ON LIE DERIVATIONS OF TRIANGULAR ALGEBRAS

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Abstract. Let $\mathcal{T}$ be a triangular algebra and $G \in \mathcal{T}$ be an arbitrary but fixed point. We say that a linear map $\phi : \mathcal{T} \to \mathcal{T}$ is Lie derivable at G if $\phi([S,T]) = [\phi(S),T] + [S,\phi(T)]$ for any $S,T \in \mathcal{T}$ with $ST = G$. In this paper, we describe the general form of ϕ and give some necessary and sufficient conditions for ϕ to be proper. These results are then applied to nest algebras.

1. INTRODUCTION

Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$. A linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if $\delta(ST) = \delta(S)T + S\delta(T)$ for all $S,T \in \mathcal{A}$. Derivations are very important maps both in theory and applications, and have been studied intensively(for example, see [5, 9, 18, 19, 20] and references therein). In recent years the problem under which conditions that a linear map becomes a derivation has attracted many mathematicians' attention. One direction in the study of this problem is to characterize derivable maps at a given point. We say that a linear map $\delta : \mathcal{A} \to \mathcal{A}$ is derivable at $G \in \mathcal{A}$ if $\delta(ST) = \delta(S)T + S\delta(T)$ for all $S,T \in \mathcal{A}$ with $ST = G$. It is obvious that the condition of δ being a derivable map at G is weaker than the condition of being a derivation. For some algebras, the cases that G is zero, the unit, nontrivial idempotents, invertible elements, and so on were discussed by several authors (for example, refer to [1, 2, 12, 13, 15, 17, 22] and the references therein). These results enhanced people's understanding of derivations on several algebras.

More generally, a linear map $\phi : \mathcal{A} \to \mathcal{A}$ is called a Lie derivation if $\phi([S,T]) = [\phi(S),T] + [S,\phi(T)]$ for all $S,T \in \mathcal{A}$, here $[S,T] = ST - TS$ is the usual Lie product and is said to be Lie derivable at $G \in \mathcal{A}$ if $\phi([S,T]) = [\phi(S),T]+[S,\phi(T)]$ for all $S,T \in \mathcal{T}$ with $ST = G$. Ji and Qi in [11] proved that, under some mild conditions, if a linear map ϕ from a triangular algebra $\mathcal{T}$ into itself is Lie derivable at 0 (resp. a standard idempotent $P \in \mathcal{T}$), then ϕ is proper, that is, $\phi = \varphi + \tau$, where φ is a derivation of $\mathcal{T}$ and τ is a linear map from $\mathcal{T}$ into the center of $\mathcal{T}$ vanishing on each commutator $[S,T]$ whenever $ST = 0$ (resp. $ST = P$). Lu and Jing [14] showed that the same is true for Lie derivable maps of $B(X)$, where $B(X)$ is the algebra of all bounded linear operators on a Banach space X of dimension greater than 2. There are similar results on generalized matrix algebras and prime rings (see [8, 16]). But, so far, to the best of our knowledge there have been no papers on the study of Lie derivable maps at a given point

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which is not 0 or a standard idempotent. So the purpose of the present paper is to characterize Lie derivable maps of triangular algebras at an arbitrary but fixed point.

Triangular algebras were firstly introduced in [6] and then studied by many authors(see, for example, [3, 4, 7, 21]). Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras with unit I_1 and I_2, respectively. Let $\mathcal{M}$ be an ($\mathcal{A}$, $\mathcal{B}$)-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module, that is, for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $a\mathcal{M} = \{0\}$ and $\mathcal{M}b = \{0\}$ imply $a = 0$ and $b = 0$, respectively. The algebra

$$\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\},$$

under the usual matrix operations is called a triangular algebra. The non-trivial idempotent element $P = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$ is said to be the standard idempotent of $\mathcal{T}$. It follows from Proposition 3 in [6] that the center of $\mathcal{T}$ is

$$\mathcal{Z}(\mathcal{T}) = \left\{ a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| am = mb \ for\ all\ m \in \mathcal{M} \right\}.$$

Define two projections $\pi_{\mathcal{A}} : \mathcal{T} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{T} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a \text{ and } \pi_{\mathcal{B}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b.$$

Then $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) \subseteq \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) \subseteq \mathcal{Z}(\mathcal{B})$. Furthermore, there exists an unique algebra isomorphism $\eta : \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T}))$ such that $am = m\eta(a)$ for every $a \in \mathcal{A}$, $m \in \mathcal{M}$.

2. Result and proof

In this section, we consider the question of characterizing Lie derivable maps of triangular algebras at an arbitrary but fixed point. Firstly, we describe the general forms of these maps as following.

Theorem 2.1. *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $G = \begin{pmatrix} a_0 & m_0 \\ 0 & b_0 \end{pmatrix}$ be an arbitrary but fixed point in $\mathcal{T}$. Assume that*

(i) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$;

(ii) For every $a \in \mathcal{A}$, there exists some integer n such that $nI_1 - a$ is invertible in $\mathcal{A}$;

(iii) For every $b \in \mathcal{B}$, there exists some integer n such that $nI_2 - b$ is invertible in $\mathcal{B}$.

If a linear map $\phi : \mathcal{T} \to \mathcal{T}$ is Lie derivable at G, then ϕ is of the form

$$\phi \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} f_{11}(a) + g_{11}(m) + h_{11}(b) & af_{12}(I_1) - f_{12}(I_1)b + g_{12}(m) \\ 0 & f_{22}(a) + g_{22}(m) + h_{22}(b) \end{pmatrix},$$

where

$$f_{11} : \mathcal{A} \to \mathcal{A},\ f_{12} : \mathcal{A} \to \mathcal{M},\ f_{22} : \mathcal{A} \to \mathcal{Z}(\mathcal{B}),\ g_{11} : \mathcal{M} \to \mathcal{Z}(\mathcal{A}),$$
$$g_{12} : \mathcal{M} \to \mathcal{M},\ g_{22} : \mathcal{M} \to \mathcal{Z}(\mathcal{B}),\ h_{11} : \mathcal{B} \to \mathcal{Z}(\mathcal{A}),\ h_{22} : \mathcal{B} \to \mathcal{B}$$

are linear maps satisfying the following conditions:

(1) $f_{12}(a) = af_{12}(I_1)$ *for all* $a \in \mathcal{A}$*;*

(2) $f_{11}(I_1) \in \mathcal{Z}(\mathcal{A})$, $h_{22}(I_2) \in \mathcal{Z}(\mathcal{B})$*;*

(3) $g_{12}(am) = ag_{12}(m) + f_{11}(a)m - mf_{22}(a) + h_{11}(I_2)am - amh_{22}(I_2)$ *for all* $m \in \mathcal{M}$ *and* $a \in \mathcal{A}$*;*

(4) $g_{12}(mb) = g_{12}(m)b + mh_{22}(b) - h_{11}(b)m + mbf_{22}(I_1) - f_{11}(I_1)mb$ *for all* $m \in \mathcal{M}$ *and* $b \in \mathcal{B}$*.*

We need the following basic fact whose proof is easy and will be skipped.

Proposition 2.2. *Let* H *be a vector space over a number field* $\mathbb{F}$*. For any fixed* $h_i \in H$, $i = 0, \pm 1, \pm 2, ..., \pm n$, *if* $\sum_{i=-n}^{n} h_i x^i = 0$, $x \in \mathbb{F}$, *has at least* $2n+1$ *distinct nonzero solutions in* $\mathbb{F}$*, then* $h_i = 0, i = 0, \pm 1, \pm 2, ..., \pm n$*.*

Proof of Theorem 2.1. Since ϕ is a linear map, we can write

$$\phi\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} f_{11}(a) + g_{11}(m) + h_{11}(b) & f_{12}(a) + g_{12}(m) + h_{12}(b) \\ 0 & f_{22}(a) + g_{22}(m) + h_{22}(b) \end{pmatrix}$$

for all $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \mathcal{T}$, where f_{11}, f_{12}, f_{22} are linear maps from $\mathcal{A}$ to $\mathcal{A}, \mathcal{M}, \mathcal{B}$, respectively; g_{11}, g_{12}, g_{22} are linear maps from $\mathcal{M}$ to $\mathcal{A}, \mathcal{M}, \mathcal{B}$, respectively; h_{11}, h_{12}, h_{22} are linear maps from $\mathcal{B}$ to $\mathcal{A}, \mathcal{M}, \mathcal{B}$, respectively.

Now we organize the proof in a series of claims

Claim 1. *For any* $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$*, the following statements hold:*

(i) $h_{12}(b) = -f_{12}(I_1)b$*;*

(ii) $f_{12}(a) = af_{12}(I_1)$*.*

For any invertible $b \in \mathcal{B}$ and any real number $\lambda > 0$, let

$$S = \begin{pmatrix} I_1 & 0 \\ 0 & \lambda b_0 b^{-1} \end{pmatrix},\ T = \begin{pmatrix} a_0 & m_0 \\ 0 & \lambda^{-1} b \end{pmatrix}.$$

Then $ST = G$, and we have

$$\begin{aligned}
&\begin{pmatrix} * & g_{12}(m_0 - \lambda m_0 b_0 b^{-1}) + h_{12}(b_0 - bb_0 b^{-1}) \\ 0 & * \end{pmatrix} \\
&= \phi(ST - TS) \\
&= \phi(S)T - T\phi(S) + S\phi(T) - \phi(T)S \\
&= \begin{pmatrix} f_{11}(I_1) + h_{11}(\lambda b_0 b^{-1}) & f_{12}(I_1) + h_{12}(\lambda b_0 b^{-1}) \\ 0 & f_{22}(I_1) + h_{22}(\lambda b_0 b^{-1}) \end{pmatrix} \begin{pmatrix} a_0 & m_0 \\ 0 & \lambda^{-1} b \end{pmatrix} \\
&- \begin{pmatrix} a_0 & m_0 \\ 0 & \lambda^{-1} b \end{pmatrix} \begin{pmatrix} f_{11}(I_1) + h_{11}(\lambda b_0 b^{-1}) & f_{12}(I_1) + h_{12}(\lambda b_0 b^{-1}) \\ 0 & f_{22}(I_1) + h_{22}(\lambda b_0 b^{-1}) \end{pmatrix} \\
&+ \begin{pmatrix} I_1 & 0 \\ 0 & \lambda b_0 b^{-1} \end{pmatrix} \\
&\cdot \begin{pmatrix} f_{11}(a_0) + g_{11}(m_0) + h_{11}(\lambda^{-1} b) & f_{12}(a_0) + g_{12}(m_0) + h_{12}(\lambda^{-1} b) \\ 0 & f_{22}(a_0) + g_{22}(m_0) + h_{22}(\lambda^{-1} b) \end{pmatrix} \\
&- \begin{pmatrix} f_{11}(a_0) + g_{11}(m_0) + h_{11}(\lambda^{-1} b) & f_{12}(a_0) + g_{12}(m_0) + h_{12}(\lambda^{-1} b) \\ 0 & f_{22}(a_0) + g_{22}(m_0) + h_{22}(\lambda^{-1} b) \end{pmatrix}
\end{aligned}$$

$$\cdot \begin{pmatrix} I_1 & 0 \\ 0 & \lambda b_0 b^{-1} \end{pmatrix} = \begin{pmatrix} * & \Delta \\ 0 & * \end{pmatrix},$$

where

$$\begin{aligned}\Delta =&(f_{11}(I_1)+h_{11}(\lambda b_0 b^{-1}))m_0+(f_{12}(I_1)+h_{12}(\lambda b_0 b^{-1}))(\lambda^{-1}b)+f_{12}(a_0)\\ &-a_0(f_{12}(I_1)+h_{12}(\lambda b_0 b^{-1}))-m_0((f_{22}(I_1)+h_{22}(\lambda b_0 b^{-1}))+g_{12}(m_0)\\ &+(h_{12}(\lambda^{-1}b)-(f_{12}(a_0)+g_{12}(m_0)+h_{12}(\lambda^{-1}b))(\lambda b_0 b^{-1}).\end{aligned}$$

It follows from the matrix equation that

$$\begin{aligned}&g_{12}(m_0-\lambda m_0 b_0 b^{-1})+h_{12}(b_0-bb_0b^{-1})\\ &=(f_{11}(I_1)+h_{11}(\lambda b_0 b^{-1}))m_0+(f_{12}(I_1)+h_{12}(\lambda b_0 b^{-1}))(\lambda^{-1}b)+f_{12}(a_0)\\ &\quad-a_0(f_{12}(I_1)+h_{12}(\lambda b_0 b^{-1}))-m_0((f_{22}(I_1)+h_{22}(\lambda b_0 b^{-1}))+g_{12}(m_0)\\ &\quad+(h_{12}(\lambda^{-1}b)-(f_{12}(a_0)+g_{12}(m_0)+h_{12}(\lambda^{-1}b))(\lambda b_0 b^{-1}).\end{aligned}$$

By Proposition 2.2, the above equality implies that $f_{12}(I_1)b+h_{12}(b)=0$ for all invertible $b\in\mathcal{B}$. For any $b\in\mathcal{B}$, by assumption (iii) of the theorem, there exists an integer n such that nI_2-b is invertible in $\mathcal{B}$. It follows from the previous fact that $f_{12}(I_1)(nI_2-b)+h_{12}(nI_2-b)=0$ for all $b\in\mathcal{B}$. Then we have

$$f_{12}(I_1)b+h_{12}(b)=0 \tag{2.1}$$

for all $b\in\mathcal{B}$.

Moreover, for any invertible $a\in\mathcal{A}$ and any real number $\lambda>0$, repeating the similar argument and considering

$$S=\begin{pmatrix} \lambda a & m_0 \\ 0 & b_0 \end{pmatrix},\ T=\begin{pmatrix} \lambda^{-1}a^{-1}a_0 & 0 \\ 0 & I_2 \end{pmatrix}.$$

one can get

$$f_{12}(a)+ah_{12}(I_2)=0 \tag{2.2}$$

for all $a\in\mathcal{A}$. Taking $b=I_2$ in Eq.(2.1), we have $h_{12}(I_2)=-f_{12}(I_1)$. Combining this and Eq.(2.2), we obtain that

$$f_{12}(a)=af_{12}(I_1)$$

for all $a\in\mathcal{A}$. The proof of the claim is completed.

Claim 2. We have the following:
(i) $h_{11}(b)\in\mathcal{Z}(\mathcal{A})$ for all $b\in\mathcal{B}$;
(ii) $f_{22}(a)\in\mathcal{Z}(\mathcal{B})$ for all $a\in\mathcal{A}$.

Taking

$$S=\begin{pmatrix} \lambda a & 0 \\ 0 & \lambda^{-1}b_0 b^{-1} \end{pmatrix},\ T=\begin{pmatrix} \lambda^{-1}a^{-1}a_0 & \lambda^{-1}a^{-1}m_0 \\ 0 & \lambda b \end{pmatrix},$$

where $a\in\mathcal{A}$, $b\in\mathcal{B}$ are invertible and $\lambda>0$, we get $ST=G$. Hence we have

$$\begin{aligned}&\begin{pmatrix} \Delta_1 & * \\ 0 & \Delta_2 \end{pmatrix}\\ &=\phi(ST-TS)\\ &=\phi(S)T-T\phi(S)+S\phi(T)-\phi(T)S\end{aligned}$$

$$= \begin{pmatrix} f_{11}(\lambda a) + h_{11}(\lambda^{-1}b_0b^{-1}) & f_{12}(\lambda a) + h_{12}(\lambda^{-1}b_0b^{-1}) \\ 0 & f_{22}(\lambda a) + h_{22}(\lambda^{-1}b_0b^{-1}) \end{pmatrix} \begin{pmatrix} \lambda^{-1}a^{-1}a_0 & \lambda^{-1}a^{-1}m_0 \\ 0 & \lambda b \end{pmatrix}$$

$$- \begin{pmatrix} \lambda^{-1}a^{-1}a_0 & \lambda^{-1}a^{-1}m_0 \\ 0 & \lambda b \end{pmatrix} \begin{pmatrix} f_{11}(\lambda a) + h_{11}(\lambda^{-1}b_0b^{-1}) & f_{12}(\lambda a) + h_{12}(\lambda^{-1}b_0b^{-1}) \\ 0 & f_{22}(\lambda a) + h_{22}(\lambda^{-1}b_0b^{-1}) \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda a & 0 \\ 0 & \lambda^{-1}b_0b^{-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b) & * \\ 0 & f_{22}(\lambda^{-1}a^{-1}a_0) + g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b) \end{pmatrix}$$

$$- \begin{pmatrix} f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b) & * \\ 0 & f_{22}(\lambda^{-1}a^{-1}a_0) + g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b) \end{pmatrix}$$

$$\cdot \begin{pmatrix} \lambda a & 0 \\ 0 & \lambda^{-1}b_0b^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \Delta_3 & * \\ 0 & \Delta_4 \end{pmatrix},$$

where

$$\Delta_1 = f_{11}(a_0 - a^{-1}a_0a) + g_{11}(m_0 - \lambda^{-2}a^{-1}m_0b_0b^{-1}) + h_{11}(b_0 - bb_0b^{-1}),$$
$$\Delta_2 = f_{22}(a_0 - a^{-1}a_0a) + g_{22}(m_0 - \lambda^{-2}a^{-1}m_0b_0b^{-1}) + h_{22}(b_0 - bb_0b^{-1}),$$

$$\begin{aligned}\Delta_3 =& (f_{11}(\lambda a) + h_{11}(\lambda^{-1}b_0b^{-1}))(\lambda^{-1}a^{-1}a_0) - (\lambda^{-1}a^{-1}a_0)(f_{11}(\lambda a) + h_{11}(\lambda^{-1}b_0b^{-1}) \\ &+ \lambda a(f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b)) - (f_{11}(\lambda^{-1}a^{-1}a_0) \\ &+ g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b))\lambda a\end{aligned}$$

and

$$\begin{aligned}\Delta_4 =& (f_{22}(\lambda a) + h_{22}(\lambda^{-1}b_0b^{-1}))(\lambda b) - (\lambda b)(f_{22}(\lambda a) + h_{22}(\lambda^{-1}b_0b^{-1})) \\ &+ (\lambda^{-1}b_0b^{-1})(f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(\lambda^{-1}a^{-1}m_0) + h_{22}(\lambda b)) \\ &- (f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(\lambda^{-1}a^{-1}m_0) + h_{22}(\lambda b))(\lambda^{-1}b_0b^{-1}).\end{aligned}$$

It follows from the matrix equation that $\Delta_1 = \Delta_3$, that is,

$$\begin{aligned}& f_{11}(a_0 - a^{-1}a_0a) + g_{11}(m_0 - \lambda^{-2}a^{-1}m_0b_0b^{-1} + h_{11}(b_0 - bb_0b^{-1})) \\ &= (f_{11}(\lambda a) + h_{11}(\lambda^{-1}b_0b^{-1}))(\lambda^{-1}a^{-1}a_0) - (\lambda^{-1}a^{-1}a_0)(f_{11}(\lambda a) + h_{11}(\lambda^{-1}b_0b^{-1}) \\ &\quad + \lambda a(f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b)) - (f_{11}(\lambda^{-1}a^{-1}a_0) \\ &\quad + g_{11}(\lambda^{-1}a^{-1}m_0) + h_{11}(\lambda b))\lambda a.\end{aligned}$$

Using Proposition 2.2, it is easy to see $h_{11}(b)a = ah_{11}(b)$ for all invertible $a \in \mathcal{A}$ and invertible $b \in \mathcal{B}$. By the assumption (iii), for any $b \in \mathcal{B}$, there exists an integer n such that $nI_2 - b$ is invertible in $\mathcal{B}$. So the preceding case yields $h_{11}(nI_2 - b)a = ah_{11}(nI_2 - b)$, which leads to $h_{11}(b)a = ah_{11}(b)$ for all invertible $a \in \mathcal{A}$ and all $b \in \mathcal{B}$. For any $a \in \mathcal{A}$, by the assumption (ii), there exists an integer n such that $nI_1 - a$ is invertible. Thus, we get $h_{11}(b)(nI_1 - a) = (nI_1 - a)h_{11}(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This implies

$$h_{11}(b)a = ah_{11}(b) \tag{2.3}$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Therefore, we obtain that $h_{11}(b) \in \mathcal{Z}(\mathcal{A})$ for all $b \in \mathcal{B}$.

Moreover, by the matrix equation, we have $\Delta_2 = \Delta_4$, that is,

$$\begin{aligned}
&f_{22}(a_0 - a^{-1}a_0 a) + g_{22}(m_0 - \lambda^{-2}a^{-1}m_0 b_0 b^{-1}) + h_{22}(b_0 - b b_0 b^{-1}) \\
&= (f_{22}(\lambda a) + h_{22}(\lambda^{-1}b_0 b^{-1}))(\lambda b) - (\lambda b)(f_{22}(\lambda a) + h_{22}(\lambda^{-1}b_0 b^{-1})) \\
&\quad + (\lambda^{-1}b_0 b^{-1})(f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(\lambda^{-1}a^{-1}m_0) + h_{22}(\lambda b) \\
&\quad - (f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(\lambda^{-1}a^{-1}m_0) + h_{22}(\lambda b))(\lambda^{-1}b_0 b^{-1}).
\end{aligned}$$

With the same argument as the proof of Eq.(2.3), we see that

$$f_{22}(a)b = b f_{22}(a)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, which means $f_{22}(a) \in \mathcal{Z}(\mathcal{B})$ for all $a \in \mathcal{A}$.

Claim 3. *Let $a \in \mathcal{A}$ and $m \in \mathcal{M}$. Then*
(i) $g_{11}(m) \in \mathcal{Z}(\mathcal{A})$;
(ii)$f_{11}(I_1) \in \mathcal{Z}(\mathcal{A})$, $h_{22}(I_2) \in \mathcal{Z}(\mathcal{B})$;
(iii) $g_{12}(am) = a g_{12}(m) + f_{11}(a)m - m f_{22}(a) + h_{11}(I_2)am - am h_{22}(I_2)$.

For any $m \in \mathcal{M}$, any invertible $a \in \mathcal{A}$ and any real number $\lambda > 0$, set

$$S = \begin{pmatrix} \lambda a & m_0 - \lambda a m \\ 0 & b_0 \end{pmatrix}, \quad T = \begin{pmatrix} \lambda^{-1}a^{-1}a_0 & m \\ 0 & I_2 \end{pmatrix}.$$

Then $ST = G$, and we have

$$\begin{aligned}
&\begin{pmatrix} \Delta_1 & \Delta_2 \\ 0 & * \end{pmatrix} \\
&= \phi(ST - TS) \\
&= \phi(S)T - T\phi(S) + S\phi(T) - \phi(T)S \\
&= \begin{pmatrix} f_{11}(\lambda a) + g_{11}(m_0 - \lambda a m) + h_{11}(b_0) & f_{12}(\lambda a) + g_{12}(m_0 - \lambda a m) + h_{12}(b_0) \\ 0 & f_{22}(\lambda a) + g_{22}(m_0 - \lambda a m) + h_{22}(b_0) \end{pmatrix} \\
&\cdot \begin{pmatrix} \lambda^{-1}a^{-1}a_0 & m \\ 0 & I_2 \end{pmatrix} \\
&- \begin{pmatrix} \lambda^{-1}a^{-1}a_0 & m \\ 0 & I_2 \end{pmatrix} \\
&\cdot \begin{pmatrix} f_{11}(\lambda a) + g_{11}(m_0 - \lambda a m) + h_{11}(b_0) & f_{12}(\lambda a) + g_{12}(m_0 - \lambda a m) + h_{12}(b_0) \\ 0 & f_{22}(\lambda a) + g_{22}(m_0 - \lambda a m) + h_{22}(b_0) \end{pmatrix} \\
&+ \begin{pmatrix} \lambda a & m_0 - \lambda a m \\ 0 & b_0 \end{pmatrix} \\
&\cdot \begin{pmatrix} f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2) & f_{12}(\lambda^{-1}a^{-1}a_0) + g_{12}(m) + h_{12}(I_2) \\ 0 & f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(m) + h_{22}(I_2) \end{pmatrix} \\
&- \begin{pmatrix} f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2) & f_{12}(\lambda^{-1}a^{-1}a_0) + g_{12}(m) + h_{12}(I_2) \\ 0 & f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(m) + h_{22}(I_2) \end{pmatrix} \\
&\cdot \begin{pmatrix} \lambda a & m_0 - \lambda a m \\ 0 & b_0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \Delta_3 & \Delta_4 \\ 0 & * \end{pmatrix},$$

where

$$\Delta_1 = f_{11}(a_0 - a^{-1}a_0a) + g_{11}(m_0 - \lambda^{-1}a^{-1}a_0m_0 + a^{-1}a_0am - mb_0),$$
$$\Delta_2 = f_{12}(a_0 - a^{-1}a_0a) + g_{12}(m_0 - \lambda^{-1}a^{-1}a_0m_0 + a^{-1}a_0am - mb_0),$$

$$\begin{aligned}\Delta_3 =&\lambda^{-1}(f_{11}(\lambda a) + g_{11}(m_0 - \lambda am) + h_{11}(b_0))a^{-1}a_0 \\ &- \lambda^{-1}a^{-1}a_0(f_{11}(\lambda a) + g_{11}(m_0 - \lambda am) + h_{11}(b_0)) \\ &+ \lambda a(f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2)) \\ &- \lambda(f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2))a\end{aligned}$$

and

$$\begin{aligned}\Delta_4 =&(f_{11}(\lambda a) + g_{11}(m_0 - \lambda am) + h_{11}(b_0))m + f_{12}(\lambda a) \\ &+ g_{12}(m_0 - \lambda am) + h_{12}(b_0) - f_{12}(\lambda^{-1}a^{-1}a_0)b_0 \\ &- \lambda^{-1}a^{-1}a_0(f_{12}(\lambda a) + g_{12}(m_0 - \lambda am) + h_{12}(b_0)) \\ &- m(f_{22}(\lambda a) + g_{22}(m_0 - \lambda am) + h_{22}(b_0)) + g_{12}(m)b_0 \\ &+ \lambda a(f_{12}(\lambda^{-1}a^{-1}a_0) + g_{12}(m) + h_{12}(I_2)) + h_{12}(I_2)b_0 \\ &+ (m_0 - \lambda am)(f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(m) + h_{22}(I_2)) \\ &- (f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2))(m_0 - \lambda am).\end{aligned}$$

Thus, by the matrix equation, we have $\Delta_1 = \Delta_3$, that is,

$$\begin{aligned}&f_{11}(a_0 - a^{-1}a_0a) + g_{11}(m_0 - \lambda^{-1}a^{-1}a_0m_0 + a^{-1}a_0am - mb_0) \\ &= \lambda^{-1}(f_{11}(\lambda a) + g_{11}(m_0 - \lambda am) + h_{11}(b_0))a^{-1}a_0 \\ &\ - \lambda^{-1}a^{-1}a_0(f_{11}(\lambda a) + g_{11}(m_0 - \lambda am) + h_{11}(b_0)) \\ &\ + \lambda a(f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2)) \\ &\ - \lambda(f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2))a,\end{aligned}$$

which implies

$$ag_{11}(m) + ah_{11}(I_2) - g_{11}(m)a - h_{11}(I_2)a = 0 \tag{2.4}$$

for all invertible $a \in \mathcal{A}$ and all $m \in \mathcal{M}$ since Proposition 2.2. Taking $m = 0$ in Eq.(2.4), we get $ah_{11}(I_2) - h_{11}(I_2)a = 0$. Combining this and Eq.(2.4) reduces to $ag_{11}(m) = g_{11}(m)a$ for all invertible $a \in \mathcal{A}$ and all $m \in \mathcal{M}$. By assumption (ii), for any $a \in \mathcal{A}$, there exists an integer n such that $nI_1 - a$ is invertible in $\mathcal{A}$. So we have $(nI_1 - a)g_{11}(m) = g_{11}(m)(nI_1 - a)$. This leads to

$$ag_{11}(m) = g_{11}(m)a$$

for all $a \in \mathcal{A}$, which means $g_{11}(m) \in \mathcal{Z}(\mathcal{A})$.

Furthermore, it follows from the matrix equation that $\Delta_2 = \Delta_4$, that is,

$$\begin{aligned}&f_{12}(a_0 - a^{-1}a_0a) + g_{12}(m_0 - \lambda^{-1}a^{-1}a_0m_0 + a^{-1}a_0am - mb_0) \\ &= (f_{11}(\lambda a) + g_{11}(m_0 - \lambda am) + h_{11}(b_0))m + f_{12}(\lambda a)\end{aligned}$$

$$
\begin{aligned}
&+ g_{12}(m_0 - \lambda am) + h_{12}(b_0) - f_{12}(\lambda^{-1}a^{-1}a_0)b_0 \\
&- (\lambda^{-1}a^{-1}a_0)(f_{12}(\lambda a) + g_{12}(m_0 - \lambda am) + h_{12}(b_0)) \\
&- m(f_{22}(\lambda a) + g_{22}(m_0 - \lambda am) + h_{22}(b_0)) + g_{12}(m)b_0 \\
&+ \lambda a(f_{12}(\lambda^{-1}a^{-1}a_0) + g_{12}(m) + h_{12}(I_2)) + h_{12}(I_2)b_0 \\
&+ (m_0 - \lambda am)(f_{22}(\lambda^{-1}a^{-1}a_0) + g_{22}(m) + h_{22}(I_2)) \\
&- (f_{11}(\lambda^{-1}a^{-1}a_0) + g_{11}(m) + h_{11}(I_2))(m_0 - \lambda am),
\end{aligned}
$$

which yields

$$
\begin{aligned}
0 =& f_{11}(a)m - g_{11}(am)m + f_{12}(a) - g_{12}(am) - mf_{22}(a) + mg_{22}(am) + ag_{12}(m) \\
&+ ah_{12}(I_2) - amg_{22}(m) - amh_{22}(I_2) + g_{11}(m)am + h_{11}(I_2)am
\end{aligned}
$$

for all invertible $a \in \mathcal{A}$ and all $m \in \mathcal{M}$. Combining this and Eq. (2.2), we have

$$
\begin{aligned}
0 =& f_{11}(a)m - g_{11}(am)m - g_{12}(am) - mf_{22}(a) + mg_{22}(am) + ag_{12}(m) \\
&- amg_{22}(m) - amh_{22}(I_2) + g_{11}(m)am + h_{11}(I_2)am
\end{aligned}
$$

for all invertible $a \in \mathcal{A}$ and all $m \in \mathcal{M}$. Replacing m with $-m$ in the above equality, we arrive at

$$
\begin{aligned}
0 =& - f_{11}(a)m - g_{11}(am)m + g_{12}(am) + mf_{22}(a) + mg_{22}(am) - ag_{12}(m) \\
&- amg_{22}(m) + amh_{22}(I_2) + g_{11}(m)am + h_{11}(I_2)am.
\end{aligned}
$$

Comparing these two equalities, we get

$$0 = f_{11}(a)m - g_{12}(am) - mf_{22}(a) + ag_{12}(m) - amh_{22}(I_2) + h_{11}(I_2)am$$

for all invertible $a \in \mathcal{A}$ and all $m \in \mathcal{M}$. By assumption (ii) again, there exists n such that $nI_1 - a$ is invertible in $\mathcal{A}$. Then one can easily check that

$$g_{12}(am) = ag_{12}(m) + f_{11}(a)m - mf_{22}(a) + h_{11}(I_2)am - amh_{22}(I_2) \tag{2.5}$$

for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$.

Moreover, letting $a = I_1$ in Eq.(2.5), we get

$$(f_{11}(I_1) + h_{11}(I_2))m = m(f_{22}(I_1) + h_{22}(I_2)) \tag{2.6}$$

for all $m \in \mathcal{M}$. By Claim 2 (i) and the assumption (i) of the theorem, there exists $\eta(h_{11}(I_2)) \in \mathcal{Z}(\mathcal{B})$ such that

$$h_{11}(I_2)m = m\eta(h_{11}(I_2)) \tag{2.7}$$

for all $m \in \mathcal{M}$. It follows from Eqs. (2.6) and (2.7) that

$$f_{11}(I_1)m = m(f_{22}(I_1) + h_{22}(I_1) - \eta(h_{11}(I_2)))$$

for all $m \in \mathcal{M}$, which implies $f_{11}(I_1) \in \mathcal{Z}(\mathcal{A})$.

Similarly, one can verify that $h_{22}(I_2) \in \mathcal{Z}(\mathcal{B})$.

Claim 4. *For any $m \in \mathcal{M}$ and $b \in \mathcal{B}$, we claim that*

(i) $g_{22}(m) \in \mathcal{Z}(\mathcal{B})$;

(ii) $g_{12}(mb) = g_{12}(m)b + mh_{22}(b) - h_{11}(b)m + mbf_{22}(I_1) - f_{11}(I_1)mb$.

For any invertible $b \in \mathcal{B}$, $m \in \mathcal{M}$ and real number $\lambda > 0$, putting

$$S = \begin{pmatrix} I_1 & m \\ 0 & \lambda^{-1} b_0 b^{-1} \end{pmatrix},\ T = \begin{pmatrix} a_0 & m_0 - \lambda m b \\ 0 & \lambda b \end{pmatrix},$$

we have $ST = G$. Then

$$\begin{aligned}
&\begin{pmatrix} * & g_{12}(m_0 - a_0 m - \lambda^{-1} m_0 b_0 b^{-1} + m b b_0 b^{-1}) + h_{12}(b_0 - b b_0 b^{-1}) \\ 0 & g_{22}(m_0 - a_0 m - \lambda^{-1} m_0 b_0 b^{-1} + m b b_0 b^{-1}) + h_{22}(b_0 - b b_0 b^{-1}) \end{pmatrix} \\
&= \phi(ST - TS) \\
&= \phi(S)T - T\phi(S) + S\phi(T) - \phi(T)S \\
&= \begin{pmatrix} f_{11}(I_1) + g_{11}(m) + h_{11}(\lambda^{-1} b_0 b^{-1}) & f_{12}(I_1) + g_{12}(m) + h_{12}(\lambda^{-1} b_0 b^{-1}) \\ 0 & f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1} b_0 b^{-1}) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} a_0 & m_0 - \lambda m b \\ 0 & \lambda b \end{pmatrix} \\
&\quad - \begin{pmatrix} a_0 & m_0 - \lambda m b \\ 0 & \lambda b \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} f_{11}(I_1) + g_{11}(m) + h_{11}(\lambda^{-1} b_0 b^{-1}) & f_{12}(I_1) + g_{12}(m) + h_{12}(\lambda^{-1} b_0 b^{-1}) \\ 0 & f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1} b_0 b^{-1}) \end{pmatrix} \\
&\quad + \begin{pmatrix} I_1 & m \\ 0 & \lambda^{-1} b_0 b^{-1} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} f_{11}(a_0) + g_{11}(m_0 - \lambda m b) + h_{11}(\lambda b) & f_{12}(a_0) + g_{12}(m_0 - \lambda m b) + h_{12}(\lambda b) \\ 0 & f_{22}(a_0) + g_{22}(m_0 - \lambda m b) + h_{22}(\lambda b) \end{pmatrix} \\
&\quad - \begin{pmatrix} f_{11}(a_0) + g_{11}(m_0 - \lambda m b) + h_{11}(\lambda b) & f_{12}(a_0) + g_{12}(m_0 - \lambda m b) + h_{12}(\lambda b) \\ 0 & f_{22}(a_0) + g_{22}(m_0 - \lambda m b) + h_{22}(\lambda b) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} I_1 & m \\ 0 & \lambda^{-1} b_0 b^{-1} \end{pmatrix} \\
&= \begin{pmatrix} * & \Delta_1 \\ 0 & \Delta_2 \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 =& (f_{11}(I_1) + g_{11}(m) + h_{11}(\lambda^{-1} b_0 b^{-1}))(m_0 - \lambda m b) \\
&+ \lambda(f_{12}(I_1) + g_{12}(m) + h_{12}(\lambda^{-1} b_0 b^{-1}))b \\
&- a_0(f_{12}(I_1) + g_{12}(m) + h_{12}(\lambda^{-1} b_0 b^{-1})) \\
&- (m_0 - \lambda m b)(f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1} b_0 b^{-1})) \\
&+ f_{12}(a_0) + g_{12}(m_0 - \lambda m b) + h_{12}(\lambda b) \\
&+ m(f_{22}(a_0) + g_{22}(m_0 - \lambda m b) + h_{22}(\lambda b)) \\
&- (f_{11}(a_0) + g_{11}(m_0 - \lambda m b) + h_{11}(\lambda b))m \\
&- \lambda^{-1}(f_{12}(a_0) + g_{12}(m_0 - \lambda m b) + h_{12}(\lambda b)) b_0 b^{-1}
\end{aligned}$$

and

$$\Delta_2 = \lambda(f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1} b_0 b^{-1}))b$$

$$\begin{aligned}
&- \lambda b(f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1}b_0b^{-1}))\\
&+ \lambda^{-1}b_0b^{-1}(f_{22}(a_0) + g_{22}(m_0 - \lambda mb) + h_{22}(\lambda b))\\
&- \lambda^{-1}(f_{22}(a_0) + g_{22}(m_0 - \lambda mb) + h_{22}(\lambda b))b_0b^{-1}.
\end{aligned}$$

By the matrix equation, we arrive at

$$\begin{aligned}
&g_{22}(m_0 - a_0m - \lambda^{-1}m_0b_0b^{-1} + mbb_0b^{-1}) + h_{22}(b_0 - bb_0b^{-1})\\
&= \lambda(f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1}b_0b^{-1}))b\\
&- \lambda b(f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1}b_0b^{-1}))\\
&+ \lambda^{-1}b_0b^{-1}(f_{22}(a_0) + g_{22}(m_0 - \lambda mb) + h_{22}(\lambda b))\\
&- \lambda^{-1}(f_{22}(a_0) + g_{22}(m_0 - \lambda mb) + h_{22}(\lambda b))b_0b^{-1}.
\end{aligned}$$

It follows from Proposition 2.2 and Claim 2 (ii) that $g_{22}(m)b = bg_{22}(m)$ for all invertible $b \in \mathcal{B}$ and all $m \in \mathcal{M}$. For any $b \in \mathcal{B}$, by assumption (iii), there exists an integer n such that $nI_2 - b$ is invertible in $\mathcal{B}$. So we have $g_{22}(m)(nI_1 - b) = (nI_1 - b)g_{22}(m)$ for all $b \in \mathcal{B}$ and $m \in \mathcal{M}$. This leads to

$$g_{22}(m)b = bg_{22}(m)$$

for all $b \in \mathcal{B}$ and $m \in \mathcal{M}$. Hence $g_{22}(m) \in \mathcal{Z}(\mathcal{B})$ for all $m \in \mathcal{M}$.

Similarly, it follows from the matrix equation that

$$\begin{aligned}
&g_{12}(m_0 - a_0m - \lambda^{-1}m_0b_0b^{-1} + mbb_0b^{-1}) + h_{12}(b_0 - bb_0b^{-1})\\
&= (f_{11}(I_1) + g_{11}(m) + h_{11}(\lambda^{-1}b_0b^{-1}))(m_0 - \lambda mb)\\
&+ \lambda(f_{12}(I_1) + g_{12}(m) + h_{12}(\lambda^{-1}b_0b^{-1}))b\\
&- a_0(f_{12}(I_1) + g_{12}(m) + h_{12}(\lambda^{-1}b_0b^{-1}))\\
&- (m_0 - \lambda mb)(f_{22}(I_1) + g_{22}(m) + h_{22}(\lambda^{-1}b_0b^{-1}))\\
&+ (f_{12}(a_0) + g_{12}(m_0 - \lambda mb) + h_{12}(\lambda b))\\
&+ m(f_{22}(a_0) + g_{22}(m_0 - \lambda mb) + h_{22}(\lambda b))\\
&- (f_{11}(a_0) + g_{11}(m_0 - \lambda mb) + h_{11}(\lambda b))m\\
&- \lambda^{-1}(f_{12}(a_0) + g_{12}(m_0 - \lambda mb) + h_{12}(\lambda b))b_0b^{-1},
\end{aligned}$$

which yields

$$\begin{aligned}
0 =& - f_{11}(I_1)mb - g_{11}(m)mb + f_{12}(I_1)b + g_{12}(m)b + mbf_{22}(I_1) + mbg_{22}(m)\\
&- g_{12}(mb) + h_{12}(b) - mg_{22}(mb) + mh_{22}(b) + g_{11}(mb)m - h_{11}(b)m.
\end{aligned}$$

Replacing m with $-m$, we arrive at

$$\begin{aligned}
0 =& f_{11}(I_1)mb - g_{11}(m)mb + f_{12}(I_1)b - g_{12}(m)b - mbf_{22}(I_1) + mbg_{22}(m)\\
&+ g_{12}(mb) + h_{12}(b) - mg_{22}(mb) - mh_{22}(b) + g_{11}(mb)m + h_{11}(b)m.
\end{aligned}$$

Comparing these two equalities, we have

$$g_{12}(mb) = g_{12}(m)b + mh_{22}(b) - h_{11}(b)m + mbf_{22}(I_1) - f_{11}(I_1)mb$$

for all invertible $b \in \mathcal{B}$ and all $m \in \mathcal{M}$. For any $b \in \mathcal{B}$, by the assumption (iii) again, there exists an integer n such that $nI_2 - b$ is invertible in $\mathcal{B}$. So we get

$$\begin{aligned} g_{12}(m(nI_2 - b)) =& g_{12}(m)(nI_2 - b) + mh_{22}(nI_2 - b) - h_{11}(nI_2 - b)m \\ &+ m(nI_2 - b)f_{22}(I_1) - f_{11}(I_1)m(nI_2 - b) \end{aligned}$$

for all $b \in \mathcal{B}$ and $m \in \mathcal{M}$, which leads to

$$g_{12}(mb) = g_{12}(m)b + mh_{22}(b) - h_{11}(b)m + mbf_{22}(I_1) - f_{11}(I_1)mb$$

for all $b \in \mathcal{B}$ and $m \in \mathcal{M}$, as desired.

By Claim 1-4, the theorem holds. □

The following proposition is from the remark of Lemma 5 in [7].

Proposition 2.3. *Let δ be a derivation of triangular algebra $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Then δ is of the form*

$$\delta \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} P_{\mathcal{A}}(a) & an - nb + g(m) \\ 0 & P_{\mathcal{B}}(b) \end{pmatrix}$$

where $n \in \mathcal{M}$ and $P_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, $g : \mathcal{M} \to \mathcal{M}$, $P_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ are linear maps satisfying the following conditions:

(1) $g(am) = P_{\mathcal{A}}(a)m + ag(m)$ for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$;

(2) $g(mb) = mP_{\mathcal{B}}(b) + g(m)b$ for all $b \in \mathcal{B}$ and $m \in \mathcal{M}$.

Proposition 2.3 has the following consequence.

Proposition 2.4. *If the derivation δ of triangular algebra $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ has the following form*

$$\delta \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} P_{\mathcal{A}}(a) & 0 \\ 0 & P_{\mathcal{B}}(b) \end{pmatrix},$$

then $\delta = 0$.

Proof. By Proposition 2.3, we see $an - nb + g(m) = 0$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}$. It follows that $g(m) = 0$ for all $m \in \mathcal{M}$. Furthermore, by Proposition 2.3 (1), we get $P_{\mathcal{A}}(a)m = 0$ for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. Thus, we have $P_{\mathcal{A}}(a) = 0$ since $\mathcal{M}$ is faithful.

Similarly, one can verify $P_{\mathcal{B}}(b) = 0$ for all $b \in \mathcal{B}$. Therefore, $\delta = 0$. □

Now we have the following necessary and sufficient conditions for Lie derivable maps of triangular algebras to be proper.

Theorem 2.5. *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $G = \begin{pmatrix} a_0 & m_0 \\ 0 & b_0 \end{pmatrix}$ is an arbitrary but fixed point in $\mathcal{T}$. Assume that*

(i) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$;

(ii) For every $a \in \mathcal{A}$, there exists some integer n such that $nI_1 - a$ is invertible in $\mathcal{A}$;

(iii) For every $b \in \mathcal{B}$, there exists some integer n such that $nI_2 - b$ is invertible in $\mathcal{B}$.

Let $\phi:\mathcal{T}\to\mathcal{T}$ is Lie derivable at G. For any $\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}\in\mathcal{T}$, we write

$$\phi\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}=\begin{pmatrix} f_{11}(a)+g_{11}(m)+h_{11}(b) & af_{12}(I_1)-f_{12}(I_1)b+g_{12}(m)\\ 0 & f_{22}(a)+g_{22}(m)+h_{22}(b)\end{pmatrix}.$$

Then the following statements are equivalent:

(1) ϕ is proper;

(2) $f_{11}(I_1)\oplus f_{22}(I_1)\in\mathcal{Z}(\mathcal{T})$, $g_{11}(m)\oplus g_{22}(m)\in\mathcal{Z}(\mathcal{T})$ for all $m\in\mathcal{M}$;

(3) $h_{11}(I_2)\oplus h_{22}(I_2)\in\mathcal{Z}(\mathcal{T})$, $g_{11}(m)\oplus g_{22}(m)\in\mathcal{Z}(\mathcal{T})$ for all $m\in\mathcal{M}$.

Proof. By Theorem 2.1, we have $f_{22}(a)\in\mathcal{Z}(\mathcal{B})$ and $h_{22}(I_2)\in\mathcal{Z}(\mathcal{B})$. Thus, there exist $\eta^{-1}(f_{22}(a))\in\mathcal{Z}(\mathcal{A})$ and $\eta^{-1}(h_{22}(I_2))\in\mathcal{Z}(\mathcal{A})$ such that $\eta^{-1}(f_{22}(a))m=mf_{22}(a)$ and $\eta^{-1}(h_{22}(I_2))m=mh_{22}(I_2)$. Hence we arrive at

$$g_{12}(am)=P_{\mathcal{A}}(a)m+ag_{12}(m) \tag{2.8}$$

for all $a\in\mathcal{A}$ and $m\in\mathcal{M}$, where $P_{\mathcal{A}}(a)=f_{11}(a)-\eta^{-1}(f_{22}(a))+h_{11}(I_2)a-\eta^{-1}(h_{22}(I_2))a$. Similarly, we get

$$g_{12}(mb)=mP_{\mathcal{B}}(b)+g_{12}(m)b \tag{2.9}$$

for all $b\in\mathcal{B}$ and $m\in\mathcal{M}$, where $P_{\mathcal{B}}(b)=h_{22}(b)-\eta(h_{11}(b))+bf_{22}(I_1)-b\eta(f_{11}(I_1))$.

Next, we define three linear maps ψ, γ and τ from $\mathcal{T}$ into itself as following.

$$\psi\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}=\begin{pmatrix} P_{\mathcal{A}}(a) & af_{12}(I_1)-f_{12}(I_1)b+g_{12}(m)\\ 0 & P_{\mathcal{B}}(b)\end{pmatrix},$$

$$\gamma\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}=\begin{pmatrix} \eta^{-1}(f_{22}(a))+h_{11}(b) & 0\\ 0 & \eta(h_{11}(b))+f_{22}(a)\end{pmatrix}$$

and

$$\tau\begin{pmatrix} a & m\\ 0 & b\end{pmatrix} =\begin{pmatrix} \eta^{-1}(h_{22}(I_2))a-h_{11}(I_2)a+g_{11}(m) & 0\\ 0 & b\eta(f_{11}(I_1))-bf_{22}(I_1)+g_{22}(m)\end{pmatrix}.$$

Clearly, $\phi=\psi+\gamma+\tau$, where γ is into $\mathcal{Z}(\mathcal{T})$. By Proposition 2.3 and Eqs.(2.8)-(2.9), one can verify that ψ is a derivation of $\mathcal{T}$.

(1) $\Rightarrow$ *(2)*: If ϕ is proper, then $\phi=\widetilde{\psi}+\widetilde{\gamma}$, where $\widetilde{\psi}$ is a derivation of $\mathcal{T}$ and $\widetilde{\gamma}:\mathcal{T}\to\mathcal{Z}(\mathcal{T})$ is a linear map vanishing at commutators $[S,T]$ whenever $ST=G$. Hence $\widetilde{\psi}+\widetilde{\gamma}=\psi+\gamma+\tau$, that is, $\widetilde{\psi}-\psi=\tau+\gamma-\widetilde{\gamma}$. Note that $\widetilde{\psi}-\psi$ is a derivation of $\mathcal{T}$. By Proposition 2.4, we have $\widetilde{\psi}-\psi=0$. This yields that τ is a linear map from $\mathcal{T}$ into $\mathcal{Z}(\mathcal{T})$. Then we conclude that

$$\tau\begin{pmatrix} 0 & m\\ 0 & 0\end{pmatrix}=\begin{pmatrix} g_{11}(m) & 0\\ 0 & g_{22}(m)\end{pmatrix}\in\mathcal{Z}(\mathcal{T})$$

and

$$\tau\begin{pmatrix} 0 & 0\\ 0 & I_2\end{pmatrix}=\begin{pmatrix} 0 & 0\\ 0 & \eta(f_{11}(I_1))-f_{22}(I_1)\end{pmatrix}\in\mathcal{Z}(\mathcal{T}),$$

which means $\eta(f_{11}(I_1))-f_{22}(I_1)=0$, that is, $f_{11}(I_1)\oplus f_{22}(I_1)\in\mathcal{Z}(\mathcal{T})$.

(2) $\Rightarrow$ *(3)*: Since $f_{11}(I_1) \oplus f_{22}(I_1) \in \mathcal{Z}(\mathcal{T})$, we have $f_{11}(I_1)m = mf_{22}(I_1)$ for all $m \in \mathcal{M}$. It follows from Eq.(2.6) that $h_{11}(I_2)m = mh_{22}(I_2)$ for all $m \in \mathcal{M}$. Therefore, $h_{11}(I_2) \oplus h_{22}(I_2) \in \mathcal{Z}(\mathcal{T})$.

(3) $\Rightarrow$ *(1)*: If $h_{11}(I_2) \oplus h_{22}(I_2) \in \mathcal{Z}(\mathcal{T})$, then $h_{11}(I_2)m = mh_{22}(I_2)$ for all $m \in \mathcal{M}$. Hence we have

$$\eta^{-1}(h_{22}(I_2)) - h_{11}(I_2) = 0. \tag{2.10}$$

It follows from Eq.(2.6) that $f_{11}(I_1)m = mf_{22}(I_1)$ for all $m \in \mathcal{M}$, which means

$$\eta(f_{11}(I_1)) - f_{22}(I_1) = 0. \tag{2.11}$$

Therefore, by Eqs.(2.10)-(2.11), we get

$$\tau\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} g_{11}(m) & 0 \\ 0 & g_{22}(m) \end{pmatrix} \in \mathcal{Z}(\mathcal{T})$$

for all $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \mathcal{T}$. Letting $\omega = \gamma + \tau$, we obtain $\phi = \psi + \omega$, where ω is a linear map from $\mathcal{T}$ into $\mathcal{Z}(\mathcal{T})$. Suppose that $ST = G$. We compute

$$\begin{aligned}
\omega([S,T]) =& \phi([S,T]) - \psi([S,T]) \\
=& [\phi(S), T] + [S, \phi(T)] - [\psi(S), T] - [S, \psi(T)] \\
=& [(\gamma+\tau)(S), T] + [S, (\gamma+\tau)(T)] \\
=& 0.
\end{aligned}$$

$\square$

Theorem 2.5 has the following consequence, which is the main result in [11] for $a_0 = 0$ and $a_0 = I_1$, respectively.

Corollary 2.6. *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $G_1 = \begin{pmatrix} a_0 & 0 \\ 0 & 0 \end{pmatrix}$, where a_0 is an arbitrary but fixed point in $\mathcal{A}$. Assume that*

(i) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$;

(ii) For every $a \in \mathcal{A}$, there exists some integer n such that $nI_1 - a$ is invertible in $\mathcal{A}$;

(iii) For every $b \in \mathcal{B}$, there exists some integer n such that $nI_2 - b$ is invertible in $\mathcal{B}$.

Then $\phi : \mathcal{T} \to \mathcal{T}$ is Lie derivable at G if and only if it has the form $\phi = \psi + \omega$, where ψ is a derivation of $\mathcal{T}$ and $\omega : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a linear map vanishing on each commutator $[S,T]$ whenever $ST = G_1$.

Proof. The "if" part is obvious. We only need to prove "only if" part.

For any $m \in \mathcal{M}$, taking

$$S = \begin{pmatrix} a_0 & m \\ 0 & 0 \end{pmatrix}, \ T = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have $ST = G_1$. Then

$$\begin{pmatrix} -g_{11}(m) & -g_{12}(m) \\ 0 & -g_{22}(m) \end{pmatrix}$$

$$\begin{aligned}
&= \phi(ST-TS)\\
&= \phi(S)T - T\phi(S) + S\phi(T) - \phi(T)S\\
&= \begin{pmatrix} f_{11}(a_0)+g_{11}(m) & f_{12}(a_0)+g_{12}(m)\\ 0 & f_{22}(a_0)+g_{22}(m)\end{pmatrix}\begin{pmatrix} I_1 & 0\\ 0 & 0\end{pmatrix}\\
&- \begin{pmatrix} I_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} f_{11}(a_0)+g_{11}(m) & f_{12}(a_0)+g_{12}(m)\\ 0 & f_{22}(a_0)+g_{22}(m)\end{pmatrix}\\
&+ \begin{pmatrix} a_0 & m\\ 0 & 0\end{pmatrix}\begin{pmatrix} f_{11}(I_1) & f_{12}(I_1)\\ 0 & f_{22}(I_1)\end{pmatrix} - \begin{pmatrix} f_{11}(I_1) & f_{12}(I_1)\\ 0 & f_{22}(I_1)\end{pmatrix}\begin{pmatrix} a_0 & m\\ 0 & 0\end{pmatrix}\\
&= \begin{pmatrix} 0 & -g_{12}(m)+mf_{22}(I_1)-f_{11}(I_1)m\\ 0 & 0\end{pmatrix}.
\end{aligned}$$

It follows from the matrix equation that

$$g_{11}(m) = g_{22}(m) = 0$$

and

$$f_{11}(I_1)m = mf_{22}(I_1)$$

for all $m \in \mathcal{M}$, which means $f_{11}(I_1) \oplus f_{22}(I_1) \in \mathcal{Z}(\mathcal{T})$. Hence, by Theorem 2.5, ϕ is proper, that is, $\phi = \psi + \omega$, where ψ is a derivation of $\mathcal{T}$ and $\omega : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a linear map vanishing on each commutator $[S, T]$ whenever $ST = G_1$. □

3. Application

In this section, we give some applications of results in Section 2 to nest algebras.

Let X be a Banach space over the complex field $\mathbb{C}$, and $B(X)$ denote the algebra of all bounded linear operators on X. A nest $\mathcal{N}$ in X is a chain of norm closed linear subspaces of X containing $\{0\}$ and X, which is closed under the formation of arbitrary closed linear span and intersection. A nest is said to be nontrivial if $\mathcal{N} \neq \{\{0\}, X\}$. The nest algebra associated to a nest $\mathcal{N}$, denoted by $alg\mathcal{N}$, is the set

$$alg\mathcal{N} = \{T \in B(X) : TN \subseteq N, \quad \forall N \in \mathcal{N}\}.$$

Theorem 2.1 and 2.5 suggest the following theorem.

Theorem 3.1. *Let $\mathcal{N}$ be a nest on a complex Banach space space. Suppose that there exists a non-trivial element in $\mathcal{N}$ which is complemented in X. Let $\phi : alg\mathcal{N} \to alg\mathcal{N}$ is Lie derivable at an arbitrary but fixed point $G \in alg\mathcal{N}$. For any $\begin{pmatrix} x & y\\ 0 & z\end{pmatrix} \in alg\mathcal{N}$, we write*

$$\phi\begin{pmatrix} x & y\\ 0 & z\end{pmatrix} = \begin{pmatrix} f_{11}(x)+g_{11}(y)+h_{11}(z) & xf_{12}(I_1)-f_{12}(I_1)z+g_{12}(y)\\ 0 & f_{22}(x)+g_{22}(y)+h_{22}(z)\end{pmatrix}.$$

Then the following statements are equivalent:

(1) ϕ is proper;

(2) $f_{11}(I_1) \oplus f_{22}(I_1) \in \mathbb{C}I$, $g_{11}(y) \oplus g_{22}(y) \in \mathbb{C}I$ for all $y \in \mathcal{M}$;

(3) $h_{11}(I_2) \oplus h_{22}(I_2) \in \mathbb{C}I$, $g_{11}(y) \oplus g_{22}(y) \in \mathbb{C}I$ for all $y \in \mathcal{M}$.

Proof of Theorem 3.1. If $\widetilde{N}$ is the non-trivial element in $\mathcal{N}$ which is complemented in X, then there exists an idempotent operator $P \in B(X)$ with $ran(P) = \widetilde{N}$, such that $X = ran(P) \oplus ker(P)$. It is clear that $P \in alg\mathcal{N}$. Set

$$\mathcal{N}_1 = \{N \cap ranP : N \in \mathcal{N}\},\ \mathcal{N}_2 = \{N \cap ker(P) : N \in \mathcal{N}\}.$$

Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are nests on Banach spaces $ran(P)$ and $ker(P)$, respectively. One can check that $PB(X)(I-P) \subseteq alg\mathcal{N}$, which implies

$$PB(X)(I-P) = Palg\mathcal{N}(I-P).$$

Then we denote

$$alg\mathcal{N} = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x \in alg\mathcal{N}_1,\ y \in B(ker(P), ran(P)),\ z \in alg\mathcal{N}_2\}.$$

Note that $B(ker(P), ran(P))$ is faithful $(alg\mathcal{N}_1, alg\mathcal{N}_2)$-bimodule. Indeed, for $x \in alg\mathcal{N}_1$, if $xy = 0$ for all $y \in B(ker(P), ran(P))$, then we have $xPB(X)(I-P) = \{0\}$. Since $B(X)$ is prime, we see $x = 0$. It follows that $B(ker(P), ran(P))$ is a faithful $(alg\mathcal{N}_1)$-left module. Similarly, one can verify that $B(ker(P), ran(P))$ is a faithful $(alg\mathcal{N}_2)$-right module.

Therefore, nest algebra $alg\mathcal{N}$ can be decomposed into a triangular algebra which satisfies conditions (i)-(iii) of Theorem 2.5. Clearly, the center of $alg\mathcal{N}$ is $\mathbb{C}I$. Hence, by Theorem 2.5, the theorem is obtained. □

Setting $G_1 = \begin{pmatrix} x_0 & 0 \\ 0 & 0 \end{pmatrix}$, where $x_0 \in alg\mathcal{N}_1$ is an arbitrary but fixed operator, we arrive at the following consequence.

Corollary 3.2. *Let $\mathcal{N}$ be a nest on a complex Banach space. Suppose that there exists a non-trivial element in $\mathcal{N}$ which is complemented in X. Then $\phi : alg\mathcal{N} \to alg\mathcal{N}$ is Lie derivable at G_1 if and only if there exists $T \in alg\mathcal{N}$ and a linear functional $\omega : alg\mathcal{N} \to \mathbb{C}$ satisfying $\omega([S,T]) = 0$ whenever $ST = G_1$ such that $\phi(A) = AT - TA + \omega(A)$ for all $A \in alg\mathcal{N}$.*

Proof. The "if" part is obvious. We only need to prove "only if" part.

By Corollary 2.6, we obtain that $\phi = \psi + \omega$, where ψ is a derivation of $alg\mathcal{N}$ and $\omega : alg\mathcal{N} \to \mathbb{C}$ is a linear functional vanishing on each commutator $[S,T]$ whenever $ST = G_1$. Using Theorem 2.2 in [10] it follows that every linear derivation of a nest algebra on a Banach space is continuous. Furthermore, by [20], every continuous linear derivation of a nest algebra on a Banach space is inner. Then there is an operator $T \in alg\mathcal{N}$ such that $\psi(A) = AT - TA$ for all $A \in alg\mathcal{N}$, so that $\phi(A) = AT - TA + \omega(A)$ for all $A \in alg\mathcal{N}$. The proof is completed. □

Since every closed subspace is complemented in Hilbert spaces, we have the following results immediately.

Corollary 3.3. *Let $\mathcal{N}$ be a nest on a complex Hilbert space. Suppose that $\phi : alg\mathcal{N} \to alg\mathcal{N}$ is Lie derivable at an arbitrary but fixed point $G \in alg\mathcal{N}$. For any*

$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in alg\mathcal{N}$, *we write*

$$\phi \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} f_{11}(x) + g_{11}(y) + h_{11}(z) & xf_{12}(I_1) - f_{12}(I_1)z + g_{12}(y) \\ 0 & f_{22}(x) + g_{22}(y) + h_{22}(z) \end{pmatrix}.$$

Then the following statements are equivalent:

(1) ϕ is proper;

(2) $f_{11}(I_1) \oplus f_{22}(I_1) \in \mathbb{C}I$, $g_{11}(y) \oplus g_{22}(y) \in \mathbb{C}I$ for all $y \in \mathcal{M}$;

(3) $h_{11}(I_2) \oplus h_{22}(I_2) \in \mathbb{C}I$, $g_{11}(y) \oplus g_{22}(y) \in \mathbb{C}I$ for all $y \in \mathcal{M}$.

Corollary 3.4. *Let $\mathcal{N}$ be a nest on a complex Hilbert space. Then $\phi : alg\mathcal{N} \to alg\mathcal{N}$ is Lie derivable at G_1 if and only if there exists $T \in alg\mathcal{N}$ and a linear functional $\omega : alg\mathcal{N} \to \mathbb{C}$ satisfying $\omega([S, T]) = 0$ whenever $ST = G_1$ such that $\phi(A) = AT - TA + \omega(A)$ for all $A \in alg\mathcal{N}$.*

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