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A NOTE ON GAPS BETWEEN HAPPY NUMBERS

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ABSTRACT. Fix a base $b \geq 2$ and an exponent $e \geq 2$. An e-power b-happy number is a positive integer that reaches 1 under iteration of the function mapping a positive integer to the sum of the eth powers of its base b digits. In this note, we answer the question of how large the gaps between e-power b-happy numbers can be.

1. Introduction

Happy numbers and generalized happy numbers have been a subject of study for over 75 years [2]. In the second edition of his book *Unsolved Problems in Number Theory* [5], Richard Guy asked how large the gaps between happy numbers can be. This was answered for traditional happy numbers and some generalized happy numbers in [4]. In this note, we answer Guy's question for all generalized happy numbers. Our results derive from those in [6] on sequences of generalized happy numbers, which includes Theorem 3, below, and variations of our initial lemmas.

2. Definitions and Preliminaries

Fix a base $b \geq 2$ and an exponent $e \geq 2$.

Define the *generalized happy function* $S = S_{e,b} : \mathbb{Z}^+ \to \mathbb{Z}^+$, by

$$S\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e,$$

where $a_n \neq 0$ and $0 \leq a_i \leq b-1$, for $0 \leq i \leq n$. For $a \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, let $S^0(a) = a$ and $S^k(a) = S(S^{k-1}(a))$. A positive integer a is an *e-power b-happy number* if $S^k(a) = 1$ for some $k \geq 0$.

It is easy to see that S has many right inverses. For example, for $s \in \mathbb{Z}^+$, let $C_s : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

$$C_s(n) = b^s \sum_{i=0}^{n-1} b^i$$

and note that for each s and $n \in \mathbb{Z}^+$, $SC_s(n) = n$. The following lemma provides a key property of S and C_s.

Lemma 1. *Given $a \in \mathbb{Z}^+$ and $k \geq 0$, for any $n \in \mathbb{Z}^+$ and for each sufficiently large $s \in \mathbb{Z}^+$,*

$$S^k(C_s^k(n) + a) = n + S^k(a). \tag{1}$$

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Proof. Fix $s \in \mathbb{Z}^+$ such that for each $0 \le i < k$, $b^s > S^i(a)$. Then, since the image of C_s is always a positive multiple of b^s, for each $0 \le i < k$ and $m \in \mathbb{Z}^+$,

$$S(C_s(m) + S^i(a)) = S(C_s(m)) + S(S^i(a)) = m + S^{i+1}(a). \tag{2}$$

Trivially, equation (1) holds for $k = 0$. By induction, assume that, for any $n' \in \mathbb{Z}^+$,

$$S^{k-1}(C_s^{k-1}(n') + a) = n' + S^{k-1}(a).$$

Letting $n' = C_s(n)$, we have

$$\begin{aligned} S^k(C_s^k(n) + a) &= S(S^{k-1}(C_s^{k-1}(C_s(n)) + a)) \\ &= S(C_s(n) + S^{k-1}(a)) \\ &= n + S^k(a), \end{aligned}$$

by equation (2). □

We next show that (as noted in [6]) all e-power b-happy numbers lie in the set $1 + P\mathbb{Z}^+$, where

$$P = P_{e,b} = \prod_{\substack{p \text{ prime} \\ p|(b-1) \\ (p-1)|(e-1)}} p.$$

Lemma 2. *For each $a \in \mathbb{Z}^+$, $S(a) \equiv a \pmod{P}$. In particular, if a is an e-power b-happy number, then $a \equiv 1 \pmod{P}$.*

Proof. Let p be a prime dividing P. Then, given $0 \le a_i \le b - 1$, with $a_n \ne 0$,

$$S\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e \equiv \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i b^i \pmod{p}.$$

Since P is a product of distinct primes, the result follows. □

Hence, the length of the largest gap between e-power b-happy number is at least $P - 1$. Further, if there exists a positive integer in $1 + P\mathbb{Z}^+$ that is not an e-power b-happy number, then the length of the largest gap is strictly larger than $P - 1$.

A *P-consecutive sequence* is an arithmetic sequence with constant difference P. In [6] it is shown that there exist P-consecutive sequences of every finite length in which every number is an e-power b-happy numbers.

Theorem 3 (Zhou & Cai)**.** *There exist arbitrarily long finite P-consecutive sequences of e-power b-happy numbers.*

3. Main Theorem

Theorem 4 demonstrates that the size of gaps between e-power b-happy numbers is determined by the size of the set

$$U_1 = \{u \in 1 + P\mathbb{Z}^+ | S^k(u) = u \text{ for some } k \in \mathbb{Z}^+\},$$

which always contains 1.

Theorem 4. *If $|U_1| > 1$, then there exist arbitrarily long finite gaps between e-power b-happy numbers. If $|U_1| = 1$, then the length of the largest gap between e-power b-happy numbers is $P-1$.*

Proof. Assume that $|U_1| > 1$. Let $v \in U_1 - \{1\}$. Let $\ell \in \mathbb{Z}^+$ be arbitrary. By Theorem 3, there exists a set, T, of ℓ P-consecutive e-power b-happy numbers. Since T is finite, there exists some $k \in \mathbb{Z}^+$ such that for each $t \in T$, $S^k(t) = 1$.

By Lemma 1, for any sufficiently large s,

$$S^k(C_s^k(v-1)+t) = (v-1)+S^k(t) = v.$$

Hence the elements of $C_s^k(v-1)+T$ form a P-consecutive sequence of numbers congruent to 1 modulo P none of which is an e-power b-happy number. Further, by Lemma 2, all e-power b-happy numbers are congruent to 1 modulo P and so none of the ℓP consecutive positive integers beginning with the smallest element of T is an e-power b-happy number. Hence there exists a gap of length at least ℓP where ℓ is arbitrary, proving the first part of the theorem.

Now assume that $|U_1| = 1$. Then $U_1 = \{1\}$ and, therefore, every positive integer congruent to 1 modulo P is an e-power b-happy number. Hence, by Lemma 2, the set of e-power b-happy number is precisely the set $1+P\mathbb{Z}^+$, and the second part of the theorem follows. □

As is well-known, and easily proven, for $e \geq 2$, every positive integer is an e-power 2-happy numbers. Hence for $b = 2$ and any $e \geq 2$, $U_1 = \{1\}$ and, since $P = 1$, Theorem 4 correctly gives that the length of the largest gap between e-power 2-happy numbers is $P-1 = 0$. This is also the case for 2-power 4-happy numbers.

Perhaps of greater interest is the case $e = 5$ and $b = 4$. Here, $P = 3$ and $U_1 = \{1\}$. Thus every positive integer in $1+3\mathbb{Z}$ is a 5-power 4-happy number and the length of the largest gap between these numbers (in fact, the length of the gap between each pair of neighboring 5-power 4-happy numbers) is $P-1 = 2$.

In Table 1, we list the numbers P and the sets U_1 for small values of e and b. (These values are straightforward to compute using [2, Theorem 1], though many can be deduced from tables in that paper or in papers cited therein.) Note that by Theorem 4, for $2 \leq e \leq 5$ and $3 \leq b \leq 7$, there exist arbitrarily long finite gaps between e-power b-happy numbers, except when $(e,b) = (2,4)$ or $(e,b) = (5,4)$, as noted above.

Finally, we provide two examples of infinite families of pairs (e,b) for which we demonstrate that there exist arbitrarily long finite gaps between e-power b-happy numbers.

Corollary 5. *For $b \geq 3$ with b odd, there exist arbitrarily long finite gaps between 2-power b-happy numbers. Further, for $b \geq 4$ with $b \equiv 1 \pmod 3$, there exist arbitrarily long finite gaps between 3-power b-happy numbers.*

Proof. Let $e = 2$ and let $b \geq 3$ and odd be given. As observed in [1, Theorem 7], the integer $(b^2+1)/2$ is a fixed point of S. Since $P = 2$ and $(b^2+1)/2 \equiv 1 \pmod 2$, $(b^2+1)/2 \in U_1$. The first result now follows from Theorem 4.

Now let $e = 3$ and let $b \geq 4$ satisfy $b \equiv 1 \pmod 3$. It is easy to verify that $(b^3+b^2+b)/3$ is a fixed point of S and that $(b^3+b^2+b)/3 \equiv 1 \pmod 3$. Hence, $(b^3+b^2+b)/3 \in U_1$ and the result follows. □

e	b	P	U_1
2	2	1	{1}
	3	2	{1,5}
	4	1	{1}
	5	2	{1,13}
	6	1	{1,5,13,17,20,25,26,29,41}
	7	2	{1,13,17,25,29,37,45}
3	2	1	{1}
	3	2	{1,17}
	4	3	{1,28,43,55}
	5	2	{1,9,35,65}
	6	1	{1,9,28,62,73,99,128,190,251}
	7	6	{1,91,133,217}
4	2	1	{1}
	3	2	{1,17,33}
	4	1	{1,3,81,83,243}
	5	2	{1,339,369,419,499,593,595,609,769,849}
	6	1	{1,3,4,17,81,82,98,114,164,256,258,259,273,288, 338,353,609,641,963,978,1218,1251,1331,1522}
	7	1	{1,1543,1753,3613,4183,4393,6493,8299,10099}
5	2	1	{1}
	3	2	{1,33,65}
	4	3	{1}
	5	2	{1,309,551,1057,1089,1543}
	6	1	{1,2081,2566,4636,5416,7276}
	7	6	{1,1543,1753,3613,4183,4393,6493,8299,10099}

TABLE 1. The values of P and U_1 for small e and b.

References

[1] Baker Swart, B., Crook, S., Grundman, H. G., Hall-Seelig, L. L., "Gaussian happy numbers," *Rocky Mountain Journal of Mathematics*, **52** (2022), 415–429.

[2] Grundman, H. G., Hall-Seelig, L. L., "Happy numbers, happy functions, and their variations: a survey," *Matematica* **1** (2022), 404–430.

[3] Grundman H. G., Teeple E. A., "Iterated sums of fifth powers of digits," *Rocky Mountain J. Math.* **38** (2008), 1139–1146.

[4] Grundman H. G., Teeple E. A., "Sequences of consecutive happy numbers," *Rocky Mountain J. Math.* **37** (2007), 1905–1916.

[5] Guy R. K., *Unsolved Problems in Number Theory,* 2nd Edn., Springer-Verlag, New York, 1994.

[6] Zhou X., Cai T., "On *e*-power *b*-happy numbers," *Rocky Mountain J. Math.* **39** (2009), 2073–2081.

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