THE EXISTENCE OF GROUND STATES TO THE DOUBLE-TYPE NONLINEAR p-LAPLACE PROBLEM INVOLVING THE SOBOLEV CRITICAL EXPONENT

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ABSTRACT. This paper extends the results of Akahori, et al [1] on ground states. We discuss the existence of ground states to the double-type nonlinear p-Laplace problem involving the Sobolev critical exponent in R^N

$$-\Delta_p u + |u|^{p-2}u = |u|^{p^*-2}u + \lambda |u|^{q-2}u, \quad u \in W^{1,p}(R^N),$$

where $N \geq 2, \lambda > 0, 1 < p < N, p < q < p^*, p^* := \frac{Np}{N-p}$ is the Sobolev critical index, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. We show that (i) if $p < q < p^*$, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists ground states; (ii) if $\max(p, p^* - \frac{p}{p-1}) < q < p^*$, there exists ground states for all $\lambda > 0$.

1. INTRODUCTION

This paper is concerned with the existence of ground states to the nonlinear p-Laplace problem

$$-\Delta_p u + |u|^{p-2}u = |u|^{p^*-2}u + \lambda |u|^{q-2}u, \quad u \in W^{1,p}(R^N), \tag{1}$$

where $N \geq 2$, $\lambda > 0$, $1 < p < N$, $p < q < p^*$, $p^* := \frac{Np}{N-p}$ is the Sobolev critical index, p-Laplace operator $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$. Ground states on the whole space will be considered in $W^{1,p}(R^N)$.

$$W^{1,p}(R^N) := \left\{u \in L^p(R^N) : \nabla u \in L^p(R^N)\right\},$$

with the norm

$$\|u\|_1 := \left(\int_{R^N} |\nabla u|^p + |u|^p dx\right)^{\frac{1}{p}}.$$

We denote $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,p}(R^N)}$, and the energy functional corresponding to (1) is

$$\varphi_\lambda(u) := \int_{R^N} \frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p - \frac{1}{p^*}|u|^{p^*} - \frac{\lambda}{q}|u|^q dx.$$

Remark: the norm in $L^p(R^N)$ is $|u|_p$, the norm in $W^{1,p}(R^N)$ is $\|u\|_1$.

Aubin [4] and Talenti [23] prove that the Sobolev inequality has an optimal embedding constant by rearranging in the Hardy-Littlewood sence and some variational methods, they also give the specificity expression of the constant. The basic solution of p-Laplace equation

$$-\Delta_p u = |u|^{p^*-2}u \quad in \quad R^N \tag{2}$$

is

$$U_\varepsilon(x) = \frac{C_{N,p}\varepsilon^{-\frac{N-p}{p}}}{[1 + (\frac{|x|}{\varepsilon})^{\frac{p}{p-1}}]^{\frac{N-p}{p}}} \quad , \quad \varepsilon > 0.$$

If $N \geq 4$, $1 < p < N$, [11, 21, 24] generalizes the result [7] when $p = 2$ and prove that there is a unique positive solution U_ε for (2), up to translations and dilations.

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Garcia and Peral [14] study the existence of solutions for the following nonlinear degenerate elliptic problems in a bounded domain $\Omega \subset R^N$

$$-\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{q-2}u, \qquad \lambda > 0 \tag{3}$$

where $u|_{\partial\Omega} = 0, p^* := \frac{Np}{N-p}$ is the Sobolev critical index. They give the existence of solutions in the following cases: If $p < q < p^*$, there exists $\lambda_0 > 0$, such that for all $\lambda > \lambda_0$, then (3) exists a nontrivial solution. If $\max(p, p^* - \frac{p}{p-1}) < q < p^*$, for all $\lambda > 0$, there exists nontrivial solution to (3).

Mercuri, Sciunzi, Squassina [17] and other researchers have studied the properties of the solution to the following problem:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u, & in \quad \Omega \\ \quad u = 0, & on \quad \partial\Omega, \end{cases} \tag{4}$$

where Ω is smooth bounded in R^N, $1 < p < N$, Solutions on the whole space will be considered in

$$D^{1,p}(R^N) := \left\{ u \in L^{p^*}(R^N) : \nabla u \in L^p(R^N) \right\},$$

endowed with the norm

$$||u|| := \left(\int_{R^N} |\nabla u|^p \right)^{\frac{1}{p}}.$$

They prove that (4) admits a nontrivial solution in annular shaped domains with sufficiently small inner hole. Clapp and Rios [8] show that when $N \geq 4$, $1 < p < N$, (4) has multiple sign-changing solutions in $D^{1,p}(R^N)$.

Barletta, Candito, et al [5] study the following questions

$$-\Delta_p u = a(x)|u|^{p-2}u + |u|^{p^*-2}u, \quad u \in D^{1,p}(R^N), \tag{5}$$

where $1 < p < N$, $a(x) \in L^{\frac{N}{p}}(R^N)$ satisfies some conditions. They prove the existence of N distinct pairs of nontrivial solutions for (5) in R^N, as well as in bounded domains. If $p \geq 2$, $a(x)$ meets certain conditions, Alves [2] gives a sufficient condition for the existence of positive solution to (5). Multiplicity results for subcritical scalar equations in R^N can be found in [9, 19].

Subsequently, Akahori, et al [1] study the existence, uniqueness and non-degenerate properties of the ground state solution to the semilinear elliptic equation. The equation is

$$-\Delta u + \omega u = |u|^{p-1}u + |u|^{\frac{4}{N-2}}u, \quad u \in W^{1,2}(R^N), \tag{6}$$

where $N \geq 3, \omega > 0, 1 < p < \frac{N+2}{N-2}$. This article introduces two infimums, and by studying the relationship and reachability between the two infimums, they eliminate the $|u|^{p-1}u$ term. Therefore, the existence of the ground state solution of (6) is proved. On this basis, we extends the case of $p = 2$ to the general case to study the existence of the ground state solution of the p-Laplace problem.

Theorem 1.1 (Existence of Ground State) *Assume $N \geq 2, \lambda > 0, 1 < p < N, p < q < p^*$, then we have the following results.*

(i) if $p < q < p^*$, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there exists ground states to (1);

(ii) if $\max(p, p^* - \frac{p}{p-1}) < q < p^*$, there exists ground states to (1) for all $\lambda > 0$.

Remark: Variational method can transform the existence of the ground state of (1) into the critical point of energy functional $\varphi_\lambda(u)$ on $W^{1,p}(R^N)$. Due to the existence of Sobolev critical index p^* and $\lambda|u|^{q-2}u$ term in (1), we can't directly use the mountain pass lemma to study the existence of the ground state of (1). To overcome these difficulties, this article introduces two infimum m_λ and M_λ and then eliminates $\lambda|u|^{q-2}u$ in M_λ, finally proves that $m_\lambda = M_\lambda$ and their minimization variables are consistent. This transforms the study of m_λ reachability into the M_λ reachability, and then we solves the difficulty caused by the interference of the $\lambda|u|^{q-2}u$ term. We also prove that the infimum m_λ satisfies the "threshold" condition $m_\lambda < \frac{1}{N}S^{\frac{N}{p}}$: (i) if $N \geq 2$, $1 < p < N$,when $p < q < p^*$, there exists $\lambda_0 > 0$, such that when $\lambda > \lambda_0$, $m_\lambda < \frac{1}{N} S^{\frac{N}{p}}$; (ii) if $N \geq 2$, $1 < p < N$, when $\max(p, p^* - \frac{p}{p-1}) < q < p^*$, for all $\lambda > 0$, we have $m_\lambda < \frac{1}{N} S^{\frac{N}{p}}$. This solves the problem of loss tightness. Finally, according to the Schwarz symmetry transformation, the function column is rearranged into radial symmetric function column. According to the good properties of radial symmetric functions, we proved the existence of the ground state solutions.

2. Preliminary

In this section, we will introduce some basic definitions and theorems.

Theorem 2.1 (Sobolev Embedding Theorem) [13] The following embedding maps are continuous:

$$W^{1,p}(R^N) \hookrightarrow L^r(R^N), \quad 1 < p < N, 1 \leq r \leq p^*.$$

Sobolev inequality holds

$$S := \inf_{u \in D^{1,p}(R^N)} \frac{|\nabla u|_p^p}{|u|_{p^*}^p} > 0,$$

where $D^{1,p}(R^N) := \left\{ u \in L^{p^*}(R^N) : \nabla u \in L^p(R^N) \right\}$.

Theorem 2.2 (Rellich-Kondrachov Embedding Theorem) [13] Assume $|\Omega| < \infty$, $\partial\Omega \in C^1$, the following embedding maps are compact:

$$W^{1,p}(\Omega) \hookrightarrow L^r(\Omega), \quad 1 < p < N, 1 \leq r < p^*.$$

Sobolev and Rellich-Kondrachov embedding theorem are very important in critical point theory.

Theorem 2.3 (Poincaré Inequality) [20] Assume $1 \leq p < \infty$, $\Omega \subset R^N$ is bounded, then we have

(i) if $u \in W_0^{1,p}(\Omega)$, then

$$\int_\Omega |u|^p dx \leq C \int_\Omega |\nabla u|^p dx.$$

(ii) In particular, if the boundary $\partial\Omega$ satisfies the Lipschitz condition,

$$\int_\Omega |u - \bar{u}|^p dx \leq C \int_\Omega |\nabla u|^p dx,$$

where $C = C(n, p, \Omega)$ is a constant, $|\Omega|$ is a measure on Ω, $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x) dx$.

Definition 2.4 ($(\mathrm{PS})_c$ condition) [25] Assume X is *Banach* space, $\varphi \in C^1(X, R)$, $c \in R$. We call $\varphi\{u_j\}$ is a (PS) sequence, if $\forall \{u_j\} \subset X$ satisfies

$$\varphi(u_j) \longrightarrow c, \quad \varphi'(u_j) \longrightarrow 0.$$

If $\varphi\{u_j\}$ still has convergent subsequence, then φ satisfies the $(\mathrm{PS})_c$ condition.

Lemma 2.5 (Moutain Pass Lemma) [25] Assume X is Banach space, functional $\varphi \in C^1(X, R)$, $e \in X$, $r > 0$ such that $\|e\| > r$, and satisfies

(i) $\inf_{|u|=r} \varphi(u) > \varphi(0) \geq \varphi(e)$,

(ii) φ satisfies $(\mathrm{PS})_c$ condition,

then c is a critical point of φ, where

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)) \quad , \quad \Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

3. m_λ AND M_λ

The problem studied in this paper constrains the nonlinear term to the form of a combined power, which is easier to prove than the general term. For the convenience of representation and calculation, let us first introduce some functional. Assuming $u \in W^{1,p}(R^N)$.

Energy functional:

$$\varphi_\lambda(u) := \frac{1}{p}|\nabla u|_p^p + \frac{1}{p}|u|_p^p - \frac{1}{p^*}|u|_{p^*}^{p^*} - \frac{\lambda}{q}|u|_q^q. \tag{7}$$

Nehari functional:

$$N_\lambda(u) := |\nabla u|_p^p + |u|_p^p - |u|_{p^*}^{p^*} - \lambda|u|_q^q. \tag{8}$$

Combining $\varphi_\lambda(u)$ and $N_\lambda(u)$, we give a positive functional:

$$\begin{aligned} T_\lambda(u) &:= \varphi_\lambda(u) - \tfrac{1}{q}N_\lambda(u) \\ &= \left(\tfrac{1}{p} - \tfrac{1}{q}\right)(|\nabla u|_p^p + |u|_p^p) + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right)|u|_{p^*}^{p^*}. \end{aligned} \tag{9}$$

The Nehari manifold is introduced to study the radially symmetric solution of the equation. When using the variational method to study the properties of solutions, the energy functional is usually required to be lower bound, so that the solution of the equation can be found directly. However, many equations correspond to the energy functional may have no lower bound on the whole space, but has a lower bound on a subset of them and the minimum value on it is just the solution of the original equation. Nehari manifolds are just a subset that satisfies this condition, the essence of it is to reduce the range of the critical point from the entire space to a certain manifold N. Therefore, we can simplify the process of finding critical points.

Definition 3.1 (Nehari Manifold) [10] For any $u, v \in W^{1,p}(R^N)$, the derivative of φ_λ is $\varphi_\lambda^{'} \in W^{-1,p}(R^N)$, which is

$$\langle \varphi_\lambda^{'}(u), v\rangle = \int_{R^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv - |u|^{p^*-2}uv - \lambda|u|^{q-2}uv dx.$$

Nehari manifold is defined as

$$N := \left\{u \in W^{1,p}(R^N) : \langle \varphi_\lambda^{'}(u), u\rangle = 0, u \neq 0\right\}.$$

Theorem 3.2 [22] Consider the following question:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u &= f(|x|, u), \quad x \in \Omega, \\ u &\in W^{1,p}(\Omega), \end{cases} \tag{10}$$

where $N \geq 2$, Ω is a rotationally symmetric domain in R^N and definied as

$$\Omega = \Omega(r_1, r_2) := \{x \in R^N : r_1 \leq |x| < r_2\} \quad , \quad F(r, u) := \int_0^u f(r, s)ds.$$

If the following assumptions hold:

(f_1) if $f \in C(\Omega \times R, R)$, when $p < N, p < q < p^*$, there exists a real number $c_o > 0$, such that

$$|f(x, u)| \leq c_o\left(1 + |u|^{q-1}\right),$$

(f_2) when $|u| \to \infty$, $\frac{F(x,u)}{|u|^p} \to \infty$ is consistent about $x \in \Omega$,

(f_3) when $u \to 0$, $f(x, u) = o(|u|^{p-1})$ is consistent about $x \in \Omega$,

(f_4) for every $r > 0$, the function $\frac{f(x,u)}{|u|^{p-1}}$ is strictly increasing with respect to u in $R\backslash\{0\}$,

then the ground states to (10) is the critical point of φ, where φ is definied as

$$\varphi(u) := \int_\Omega \frac{1}{p}|\nabla u|^p + \frac{1}{p}|u|^p - F(|x|, u)dx.$$

When Ω is a full space, the conclusion is obviously also true. In this way, we can transform the existence of the ground state of (1) into the existence of extreme point of the energy functional $\varphi_\lambda(u)$ on Nehari manifold N.

In order to explore the relationship between m_λ and M_λ, we should prove two basic lemmas first.

Lemma 3.3 *Assume $u \in W^{1,p}(R^N)\backslash\{0\}$, then the following functions are non-decreasing*

$$T_\lambda : \begin{array}{ll} R^+ & \to R \\ t & \to T_\lambda(tu). \end{array} \tag{11}$$

Proof. By taking the derivative of $T_\lambda(tu)$ with respect to t, we can get

$$\langle T_\lambda'(tu), u\rangle = p\left(\frac{1}{p} - \frac{1}{q}\right) t^{p-1}\left(|\nabla u|_p^p + |u|_p^p\right) + p^*\left(\frac{1}{q} - \frac{1}{p^*}\right) t^{p^*-1} |u|_{p^*}^{p^*}.$$

Since $1 \le p < q < p^*$, we can easily get $\frac{1}{p} - \frac{1}{q} > 0$, $\frac{1}{q} - \frac{1}{p^*} > 0$. When $t > 0$, we have $\langle T_\lambda'(tu), u\rangle > 0$. Hence, $T_\lambda(tu)$ is non-decreasing with respect to t. □

Lemma 3.4 *Assume $u \in W^{1,p}(R^N)\backslash\{0\}$, then there is a unique $t(u) > 0$, such that*

$$N_\lambda(tu)\begin{cases} > 0, & 0 < t < t(u), \\ = 0, & t = t(u), \\ < 0, & t > t(u). \end{cases} \tag{12}$$

Proof. Let $y_\lambda(t) = \varphi_\lambda(tu)$, $t \in [0, \infty)$, then we have

$$\begin{aligned} y_\lambda'(t) = 0 \quad &\Leftrightarrow \quad \langle \varphi_\lambda'(tu), u\rangle = 0 \\ &\Leftrightarrow \quad N_\lambda(tu) = 0. \end{aligned}$$

We easily get $y_\lambda(0) = 0$. It is also known from $1 \le p < q < p^*$ that (i) when t is small enough and $t > 0$, we have $y_\lambda(t) > 0$; (ii) when t is large enough and $t > 0$, we have $y_\lambda(t) < 0$.

Hence, $y_\lambda(t)$ attains its maximum only at the point $t(u) > 0$, that is $y_\lambda'(t(u)) = 0$. When $0 < t < t(u)$, $y_\lambda(t)$ is increasing; when $t > t(u)$, $y_\lambda(t)$ is decreasing. □

Remark: From the proof of Lemma 3.4, we know more that $t(u) \in (0, 1]$.

3.1. **The relationship between m_λ and M_λ.**

We introduce several variational values:

$$S := \inf_{u\in D^{1,p}(R^N)} \frac{|\nabla u|_p^p}{|u|_{p^*}^p}, \tag{13}$$

$$m_\lambda := \inf_{N_\lambda(u)=0} \varphi_\lambda(u), \tag{14}$$

$$M_\lambda := \inf_{N_\lambda(u)\le 0} T_\lambda(u). \tag{15}$$

Theorem 3.5 *If $1 < p < N$, $N \ge 2$, $1 < p < q < p^*$, then for every $\lambda > 0$, we have the following results:*

(i) $m_\lambda = M_\lambda > 0$,

(ii) m_λ and M_λ have the same minimum element.

Proof. (i) Let's prove in three steps.

Step 1: prove $m_\lambda \ge M_\lambda$.

Assume:

$$A := \{u \in W^{1,p}(R^N) : N_\lambda(u) = 0\},$$
$$B := \{u \in W^{1,p}(R^N) : N_\lambda(u) \le 0\}.$$

For every $u \in A$, we have $u \in B$, then $A \subseteq B$. Using (14) and (15), we obtain

$$m_\lambda = \inf_{u\in A} \varphi_\lambda(u) \ge \inf_{u\in B} T_\lambda(u) = M_\lambda,$$

that is

$$m_\lambda \geq M_\lambda.$$

Step 2: prove $m_\lambda \leq M_\lambda$.

Fix $u \in B$, by Lemma 3.4, we show that there exists $t \in (0,1)$ such that $N_\lambda(tu) = 0$. By Lemma 3.3, we obtain that $N_\lambda(tu)$ is non-decreasing with respect to t, then

$$m_\lambda \leq \varphi_\lambda(tu) = T_\lambda(tu) \leq T_\lambda(u).$$

when we take u over B, we show that

$$m_\lambda \leq M_\lambda.$$

In summary, we prove that $m_\lambda = M_\lambda$.

Step 3: prove $M_\lambda > 0$.

For every $u \in B$, it shows

$$\|u\|_1^p \leq |u|_{p^*}^{p^*} + \lambda |u|_q^q.$$

It follows from (13) that

$$S|u|_{p^*}^{p^*} \leq |\nabla u|_p^{p^*} \leq \|u\|_p^{p^*},$$

by Embedding Theorem, we have

$$|u|_q^q \leq \|u\|_1^q.$$

Then, there exists a constant $C_0 > 0$ such that

$$\|u\|_1^p \leq C_0\left(\|u\|_1^q + \|u\|_1^{p^*}\right).$$

This implies that there exists a constant $C_1(\lambda) > 0$ such that

$$\|u\|_1^p \geq C_1(\lambda).$$

By the definition of T_λ, there also exists a constant $C_2(\lambda) > 0$ that satisfies

$$T_\lambda(u) \geq C_2(\lambda) > 0,$$

then

$$M_\lambda = \inf_{u\in B} T_\lambda(u) > 0.$$

(ii) Let's prove in two steps.

Step 1: any minimizer for M_λ is also a minimizer for m_λ.

Claim: any minimizer U_λ for M_λ satisfies that $N_\lambda(U_\lambda) = 0$.

Let's proof by contradiction. If the assumption doesn't hold, then we can find a minimizer V_λ for M_λ, but $N_\lambda(V_\lambda) < 0$. It follows from Lemma 3.4 that there exists $t \in (0,1)$ such that $N_\lambda(tV_\lambda) = 0$. By the definition of T_λ, we have

$$M_\lambda \leq T_\lambda(tV_\lambda) < T_\lambda(V_\lambda) = m_\lambda.$$

However, it is a contradiction. Therefore, the claim is proved.

Combining the definitions of m_λ and M_λ and conclusion (i),we obtain that

$$m_\lambda = M_\lambda = T_\lambda(U_\lambda) = \varphi_\lambda(U_\lambda) - \frac{1}{q}N_\lambda(U_\lambda) = \varphi_\lambda(U_\lambda).$$

Hence, any minimizer for M_λ is a minimizer for m_λ.

Step 2: any minimizer for M_λ is a minimizer for m_λ.

Take any minimizer u_λ for m_λ, it is easily prove that $N_\lambda(u_\lambda) = 0$. It follows from (i) that

$$M_\lambda = m_\lambda = \varphi_\lambda(u_\lambda) = T_\lambda(u_\lambda).$$

Hence, u_λ is also a minimizer for M_λ.

Thus, we have proved (ii). □

3.2. **Inequalities about m_λ.**

Theorem 3.6 *Assume* $1 < p < N, p < q < p^*$, *then there exists* $\lambda_0 > 0$ *such that when* $\lambda > \lambda_0$, *we have*

$$m_\lambda < \frac{1}{N} S^{\frac{N}{p}}.$$

Proof. Take $u_0 \in W^{1,p}(R^N)$ such that $|u_0|_{p^*} = 1$. Let $y_\lambda(t) := \varphi_\lambda(tu_0)$, we have

$$y_\lambda(0) = 0, \quad \lim_{t\to\infty} y_\lambda(t) = -\infty. \tag{16}$$

When t is small enough, we prove that $y_\lambda(t) > 0$, therefore there exists $t_\lambda > 0$ such that

$$y_\lambda(t_\lambda) = \sup_{t\geq 0} y_\lambda(t), \quad y_\lambda^{'}(t_\lambda) = 0.$$

Hence,

$$\begin{aligned} 0 &= y_\lambda^{'}(t_\lambda) \\ &= t_\lambda^{p-1}\left(|\nabla u_0|_p^p + |u_0|_p^p\right) - t_\lambda^{p^*-1} - \lambda t_\lambda^{q-1}|u_0|_q^q. \end{aligned}$$

That is

$$0 = t_\lambda^{q-1}\{t_\lambda^{p-q}\left(|\nabla u_0|_p^p + |u_0|_p^p\right) - t_\lambda^{p^*-q} - \lambda|u_0|_q^q\}.$$

Because

$$\lim_{\lambda\to+\infty}\left(t_\lambda^{p^*-q} + \lambda|u_0|_q^q\right) = 0,$$

substitute it into the original formula, we prove that $\lim_{\lambda\to\infty} t_\lambda = 0$.

It can be seen from the continuity of φ_λ that

$$\lim_{\lambda\to\infty}\sup_{t\geq 0}\varphi_\lambda(tu_0) = \lim_{\lambda\to\infty}\varphi_\lambda(t_\lambda u_0) = 0.$$

Hence, there exists $\lambda_0 > 0$ such that when $\lambda \geq \lambda_0$, we have $\sup_{t\geq 0}\varphi_\lambda(tu_0) < \frac{1}{N}S^{\frac{N}{p}}$.

Let $v_0 = tu_0$, by (16) we know when t is large enough, there have $\varphi_\lambda(v_0) < 0$. Now we construct a road γ starting from 0 and ending from v_0,

$$\gamma \in ([0,1], W^{1,p}\left(R^N\right)) \quad , \quad \gamma(s) = sv_0.$$

From the Moutain Pass Lemma, we have

$$m_\lambda \leq \sup_{s\in[0,1]} \varphi_\lambda(\gamma(s)).$$

Moreover,

$$m_\lambda \leq \sup_{t\geq 0}\varphi_\lambda(tu_0) < \frac{1}{N}S^{\frac{N}{p}}.$$

□

If we make further restrictions on q, we will get a stronger result in the next Theorem.

Theorem 3.7 *Assume* $1 < p < N$, *if* $\max(p, p^* - \frac{p}{p-1}) < q < p^*$, *then for every* $\lambda > 0$, *we have*

$$m_\lambda < \frac{1}{N} S^{\frac{N}{p}}.$$

Proof. The proof will expand from the following five parts.

Step 1: introduce the basic solutions.

Assume $u \in W^{1,p}(R^N)$,

$$-\Delta_p u = |u|^{p^*-2}u.$$

From Talenti [23] and Aubin [4] we know that its basic solution is

$$U(x) = \frac{C_{N,p}}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}, \quad C_{N,p} = N^{\frac{N-p}{p^2}}\left(\frac{N-p}{p-1}\right)^{\frac{(p-1)(N-p)}{p^2}} > 0. \tag{17}$$

After the following stretching, the solution of above equation can also be a family of solutions of the form $((\varepsilon > 0))$

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-p}{p}} U\left(\frac{x}{\varepsilon}\right) = \frac{C_{N,p}\varepsilon^{-\frac{N-p}{p}}}{\left[1+\left(\frac{|x|}{\varepsilon}\right)^{\frac{p}{p-1}}\right]^{\frac{N-p}{p}}}. \tag{18}$$

and satisfies

$$\begin{cases} |\nabla U|_p^p = |\nabla U_\varepsilon|_p^p, \\ |U|_{p^*}^{p^*} = |U_\varepsilon|_{p^*}^{p^*}. \end{cases} \tag{19}$$

We can easily prove that $S^{\frac{N}{p}} = |\nabla U_\varepsilon|_p^p = |U_\varepsilon|_{p^*}^{p^*}$.

Step 2: smooth and normalize the basic solution.

Firstly, the function defined in the whole space is transformed into a function with compact support by using the truncation function, and then it is proved that these functions with compact support converge to S according to the norm. The specific method is: take the smooth truncation function $\eta : [0,\infty) \to [0,1]$, such that $\eta(r)$ is non-increasing and satisfies ($\delta > 0$, $B_\delta := \{x \in R^N : |x| < \delta\}$)

$$\eta(r) = \begin{cases} 1 & , \quad 0 < r \le \frac{\delta}{2}, \\ 0 \sim 1 & , \quad \frac{\delta}{2} < r < \delta, \\ 0 & , \quad r \ge \delta. \end{cases}$$

After smoothing and normalization, we get new functions

$$v_\varepsilon(x) = \eta(|x|) * U_\varepsilon(x), \quad u_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon|_{p^*}}.$$

In this way, when $\varepsilon \to 0$, u_ε and U_ε have similar properties. Hence, we only need to estimate u_ε.

Step 3: preliminary estimates.

When $\varepsilon \to 0$, the following estimates can be obtained from [3, 12, 15],the proof will be given in this paper.

$$|\nabla u_\varepsilon|_p^p = S + O(\varepsilon^{\frac{N-p}{p-1}}). \tag{20}$$

$$|u_\varepsilon|_{p^*}^{p^*} = 1. \tag{21}$$

$$|u_\varepsilon|_q^q = \begin{cases} O(\varepsilon^{N-\frac{q(N-p)}{p}}) & , \quad q > p^*\left(1-\frac{1}{p}\right), \\ O(\varepsilon^{\frac{N}{p}}|\log\varepsilon|) & , \quad q = p^*\left(1-\frac{1}{p}\right), \\ O(\varepsilon^{\frac{q(N-p)}{p(p-1)}}) & , \quad 1 < q < p^*\left(1-\frac{1}{p}\right). \end{cases} \tag{22}$$

$$|u_\varepsilon|_p^p = \begin{cases} O(\varepsilon^p) & , \quad N > p^2 \quad (p > p^*\left(1-\frac{1}{p}\right)), \\ O(\varepsilon^p|\log\varepsilon|) & , \quad N = p^2 \quad (p = p^*\left(1-\frac{1}{p}\right)), \\ O(\varepsilon^{\frac{N-p}{p-1}}) & , \quad N < p^2 \quad (1 < p < p^*\left(1-\frac{1}{p}\right)). \end{cases} \tag{23}$$

When $p < q < p^*$,

$$\lim_{\varepsilon\to 0} |u_\varepsilon|_q^q = 0. \tag{24}$$

It is easy to obtain (21) and (23) is a special case of (22), so we only need to proof (20) and (22).
The detailed process of v_ε estimation will be given first and we take $C_{N,p}$ as 1 in the following.

$$v_\varepsilon(x) = \frac{\varepsilon^{-\frac{N-p}{p}}\eta(|x|)}{\left(1+(\frac{|x|}{\varepsilon})^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}},$$

$$|v_\varepsilon|_{p^*}^{p^*} = \int_{B_\delta} |v_\varepsilon|^{p^*} dx = \int_{R^N} \frac{\varepsilon^{-N}(\eta(|x|))^{p^*}}{\left(1+(\frac{|x|}{\varepsilon})^{\frac{p}{p-1}}\right)^N} dx.$$

Let $x = \varepsilon y$ be a variable substitution, we have

$$\int_{R^N} \frac{\varepsilon^{-N}}{\left(1+(\frac{|x|}{\varepsilon})^{\frac{p}{p-1}}\right)^N} dx = |U|_{p^*}^{p^*} = S^{\frac{N}{p}}.$$

Hence,

$$\begin{aligned} 0 &\le |U|_{p^*}^{p^*} - |v_\varepsilon|_{p^*}^{p^*} \\ &= \int_{R^N\setminus B_{\frac{\delta}{2}}} \frac{1-\eta^{p^*}(|x|)}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^N} dx \\ &\le \int_{R^N\setminus B_{\frac{\delta}{2}}} \frac{1}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^N} dx \\ &\le \int_{R^N\setminus B_{\frac{\delta}{2}}} \frac{\varepsilon^{\frac{p-1}{N}}}{|x|^{\frac{Np}{p-1}}} dx \\ &= \varepsilon^{\frac{N}{p-1}} N\omega_N \int_{\frac{\delta}{2}}^{\infty} \frac{r^{N-1}}{r^{\frac{Np}{p-1}}} dr \\ &= C_1 \varepsilon^{\frac{N}{p-1}}. \end{aligned}$$

Therefore, we obtain that

$$0 \le 1 - \frac{|v_\varepsilon|_{p^*}^{p^*}}{|U|_{p^*}^{p^*}} \le \frac{C_1\varepsilon^{\frac{N}{p-1}}}{|U|_{p^*}^{p^*}}, \quad 0 \le 1 - \frac{|v_\varepsilon|_{p^*}^{p^*}}{S^{\frac{N}{p}}} \le \frac{C_1\varepsilon^{\frac{N}{p-1}}}{S^{\frac{N}{p}}},$$

$$1 - \frac{C_1\varepsilon^{\frac{N}{p-1}}}{S^{\frac{N}{p}}} \le \frac{|v_\varepsilon|_{p^*}^{p^*}}{S^{\frac{N}{p}}} \le 1, \quad S^{\frac{N}{p}} - C_1\varepsilon^{\frac{N}{p-1}} \le |v_\varepsilon|_{p^*}^{p^*} \le S^{\frac{N}{p}}.$$

When $\varepsilon \to 0$, we have $|v_\varepsilon|_{p^*}^{p^*} = S^{\frac{N}{p}} + O(\varepsilon^{\frac{N}{p-1}})$.
Proof (20):

$$\nabla v_\varepsilon(x) = \frac{\nabla\eta(|x|)}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}} + \frac{p-N}{p-1}\frac{x\eta(|x|)\varepsilon^{\frac{-1}{p-1}}}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^{\frac{N}{p}}|x|^{\frac{p-2}{p-1}}}.$$

Because $\eta(|x|) \equiv 1$ holds near 0, a similar estimates can be obtained from [6] that

$$\begin{aligned} |\nabla v_\varepsilon|_p^p &= \int_{B_\delta} |\nabla v_\varepsilon|^p dx \\ &= \left(\tfrac{N-p}{p-1}\right)^p \int_{R^N} \frac{|x|^{\frac{p}{p-1}}\eta^p(|x|)\,\varepsilon^{\frac{-p}{p-1}}}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^N} dx + O(1) \\ &= I_1 + I_2 + O(1), \end{aligned}$$

where

$$I_1=\int_{R^N}\frac{|x|^{\frac{p}{p-1}}\varepsilon^{\frac{-p}{p-1}}}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^N}dx=S^{\frac{N}{p}},\quad I_2=\int_{R^N\setminus B_{\frac{\delta}{2}}(0)}\frac{(\eta^p(|x|)-1)|x|^{\frac{p}{p-1}}\varepsilon^{\frac{-p}{p-1}}}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^N}dx.$$

$$\begin{aligned}|I_2| &= \left|\int_{R^N\setminus B_{\frac{\delta}{2}}(0)}\frac{(\eta^p(|x|)-1)|x|^{\frac{p}{p-1}}\varepsilon^{\frac{-p}{p-1}}}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^N}dx\right|\le\int_{R^N\setminus B_{\frac{\delta}{2}}(0)}\frac{|x|^{\frac{p}{p-1}}\varepsilon^{\frac{-p}{p-1}}}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^N}dx\\ &= \int_{R^N\setminus B_{\frac{\delta}{2}}(0)}\frac{1}{\left(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}}\right)^N}dx=N\omega_N\int_{\frac{\delta}{2\varepsilon}}^{\infty}\frac{t^{N-1+\frac{p}{p-1}}}{\left(1+t^{\frac{p}{p-1}}\right)^N}dr\\ &\le N\omega_N\frac{p-1}{N-p}\left(\frac{2}{\delta}\right)^{\frac{N-p}{p-1}}\varepsilon^{\frac{N-p}{p-1}}\\ &= O(\varepsilon^{\frac{N-p}{p-1}}),\quad \varepsilon\to 0.\end{aligned}$$

When $\varepsilon\to 0$, there is $|\nabla u_\varepsilon|_p^p=\frac{|\nabla v_\varepsilon|_p^p}{|v_\varepsilon|_{p^*}^p}=S+O(\varepsilon^{\frac{N-p}{p-1}})$.

Proof (22):

$$|v_\varepsilon|_q^q=\int_{B_\delta}\frac{\eta^q(|x|)}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx=J_1+J_2.$$

where

$$J_1=\int_{B_\delta}\frac{\eta^q(|x|)-1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx,\quad J_2=\int_{B_\delta}\frac{1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx.$$

$$\begin{aligned}|J_1| &\le \int_{B_\delta}\frac{|\eta^q(|x|)-1|}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx\\ &\le \int_{B_\delta\setminus B_{\frac{\delta}{2}}}\frac{1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx\\ &= N\omega_N\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}}\frac{\varepsilon^{N-\frac{q(N-p)}{p}}r^{N-1}}{r^{\frac{q(N-p)}{p}}}dr,\end{aligned}$$

$$|J_1|\le\begin{cases}-K_1(\varepsilon^{\frac{q(N-p)}{p(p-1)}}) &, \quad q>p^*\left(1-\frac{1}{p}\right),\\ K_2(\varepsilon^{N-\frac{N}{p}}) &, \quad q=p^*\left(1-\frac{1}{p}\right),\\ K_3(\varepsilon^{\frac{q(N-p)}{p(p-1)}}) &, \quad 1<q<p^*\left(1-\frac{1}{p}\right).\end{cases}$$

(a) If $q>p^*\left(1-\frac{1}{p}\right)$, then

$$\int_{R^N}\frac{1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx=\varepsilon^{N-\frac{q(N-p)}{p}}|U|_q^q,$$

$$\left|J_2-\varepsilon^{N-\frac{q(N-p)}{p}}|U|_q^q\right|\le\int_{R^N\setminus B_\delta}\frac{1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx\le O(\varepsilon^{N-\frac{q(N-p)}{p}}).$$

Therefore, we obtain that $|u_\varepsilon|_q^q=O(\varepsilon^{N-\frac{q(N-p)}{p}})$.

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(b) If $q = p^*\left(1-\frac{1}{p}\right)$, then

$$\begin{aligned} J_2 &\le \int_{B_{2\delta}} \frac{1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx \\ &= N\omega_N\varepsilon^{\frac{N}{p}}\int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{N-1}}{(1+r^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dr \\ &\le N\omega_N\varepsilon^{\frac{N}{p}}\left|\log\varepsilon\right| + K_4, \qquad 0<\varepsilon<1. \end{aligned}$$

Calculate as above, we obtain that

$$J_2 \ge \int_{B_{2\delta}} \frac{1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx = N\omega_N\varepsilon^{\frac{N}{p}}\left|\log\varepsilon\right| + K_5, \quad 0<\varepsilon<1.$$

Then

$$\left|J_2 - N\omega_N\varepsilon^{\frac{N}{p}}\left|\log\varepsilon\right|\right| \le K_6.$$

In summary, we obtain that $|u_\varepsilon|_q^q = O(\varepsilon^{\frac{N}{p}}\left|\log\varepsilon\right|)$.

(c) If $1<q<p^*\left(1-\frac{1}{p}\right)$, then

$$J_2 \le \int_{B_{2\delta}} \frac{1}{(\varepsilon+\varepsilon^{\frac{-1}{p-1}}|x|^{\frac{p}{p-1}})^{\frac{q(N-p)}{p}}}dx \le N\omega_N\varepsilon^{\frac{q(N-p)}{p(p-1)}} = K_7\varepsilon^{\frac{q(N-p)}{p(p-1)}}.$$

In summary, we have $|u_\varepsilon|_q^q = O(\varepsilon^{\frac{q(N-p)}{p(p-1)}})$ (K_i $(i\in Z)$ are normal numbers).

Now we take a further exploration, for any given $\varepsilon\in(0,1)$, define the function $h_\varepsilon : (0,\infty)\to R$, satisfying:

$$h_\varepsilon(t) := \frac{1}{p}t^p\left(|\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p\right) - \frac{1}{p^*}t^{p^*}.$$

Let $h'_\varepsilon(t)=0$, it is easy to find that function $h_\varepsilon(t)$ attains its maximum only at the point

$$t_\varepsilon = (|\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p)^{\frac{N-p}{p^2}},$$

then

$$h_\varepsilon(t_\varepsilon) = \frac{1}{N}(|\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p)^{\frac{N}{p}} = \begin{cases} \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^p) &, \ N>p^2, \\ \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^p\left|\log\varepsilon\right|) &, \ N=p^2, \\ \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^{\frac{p-1}{N-p}}) &, \ N<p^2. \end{cases} \tag{25}$$

Step 4: constructor further estimation.

The estimation is similar to Theorem 3.6, with only slight changes. Let $y_\varepsilon(t) = \varphi_\lambda(tu_\varepsilon)$, it is clear to obtain that $y_\varepsilon(0)=0$, $\lim_{t\to\infty} y_\varepsilon(t) = -\infty$. Thus, there must exist $t_0>0$ such that $\varphi_\lambda(t_0u_\varepsilon) = \sup_{t\ge0}\varphi_\lambda(tu_\varepsilon)$, hence,

$$\begin{aligned} 0 &= y'_\varepsilon(t_0) \\ &= {t_0}^{p-1}\left(|\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p\right) - {t_0}^{p^*-1} - \lambda {t_0}^{q-1}|u_\varepsilon|_q^q. \end{aligned}$$

Simplified to get

$${t_0}^{p^*-p} < {t_0}^{p^*-p} + \lambda {t_0}^{q-p}|u_\varepsilon|_q^q = |\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p,$$

then

$$t_0 \le (|\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p)^{\frac{1}{p^*-p}}.$$

Substitute it into above formula to get

$$|\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p \le {t_0}^{p^*-p} + \lambda(|\nabla u_\varepsilon|_p^p + |u_\varepsilon|_p^p)^{\frac{q-p}{p^*-p}}|u_\varepsilon|_q^q.$$

When ε is small enough, it follows from (20) and (24) that

$$t_0 \geq \left(\frac{S}{2}\right)^{\frac{1}{p^*-p}}.$$

This yields a lower bound on t_0 and is independent of ε.

Step 5: show $m_\lambda < \frac{1}{N}S^{\frac{N}{p}}$.

Combining the results obtained in the above four steps, now we will prove the final conclusion. Obviously we see from the definition of m_λ that

$$\begin{aligned} m_\lambda &\leq \varphi_\lambda(t_0 u_\varepsilon) = y_\varepsilon(t_0) \\ &= h_\varepsilon(t_0) - \tfrac{\lambda}{q} t_0{}^q |u_\varepsilon|_q^q \\ &\leq h_\varepsilon(t_\varepsilon) - \tfrac{\lambda}{q}\left(\tfrac{S}{2}\right)^{\frac{1}{p^*-p}} |u_\varepsilon|_q^q. \end{aligned}$$

(a) If $N < p^2$, that is $q > p^*(1-\frac{1}{p})$, then

$$m_\lambda \leq \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p-1}}) - C\lambda\varepsilon^{N-\frac{q(N-p)}{p}}.$$

By $q > p^*(1-\frac{1}{p})$, we have $\frac{N-p}{p-1} > N - \frac{q(N-p)}{p}$. Thus, when ε is small enough, we obtain $m_\lambda < \frac{1}{N}S^{\frac{N}{p}}$.

(b) If $N = p^2$, that is $q > p = p^*(1-\frac{1}{p})$, then

$$m_\lambda \leq \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^p|\log\varepsilon|) - C\lambda\varepsilon^{N-\frac{q(N-p)}{p}}.$$

By $q > p = p^*(1-\frac{1}{p})$, we see $p > N - \frac{q(N-p)}{p}$. Thus, when ε is small enough,we have $m_\lambda < \frac{1}{N}S^{\frac{N}{p}}$.

(c) If $N > p^2$, that is $q > p = p^*(1-\frac{1}{p})$, then

$$m_\lambda \leq \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^p) - C\lambda\varepsilon^{N-\frac{q(N-p)}{p}}.$$

By $q > p^*(1-\frac{1}{p})$, we attain that $\frac{N-p}{p-1} > N - \frac{q(N-p)}{p}$. Thus, when ε is small enough, we have $m_\lambda < \frac{1}{N}S^{\frac{N}{p}}$.

In summary, when $\max(p, p^* - \frac{p}{p-1}) < q < p^*$, for every $\lambda > 0$, there always have $m_\lambda < \frac{1}{N}S^{\frac{N}{p}}$. □

4. Proof of Theorem 1.1

In this section, we also use the Schwarz symmetric transformation to prove Theorem 1.1. This transformation rearranges the non-radially symmetric function to make it radially symmetric. Let's talk the basic knowledge of Schwarz symmetry transformation.

Definition 4.1 (Schwarz Symmetry Transformation) [16] Fix a Lebesgue measurable function u, such that $u \in R^n \to R^+$ with measure μ, if $u^* \in R^N \to R^+$ satisfies

(i) for any $t \in R$, the level set $u^{*-1}(t, \infty)$ is a open ball with the origin 0 as the center,

(ii) $\mu(u^{*-1}(t,\infty)) = \mu(u^{-1}(t,\infty))$.

then we call u^* the Schwarz symmetric transformation of u.

In particular, this symmetry transformation exists and is unique.

Theorem 4.2 (Pólya-Szegö Inequality) [18] Assume $u \in W^{1,p}(R^N)$, u^* is the Schwarz symmetry transformation of u, then $u^* \in W^{1,p}(R^N)$ and satisfies

$$\begin{cases} |\nabla u_j^*|_p \leq |\nabla u_j|_p, \\ |u_j^*|_r = |u_j|_r, \quad r \in [1, p^*]. \end{cases}$$

Pólya-Szegö inequality describes that the Sobolev energy of the function will not decrease after symmetric rearrangement in the Sobolev space, that is, the rearrangement preserves the original L^p-norm of the function, and reduces only the L^p-norm of the gradient.

Lemma 4.3 (Brézis-Lieb Lemma) [25] Assume $\Omega \subset R^N$ is open, $\{u_j\} \subset L_p(\Omega)$, $1 \leq p < \infty$, if
(i) $\{u_j\}$ is norm bounded in $L_p(\Omega)$,
(ii) $u_j \to u$ a.e in Ω,
then

$$\lim_{j\to\infty}\left(|u_j|_p^p - |u_j - u|_p^p\right) = |u|_p^p.$$

Brézis-Lieb lemma is a refinement of Fatou's lemma, which not only equalizes the inequality relation in Fatou's lemma, but also gives the concrete expression of the difference value.

Now, we are ready to prove Theorem 1.1.

Proof. By Theorem 3.5, it just need to prove the existence of minimizer for M_λ. Now we take the sequence $\{u_j\} \subset B$, such that $\lim\limits_{j\to\infty} T_\lambda(u_j) = M_\lambda$, by the definiition of φ_λ, we know that

$$\varphi_\lambda(u_j) = T_\lambda(u_j) + \frac{1}{q}N_\lambda(u_j),$$

then

$$c := \sup_j \varphi_\lambda(u_j) < \infty.$$

When j is large enough, we have

$$\begin{array}{rcl} c + 1 + \|u_j\|_1 & \geq & \varphi_\lambda(u_j) - \frac{1}{q}N_\lambda(u_j) \\ & = & T_\lambda(u_j) \\ & \geq & \left(\frac{1}{p} - \frac{1}{q}\right)\|u_j\|_1^p. \end{array}$$

Hence, the sequence $\{u_j\}$ is bounded in $W^{1,p}(R^N)$.

For each $j \geq 1$, denote that u_j^* is the Schwarz symmetric transformation of u_j. According to the Pólya-Szegö inequality, we have

$$\begin{cases} |\nabla u_j^*|_p & \leq & |\nabla u_j|_p, \\ |u_j^*|_r & = & |u_j|_r \quad , \quad r \in [1, p^*]. \end{cases}$$

It can be seen that the new sequence $\{u_j^*\}$ is also bounded in $W^{1,p}(R^N)$ and satisfies

$$N_\lambda(u_j^*) \leq 0 \quad , \quad \forall j \geq 1. \tag{26}$$

$$|\nabla u_j^*|_p < \infty \quad , \quad \forall j \geq 1. \tag{27}$$

$$\lim_{j\to\infty} T_\lambda\left(u_j^*\right) = M_\lambda. \tag{28}$$

Combining the definition of $\{u_j^*\}$ and (27), we obtain that $\{u_j^*\}$ is radially symmetric and bounded in $W^{1,p}(R^N)$. Hence, there exists a radially symmetric function $u \in W^{1,p}(R^N)$ such that, passing to some subsequence, and satisfies

$$\begin{cases} u_j^* \rightharpoonup u \quad , \quad W^{1,p}(R^N), \\ u_j^* \to u \quad , \quad L^r(R^N), \\ u_j^* \to u \quad , \quad a.e \quad R^N, \end{cases} \tag{29}$$

where $r \in [1, p^*)$, $u_j^* \rightharpoonup u$ denotes u_j^* weakly converges to u.

Next, we shall prove that the radially symmetric function u is the minimizer for m_λ.

Claim 1: $u \neq 0$.

Suppose the contrary that $u \equiv 0$. From the convergence of the $\{u_j^*\}$, passing to some subsequence, we have

$$\lim_{j\to\infty}\left(\left|\nabla u_j^*\right|_p^p - |u_j^*|_{p^*}^{p^*}\right) \le \lim_{j\to\infty} N_\lambda\left(u_j^*\right) \le 0. \tag{30}$$

By Poincaré inequality, if $\lim_{j\to\infty} |\nabla u_j^*|_p^p = 0$, then $\lim_{j\to\infty} |u_j^*|_r^r = 0$, where $r \in [1, p^*]$. From (28), we have $M_\lambda = 0$, this contradicts Theorem 3.5. Therefore, $\lim_{j\to\infty} |\nabla u_j^*|_p^p > 0$.

With the Sobolev inequality (13) and (30), we have

$$\begin{aligned}\lim_{j\to\infty} |\nabla u_j^*|_p^p &\ge S \lim_{j\to\infty} |u_j^*|_{p^*}^{p^*} \\ &\ge S \lim_{j\to\infty} |\nabla u_j^*|_p^{\frac{p(N-p)}{N}}.\end{aligned}$$

That is

$$S \le \lim_{j\to\infty} |\nabla u_j^*|_p^{\frac{p^2}{N}}. \tag{31}$$

Therefore,

$$\begin{aligned}M_\lambda &= \lim_{j\to\infty} T_\lambda\left(u_j^*\right) \\ &= \lim_{j\to\infty}\{(\tfrac{1}{p}-\tfrac{1}{q})(|\nabla u_j^*|_p^p + |u_j^*|_p^p) + (\tfrac{1}{q}-\tfrac{1}{p^*})|u_j^*|_{p^*}^{p^*}\} \\ &\ge \lim_{j\to\infty}(\tfrac{1}{p}-\tfrac{1}{p^*})|\nabla u_j^*|_p^p \\ &\ge \tfrac{1}{N} S^{\frac{N}{p}}.\end{aligned}$$

However, this contradicts Theorem 3.6 and Theorem 3.7.

Claim 2: $N_\lambda(u) = 0$.

Using Brézis-Lieb Lemma, we obtain that

$$\begin{cases}\lim_{j\to\infty} & (|u_j^*|_r^r - |u_j^* - u|_r^r) = |u|_r^r \quad , \qquad r \in [1, p^*], \\ \lim_{j\to\infty} & (|\nabla u_j^*|_p^p - |\nabla u_j^* - u|_p^p) = |u|_p^p.\end{cases}$$

From the definition of T_λ, there are

$$\lim_{j\to\infty}(T_\lambda(u_j^*) - T_\lambda(u_j^* - u)) = T_\lambda(u), \tag{32}$$

$$\lim_{j\to\infty}(N_\lambda(u_j^*) - N_\lambda(u_j^* - u)) = N_\lambda(u). \tag{33}$$

By (28) and the non-negativity of T_λ,we have

$$T_\lambda(u) \le \lim_{i\to\infty} T_\lambda(u_j^*) = M_\lambda. \tag{34}$$

(a) Assume $N_\lambda(u) > 0$. For $\forall j \ge 1$, we have $N_\lambda(u_j^*) \le 0$. Using (33), we know that there exists $K \in N^+$, when $j > K$, $N_\lambda(u_j^* - u) < 0$. The lemma 3.4 implies that there exists $t_j \in (0,1)$, such that $N_\lambda(t_j(u_j^* - u)) = 0$. Furthermore, Lemma 3.3 together with (32) and the non-negativity of T_λ show that

$$\begin{aligned}M_\lambda &\le \lim_{j\to\infty} T_\lambda(t_j(u_j^* - u)) \\ &< \lim_{j\to\infty} T_\lambda(u_j^* - u) \\ &= \lim_{j\to\infty}(T_\lambda(u_j^*) - T_\lambda(u)) \\ &= M_\lambda - T_\lambda(u) \\ &< M_\lambda.\end{aligned}$$

This is a contradiction.

(b) Assume $N_\lambda(u) < 0$. Take any function u form the set B, such that $M_\lambda \le T_\lambda(u)$. Note that (34) implies that $M_\lambda = T_\lambda(u)$. From the Lemma 3.4, there exists $t \in (0,1)$, such that $N_\lambda(tu) = 0$. Hence,

$$M_\lambda = T_\lambda(u) > T_\lambda(tu) \ge M_\lambda.$$

This is a contradiction. Hence, we prove that $N_\lambda(u) = 0$.

Claim 3: u is a minimizer for m_λ.

It follows from (34) and the definition of M_λ that $M_\lambda = T_\lambda(u)$. Hence, we prove that u is a minimizer for M_λ. Furthermore, we see from $N_\lambda(u) = 0$ that

$$m_\lambda = M_\lambda = T_\lambda(u) = \varphi_\lambda(u).$$

Hence, we conclude that u is a minimizer for m_λ.

Thus, Theorem 1.1 have been proved.

□

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References

[1] T. Akahori, S. Ibrahim, N. Ikoma, H. Kikuchi, and H. Nawa, *Uniqueness and nondegeneracy of ground states to nonlinear scalar field equations involving the sobolev critical exponent in their nonlinearities for high frequencies*, Calculus of Variations and Partial Differential Equations, 58 (2019), pp. 1–32.

[2] C. O. Alves, *Existence of positive solutions for a problem with lack of compactness involving the p-laplacian*, Nonlinear Analysis: Theory, Methods & Applications, 51 (2002), pp. 1187–1206.

[3] G. Arioli and F. Gazzola, *Some results on p-laplace equations with a critical growth term*, Differential and Integral Equations, 11 (1998), pp. 311–326.

[4] T. Aubin, *Problemes isopérimétriques et espaces de sobolev*, Journal of differential geometry, 11 (1976), pp. 573–598.

[5] G. Barletta, P. Candito, S. A. Marano, and K. Perera, *Multiplicity results for critical p-laplacian problems*, Annali di Matematica Pura ed Applicata, 196 (2017), pp. 1431–1440.

[6] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical sobolev exponents*, Communications on Pure and Applied Mathematics, 36 (1983), pp. 437–477.

[7] L. A. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth*, Communications on Pure and Applied Mathematics, 42 (1989), pp. 271–297.

[8] M. Clapp and L. L. Rios, *Entire nodal solutions to the pure critical exponent problem for the p-laplacian*, Journal of Differential Equations, 265 (2018), pp. 891–905.

[9] M. Clapp and T. Weth, *Multiple solutions of nonlinear scalar field equations*, Communications in Partial Differential Equations, 29 (2005), pp. 1533–1554.

[10] D. G. Costa, *An invitation to variational methods in differential equations*, Springer Science & Business Media, 2010.

[11] L. Damascelli, S. Merchán, L. Montoro, and B. Sciunzi, *Radial symmetry and applications for a problem involving the-$\triangle_p(\cdot)$ operator and critical nonlinearity in R^N*, Advances in Mathematics, 265 (2014), pp. 313–335.

[12] M. Degiovanni and S. Lancelotti, *Linking solutions for p-laplace equations with nonlinearity at critical growth*, Journal of Functional Analysis, 256 (2009), pp. 3643–3659.

[13] L. C. Evans, *Partial differential equations*, vol. 19, American Mathematical Society, 2010.

[14] J. Garcia Azorero and I. Peral Alonso, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Transactions of the American Mathematical Society, 323 (1991), pp. 877–895.

[15] F. Gazzola and B. Ruf, *Lower-order perturbations of critical growth nonlinearities in semilinear elliptic equations*, Advances in Differential Equations, 2 (1997), pp. 555–572.

[16] S. Kesavan, *Symmetrization and applications*, vol. 3, World Scientific, 2006.

[17] C. Mercuri, B. Sciunzi, and M. Squassina, *On coron's problem for the p-laplacian*, Journal of Mathematical Analysis and Applications, 421 (2015), pp. 362–369.

[18] M. Moussa, *Schwarz rearrangement does not decrease the energy for the pseudo p-laplacian operator*, Bol. Soc. Parana. Mat.(3), 29 (2011), pp. 49–53.

[19] K. Perera, *A multiplicity result for the scalar field equation*, Advances in Nonlinear Analysis, 3 (2014), pp. 47–54.

[20] M. H. Poincar, *Sur les équations de la physique mathmatique*, Rendiconti del Circolo Matematico di Palermo (1884-1940), 8 (2008), pp. 57–155.

[21] B. Sciunzi, *Classification of positive $D^{1,p}(R^N)$-solutions to the critical p-laplace equation in R^N*, Advances in Mathematics, 291 (2016), pp. 12–23.
[22] A. Szulkin and T. Weth, *The method of nehari manifold*, Handbook of nonconvex analysis and applications, (2010), pp. 597–632.
[23] G. Talenti, *Best constant in sobolev inequality*, Annali di Matematica pura ed Applicata, 110 (1976), pp. 353–372.
[24] J. Vétois, *A priori estimates and application to the symmetry of solutions for critical p-laplace equations*, Journal of Differential Equations, 260 (2016), pp. 149–161.
[25] M. Willem, *Minimax theorems*, vol. 24, Springer Science & Business Media, 1997.

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