$. Thus, M is a strongly $\mathcal{E}$-Gorenstein injective left R-module.

(2) The proof is similar to that of (1). □

Theorem 3.4 *A left R-module is $\mathcal{E}$-Gorenstein injective if and only if it is a direct summand of a strongly $\mathcal{E}$-Gorenstein injective left R-module.*

Proof. Since the subcategory of $\mathcal{E}$-Gorenstein injective left R-modules is closed under direct summands by [22, Theorem 2.7], we only need to prove the direct implication.

Let M be an $\mathcal{E}$-Gorenstein injective left R-module. Then, there exists an exact sequence of injective left R-modules

$$\mathbf{E} = \ \cdots \xrightarrow{d_{m+1}} E_m \xrightarrow{d_m} \cdots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} \cdots$$

with $M \cong \operatorname{Im}(d_0)$ such that $\operatorname{Hom}_R(I, -)$ leaves the sequence $\mathbf{E}$ exact for any $I \in \mathcal{E}$. Put $E = \prod_{i=-\infty}^{+\infty} E_i$ and $f : E \to E$, where $f([x_i]) = [d_i(x_i)]$, $x_i \in E_i$ for all $i \in \mathbb{Z}$. Then, $f^2 = 0$ and E is injective. Let $N = \operatorname{Im} f$. Then, we have $N \subseteq \operatorname{Ker} f$. For the converse inclusion, we suppose $f([x_i]) = [d_i(x_i)] = 0$. Then, one has that $d_i(x_i) = 0$. Thus, there exists $y_{i+1} \in E_{i+1}$ such that $d_{i+1}(y_{i+1}) = x_i$. Since $y = [y_{i+1}] \in E$ and $f(y) = [x_i]$, it follows that $\operatorname{Ker} f \subseteq \operatorname{Im} f$, and hence $N = \operatorname{Ker} f$. Thus, we get the following exact sequence of injective left R-modules

$$\prod_{i\in\mathbb{Z}} \mathbf{E} = \ \cdots \xrightarrow{f} \prod_{i=-\infty}^{+\infty} E_i \xrightarrow{f} \prod_{i=-\infty}^{+\infty} E_i \xrightarrow{f} \prod_{i=-\infty}^{+\infty} E_i \xrightarrow{f} \cdots .$$

By [28, Theorem 2.6], we have $\operatorname{Hom}_R(I, \prod_{i\in Z} \mathbf{E}) \cong \prod_{i\in Z} \operatorname{Hom}_R(I, \mathbf{E})$ is exact for any $I \in \mathcal{E}$. Therefore, N is a strongly $\mathcal{E}$-Gorenstein injective left R-module. Notice that

$$N = \ \operatorname{Im} f = \ \operatorname{Im}(\prod_{i=-\infty}^{+\infty} d_i) \cong \prod_{i=-\infty}^{+\infty} \operatorname{Im}\ d_i,$$

it follows that M is a direct summand of N. □

Theorem 3.5 *If a right R-module is $\mathcal{E}$-Gorenstein flat, then it is a direct summand of a strongly $\mathcal{E}$-Gorenstein flat right R-module.*

Proof. Let M be an $\mathcal{E}$-Gorenstein flat right R-module. Then, there exists an exact sequence of flat right R-modules

$$\mathbf{F} = \ \cdots \xrightarrow{g_{m+1}} F_m \xrightarrow{g_m} \cdots \xrightarrow{g_2} F_1 \xrightarrow{g_1} F_0 \xrightarrow{g_0} F_{-1} \xrightarrow{g_{-1}} \cdots$$

with $M \cong \operatorname{Im} g_0$ such that $\mathbf{F} \otimes_R I$ is still exact for any $I \in \mathcal{E}$. Put $Q = \bigoplus_{i=-\infty}^{+\infty} F_i$ and $g : Q \to Q$, where $g([x_i]) = [g_i(x_i)]$, $x_i \in F_i$ for all $i \in \mathbb{Z}$. Then, $g^2 = 0$ and Q is flat. Let $L = \operatorname{Im} g$. Then, we have $L \subseteq \operatorname{Ker} g$. Now, let $g([x_i]) = [g_i(x_i)] = 0$. Then, we have

$g_i(x_i) = 0$. Thus, there exists $z_{i+1} \in F_{i+1}$ such that $g_{i+1}(z_{i+1}) = x_i$. Since $z = [z_{i+1}] \in Q$ and $g(z) = [x_i]$, we have $\operatorname{Ker} g \subseteq \operatorname{Im} g$, and so $L = \operatorname{Ker} g$. Thus, we obtain the following exact sequence of flat right R-modules

$$\bigoplus_{i\in Z} \mathbf{F} = \ \cdots \xrightarrow{g} \bigoplus_{i=-\infty}^{+\infty} F_i \xrightarrow{g} \bigoplus_{i=-\infty}^{+\infty} F_i \xrightarrow{g} \bigoplus_{i=-\infty}^{+\infty} F_i \xrightarrow{g} \cdots .$$

It follows from [28, Theorem 2.8] that $(\bigoplus_{n\in Z} \mathbf{F}) \bigotimes_R I \cong \bigoplus_{n\in Z} (\mathbf{F} \otimes_R I)$ is exact for any $I \in \mathcal{E}$, and hence L is a strongly $\mathcal{E}$-Gorenstein flat right R-module. Now, since

$$L = \operatorname{Im} g = \operatorname{Im}(\bigoplus_{i=-\infty}^{+\infty} g_i) \cong \bigoplus_{i=-\infty}^{+\infty} \operatorname{Im} g_i,$$

one gets that M is a direct summand of L, and the assertion follows. $\square$

Recall from [20] that the $\mathcal{E}$-dimension of a left R-module M, denoted by $\mathcal{E}$-dim(M), is defined to be the smallest $n \geq 0$ such that there is an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ with all E_i in $\mathcal{E}$. Set $\mathcal{E}$-dim$(M) = \infty$ if no such integer exists. The next result gives a simple characterization of the strongly $\mathcal{E}$-Gorenstein flat modules.

Proposition 3.6 *The following statements are equivalent for a right R-module M.*

(1) *$M \in \mathcal{SGF}_{\mathcal{E}}(R)$.*

(2) *There is an exact sequence of right R-modules $0 \to M \to F \to M \to 0$ with F flat such that $0 \to M \otimes_R I \to F \otimes_R I \to M \otimes_R I \to 0$ is exact for any $I \in \mathcal{E}$.*

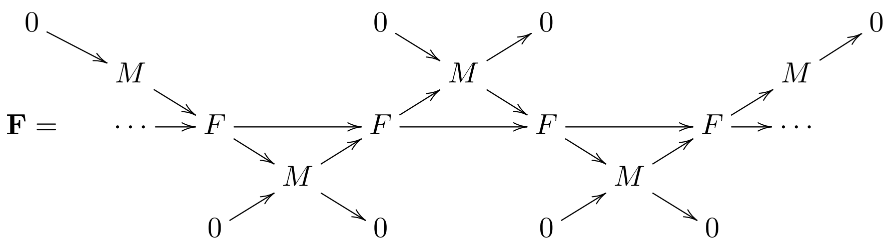
(3) *There is an exact sequence of right R-modules $0 \to M \to F \to M \to 0$ with F flat such that $0 \to M \otimes_R I' \to F \otimes_R I' \to M \otimes_R I' \to 0$ is exact for any left R-module I' with finite $\mathcal{E}$-dimension.*

(4) *There is an exact sequence of right R-modules $0 \to M \to F \to M \to 0$ with F flat such that $\operatorname{Tor}^R_{i\geq 1}(M, I) = 0$ for any $I \in \mathcal{E}$.*

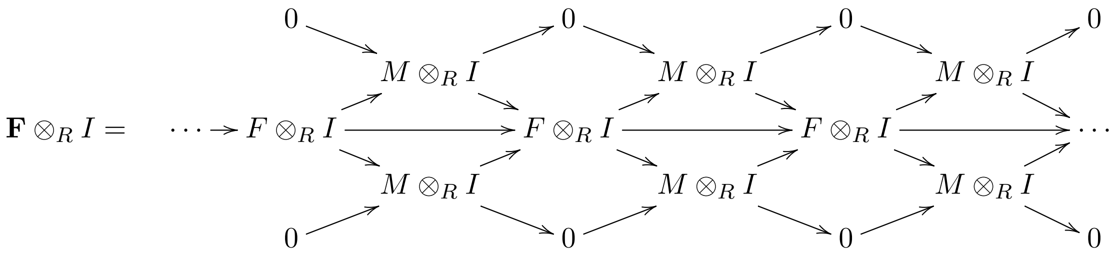
(5) *There is an exact sequence of right R-modules $0 \to M \to F \to M \to 0$ with F flat such that $\operatorname{Tor}^R_{i\geq 1}(M, I') = 0$ for any left R-module I' with finite $\mathcal{E}$-dimension.*

Proof. (5) ⇒ (3) ⇒ (2) are trivial. By the definition of strongly $\mathcal{E}$-Gorenstein flat modules, we immediately get that (1) ⇒ (4) ⇒ (2). Next we will show that (2) ⇒ (1) and (4) ⇒ (5).

(2) ⇒ (1) Since the sequence $0 \to M \to F \to M \to 0$ is exact with F flat, one easily gets the following exact commutative diagram:



By assumption, we have $0 \to M \otimes_R I \to F \otimes_R I \to M \otimes_R I \to 0$ is exact for any $I \in \mathcal{E}$. Thus, we get the following exact commutative diagram:



Therefore, $\mathbf{F} \otimes_R I$ is exact, and so M is strongly $\mathcal{E}$-Gorenstein flat, as desired.

(4) $\Rightarrow$ (5) It suffices to show that if $\mathrm{Tor}^R_{i\geq 1}(M, I) = 0$ for any $I \in \mathcal{E}$, then $\mathrm{Tor}^R_{i\geq 1}(M, I') = 0$ for any left R-module I' with finite $\mathcal{E}$-dimension. Suppose that I' is a left R-module with $\mathcal{E}\text{-dim}(I') \leq n < \infty$. Then, by [20, Corollary 2.5], there exists an exact sequence of left R-modules $0 \to I' \to E_0 \to E_1 \to \cdots \to E_{n-1} \to I_n \to 0$ with all E_i injective and $I_n \in \mathcal{E}$. One easily gets that $\mathrm{Tor}^R_i(M, I') \cong \mathrm{Tor}^R_{i+n}(M, I_n) = 0$ for all $i \geq 1$, and (5) holds. □

By analogy with the proof of Proposition 3.6, we have the following result.

Proposition 3.7 *The following statements are equivalent for a left R-module M.*

(1) $M \in \mathcal{SGI}_{\mathcal{E}}(R)$.

(2) *There is an exact sequence of left R-modules $0 \to M \to E \to M \to 0$ with E injective such that $0 \to \mathrm{Hom}_R(I, M) \to \mathrm{Hom}_R(I, E) \to \mathrm{Hom}_R(I, M) \to 0$ is exact for any $I \in \mathcal{E}$.*

(3) *There is an exact sequence of left R-modules $0 \to M \to E \to M \to 0$ with E injective such that $0 \to \mathrm{Hom}_R(I', M) \to \mathrm{Hom}_R(I', E) \to \mathrm{Hom}_R(I', M) \to 0$ is exact for any left R-module I' with finite $\mathcal{E}$-dimension.*

(4) *There is an exact sequence of left R-modules $0 \to M \to E \to M \to 0$ with E injective such that $\mathrm{Ext}^{i\geq 1}_R(I, M) = 0$ for any $I \in \mathcal{E}$.*

(5) *There is an exact sequence of left R-modules $0 \to M \to E \to M \to 0$ with E injective such that $\mathrm{Ext}^{i\geq 1}_R(I, M) = 0$ for any left R-module I' with finite $\mathcal{E}$-dimension.*

Proposition 3.8 (1) *A strongly $\mathcal{E}$-Gorenstein flat right R-module is flat if and only if it has finite flat dimension.*

(2) *A strongly $\mathcal{E}$-Gorenstein injective left R-module is injective if and only if it has finite injective dimension.*

Proof. The assertions follow directly from Propositions 3.6 and 3.7. □

Recall the (weak) global dimension of a ring R in [28], $D(R)$ ($wD(R)$), is defined by

$$D(R) = \sup\{\mathrm{pd}_R(M) \mid M \text{ is a left } R\text{-module}\},$$
$$wD(R) = \sup\{\mathrm{fd}_R(M) \mid M \text{ is a left } R\text{-module}\}.$$

Corollary 3.9 (1) *If R has finite weak global dimension, then a right R-module is $\mathcal{E}$-Gorenstein flat if and only if it is flat.*

3 Nov 2022 18:24:49 PDT
220430-ZenghuiGao Version 4 - Submitted to Rocky Mountain J. Math.

(2) *If R has finite global dimension, then a left R-module is $\mathcal{E}$-Gorenstein injective if and only if it is injective.*

Proof. It is immediate by Theorems 3.4, 3.5 and Proposition 3.8. □

Recall that a ring R is said to be left coherent [28] if every finitely generated left ideal is finitely presented.

Proposition 3.10 *Suppose that R is a left coherent ring and every left R-module in $\mathcal{E}$ has a finite flat dimension. If M is a strongly $\mathcal{E}$-Gorenstein injective left R-module, then M^+ is a strongly $\mathcal{E}$-Gorenstein flat right R-module.*

Proof. Let M be a strongly $\mathcal{E}$-Gorenstein injective left R-module. Then, there exists a short exact sequence $0 \to M \to E \to M \to 0$, where E is injective. It follows that $0 \to M^+ \to E^+ \to M^+ \to 0$ is exact and E^+ is a flat right R-module by [11, Theorem 1] since R is left coherent. For every left R-module $I \in \mathcal{E}$, we have

$$\mathrm{Tor}_i^R(M^+, I) \cong \mathrm{Tor}_{i+1}^R(M^+, I) \text{ for all } i \geq 1.$$

By the hypothesis, $\mathrm{fd}_R(I) < \infty$, one easily gets that $\mathrm{Tor}_i^R(M^+, I) = 0$ for all $i \geq 1$. Therefore, M^+ is a strongly $\mathcal{E}$-Gorenstein flat right R-module by Proposition 3.6. □

In [29], Šaroch and Šťovíček proved that the class of Gorenstein flat modules is always closed under extensions, regardless of the ring R. The following result extends that fact for the subcategory $\mathcal{GF}_{\mathcal{E}}(R)$ to some extent. Also, it is not known whether the subcategory $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under extensions for any ring.

Lemma 3.11 *Suppose that R is a ring such that every left R-module in $\mathcal{E}$ has a finite flat dimension. Then, the subcategory $\mathcal{GF}_{\mathcal{E}}(R)$ of $\mathcal{E}$-Gorenstein flat right R-modules is closed under extensions.*

Proof. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of right R-modules with $M_1, M_3 \in \mathcal{GF}_{\mathcal{E}}(R)$. Then, we have M_1, M_3 are Gorenstein flat by Remark 3.2(1), and so that M_2 is Gorenstein flat since the $\mathcal{GF}(R)$ is closed under extensions. Thus, there exists an exact sequence of flat right R-modules

$$\mathbb{F} = \ \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M_2 \cong \mathrm{Ker}(F^0 \to F^1)$. Now, it remains to show that $\mathbb{F} \otimes_R I$ is exact for any $I \in \mathcal{E}$. By assumption, we may assume that $\mathrm{fd}_R(I) = m < \infty$. We proceed by induction on m. The case $m = 0$ is clear. Let $m \geq 1$. Then, there exists an exact sequence of left R-modules

$$0 \to K \to Q \to I \to 0,$$

where Q is flat and $\mathrm{fd}_R(K) \leq m-1$. So, we have an exact sequence of complexes:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes_R K & \longrightarrow & F_1 \otimes_R Q & \longrightarrow & F_1 \otimes_R I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes_R K & \longrightarrow & F_0 \otimes_R Q & \longrightarrow & F_0 \otimes_R I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^0 \otimes_R K & \longrightarrow & F^0 \otimes_R Q & \longrightarrow & F^0 \otimes_R I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^1 \otimes_R K & \longrightarrow & F^1 \otimes_R Q & \longrightarrow & F^1 \otimes_R I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{F} \otimes_R K & \longrightarrow & \mathbb{F} \otimes_R Q & \longrightarrow & \mathbb{F} \otimes_R I & \longrightarrow & 0
\end{array}
$$

Since Q is flat, we have $\mathbb{F} \otimes_R Q$ is exact and $\mathbb{F} \otimes_R K$ is exact by induction hypothesis. Then, $\mathbb{F} \otimes_R I$ is exact by [28, Theorem 6.3], and so M_2 is $\mathcal{E}$-Gorenstein flat. □

Proposition 3.12 *Let R be a left coherent ring such that every left R-module in $\mathcal{E}$ has a finite flat dimension. If M is an $\mathcal{E}$-Gorenstein injective left R-module, then M^+ is an $\mathcal{E}$-Gorenstein flat right R-module.*

Proof. Let M be an $\mathcal{E}$-Gorenstein injective left R-module. Then, by Theorem 3.4 there exists a strongly $\mathcal{E}$-Gorenstein injective left R-module N such that $N = M \oplus M'$. Since R is left coherent, we have $N^+ = M^+ \oplus M'^+$ is strongly $\mathcal{E}$-Gorenstein flat by Proposition 3.10. Thus, N^+ is $\mathcal{E}$-Gorenstein flat. Notice that the subcategory of $\mathcal{E}$-Gorenstein flat right R-modules is closed under extensions by Lemma 3.11, it follows that M^+ is $\mathcal{E}$-Gorenstein flat by [23, Theorem 2.7]. □

Recall that a ring R is said to be an n-FC ring in [14] if R is a left and right coherent ring with FP-id$({}_RR) \leq n$ and FP-id$(R_R) \leq n$ for some integer $n \geq 0$. By taking $\mathcal{E} = \mathcal{I}(R)$ in Proposition 3.12 and using [13, Theorem 3.8], we get immediately the following result.

Corollary 3.13 [14, Proposition 12] *Let R be an n-FC ring with $n \geq 0$. If G is a Gorenstein injective left R-module, then G^+ is a Gorenstein flat right R-module.*

In the following, we discuss how the property of being a strongly $\mathcal{E}$-Gorenstein injective (resp., flat) module can be inherited by its direct summands under certain condition.

Theorem 3.14 *Let $0 \to E \to M \to N \to 0$ be an exact sequence of left R-modules, where E is injective. Then, $N \in \mathcal{SGI}_{\mathcal{E}}(R)$ if and only if $M \in \mathcal{SGI}_{\mathcal{E}}(R)$.*

Proof. We first prove the "only if part". Let N be a strongly $\mathcal{E}$-Gorenstein injective left R-module. Since E is injective, we have $M \cong N \oplus E$ is strongly $\mathcal{E}$-Gorenstein injective by Proposition 3.3(1).

3 Nov 2022 18:24:49 PDT
220430-ZenghuiGao Version 4 - Submitted to Rocky Mountain J. Math.

For the "if part", suppose that $M \cong N \oplus E$ is strongly $\mathcal{E}$-Gorenstein injective. Then, we have an exact sequence $0 \to N \oplus E \to I \to N \oplus E \to 0$ with I injective by Proposition 3.7. Consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N \oplus E & = & N \oplus E & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Q & \to & I & \to & E & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N & \to & N \oplus E & \to & E & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By Theorem 3.4, N is $\mathcal{E}$-Gorenstein injective. Now, since $N \oplus E$ is $\mathcal{E}$-Gorenstein injective by Remark 3.2(1), we have Q is $\mathcal{E}$-Gorenstein injective by [22, Theorem 2.7]. It follows that $\mathrm{Ext}^1_R(E, Q) = 0$, and so the sequence $0 \to Q \to I \to E \to 0$ is split. Hence, Q is injective. Now, consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & E & \to & N \oplus E & \to & N & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & E & \to & Q & \to & H & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N & = & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Then, the middle row gives that H is injective, and we have an exact sequence $0 \to N \to H \to N \to 0$ with H injective. Now, since N is $\mathcal{E}$-Gorenstein injective, one gets that $\mathrm{Ext}^i_R(E', N) = 0$ for all $E' \in \mathcal{E}$ and all $i \geq 1$. Hence, N is strongly $\mathcal{E}$-Gorenstein injective by Proposition 3.7. □

Theorem 3.15 *Let M be a right R-module and F be a flat right R-module. Then, $M \in \mathcal{SGF}_{\mathcal{E}}(R)$ if and only if $M \oplus F \in \mathcal{SGF}_{\mathcal{E}}(R)$.*

Proof. We first prove the "only if part". Let M be a strongly $\mathcal{E}$-Gorenstein flat right R-module. Then, $M \oplus F$ is strongly $\mathcal{E}$-Gorenstein flat by Proposition 3.3(2).

For the "if part", suppose that $M \oplus F$ is a strongly $\mathcal{E}$-Gorenstein flat right R-module. Then, by Proposition 3.6 we have an exact sequence $0 \to M \oplus F \to Q \to M \oplus F \to 0$ with Q flat. Then, $(M \oplus F)^+$ is $\mathcal{E}$-Gorenstein injective by [23, Proposition 2.5], and so M^+ is

$\mathcal{E}$-Gorenstein injective by [22, Theorem 2.7]. We consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & F & \to & M \oplus F & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & F & \to & Q & \to & Q' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M \oplus F & = & M \oplus F & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

which gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (M \oplus F)^+ & = & (M \oplus F)^+ & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Q'^+ & \to & Q^+ & \to & F^+ & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M^+ & \to & (M \oplus F)^+ & \to & F^+ & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the first column $0 \to (M \oplus F)^+ \to Q'^+ \to M^+ \to 0$, we obtain that Q'^+ is $\mathcal{E}$-Gorenstein injective by [22, Theorem 2.7]. Thus, $\mathrm{Ext}^1_R(F^+, Q'^+) = 0$ since $F^+ \in \mathcal{E}$, and so the sequence $0 \to Q'^+ \to Q^+ \to F^+ \to 0$ splits. Notice that Q is flat, we have Q^+ is injective since the fact that a module is flat if and only if its character module is injective [28, Theorem 3.52]. It follows that Q'^+ is injective, and so Q' is flat. Now, consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Q'' & \to & Q' & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & M \oplus F & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Notice that Q' is flat, so the middle row gives that Q'' is flat. Then, $0 \to M \to Q'' \to M \to 0$ is exact with Q'' flat. For any $I \in \mathcal{E}$, we have

$$0 = \mathrm{Tor}^R_{i+1}(F, I) \to \mathrm{Tor}^R_i(M, I) \to \mathrm{Tor}^R_i(M \oplus F, I) = 0$$

is exact for all $i \geq 1$. It follows that $\mathrm{Tor}_i^R(M, I) = 0$ for all $i \geq 1$, and hence M is strongly $\mathcal{E}$-Gorenstein flat by Proposition 3.6. □

Proposition 3.16 *For any right R-module M, we consider the following conditions.*

(1) M is a strongly $\mathcal{E}$-Gorenstein flat right R-module.

(2) M^+ is a strongly $\mathcal{E}$-Gorenstein injective left R-module.

Then, (1) $\Rightarrow$ (2). If R is left coherent, then also (2) $\Rightarrow$ (1), and hence the two conditions are equivalent.

Proof. (1) $\Rightarrow$ (2) Since M is a strongly $\mathcal{E}$-Gorenstein flat right R-module, there exists an exact sequence of right R-modules $0 \to M \to F \to M \to 0$ with F flat. Then $0 \to M^+ \to F^+ \to M^+ \to 0$ is an exact sequence of left R-modules with F^+ injective. For any $I \in \mathcal{E}$, we have $\mathrm{Ext}_R^i(I, M^+) \cong \mathrm{Tor}_i^R(M, I)^+ = 0$ for all $i \geq 1$ by [17, Theorem 3.2.1], and hence M^+ is strongly $\mathcal{E}$-Gorenstein injective by Proposition 3.7.

(2) $\Rightarrow$ (1) Let M^+ is a strongly $\mathcal{E}$-Gorenstein injective left R-module. Then, there is an exact sequence $0 \to M^+ \to E \to M^+ \to 0$ of left R-modules with E injective. Since R is a left coherent ring, we have E^{++} is injective by [11, Theorem 1]. We notice that E is a pure submodule E^{++} by [31, Exercise 41, p. 48], there is an injective left R-module E' such that $E \oplus E' = E^{++}$. Let $H = (E' \oplus E)^{\mathbb{N}} = (E^{+(\mathbb{N})})^+$. Consider the following exact sequence

$$0 \to M^+ \oplus H \to E \oplus H \oplus H \to M^+ \oplus H \to 0.$$

Notice that

$$\begin{aligned} M^+ \oplus H &= \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(E^{+(\mathbb{N})}, \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(M \oplus E^{+(\mathbb{N})}, \mathbb{Q}/\mathbb{Z}) \\ E \oplus H \oplus H &\cong \mathrm{Hom}_{\mathbb{Z}}(E^{+(\mathbb{N})}, \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(E^{+(\mathbb{N})}, \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})}, \mathbb{Q}/\mathbb{Z}), \end{aligned}$$

it follows that

$$0 \to M \oplus E^{+(\mathbb{N})} \to E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})} \to M \oplus E^{+(\mathbb{N})} \to 0$$

is exact and $E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})}$ is flat. Since all $\mathcal{E}$-Gorenstein injective modules are $\mathcal{E}$-injective by [22, Remark 3.9(2)], we have M^+ is $\mathcal{E}$-injective. For any $I \in \mathcal{E}$, one gets easily that

$$\mathrm{Tor}_i^R(M \oplus E^{+(N)}, I) \cong \mathrm{Tor}_i^R(M, I) \oplus \mathrm{Tor}_i^R(E^{+(\mathbb{N})}, I) = 0$$

for all $i \geq 1$ since M is $\mathcal{E}$-flat by [20, Remark 3.2(vii)], and hence $M \oplus E^{+(\mathbb{N})}$ is strongly $\mathcal{E}$-Gorenstein flat. Therefore, M is strongly $\mathcal{E}$-Gorenstein flat by Theorem 3.15. □

By Theorem 3.14 and Proposition 3.16, we immediately get the following result.

Proposition 3.17 *Let R be a left coherent ring and $0 \to N \to M \to F \to 0$ be an exact sequence of right R-modules where F is flat. Then, $N \in \mathcal{SGF}_{\mathcal{E}}(R)$ if and only if $M \in \mathcal{SGF}_{\mathcal{E}}(R)$.*

3 Nov 2022 18:24:49 PDT
220430-ZenghuiGao Version 4 - Submitted to Rocky Mountain J. Math.

Recall that a submodule T of an R-module N is said to be a pure submodule in [17] if $0 \to A \otimes_R T \to A \otimes_R N$ is exact for all right R-modules A, or equivalently, if $\mathrm{Hom}_R(A, N) \to \mathrm{Hom}_R(A, N/T) \to 0$ is exact for all finitely presented R-modules A.

Theorem 3.18 *Let R be a left coherent ring such that every left R-module in $\mathcal{E}$ has a finite flat dimension. Then, the following statements are equivalent:*

(1) *$\mathcal{GI}_{\mathcal{E}}(R)$ is closed under pure submodules.*

(2) *$M \in \mathcal{GI}_{\mathcal{E}}(R)$ if and only if $M^+ \in \mathcal{GF}_{\mathcal{E}}(R)$ for any left R-module M.*

(3) *The pair $(\mathcal{GI}_{\mathcal{E}}(R), \mathcal{GF}_{\mathcal{E}}(R))$ is a duality pair.*

Proof. (1) $\Rightarrow$ (2) Assume that M is an $\mathcal{E}$-Gorenstein injective left R-module. Then, M^+ is $\mathcal{E}$-Gorenstein flat by Proposition 3.12. Conversely, if M^+ is an $\mathcal{E}$-Gorenstein flat right R-module, then there exists a strongly $\mathcal{E}$-Gorenstein flat right R-module N such that $N = M^+ \oplus L$ by Theorem 3.5. It follows that $N^+ = M^{++} \oplus L^+$ is strongly $\mathcal{E}$-Gorenstein injective by Proposition 3.16, and hence N^+ is $\mathcal{E}$-Gorenstein injective. By [22, Theorem 2.7], the subcategory of $\mathcal{E}$-Gorenstein injective left R-modules is closed under direct summands, and so M^{++} is $\mathcal{E}$-Gorenstein injective. By (1), the subcategory of $\mathcal{E}$-Gorenstein injective left R-modules is closed under pure submodules. So, M is $\mathcal{E}$-Gorenstein injective due to it is a pure submodule of M^{++}.

(2) $\Rightarrow$ (3) For any left R-module M, we have $M \in \mathcal{GI}_{\mathcal{E}}(R)$ if and only if $M^+ \in \mathcal{GF}_{\mathcal{E}}(R)$ by assumption. Now, since $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under extensions by Lemma 3.11, we have $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under direct summands by [23, Theorem 2.7]. From [23, Remark 2.2(2)], one has that $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under finite direct sums. Therefore, $(\mathcal{GI}_{\mathcal{E}}(R), \mathcal{GF}_{\mathcal{E}}(R))$ is a duality pair.

(3) $\Rightarrow$ (1) This follows immediately from Lemma 2.8. □

By Theorem 3.18 and [13, Theorem 3.8], we immediately get the following result.

Corollary 3.19 [7, Theorem 3.19] *Let R be an n-FC ring. Then the following statements are equivalent:*

(1) *$\mathcal{GI}(R)$ is closed under pure submodules.*

(2) *The pair $(\mathcal{GI}(R), \mathcal{GF}(R))$ is a duality pair.*

Corollary 3.20 *Let R be a coherent ring such that every left R-module in $\mathcal{E}$ has a finite flat dimension. If $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under pure submodules, then $\mathcal{GI}_{\mathcal{E}}(R)$ is a covering and a preenveloping subcategory.*

Proof. Since $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under pure submodules, the pair $(\mathcal{GI}_{\mathcal{E}}(R), \mathcal{GF}_{\mathcal{E}}(R))$ is a duality pair by Theorem 3.18. We notice that any direct sum (i.e., coproduct) is a pure submodule of a direct product by [11, Lemma 1] and that $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under direct products by [22, Remark 2.3(1)], it follows that $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under coproducts, and hence $(\mathcal{GI}_{\mathcal{E}}(R), \mathcal{GF}_{\mathcal{E}}(R))$ is a coproduct-closed duality pair. By [26, Theorem 3.1(b)], one gets that $\mathcal{GI}_{\mathcal{E}}(R)$ is a covering subcategory.

On the other hand, since $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under direct products by [22, Remark 2.3(1)], we have the duality pair $(\mathcal{GI}_{\mathcal{E}}(R), \mathcal{GF}_{\mathcal{E}}(R))$ is product-closed. Hence, $\mathcal{GI}_{\mathcal{E}}(R)$ is a preenveloping subcategory by Lemma 2.8(2). □

Theorem 3.21 *For any right R-module M, we consider the following conditions.*

(1) *M is an $\mathcal{E}$-Gorenstein flat right R-module.*

(2) *M^+ is an $\mathcal{E}$-Gorenstein injective left R-module.*

Then, (1) $\Rightarrow$ (2)*. If R is left coherent, then also* (2) $\Rightarrow$ (1)*, and hence the two conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2) follows directly by [23, Proposition 2.5]

(2) $\Rightarrow$ (1) Let M^+ be an $\mathcal{E}$-Gorenstein injective left R-module. Then, by [22, Proposition 2.6], there is an exact sequence $0 \to C \to E \overset{\partial}{\to} M^+ \to 0$ of left R-modules where E is injective and C is $\mathcal{E}$-Gorenstein injective, which gives rise to the exactness of

$$0 \to M^{++} \overset{\partial^+}{\to} E^+ \to C^+ \to 0,$$

where E^+ is flat since R is left coherent. Notice that, since $\delta_M : M \to M^{++}$ is injective we have an injective map $\nu = \partial^+\delta_M$ from M to E^+. Since R is left coherent, every right R-module has a flat preenvelope by [32, Theorem 2.5.1]. Let $\phi : M \to F^0$ be a flat preenvelope of M. Then, ϕ is injective by [10, Lemma 4.3.3]. Putting $L = \mathrm{Coker}(\phi)$, one gets the following exact sequence

$$0 \to M \overset{\phi}{\to} F^0 \to L \to 0$$

of right R-modules, which gives the following exact sequence of left R-modules

$$0 \to L^+ \to (F^0)^+ \overset{\phi^+}{\longrightarrow} M^+ \to 0, \qquad (*)$$

where $(F^0)^+$ is injective and M^+ is $\mathcal{E}$-Gorenstein injective by assumption. We will show that L^+ is $\mathcal{E}$-Gorenstein injective. For any injective left R-module E, applying $\mathrm{Hom}_R(E, -)$ to the sequence $(*)$, we have the following exact sequence

$$\mathrm{Hom}_R(E, (F^0)^+) \overset{\mathrm{Hom}_R(E,\phi^+)}{\longrightarrow} \mathrm{Hom}_R(E, M^+) \longrightarrow \mathrm{Ext}^1_R(E, L^+) \to 0.$$

Now, we consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(E, (F^0)^+) & \xrightarrow{\mathrm{Hom}_R(E,\phi^+)} & \mathrm{Hom}_R(E, M^+) \\ \downarrow \varphi & & \downarrow \psi \\ \mathrm{Hom}_R(F^0, E^+) & \xrightarrow{\mathrm{Hom}_R(\phi,E^+)} & \mathrm{Hom}_R(M, E^+) \end{array}$$

By [28, Theorem 2.11] we have φ and ψ are isomorphisms. Notice that ϕ is a flat preenvelope of M, and E^+ is flat since R is left coherent, one gets that

$$\mathrm{Hom}_R(\phi, E^+) : \mathrm{Hom}_R(F^0, E^+) \to \mathrm{Hom}_R(M, E^+)$$

is surjective. It follows that

$$\mathrm{Hom}_R(E, (F^0)^+) \longrightarrow \mathrm{Hom}_R(E, M^+) \longrightarrow 0$$

is exact, and hence $\mathrm{Ext}^1_R(E, L^+) = 0$. Then, by [22, Proposition 2.15], L^+ is $\mathcal{E}$-Gorenstein injective. By repeating the above step to L^+ and so on, one gets the exact sequences

$$0 \longrightarrow L \longrightarrow F^1 \longrightarrow L^1 \longrightarrow 0$$
$$0 \longrightarrow L^1 \longrightarrow F^2 \longrightarrow L^2 \longrightarrow 0$$
$$\vdots$$

where each F^i is flat and each $(L^i)^+$ is $\mathcal{E}$-Gorenstein injective. Thus, we can construct a right flat resolution of M:

$$0 \to M \to F^0 \to F^1 \to \cdots . \qquad (**)$$

Let $I \in \mathcal{E}$. Then, $\mathrm{Tor}^R_i(M, I)^+ \cong \mathrm{Ext}^i_R(I, M^+) = 0$ for any $i \geq 1$, and so $\mathrm{Tor}^R_i(M, I) = 0$ for any $i \geq 1$. It follows that $- \otimes_R I$ leaves the sequence $(**)$ exact. Therefore, M is $\mathcal{E}$-Gorenstein flat by [23, Proposition 2.6]. □

By taking $\mathcal{E} = \mathcal{I}(R)$ in Theorem 3.21, we have directly the following result.

Corollary 3.22 [25, Theorem 3.6] *For a right R-module M, we consider the following conditions.*

(1) *M is a Gorenstein flat right R-module.*

(2) *M^+ is a Gorenstein injective left R-module.*

Then, (1) $\Rightarrow$ (2)*. If R is left coherent, then also* (2) $\Rightarrow$ (1)*, and hence the two conditions are equivalent.*

Corollary 3.23 *Let R be a left coherent ring. Then, the pair $(\mathcal{GF}_{\mathcal{E}}(R), \mathcal{GI}_{\mathcal{E}}(R))$ is a duality pair and $\mathcal{GF}_{\mathcal{E}}(R)$ is a covering subcategory. Furthermore, if every left R-module in $\mathcal{E}$ has a finite flat dimension, then the pair $(\mathcal{GF}_{\mathcal{E}}(R), \mathcal{GF}_{\mathcal{E}}(R)^{\perp})$ is a perfect cotorsion pair.*

Proof. By Theorem 3.21 we have $M \in \mathcal{GI}_{\mathcal{E}}(R)$ if and only if $M^+ \in \mathcal{GF}_{\mathcal{E}}(R)$ for any right R-module M. By [22, Theorem 2.7] we have that $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under direct summands. Notice that $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under direct products by [22, Remark 2.3(1)], it follows that $\mathcal{GI}_{\mathcal{E}}(R)$ is closed under finite direct sums. Therefore, $(\mathcal{GF}_{\mathcal{E}}(R), \mathcal{GI}_{\mathcal{E}}(R))$ is a duality pair.

Now, since $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under coproducts (i.e., direct sums) by [23, Remark 2.2(2)] we have the duality pair $(\mathcal{GF}_{\mathcal{E}}(R), \mathcal{GI}_{\mathcal{E}}(R))$ is coproduct-closed. It follows that $\mathcal{GF}_{\mathcal{E}}(R)$ is a covering subcategory by Lemma 2.8.

By Lemma 3.11 $\mathcal{GF}_{\mathcal{E}}(R)$ is closed under extensions. Now, since $R \in \mathcal{GF}_{\mathcal{E}}(R)$ we have $(\mathcal{GF}_{\mathcal{E}}(R),\mathcal{GI}_{\mathcal{E}}(R))$ is a perfect duality pair. Thus, $(\mathcal{GF}_{\mathcal{E}}(R), \mathcal{GF}_{\mathcal{E}}(R)^{\perp})$ is a perfect cotorsion pair by Lemma 2.8. □

The next proposition gives a description of the rings over which all $\mathcal{E}$-Gorenstein injective left R-modules are injective. Before that, recall the $\mathcal{GI}_{\mathcal{E}}(R)$-resolution dimension of a left R-module M in [22], denoted by $\mathrm{res.dim}_{\mathcal{GI}_{\mathcal{E}}} M$, is defined as $\inf\{n \geq 0 \mid$ there exists an exact sequence of left R-modules $0 \to M \to G_0 \to G_1 \to \cdots \to G_n \to 0$ with all G_i $\mathcal{E}$-Gorenstein injective$\}$. Set $\mathrm{res.dim}_{\mathcal{GI}_{\mathcal{E}}} M = \infty$ if no such integer exists.

Proposition 3.24 *The following conditions are equivalent:*

(1) *All $\mathcal{E}$-Gorenstein injective left R-modules are injective.*

(2) *All strongly $\mathcal{E}$-Gorenstein injective left R-modules are injective.*

(3) *For every left R-module M,* $\text{res.dim}_{\mathcal{GI}_{\mathcal{E}}}(M) = \text{id}_R(M)$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1) This follows immediately from Theorem 3.4.

(1) $\Rightarrow$ (3) Let M be a left R-module. It is obvious that $\text{res.dim}_{\mathcal{GI}_{\mathcal{E}}}(M) \leq \text{id}_R(M)$. Now, it remains to prove $\text{id}_R(M) \leq \text{res.dim}_{\mathcal{GI}_{\mathcal{E}}}(M)$. We assume that $\text{res.dim}_{\mathcal{GI}_{\mathcal{E}}}(M) = t < \infty$ for some positive integer t. Then, M admits a $\mathcal{E}$-Gorenstein injective resolution of length t. Notice that all $\mathcal{E}$-Gorenstein injective left R-modules are injective by assumption, we have $\text{id}_R(M) \leq t = \text{res.dim}_{\mathcal{GI}_{\mathcal{E}}}(M)$. It follows that $\text{res.dim}_{\mathcal{GI}_{\mathcal{E}}}(M) = \text{id}_R(M)$. $\square$

Recall that a ring R is called quasi-Frobenius (a QF-ring for short) if R is left Noetherian and ${}_RR$ is injective.

Proposition 3.25 *The following statements are equivalent for a ring R:*

(1) *Every left R-module is $\mathcal{E}$-Gorenstein injective.*

(2) *Every left R-module in $\mathcal{E}$ is projective.*

If the above equivalence conditions are satisfied, then R is quasi-Frobenius.

Proof. Since every $\mathcal{E}$-Gorenstein injective left R-module is Gorenstein injective, we easily obtain that if one of the equivalence conditions are satisfied then R is quasi-Frobenius by [5, Proposition 2.6].

(1) $\Rightarrow$ (2) Assume that every left R-module is $\mathcal{E}$-Gorenstein injective and M is a left R-module. It follows from [22, Proposition 2.6] that $\text{Ext}^i_R(I, M) = 0$ for any $I \in \mathcal{E}$ and any $i \geq 1$. Thus, every I in $\mathcal{E}$ is projective.

(2) $\Rightarrow$ (1) Let M be a left R-module and let

$$\cdots \to P_1 \to P_0 \to M \to 0 \text{ and } 0 \to M \to E_0 \to E_1 \to \cdots$$

be projective and injective resolutions of M, respectively. Since every injective left R-module is projective by (2), we have that R is quasi-Frobenius. By [1, Theorem 31.9] every projective left R-module is injective. Then, the above projective resolution is a left injective resolution of M. Now, assembling the two above resolutions we have the following exact sequence:

$$\mathbb{E} = \cdots \to P_1 \to P_0 \to I_0 \to I_1 \to \cdots .$$

Since every left R-module in $\mathcal{E}$ is projective by assumption, we have that $\text{Hom}_R(I, \mathbb{E})$ is exact for any $I \in \mathcal{E}$. Hence, M is $\mathcal{E}$-Gorenstein injective, as desired. $\square$

We finish the paper by giving some characterizations of FC rings in terms of the strongly $\mathcal{E}$-Gorenstein injective and flat modules. Recall a ring R is called left IF [12] if every injective left R-module is flat; and a left R-module M is called FP-injective in [30] if $\text{Ext}^1_R(F, M) = 0$ for any finitely presented left R-module F.

Theorem 3.26 *The following are equivalent for a two-sided coherent ring R:*

(1) *R is an FC ring.*

(2) *Every strongly $\mathcal{E}$-Gorenstein injective R-module (left and right) is strongly Gorenstein flat.*

(3) *Every injective R-module (left and right) is strongly Gorenstein flat.*

(4) *Every strongly $\mathcal{E}$-Gorenstein flat R-module (left and right) is strongly Gorenstein FP-injective.*

(5) *Every flat R-module (left and right) is strongly Gorenstein FP-injective.*

(6) *Every projective R-module (left and right) is strongly Gorenstein FP-injective.*

Proof. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are trivial, (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows directly by [19, Theorem 2.14]. Next, it remains to show that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4).

(1) $\Rightarrow$ (2) Let M be a strongly $\mathcal{E}$-Gorenstein injective left R-module. Then, there is a short exact sequence $0 \to M \to E \to M \to 0$ with E injective by Proposition 3.7. Since R is an FC ring, we have R is an IF ring by [13, Corollary 3.14]. So, E is flat by [13, Theorem 3.5]. For any injective right R-module I, we have that I is also flat. It follows that $\mathrm{Tor}_i^R(I, M) = 0$ for all $i \geq 1$. Thus, M is strongly Gorenstein flat by [4, Proposition 3.6]. Similarly, we can prove the case of right R-modules.

(1) $\Rightarrow$ (4) Let M be a strongly $\mathcal{E}$-Gorenstein flat left R-module. By Proposition 3.6, there exists a short exact sequence $0 \to M \to F \to M \to 0$, where F is flat. Notice that R is an FC ring, F is FP-injective by [13, Theorem 3.8]. Now, since R is left and right coherent, it follows from [19, Corollary 2.9] that M is strongly Gorenstein FP-injective. $\square$

Acknowledgements. This work was partially supported by National Natural Science Foundation of China (Grant No. 11971225). The authors would like to thank the referee for his/her many helpful suggestions and corrections, which have improved this paper greatly.

References

[1] Anderson, F. W., Fuller, K. R.: Rings and Categories of modules (2nd edition), Graduate Texts Mathematics, vol.13, New York: Springer-Verlag, 1992.

[2] Auslander, M., Bridger, M.: *Stable Module Theory*, Memoirs Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969.

[3] Bennis, D.: Rings over which the class of Gorenstein flat modules is closed under extensions, Comm. Algebra **37**(2009), 855–868.

[4] Bennis, D., Mahdou, N.: Strongly Gorenstein projective, injective, and flat modules, J. Pure Appl. Algebra **210**(2007), 437–445.

[5] Bennis, D., Mahdou, N.: Global Gorenstein dimensions, Proc. Amer. Math. Soc. **138**(2010), 461–465.

[6] Bennis, D., Mahdou, N. and Ouarghi, K.: Rings over which all modules are strongly Gorenstein projective, Rocky Mountain J. Math. **40**(2010),749–759.

[7] Bravo, D., Estrada, S., Iacob, A.: FP_n-injective, FP_n-flat covers and preenvelopes, and Gorenstein AC-flat covers, Algebra Colloquium **25**(2018), 319–334.

[8] Bravo, D., Gillespie, J.: Absolutely clean, level, and Gorenstein AC-injective complexes, Comm. Algebra **44**(2016), 2213–2233.

[9] Bravo, D., Gillespie, J., Hovey, M.: The stable module category of a general ring, arXiv:1405.5768v1.

[10] Christensen, L. W.: Gorenstein Dimensions, Lecture Notes in Math., vol. **1747**, Berlin: Springer-Verlag, 2000.

[11] Cheatham, T. J., Stone, D. R.: Flat and projective character modules, Proc. Amer. Math. Soc. **81**(1981), 175–177.

[12] Colby, R. R.: Rings which have flat injective modules, J. Algebra **35**(1975), 239–252.

[13] Ding, N., Chen, J.: The flat dimensions of injective modules, Manuscripta Math. **78**(1993), 165–177.

[14] Ding, N., Chen, J.: Coherent rings with finite self-FP-injective dimension, Comm. Algebra **24**(1996),2963–2980.

[15] Ding, N., Li, Y., Mao, L.: Strongly Gorenstein flat modules, J. Aust. Math. Soc. **86**(2009), 323–338.

[16] Enochs, E. E., Jenda, O. M. G.: Gorenstein injective and projective modules, Math. Z. **220**(1995), 611–633.

[17] Enochs, E. E., Jenda, O. M. G.: Relative Homological Algebra, de Gruyter Exp. Math., vol. **30**, de Gruyter, Berlin, 2000.

[18] E. E. Enochs, O. M. G. Jenda and B. Torrecillas, Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan **10**(1993), 1–9.

[19] Gao, Z.: On strongly Gorenstein FP-injective modules, Comm. Algebra **41**(2013), 3035–3044.

[20] Gao, Z.: Homological properties relative to injectively resolving subcategories, Canad. Math. Bull. **58**(2015), 741–756.

[21] Gao, Z., Wang, F.: Weak injective and weak flat modules, Comm. Algebra **43**(2015), 3857–3868.

[22] Gao, Z., Xu, L.: Gorenstein coresolving categories, Comm. Algebra **45**(2017), 4477–4491.

[23] Gao, Z., Wu, W.: Gorenstein flat modules relative to injectively resolving subcategories, J. Algebra Appl. **21**(2022), 2250099 (19 pages).

[24] Gillespie, J.: Gorenstein complexes and recollements from cotorsion pairs, Adv. Math. **291**(2016), 859–911.

[25] Holm, H.: Gorenstein homological dimensions, J. Pure Appl. Algebra **189**(2004), 167–193.

[26] Holm, H., Jørgensen, P.: Cotorsion pairs induced by duality pairs, J. Commut. Algebra **1**(2009), 621–633.

[27] Mao, L., Ding, N.: Gorenstein FP-injective and Gorenstein flat modules, J. Algebra Appl. **7**(2008), 491–506.

[28] Rotman, J. J.: An Introduction to Homological Algebra, New York: Academic Press, 1979.

[29] Šaroch, J. and Šťovíček, J.: Singular compactness and definability for Σ-cotorsion and Gorenstein modules, Selecta Math. (N. S.) **26**(2020), no. 2, Paper No. 23.

[30] Stenström, B.: Coherent rings and FP-injective modules, J. London Math. Soc. **2**(1970), 323–329.

[31] Stenström, B.: Rings of Quotients, New York: Springer-Verlag, 1975.

[32] Xu, J. Z.: Flat Covers of Modules, Lecture Notes in Math., **1634**, Berlin: Springer-Verlag, 1996.

[33] Yang, X., Liu, Z.: Strongly Gorenstein projective, injective and flat modules, J. Algebra **320**(2008), 2659–2674.

3 Nov 2022 18:24:49 PDT
220430-ZenghuiGao Version 4 - Submitted to Rocky Mountain J. Math.