# ON THE $N$-HYPERCONTRACTIONS AND SIMILARITY OF MULTIVARIABLE WEIGHTED SHIFTS

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Abstract. In [35], A. L. Shields proved a well-known theorem for the similarity of unilateral weighted shift operators. By using the generalization of this theorem for multivariable weighted shifts and the curvature of holomorphic bundles, we give a necessary and sufficient condition for the similarity of $m$-tuples in the Cowen-Douglas class. We also present a necessary condition for commuting $m$-tuples of backward weighted shift operators to be $n$-hypercontractive in terms of the weight sequences.

## 1. Introduction

For $m \geq 1$, let $\Omega$ be a bounded connected open subset of $m$-dimensional complex space $\mathbb{C}^m$, $\mathcal{L}(\mathcal{H})^m$ be the space of all commuting $m$-tuples $\mathbf{T} = (T_1, \cdots, T_m)$ of bounded linear operators on the complex separable Hilbert space $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})^1$ be written as $\mathcal{L}(\mathcal{H})$. Let $\mathbf{T} = (T_1, \cdots, T_m), \mathbf{S} = (S_1, \cdots, S_m) \in \mathcal{L}(\mathcal{H})^m$. If there is a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $UT_i = S_iU, 1 \leq i \leq m$, then $\mathbf{T}$ and $\mathbf{S}$ are unitarily equivalent (denoted by $\mathbf{T} \sim_u \mathbf{S}$). If there is an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $XT_i = S_iX, 1 \leq i \leq m$, then $\mathbf{T}$ and $\mathbf{S}$ are similar (denoted by $\mathbf{T} \sim_s \mathbf{S}$). However, it is not easy to characterize the similarity (unitary equivalence) of any two commuting tuples of operators. Thus one can only consider similarity classification in some special classes.

In [8, 9], Cowen and Douglas introduced a subclass $\mathcal{B}_n^m(\Omega)$ of $\mathcal{L}(\mathcal{H})^m$ and proved that two $m$-tuples $\mathbf{T} = (T_1, \cdots, T_m)$ and $\mathbf{S} = (S_1, \cdots, S_m)$ in $\mathcal{B}_n^m(\Omega)$ are unitarily equivalent if and only if their corresponding complex bundles $E_{\mathbf{T}}$ and $E_{\mathbf{S}}$ are equivalent as Hermitian holomorphic vector bundles. They also provide a large number of local criteria for holomorphic bundle equivalence. For Hermitian holomorphic vector bundles associated with the Cowen-Douglas class, unitary operators preserve their local rigidity, so the local properties of the bundle are also valid globally. In particular, the curvature and its covariant partial derivatives of Hermitian holomorphic vector bundles corresponding to tuples in $\mathcal{B}_n^m(\Omega)$ are shown as a set of complete unitary invariants. Thus the curvature contains a lot of information about commuting $m$-tuples and is closely related to the unitary classification of commuting $m$-tuples.

An effective way to study operators is to find an operator model for them with some common properties. There is a rich theory of normal operators and subnormal operators, such as the spectrum theorem, von Neumann-Wold theorem, etc. In order to generalize a famous model theorem on contractions given by Sz.-Nagy and Foias, Alger in [1] put forward the concept of $n$-hypercontraction, which is stronger than contraction. For a contraction $T \in \mathcal{B}_1^1(\mathbb{D})$, the curvature of $T$ is dominated by the curvature associated with the adjoint of the multiplication operator on Hardy space in [28].

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2000 *Mathematics Subject Classification.* Primary 47C15, 47B13; Secondary 47B48, 47L40.

*Key words and phrases.* The class $\mathcal{B}_n^m(\Omega)$; Similarity; $N$-hypercontraction; Curvature.

This work was supported by the National Natural Science Foundation of China, Grant No. 11922108 and 12001159.

Then this result is extended to the case of commuting $m$-tuples in [30, 31]. Based on Sz.-Nagy-Foias theory, in 1993, Müller and Vasilescu gave the following theorem in [32] to decide when a commuting $m$-tuple of operators is unitarily equivalent to the restriction of certain commuting $m$-tuple of backward weighted shifts to an invariant subspace.

**Theorem 1.1.** [32] *Let $\mathbf{T}=(T_1,\cdots,T_m)\in\mathcal{L}(\mathcal{H})^m$ be a commuting $m$-tuple of operators and $n\geq 1$ be an integer. Then there exist a Hilbert space $E$ and an $\mathbf{S}^*_{n,E}$-invariant subspace $K$ of $H^2_{n,E}$ such that $\mathbf{T}$ is unitarily equivalent to $\mathbf{S}^*_{n,E}|K$ if and only if $\mathbf{T}$ is an $n$-hypercontraction with $\lim\limits_{k\to\infty}\mathbf{M}^k_{\mathbf{T}}(1)=0$ in the strong operator topology.*

In [13], Curto and Salinas explicitly pointed out that every $m$-tuple $\mathbf{T}\in\mathcal{B}^m_n(\Omega)$ can be realized as the adjoint of a commuting $m$-tuple of multiplication operators by coordinate functions on some Hilbert space of holomorphic functions. In the past four decades, Cowen, Douglas, Misra and other mathematicians have done a lot of work on the classification of this class of operators (see [10, 14, 17, 19, 20, 21, 22, 23, 28, 36]).

The class $\mathcal{B}^1_1(\mathbb{D})$ contains many the adjoint of unilateral weighted shift operators. Shields provided a large number of properties and results about these shifts in [35], including the equivalent condition for the similarity of such operators.

**Lemma 1.2.** [35] *Let $T_1$ and $T_2$ be unilateral weighted shifts with weight sequences $\{\lambda_j\}_{j=0}^\infty$ and $\{\tilde{\lambda}_j\}_{j=0}^\infty$, respectively. Then $T_1\sim_s T_2$ if and only if there exist positive constants $C_1$ and $C_2$ such that $0<C_1\leq\left|\frac{\lambda_k\lambda_{k+1}\cdots\lambda_l}{\tilde{\lambda}_k\tilde{\lambda}_{k+1}\cdots\tilde{\lambda}_l}\right|\leq C_2$ for all $0\leq k\leq l$.*

In order to find new similarity invariants of unilateral shift operators, Clark and Misra first studied the similarity of some backward weighted shift operators by using the quotient of the metric of associated holomorphic bundles in [10, 11, 12]. In a way, this result can be seen as a geometric version of the similar result due to Shields (Lemma 1.2). Subsequently, the study of geometric similarity invariants of weighted shift operators has become an important branch of research in this subject topic (see [14, 17, 21, 23]). The notion of a weighted shift has a natural generalization to commuting tuples of operators. However, there are few related studies on the similarity of tuples.

Inspired by the result of single variable, Lemma 1.2 is extended to the case of commuting $m$-tuples of operators (see Theorem 3.1). By using this theorem and model theorem, we characterize the similarity of $n$-hypercontractive commuting $m$-tuples in $\mathcal{B}^m_1(\Omega)$ in terms of the difference of the curvature (see Theorem 3.3). Thus a natural question is the following: when is an $m$-tuple $n$-hypercontractive? In the last section, we use the weight sequence to give a necessary condition for some $m$-tuple to be $n$-hypercontractive (see Theorem 4.2), and show that the $n$-hypercontraction assumption in Theorem 3.3 is also necessary (see Example 4.5). The following are the two main theorems of this paper.

**Theorem 1.3.** *Let $\mathbf{T}=(T_1,\cdots,T_m),\mathbf{S}^*=(S^*_1,\cdots,S^*_m)\in\mathcal{B}^m_1(\mathbb{B}^m)$ be the adjoint of the multiplication tuples on $\widehat{\mathcal{H}}$ and $H^2_n$, respectively. If $\mathbf{T}$ is $n$-hypercontractive and $\lim\limits_j f_j(\mathbf{T}^*,\mathbf{T})h=0, h\in\widehat{\mathcal{H}}$, where $f_j(z,w)=\sum\limits_{i=j}^\infty\mathbf{e}_i(z)(1-\langle z,w\rangle)^k\mathbf{e}_i(w)^*$ and $\{\mathbf{e}_i\}_{i=0}^\infty$ is an orthonormal basis of $H^2_n$. Then $\mathbf{T}$ is similar*

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*to* $\mathbf{S}^*$ *if and only if there exists a bounded plurisubharmonic function* $\psi$ *on* $\mathbb{B}^m$ *such that*

$$\mathcal{K}_{\mathbf{S}^*}(w)-\mathcal{K}_{\mathbf{T}}(w)=\sum_{i,j=1}^{m}\frac{\partial^2\psi(w)}{\partial w_i\partial\overline{w}_j}dw_i\wedge d\overline{w}_j,\quad w\in\mathbb{B}^m.$$

**Theorem 1.4.** *Let* $m\geq 2$ *be a positive integer and* $\mathbf{T}=(T_1,\cdots,T_m)$ *be a commuting* $m$*-tuple of backward weighted shifts on Hilbert space* $\mathcal{H}$ *with reproducing kernel* $K(z,w)=\sum\limits_{\alpha\in\mathbf{Z}_+^m}\rho(\alpha)z^\alpha\overline{w}^\alpha$, *where* $\rho(\alpha)>0$ *and* $z,w\in\Omega$. *If* $\mathbf{T}$ *is* $n$*-hypercontractive, then for any* $\alpha\in\mathbf{Z}_+^m$,

$$\sum_{\substack{\beta\,\in\,\mathbf{Z}_+^m,\,\beta\,\leq\,\alpha\\ |\alpha-\beta|\,=\,1}}\frac{\rho(\beta)}{\rho(\alpha)}\leq\frac{|\alpha|}{|\alpha|+n-1}.$$

## 2. Preliminaries

Let $\mathbf{T}=(T_1,\cdots,T_m)\in\mathcal{L}(\mathcal{H})^m$ be the commuting $m$-tuple of operators in $\mathcal{L}(\mathcal{H})$, i.e., $T_i\in\mathcal{L}(\mathcal{H})$ and $T_iT_j=T_jT_i, 1\leq i,j\leq m$. Let $\mathbf{Z}_+^m$ be the collection of $m$-tuples of nonnegative integers. For $\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbf{Z}_+^m$ and $\mathbf{T}$ mentioned above, we write $\theta=(0,\cdots,0)$, $|\alpha|=|\alpha_1|+\cdots+|\alpha_m|$, $\alpha!=\alpha_1!\cdots\alpha_m!$, $\mathbf{T}^\alpha=T_1^{\alpha_1}\cdots T_m^{\alpha_m}$ and $\mathbf{T}^*=(T_1^*,\cdots,T_m^*)$. For $\alpha,\beta\in\mathbf{Z}_+^m$, we define $\alpha+\beta=(\alpha_1+\beta_1,\cdots,\alpha_m+\beta_m)$ and $\alpha\leq\beta$ whenever $\alpha_i\leq\beta_i, 1\leq i\leq m$.

2.1. **The Cowen-Douglas class $\mathcal{B}_n^m(\Omega)$.** For each $\mathbf{T}=(T_1,\cdots,T_m)\in\mathcal{L}(\mathcal{H})^m$, we will associate with a bounded linear operator $\mathscr{D}_{\mathbf{T}}:\mathcal{H}\longrightarrow\mathcal{H}\oplus\cdots\oplus\mathcal{H}$ by

$$\mathscr{D}_{\mathbf{T}}h=(T_1h,\cdots,T_mh),$$

for $h\in\mathcal{H}$. Let $\Omega$ be a connected open subset of $\mathbb{C}^m$. For any $w=(w_1,\cdots,w_m)\in\Omega$, $\mathbf{T}-w$ means the $m$-tuple $(T_1-w_1,\cdots,T_m-w_m)$. In the following, $n$ is considered as a positive integer.

**Definition 2.1.** [8, 9] *Let* $\mathcal{B}_n^m(\Omega)$ *denote the Cowen-Douglas class of commuting* $m$*-tuples* $\mathbf{T}=(T_1,\cdots,T_m)\in\mathcal{L}(\mathcal{H})^m$ *satisfying:*

(1) $ran\mathscr{D}_{\mathbf{T}-w}$ *is closed for all* $w$ *in* $\Omega$;
(2) $\bigvee\limits_{w\in\Omega}\ker\mathscr{D}_{\mathbf{T}-w}=\mathcal{H}$; *and*
(3) $\dim\ker\mathscr{D}_{\mathbf{T}-w}=n$ *for all* $w$ *in* $\Omega$,

*where* $\ker\mathscr{D}_{\mathbf{T}-w}$ *means the joint kernel* $\bigcap\limits_{i=1}^{m}\ker(T_i-w_i)$ *for* $w=(w_1,\cdots,w_m)$ *and* $\bigvee$ *stands for the closed linear span.*

For a tuple $\mathbf{T}$ in $\mathcal{B}_n^m(\Omega)$, let $(E_{\mathbf{T}},\pi)$ denote the sub-bundle of the trivial bundle $\Omega\times\mathcal{H}$ defined by

$$E_{\mathbf{T}}=\{(w,x)\in\Omega\times\mathcal{H}:x\in\ker\mathscr{D}_{\mathbf{T}-w}\},\quad\pi(w,x)=w.$$

It is shown in [8, 9] that two commuting $m$-tuples $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ in $\mathcal{B}_n^m(\Omega)$ are unitarily equivalent if and only if the complex bundles $E_{\mathbf{T}}$ and $E_{\widetilde{\mathbf{T}}}$ are equivalent as Hermitian holomorphic vector bundles. Since $\dim\ker\mathscr{D}_{\mathbf{T}-w}=n$ for all $w$ in $\Omega$, the rank of the holomorphic vector bundle $E_{\mathbf{T}}$ is

$n$. Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be the holomorphic frame of $E_{\mathbf{T}}$ and form the metric of inner products $h(w) := (\langle \sigma_j(w), \sigma_i(w)\rangle)_{i,j=1}^n$, then the curvature $\mathcal{K}_{\mathbf{T}}$ of the bundle $E_{\mathbf{T}}$ is given by the following formula

$$\mathcal{K}_{\mathbf{T}}(w) := \sum_{i,j=1}^{m} \frac{\partial}{\partial \overline{w}_j} \left( h^{-1}(w) \frac{\partial}{\partial w_i} h(w) \right) d\overline{w}_j \wedge dw_i. \tag{2.1}$$

In fact, $\mathcal{K}_{\mathbf{T}}(w)$ depends on the choice of holomorphic frame $\sigma$. We also use $\mathcal{K}_{\mathbf{T}}(\sigma)(w)$ to represent the curvature function of $E_{\mathbf{T}}$ associated with the frame $\sigma$. In particular, when $\mathbf{T} \in \mathcal{B}_1^m(\Omega)$, the curvature of the bundle $E_{\mathbf{T}}$ can be defined as $\mathcal{K}_{\mathbf{T}}(w) = -\sum\limits_{i,j=1}^{m} \frac{\partial^2 \log \|\gamma(w)\|^2}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j$, where $\gamma$ is a non-vanishing holomorphic section of $E_{\mathbf{T}}$.

Let $\mathbf{T} = (T_1, \cdots, T_m) \in \mathcal{B}_n^m(\Omega)$. Then $m$-tuple $\mathbf{T}$ is unitarily equivalent to the adjoint $\mathbf{M}_z^* = (M_{z_1}^*, \cdots, M_{z_m}^*)$ of an $m$-tuple of multiplication operators by coordinate functions on a Hilbert space $\mathcal{H}_K$ of holomorphic functions on $\Omega^* = \{w \in \mathbb{C}^m : \overline{w} \in \Omega\}$ with reproducing kernel $K$ in [8, 9, 13] and $\ker(\mathbf{M}_z^* - w) = \{K(\cdot, \overline{w})\xi, \xi \in \mathbb{C}^n\}$. If $K(z, w) = \sum\limits_{\alpha \,\in\, \mathbf{Z}_+^m} \rho(\alpha) z^\alpha w^\alpha$, where $\rho(\alpha) > 0$, then an orthonormal basis of $\mathcal{H}$ can be determined as $\left\{\mathbf{e}_\alpha : \mathbf{e}_\alpha(z) = \sqrt{\rho(\alpha)} z^\alpha\right\}_{\alpha \in \mathbf{Z}_+^m}$. For any $z = (z_1, \cdots, z_m) \in \Omega$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{Z}_+^m$, we have

$$M_{z_i}\mathbf{e}_\alpha(z) = \sqrt{\rho(\alpha)} z^{\alpha + e_i} = \sqrt{\frac{\rho(\alpha)}{\rho(\alpha + e_i)}} \sqrt{\rho(\alpha + e_i)} z^{\alpha + e_i} = \sqrt{\frac{\rho(\alpha)}{\rho(\alpha + e_i)}} \mathbf{e}_{\alpha + e_i}(z).$$

It follows that $\mathbf{T}$ can be realized as a commuting $m$-tuple of backward weighted shift up to unitary equivalence.

2.2. *$N$*-**hypercontractivity of commuting tuples.** In the case of a single operator, it can be seen from the model theorem [37] and similarity theorems [23, 14] that the $n$-hypercontraction of the operator is closely related to the similarity of the operator. Therefore, when considering the similarity of commuting $m$-tuples, it is necessary to introduce the following concept.

We will use the symbols in [32] to define the $n$-hypercontractive of commuting tuples. Let $\mathbf{T} = (T_1, \cdots, T_m) \in \mathcal{L}(\mathcal{H})^m$. Define the operator $\mathbf{M}_{\mathbf{T}} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by

$$\mathbf{M}_{\mathbf{T}}(X) := \sum_{i=0}^{m} T_i^* X T_i$$

for any operator $X$ in $\mathcal{L}(\mathcal{H})$. Note that $\mathbf{M}_{\mathbf{T}}^k(X) = \sum\limits_{\substack{\alpha \in \mathbf{Z}_+^m \\ |\alpha| = k}} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} X \mathbf{T}^\alpha$ for any nonnegative integer $k$.

Then we also define $\triangle_{\mathbf{T}}^{(k)}$,

$$\triangle_{\mathbf{T}}^{(k)} := (I - \mathbf{M}_{\mathbf{T}})^k(I) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathbf{M}_{\mathbf{T}}^j(I) = \sum_{\alpha \in \mathbf{Z}_+^m, |\alpha| \le k} (-1)^{|\alpha|} \frac{k!}{\alpha!(k - |\alpha|)!} \mathbf{T}^{*\alpha} \mathbf{T}^\alpha.$$

**Definition 2.2.** *An $m$-tuple $\mathbf{T} = (T_1, \cdots, T_m) \in \mathcal{L}(\mathcal{H})^m$ is called $n$-hypercontractive, if*

$$\triangle_{\mathbf{T}}^{(k)} = \sum_{\alpha \in \mathbf{Z}_+^m, |\alpha| \le k} (-1)^{|\alpha|} \frac{k!}{\alpha!(k - |\alpha|)!} \mathbf{T}^{*\alpha} \mathbf{T}^\alpha$$

*is non-negative for $1 \le k \le n$. The special case of 1-hypercontraction corresponds to the usual (row) contraction.*

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Let $\mathbb{B}^m$ be the open unit ball $\{w : |w| < 1\}$ in $\mathbb{C}^m$. Let $K$ denote the positive definite kernel $K(z,w) = \frac{1}{1-\langle z,w\rangle}$ on $\mathbb{B}^m \times \mathbb{B}^m$, where $\langle z,w\rangle = z_1\overline{w}_1 + \cdots + z_m\overline{w}_m$. Then

$$K^n(z,w) = \frac{1}{(1-\langle z,w\rangle)^n} = \sum_{\alpha\in\mathbf{Z}_+^m} \frac{(n+|\alpha|-1)!}{\alpha!(n-1)!} z^\alpha \bar{w}^\alpha, \quad z,w \in \mathbb{B}^m,$$

is also a positive definite kernel on $\mathbb{B}^m \times \mathbb{B}^m$. Let $H_n^2$ be the reproducing kernel Hilbert space determined by the kernel function $K^n$. Now, the $n$-hypercontractivity of a commuting $m$-tuple is the same as requiring $K^{-k}(\mathbf{T}^*,\mathbf{T})$ is non-negative for $1 \leq k \leq n$. Obviously, the adjoint of the multiplication tuple $\mathbf{M}_z = (M_{z_1},\cdots,M_{z_m})$ on $H_n^2$ is $n$-hypercontractive. Letting $\rho_n(\alpha) = \frac{(n+|\alpha|-1)!}{\alpha!(n-1)!}$, we obtain an orthonormal basis of space $H_n^2$ as $\{\mathbf{e}_\alpha : \mathbf{e}_\alpha(z) = \sqrt{\rho_n(\alpha)}z^\alpha\}_{\alpha\in\mathbf{Z}_+^m}$. In particular, when $n = 1$, space $H_1^2$ is the Drury-Arveson space.

2.3. **Plurisubharmonic function.** The following are some basics of plurisubharmonic functions, which will be used to describe the similarity of certain $m$-tuples in $\mathcal{B}_1^m(\mathbb{B}^m)$. The space $\mathcal{C}^2$ consists of complex functions whose 2th-order partial derivatives are continuous.

**Definition 2.3.** [4] *Let $\Omega$ be a bounded domain of $\mathbb{C}^m, m > 1$. A function $u \in \mathcal{C}^2(\Omega)$ is said to be pluriharmonic if it satisfies the $m^2$ differential equations $\frac{\partial^2 u}{\partial w_i \partial \bar{w}_j} = 0$ for $1 \leq i,j \leq m$.*

**Definition 2.4.** [26, 33] *A real-valued function $u : \Omega \to \mathbf{R} \cup \{-\infty\}$ ($u \not\equiv -\infty$) is plurisubharmonic if it satisfies the following conditions:*

(1) *$u(z)$ is upper-semicontinuous on $\Omega$;*
(2) *For any arbitrary $z_0 \in \Omega$ and some $z_1 \in \mathbb{C}^m$ determined by $z_0$, $u(z_0 + \lambda z_1)$ is subharmonic with respect to $\lambda \in \mathbb{C}$.*

**Lemma 2.5.** *A real-valued function $f \in \mathcal{C}^2(\Omega)$ is plurisubharmonic if and only if $\left(\frac{\partial^2 f(w)}{\partial w_i \partial \bar{w}_j}\right)_{i,j=1}^m$ is non-negative for every $w \in \Omega$.*

## 3. The similarity of commuting tuples of weighted shifts

In this section, we focus on investigating how to describe the similarity of commuting $m$-tuples of unilateral weighted shifts. In [35], Shields provided a necessary and sufficient condition for the similarity of unilateral weighted shift operators by using the weight sequences. In the following, we list a generalization of Shields's result on commuting $m$-tuples of unilateral weighted shifts, and prove that the similarity invariants of certain $n$-hypercontrative tuples can be characterized by the difference of the curvature of commuting $m$-tuples in the Cowen-Douglas class.

Let $H$ be a separable Hilbert space, and $\{\mathbf{e}_\alpha\}_{\alpha\in\mathbf{Z}_+^m}$ be an orthonormal basis of the Hilbert space $\mathcal{H} = \ell^2(\mathbf{Z}_+^m, H)$ composed of functions $f$ satisfying $\|f\|^2 = \sum\limits_{\alpha\in\mathbf{Z}_+^m} |f(\alpha)|^2 < \infty$. Let $\mathbf{T} = (T_1,\cdots,T_m)$ and $\mathbf{S} = (S_1,\cdots,S_m)$ be commuting $m$-tuples of unilateral weighted shift operators on $\mathcal{H}$, and their nonzero weight sequences are $\{\lambda_\alpha^{(1)},\cdots,\lambda_\alpha^{(m)}\}_{\alpha\in\mathbf{Z}_+^m}$ and $\{\tilde{\lambda}_\alpha^{(1)},\cdots,\tilde{\lambda}_\alpha^{(m)}\}_{\alpha\in\mathbf{Z}_+^m}$, respectively. That is,

$$T_i\mathbf{e}_\alpha = \lambda_\alpha^{(i)}\mathbf{e}_{\alpha+e_i} \quad \text{and} \quad S_i\mathbf{e}_\alpha = \widetilde{\lambda}_\alpha^{(i)}\mathbf{e}_{\alpha+e_i},$$

where $1 \leq i \leq m$ and $e_i = (0,\cdots,0,1,0,\cdots,0) \in \mathbf{Z}_+^m$ with 1 on the $i$th position.

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Jewell and Lubin provide a criterion for an operator $X$ to commute between two commuting $m$-tuples of weighted shifts by using weight sequences in Proposition 7 of [18]. Meanwhile, a necessary and sufficient condition for the unitary equivalence of two commuting $m$-tuples is given by them. A natural question is how to characterize the similarity of commuting $m$-tuples of weighted shifts using weight sequences. The following theorem is the generalization of Shields's result on commuting $m$-tuples of unilateral weighted shifts, which were given by Kumar and Pilidi in [24] and [34], respectively.

**Theorem 3.1.** *Let* $\mathbf{T}=(T_1,\cdots,T_m),\mathbf{S}=(S_1,\cdots,S_m)\in\mathcal{L}(\mathcal{H})^m$ *be commuting tuples of unilateral weighted shifts, and their nonzero weight sequences are* $\{\lambda_\alpha^{(1)},\cdots,\lambda_\alpha^{(m)}\}_{\alpha\in\mathbf{Z}_+^m}$ *and* $\{\tilde{\lambda}_\alpha^{(1)},\cdots,\tilde{\lambda}_\alpha^{(m)}\}_{\alpha\in\mathbf{Z}_+^m}$*, respectively. Then* $\mathbf{T}$ *and* $\mathbf{S}$ *are similar if and only if there exist positive constants* $C_1$ *and* $C_2$ *such that*

$$0<C_1\leq\left|\frac{\prod\limits_{k=0}^{l}\lambda_{\alpha+ke_i}^{(i)}}{\prod\limits_{k=0}^{l}\tilde{\lambda}_{\alpha+ke_i}^{(i)}}\right|\leq C_2$$

*for any nonnegative integer* $l$, $\alpha\in\mathbf{Z}_+^m$ *and* $1\leq i\leq m$.

We will give a result to illustrate an application of the theorem above to the similarity of $m$-tuples in $\mathcal{B}_1^m(\Omega)$. Before introducing this theorem, we need to prove the following lemma.

**Lemma 3.2.** *Let* $\beta=(\beta_1,\beta_2,\cdots,\beta_m)\in\mathbf{Z}_+^m$ *and* $i$ *be a nonnegative integer. If* $i\leq|\beta|$*, then*

$$\sum_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha\leq\beta,|\alpha|=i}}\frac{|\alpha|!\beta!(|\beta-\alpha|)!}{\alpha!|\beta|!(\beta-\alpha)!}=1.$$

*Proof.* In order to prove that $\sum\limits_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha\leq\beta,|\alpha|=i}}\frac{|\alpha|!\beta!(|\beta-\alpha|)!}{\alpha!|\beta|!(\beta-\alpha)!}=\sum\limits_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha\leq\beta,|\alpha|=i}}\frac{i!\beta!(|\beta|-i)!}{\alpha!|\beta|!(\beta-\alpha)!}=1$, we just need to verify that $\sum\limits_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha\leq\beta,|\alpha|=i}}\frac{\beta!}{\alpha!(\beta-\alpha)!}=\frac{|\beta|!}{i!(|\beta|-i)!}$. Since $\prod\limits_{i=1}^{m}(1+x)^{\beta_i}=(1+x)^{|\beta|}$, by comparing the coefficients of the above $x^i, i\leq|\beta|$, we have that $\sum\limits_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha\leq\beta,|\alpha|=i}}\binom{\beta_1}{\alpha_1}\binom{\beta_2}{\alpha_2}\cdots\binom{\beta_m}{\alpha_m}=\binom{|\beta|}{i}$. This completes the proof. $\square$

Let $\widehat{\mathcal{H}}$ be the reproducing kernel Hilbert space determined by the kernel function $\widehat{K}(z,w)=\sum\limits_{i=0}^{\infty}a(i)(z_1\overline{w}_1+\cdots+z_m\overline{w}_m)^i, a(i)>0$ for $z,w\in\mathbb{B}^m$, and let $\{\widehat{\mathbf{e}_\alpha}\}_{\alpha\in\mathbf{Z}_+^m}$ and $\{\mathbf{e}_\alpha\}_{\alpha\in\mathbf{Z}_+^m}$ be the orthonormal basis of spaces $\widehat{\mathcal{H}}$ and $H_n^2$, respectively. The following theorem is based on the assumption of the above notations.

**Theorem 3.3.** *Let* $\mathbf{T}=(T_1,\cdots,T_m),\mathbf{S}^*=(S_1^*,\cdots,S_m^*)\in\mathcal{B}_1^m(\mathbb{B}^m)$ *be the adjoint of the multiplication tuples on* $\widehat{\mathcal{H}}$ *and* $H_n^2$*, respectively. If* $\mathbf{T}$ *is* $n$*-hypercontractive and* $\lim\limits_{j}f_j(\mathbf{T}^*,\mathbf{T})h=0, h\in\widehat{\mathcal{H}}$*, where* $f_j(z,w)=\sum\limits_{i=j}^{\infty}\mathbf{e}_i(z)(1-\langle z,w\rangle)^k\mathbf{e}_i(w)^*$ *and* $\{\mathbf{e}_i\}_{i=0}^{\infty}$ *is an orthonormal basis of* $H_n^2$*. Then* $\mathbf{T}$ *is similar to* $\mathbf{S}^*$ *if and only if there exists a bounded plurisubharmonic function* $\psi$ *on* $\mathbb{B}^m$ *such that*

$$\mathcal{K}_{\mathbf{S}^*}(w)-\mathcal{K}_{\mathbf{T}}(w)=\sum_{i,j=1}^{m}\frac{\partial^2\psi(w)}{\partial w_i\partial\overline{w}_j}dw_i\wedge d\overline{w}_j,\quad w\in\mathbb{B}^m.$$

*Proof.* Let $\widehat{\gamma}(w) = \widehat{K}(\cdot,\overline{w})$, $\gamma(w) = K^n(\cdot,\overline{w})$ for all $w \in \mathbb{B}^m$. Then $\widehat{\gamma}$ and $\gamma$ are non-vanishing holomorphic sections of $E_{\mathbf{T}}$ and $E_{\mathbf{S}^*}$, respectively. Thus,

$$h_{\mathbf{T}}(w) = \|\widehat{\gamma}(w)\|^2 = \sum_{\alpha\in\mathbf{Z}_+^m} \widehat{\rho}(\alpha) w^\alpha \overline{w}^\alpha \quad \text{and} \quad h_{\mathbf{S}^*}(w) = \|\gamma(w)\|^2 = \sum_{\alpha\in\mathbf{Z}_+^m} \rho(\alpha) w^\alpha \overline{w}^\alpha, \tag{3.1}$$

where $\widehat{\rho}(\alpha) = a(|\alpha|)\frac{|\alpha|!}{\alpha!}$ and $\rho(\alpha) = \frac{(n+|\alpha|-1)!}{\alpha!(n-1)!}$. Since the reproducing kernel on a Hilbert space is unique, from the relationship between multiplication tuple and reproducing kernel, we have

$$T_i^*\widehat{\mathbf{e}}_\alpha = \sqrt{\frac{\widehat{\rho}(\alpha)}{\widehat{\rho}(\alpha+e_i)}}\widehat{\mathbf{e}}_{\alpha+e_i}, \qquad T_i\widehat{\mathbf{e}}_\alpha = \sqrt{\frac{\widehat{\rho}(\alpha-e_i)}{\widehat{\rho}(\alpha)}}\widehat{\mathbf{e}}_{\alpha-e_i},$$

and

$$S_i\mathbf{e}_\alpha = \sqrt{\frac{\rho(\alpha)}{\rho(\alpha+e_i)}}\mathbf{e}_{\alpha+e_i}, \qquad S_i^*\mathbf{e}_\alpha = \sqrt{\frac{\rho(\alpha-e_i)}{\rho(\alpha)}}\mathbf{e}_{\alpha-e_i},$$

where $1 \le i \le m$ and $e_i = (0,\cdots,0,1,0,\cdots,0) \in \mathbf{Z}_+^m$ with 1 on the $i$th position.

On the one hand, suppose that $\mathbf{T}$ and $\mathbf{S}^*$ are similar. By Theorem 3.1, there are positive constants $C_1$ and $C_2$ such that

$$0 < C_1 \le \frac{\prod\limits_{k=0}^{l}\sqrt{\frac{\rho(\alpha+ke_i)}{\rho(\alpha+(k+1)e_i)}}}{\prod\limits_{k=0}^{l}\sqrt{\frac{\widehat{\rho}(\alpha+ke_i)}{\widehat{\rho}(\alpha+(k+1)e_i)}}} = \frac{\sqrt{\frac{\rho(\alpha)}{\rho(\alpha+(l+1)e_i)}}}{\sqrt{\frac{\widehat{\rho}(\alpha)}{\widehat{\rho}(\alpha+(l+1)e_i)}}} \le C_2$$

for any nonnegative integers $l$, $1 \le i \le m$ and $\alpha \in \mathbf{Z}_+^m$. For a fixed but arbitrary $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_m) \in \mathbf{Z}_+^m$, if $\alpha_i \ne 0, 1 \le i \le m$, we obtain

$$\begin{cases} 0 < C_1^2 \le \dfrac{\frac{\rho(\alpha-\alpha_1e_1)}{\rho(\alpha)}}{\frac{\widehat{\rho}(\alpha-\alpha_1e_1)}{\widehat{\rho}(\alpha)}} \le C_2^2, \\ 0 < C_1^2 \le \dfrac{\frac{\rho(\alpha-\alpha_1e_1-\alpha_2e_2)}{\rho(\alpha-\alpha_1e_1)}}{\frac{\widehat{\rho}(\alpha-\alpha_1e_1-\alpha_2e_2)}{\widehat{\rho}(\alpha-\alpha_1e_1)}} \le C_2^2, \\ \cdots\cdots \\ 0 < C_1^2 \le \dfrac{\frac{\rho(\theta)}{\rho(\alpha-\alpha_1e_1-\cdots-\alpha_{m-1}e_{m-1})}}{\frac{\widehat{\rho}(\theta)}{\widehat{\rho}(\alpha-\alpha_1e_1-\cdots-\alpha_{m-1}e_{m-1})}} \le C_2^2. \end{cases}$$

Further, we have $0 < C_1^{2m} \le \frac{\rho(\theta)}{\widehat{\rho}(\theta)}\frac{\widehat{\rho}(\alpha)}{\rho(\alpha)} \le C_2^{2m}$. Otherwise, there is a positive integer $p$, $1 \le p < m$, such that $0 < C_1^{2p} \le \frac{\rho(\theta)}{\widehat{\rho}(\theta)}\frac{\widehat{\rho}(\alpha)}{\rho(\alpha)} \le C_2^{2p}$. For convenience, we write $r$ as $m$ and $p$ in the above two cases. From $\rho(\theta) = \frac{(n+|\theta|-1)!}{\theta!(n-1)!} = 1$, we obtain

$$0 < C_1^{2r}\widehat{\rho}(\theta)\rho(\alpha) \le \widehat{\rho}(\alpha) \le C_2^{2r}\widehat{\rho}(\theta)\rho(\alpha), \quad \alpha \in \mathbf{Z}_+^m.$$

So

$$0 < C_1^{2r}\widehat{\rho}(\theta)\sum_{\alpha\in\mathbf{Z}_+^m}\rho(\alpha)w^\alpha\overline{w}^\alpha \le \sum_{\alpha\in\mathbf{Z}_+^m}\widehat{\rho}(\alpha)w^\alpha\overline{w}^\alpha \le C_2^{2r}\widehat{\rho}(\theta)\sum_{\alpha\in\mathbf{Z}_+^m}\rho(\alpha)w^\alpha\overline{w}^\alpha.$$

Let $M_1 := C_1^{2r}\widehat{\rho}(\theta)$ and $M_2 := C_2^{2r}\widehat{\rho}(\theta)$. By (3.1), we have $0 < M_1 \le \frac{h_{\mathbf{T}}(w)}{h_{\mathbf{S}^*}(w)} = \frac{\|\widehat{\gamma}(w)\|^2}{\|\gamma(w)\|^2} \le M_2 < \infty$ and $\log M_1 \le \log\frac{\|\widehat{\gamma}(w)\|^2}{\|\gamma(w)\|^2} \le \log M_2$ for any $w \in \mathbb{B}^m$. This means that $\log\frac{\|\widehat{\gamma}(w)\|^2}{\|\gamma(w)\|^2}$ is a bounded function.

Since $\mathbf{T}$ is $n$-hypercontractive and $\lim_j f_j(\mathbf{T}^*, \mathbf{T})h = 0$ for any $h \in \widehat{\mathcal{H}}$, by Corollary 15 and equation (6) of [3], we obtain that $\|\widehat{\gamma}(w)\|^2 = \|\gamma(w)\|^2 \left\| \left(\triangle_{\mathbf{T}}^{(n)}\right)^{\frac{1}{2}} \widehat{\gamma}(w) \right\|^2$. It follows that

$$\mathcal{K}_{\mathbf{S}^*}(w) - \mathcal{K}_{\mathbf{T}}(w) = \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \overline{w}_j} \log \left\| \left(\triangle_{\mathbf{T}}^{(n)}\right)^{\frac{1}{2}} \widehat{\gamma}(w) \right\|^2 dw_i \wedge d\overline{w}_j, \quad w \in \mathbb{B}^m.$$

By Proposition 2.1 of [7], we know that $\left( \frac{\partial^2}{\partial w_i \partial \overline{w}_j} \log \left\| \left(\triangle_{\mathbf{T}}^{(n)}\right)^{\frac{1}{2}} \widehat{\gamma}(w) \right\|^2 \right)_{i,j=1}^{m} \geq 0$. It follows from Lemma 2.5 that $\psi(w) := \log \left\| \left(\triangle_{\mathbf{T}}^{(n)}\right)^{\frac{1}{2}} \widehat{\gamma}(w) \right\|^2$ is a bounded plurisubharmonic function.

On the other hand, if there is a bounded plurisubharmonic function $\psi$ such that

$$\mathcal{K}_{\mathbf{S}^*}(w) - \mathcal{K}_{\mathbf{T}}(w) = \sum_{i,j=1}^{m} \frac{\partial^2 \psi(w)}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j, \quad w \in \mathbb{B}^m.$$

It follows that $\frac{\partial^2 \psi(w)}{\partial w_i \partial \overline{w}_j} = \frac{\partial^2 \log \frac{\|\widehat{\gamma}(w)\|^2}{\|\gamma(w)\|^2}}{\partial w_i \partial \overline{w}_j}$, that is, $\frac{\partial^2}{\partial w_i \partial \overline{w}_j} \log \frac{\|\widehat{\gamma}(w)\|^2}{\|\gamma(w)\|^2 e^{\psi(w)}} = 0$ for $1 \leq i, j \leq m$. Therefore, there exists a nonzero holomorphic function $\phi$ such that $\frac{\|\widehat{\gamma}(w)\|^2}{\|\phi(w)\gamma(w)\|^2} = e^{|\psi(w)|}$. Since $\psi$ is bounded, there are constants $C_3$ and $C_4$ such that $C_3 \leq |\psi(w)| \leq C_4$. Set $\widetilde{m} := e^{C_3}$ and $\widetilde{M} := e^{C_4}$, then

$$0 < \widetilde{m} \leq \frac{\|\widehat{\gamma}(w)\|^2}{\|\phi(w)\gamma(w)\|^2} \leq \widetilde{M}, \quad w \in \mathbb{B}^m. \tag{3.2}$$

Note that for all $\alpha \in \mathbf{Z}_+^m$, $\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\widehat{\mathbf{e}}_\beta = \begin{cases} \widehat{\rho}(\beta)^{-1}\widehat{\rho}(\beta-\alpha)\widehat{\mathbf{e}}_\beta, & \text{if } \alpha \leq \beta, \\ 0, & \text{if } \alpha \nleq \beta. \end{cases}$ Since $\mathbf{T} = (T_1, \cdots, T_m)$ is an $n$-hypercontraction and $\widehat{\rho}(\alpha) = a(|\alpha|)\frac{|\alpha|!}{\alpha!}$, by Lemma 3.2, we have for $1 \leq k \leq n$ and $\beta \in \mathbf{Z}_+^m$,

$$\begin{aligned}
\langle \triangle_{\mathbf{T}}^{(k)} \widehat{\mathbf{e}}_\beta, \widehat{\mathbf{e}}_\beta \rangle &= \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \frac{k!}{\alpha!(k-|\alpha|)!} \langle \mathbf{T}^{\alpha}\widehat{\mathbf{e}}_\beta, \mathbf{T}^{\alpha}\widehat{\mathbf{e}}_\beta \rangle \\
&= \sum_{\substack{|\alpha| \leq k \\ \alpha \leq \beta}} (-1)^{|\alpha|} \frac{k!}{\alpha!(k-|\alpha|)!} \frac{\widehat{\rho}(\beta-\alpha)}{\widehat{\rho}(\beta)} \\
&= \sum_{\substack{|\alpha| \leq k \\ \alpha \leq \beta}} (-1)^{|\alpha|} \frac{k!}{\alpha!(k-|\alpha|)!} \frac{a(|\beta-\alpha|)|\beta-\alpha|!}{(\beta-\alpha)!} \frac{\beta!}{a(|\beta|)|\beta|!} \\
&= \frac{1}{a(|\beta|)} \sum_{\substack{|\alpha| \leq k \\ \alpha \leq \beta}} (-1)^{|\alpha|} \frac{k!}{|\alpha|!(k-|\alpha|)!} a(|\beta|-|\alpha|) \frac{|\beta-\alpha|!}{(\beta-\alpha)!} \frac{|\alpha|!\beta!}{\alpha!|\beta|!} \\
&= \frac{1}{a(|\beta|)} \sum_{i=0}^{k} \left[ (-1)^i \frac{k!}{i!(k-i)!} a(|\beta|-i) \sum_{\substack{|\alpha|=i \\ \alpha \leq \beta}} \frac{(|\beta|-i)!i!\beta!}{(\beta-\alpha)!\alpha!|\beta|!} \right] \\
&= \frac{1}{a(|\beta|)} \sum_{i=0}^{k} (-1)^i \frac{k!}{i!(k-i)!} a(|\beta|-i) \\
&\geq 0,
\end{aligned}$$

then

$$\sum_{i=0}^{k}(-1)^i\frac{k!}{i!(k-i)!}a(|\beta|-i)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}a(|\beta|-i)\geq 0,\quad |\beta|\geq k. \tag{3.3}$$

Note that

$$\begin{aligned}\frac{h_{\mathbf{T}}(w)}{h_{\mathbf{S}^*}(w)}&=(1-|w|^2)^n\sum_{i=0}^{\infty}a(i)(|w|^2)^i\\&=\sum_{l=0}^{n}[\sum_{j=0}^{l}(-1)^j\binom{n}{j}a(l-j)|w|^{2l}]+\sum_{l=n+1}^{\infty}[\sum_{j=0}^{n}(-1)^j\binom{n}{j}a(l-j)|w|^{2l}].\end{aligned}$$

From the known fact $\binom{n}{i}=\binom{n-1}{i}+\binom{n-1}{i-1}$ for $n\geq 2$ and $i<n$, we obtain that

$$\sum_{i=0}^{m}(-1)^i\binom{n}{i}=(-1)^m\binom{n-1}{m},\quad 0\leq m\leq n-1. \tag{3.4}$$

From (3.2)-(3.4), we have that

$$\begin{aligned}\max_{|w|\leq 1}\frac{h_{\mathbf{T}}(w)}{h_{\mathbf{S}^*}(w)}&=\sum_{l=0}^{n}[\sum_{j=0}^{l}(-1)^j\binom{n}{j}a(l-j)]+\sum_{l=n+1}^{\infty}[\sum_{j=0}^{n}(-1)^j\binom{n}{j}a(l-j)]\\&=\lim_{k\to\infty}[a(k)+(\sum_{i=0}^{1}(-1)^i\binom{n}{i})a(k-1)+\cdots+(\sum_{i=0}^{n-1}(-1)^i\binom{n}{i})a(k-n+1)]\\&=\lim_{k\to\infty}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}a(k-i).\end{aligned}$$

Then there are positive constants $m':=a(0)$ and $M'<\infty$ such that

$$0<m'\leq\lim_{k\to\infty}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}a(k-i)\leq M' \tag{3.5}$$

and

$$0<m'\leq\frac{h_{\mathbf{T}}(w)}{h_{\mathbf{S}^*}(w)}=\frac{\|\widehat{\gamma}(w)\|^2}{\|\gamma(w)\|^2}\leq M',\quad w\in\mathbb{B}^m.$$

Otherwise, we have $\frac{\|\widehat{\gamma}(w)\|^2}{\|\gamma(w)\|^2}\to\infty$ as $|w|\to 1$. From (3.2), we know that $\frac{1}{|\phi(w)|}\to 0$ as $|w|\to 1$. By the maximum modulus principle of holomorphic functions, we have that $\frac{1}{|\phi(w)|}=0$ for any $w\in\mathbb{B}^m$. This is a contradiction.

**Claim:** For any positive integer $n$ and any positive sequence $\{a(k)\}_{k\geq n-1}$, if $n$ and $\{a(k)\}_{k\geq n-1}$ satisfy inequality (3.5), then there exists non-zero constants $c_1$ and $c_2$ such that

$$0<c_1\leq\lim_{k\to\infty}\frac{a(k)}{k^{n-1}}\leq c_2,$$

denoted by $a(k)=O(k^{n-1})(k\to\infty)$.

In the case of $n=1$, it is obviously true.

When $n=2$, from (3.5), we know that $m'\leq a(k)-a(k-1)\leq M'$. Furthermore,

$$km'+a(0)\leq a(k)\leq kM'+a(0),$$

which means that $a(k)=O(k)(k\to\infty)$ holds.

Assume that the Claim is valid for $n = t$. That is, $a(k) = O(k^{t-1})(k \to \infty)$. We will prove that the Claim is also valid when $n = t + 1$. Set

$$u(k) := \begin{cases} a(k) - a(k-1), & k \geq 1, \\ a(0), & k = 0. \end{cases}$$

Since $\mathbf{T}$ is a contraction, the positive sequence $\{a(k)\}_{k\geq 0}$ is increasing, $u(k) \geq 0$ for any nonnegative integer $k$. By (3.5) and $\binom{t}{i} = \binom{t-1}{i} + \binom{t-1}{i-1}$ for $t \geq 2$ and $i < t$, we have

$$0 < m' \leq \lim_{k\to\infty} \sum_{i=0}^{t} (-1)^i \binom{t}{i} a(k-i) = \lim_{k\to\infty} \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} u(k-i) \leq M'.$$

Therefore, $u(k) = O(k^{t-1})(k \to \infty)$. That is, there exist $m_1$ and $M_1$ such that

$$0 < m_1 k^{t-1} \leq u(k) = a(k) - a(k-1) \leq M_1 k^{t-1}.$$

So $m_1 \sum\limits_{i=1}^{k} i^{t-1} + a(0) \leq a(k) \leq M_1 \sum\limits_{i=1}^{k} i^{t-1} + a(0)$. Set $l_{t-1} := \sum\limits_{i=1}^{k} i^{t-1}$. It is well-known that $l_{t-1} = \frac{1}{t}\Big[(1+k)^t - (1+k) - (\sum\limits_{i=2}^{t-1} \binom{t}{i} l_{t-i})\Big]$. Then

$$\sum_{i=1}^{k} i^{t-1} = O(k^t)(k \to \infty) \quad \text{and} \quad a(k) = O(k^t)(k \to \infty).$$

This proves that the Claim is true.

By the Claim, (3.5) and $\widehat{\rho}(\alpha) = a(|\alpha|)\frac{|\alpha|!}{\alpha!}$, we have $\widehat{\rho}(\alpha) = O(|\alpha|^{n-1}\frac{|\alpha|!}{\alpha!})(|\alpha| \to \infty)$. Thus, for any $\alpha \in \mathbf{Z}_+^m$, $1 \leq i \leq m$ and any positive integer $l$,

$$\frac{\prod\limits_{k=0}^{l} \sqrt{\frac{\rho(\alpha+ke_i)}{\rho(\alpha+(k+1)e_i)}}}{\prod\limits_{k=0}^{l} \sqrt{\frac{\widehat{\rho}(\alpha+ke_i)}{\widehat{\rho}(\alpha+(k+1)e_i)}}} = \sqrt{\frac{\rho(\alpha)\widehat{\rho}(\alpha+(l+1)e_i)}{\widehat{\rho}(\alpha)\rho(\alpha+(l+1)e_i)}}$$

$$= O\left(\sqrt{\frac{(n+|\alpha|-1)!}{(n+|\alpha|+l)!} \frac{|\alpha+(l+1)e_i|^{n-1}}{|\alpha|^{n-1}} \frac{|\alpha+(l+1)e_i|!}{|\alpha|!}}\right), \quad |\alpha| \to \infty.$$

Note that for any positive integer $l$,

$$\frac{(n+|\alpha|-1)!}{(n+|\alpha|+l)!} \frac{|\alpha+(l+1)e_i|^{n-1}}{|\alpha|^{n-1}} \frac{|\alpha+(l+1)e_i|!}{|\alpha|!} \longrightarrow 1, \quad |\alpha| \to \infty.$$

Therefore, for any positive integer $l$ and $\varepsilon_0$, $0 < \varepsilon_0 < 1$, there is an integer $N_1$, so that when $|\alpha| \geq N_1$, we have

$$1 - \varepsilon_0 \leq \sqrt{\frac{(n+|\alpha|-1)!}{(n+|\alpha|+l)!} \frac{|\alpha+(l+1)e_i|^{n-1}}{|\alpha|^{n-1}} \frac{|\alpha+(l+1)e_i|!}{|\alpha|!}} \leq 1 + \varepsilon_0.$$

Then for positive integer $l$, we will find positive constants $m_2$ and $M_2$ satisfy

$$0 < (1-\varepsilon_0)m_2 \leq \sqrt{\frac{\rho(\alpha)\widehat{\rho}(\alpha+(l+1)e_i)}{\widehat{\rho}(\alpha)\rho(\alpha+(l+1)e_i)}} \leq (1+\varepsilon_0)M_2, \quad |\alpha| \geq N_1.$$

When $|\alpha| < N_1$, we have

$$\sqrt{\frac{\rho(\alpha)\widehat{\rho}(\alpha+(l+1)e_i)}{\widehat{\rho}(\alpha)\rho(\alpha+(l+1)e_i)}} = O\left(\sqrt{\frac{\rho(\alpha)}{\widehat{\rho}(\alpha)}}\sqrt{\frac{(|\alpha|+l+1)^{n-1}(n-1)!}{(|\alpha|+l+2)(|\alpha|+l+3)\cdots(|\alpha|+l+n)}}\right), \quad l \to \infty$$

and $\frac{(|\alpha|+l+1)^{n-1}}{(|\alpha|+l+2)(|\alpha|+l+3)\cdots(|\alpha|+l+n)} = O(1)(l \to \infty)$. Let

$$m_3 := \min\left\{\sqrt{\frac{(n-1)!\rho(\alpha)}{\widehat{\rho}(\alpha)}} : |\alpha| < N_1, \alpha \in \mathbf{Z}_+^m\right\},\ M_3 := \max\left\{\sqrt{\frac{(n-1)!\rho(\alpha)}{\widehat{\rho}(\alpha)}} : |\alpha| < N_1, \alpha \in \mathbf{Z}_+^m\right\}.$$

Then there are $\varepsilon_1 (0 < \varepsilon_1 < 1)$ and positive integer $N_2$ such that

$$0 < m_3(1-\varepsilon_1) \leq \sqrt{\frac{\rho(\alpha)}{\widehat{\rho}(\alpha)}}\sqrt{\frac{(|\alpha|+l+1)^{n-1}(n-1)!}{(|\alpha|+l+2)(|\alpha|+l+3)\cdots(|\alpha|+l+n)}} \leq M_3(1+\varepsilon_1),$$

when $|\alpha| < N_1$ and $l \geq N_2$. Thus there are positive constants $m_4$ and $M_4$ such that

$$0 < m_3(1-\varepsilon_1)m_4 \leq \sqrt{\frac{\rho(\alpha)\widehat{\rho}(\alpha+(l+1)e_i)}{\widehat{\rho}(\alpha)\rho(\alpha+(l+1)e_i)}} \leq M_3(1+\varepsilon_1)M_4, \quad |\alpha| < N_1, l \geq N_2.$$

Let $m_5$ and $M_5$ be the minimum and maximum values of $\left\{\sqrt{\frac{\rho(\alpha)\widehat{\rho}(\alpha+(l+1)e_i)}{\widehat{\rho}(\alpha)\rho(\alpha+(l+1)e_i)}} : |\alpha| < N_1, l < N_2\right\}$, respectively. Let

$$C_1' = \min\left\{(1-\varepsilon_0)m_2, m_3(1-\varepsilon_1)m_4, m_5\right\} \quad \text{and} \quad C_2' = \max\left\{(1+\varepsilon_0)M_2, M_3(1+\varepsilon_1)M_4, M_5\right\}.$$

Thus, we infer that

$$0 < C_1' \leq \frac{\prod_{k=0}^{l}\sqrt{\frac{\rho(\alpha+ke_i)}{\rho(\alpha+(k+1)e_i)}}}{\prod_{k=0}^{l}\sqrt{\frac{\widehat{\rho}(\alpha+ke_i)}{\widehat{\rho}(\alpha+(k+1)e_i)}}} \leq C_2', 1 \leq i \leq m$$

for any nonnegative integer $l$ and $\alpha \in \mathbf{Z}_+^m$. It is obtained from Theorem 3.1 that $\mathbf{T}$ is similar to $\mathbf{S}^*$. $\square$

## 4. The $N$-hypercontractive of commuting $m$-tuples of backward weighted shifts

In this section, we mainly consider $n$-hypercontractive descriptions of some commuting $m$-tuples. In the following, we give a necessary condition for a commuting $m$-tuple of backward weighted shifts to be $n$-hypercontractive, and use some specific examples to show that the $n$-hypercontractivity in Theorem 3.3 is indispensable.

Before presenting the main theorem of this section, we need to prove the following lemma.

**Lemma 4.1.** *Let $n \geq 2$ be an integer. For integers $k, l$, if $2 \leq k \leq n$ and $l > n$, then the following equations hold:*

(1) $\sum_{i=0}^{k}(-1)^{-i}\binom{n-2+i}{i}\binom{n}{k-i} = 0$ *and* $\sum_{i=1}^{k}(-1)^{-i}\binom{n-2+i}{i}\binom{n}{k-i}i = 0$;

(2) $\sum_{i=l-n}^{l}(-1)^{-i}\binom{n-2+i}{i}\binom{n}{l-i} = 0$ *and* $\sum_{i=l-n}^{l}(-1)^{-i}\binom{n-2+i}{i}\binom{n}{l-i}i = 0.$

*Proof.* Since $(1-x)^n = \sum\limits_{i=0}^{n} \binom{n}{i}(-x)^i$ and $\frac{1}{(1-x)^n} = \sum\limits_{i=0}^{\infty} \binom{n-1+i}{i} x^i$ for $|x|<1$, we have that

$$\begin{aligned}
1-x &= (1-x)^n \frac{1}{(1-x)^{n-1}} \\
&= \Big[\sum_{j=0}^{n} \binom{n}{j}(-x)^j\Big]\Big[\sum_{i=0}^{\infty} \binom{n-2+i}{i} x^i\Big] \\
&= \sum_{k=0}^{n} \Big[\sum_{i=0}^{k} (-1)^{k-i} \binom{n-2+i}{i}\binom{n}{k-i}\Big] x^k + \sum_{k=n+1}^{\infty} \Big[\sum_{i=k-n}^{k} (-1)^{k-i} \binom{n-2+i}{i}\binom{n}{k-i}\Big] x^k
\end{aligned}$$

and

$$\begin{aligned}
n-1 &= (1-x)^n \Big(\frac{1}{(1-x)^{n-1}}\Big)' \\
&= \Big[\sum_{j=0}^{n} \binom{n}{j}(-x)^j\Big]\Big[\sum_{i=1}^{\infty} \binom{n-2+i}{i} i x^{i-1}\Big] \\
&= \sum_{k=1}^{n} \Big[\sum_{i=1}^{k} (-1)^{k-i} i \binom{n-2+i}{i}\binom{n}{k-i}\Big] x^{k-1} + \sum_{k=n+1}^{\infty} \Big[\sum_{i=k-n}^{k} (-1)^{k-i} i \binom{n-2+i}{i}\binom{n}{k-i}\Big] x^{k-1}.
\end{aligned}$$

This Lemma can be proved by comparing the coefficients of $x^k$ in the above two equations. □

**Theorem 4.2.** *Let $m \geq 2$ be a positive integer and $\mathbf{T} = (T_1, \cdots, T_m)$ be a commuting $m$-tuple of backward weighted shifts on Hilbert space $\mathcal{H}$ with reproducing kernel $K(z,w) = \sum\limits_{\alpha \in \mathbf{Z}_+^m} \rho(\alpha) z^\alpha \overline{w}^\alpha$, where $\rho(\alpha) > 0$ and $z, w \in \Omega$. If $\mathbf{T}$ is $n$-hypercontractive, then for any $\alpha \in \mathbf{Z}_+^m$,*

$$\sum_{\substack{\beta \in \mathbf{Z}_+^m, \beta \leq \alpha \\ |\alpha-\beta|=1}} \frac{\rho(\beta)}{\rho(\alpha)} \leq \frac{|\alpha|}{|\alpha|+n-1}.$$

*Proof.* It is easy to see that $\{\mathbf{e}_\alpha : \mathbf{e}_\alpha(w) = \sqrt{\rho(\alpha)} w^\alpha\}_{\alpha \in \mathbf{Z}_+^m}$ is an orthonormal basis of the Hilbert space $\mathcal{H}$. If $\mathbf{T}$ is $n$-hypercontractive, we know that $\triangle_{\mathbf{T}}^{(k)}$ is non-negative definite for all $1 \leq k \leq n$. Note that

$$\triangle_{\mathbf{T}}^{(k)} \mathbf{e}_\alpha = \sum_{\substack{\beta \in \mathbf{Z}_+^m \\ |\beta| \leq k}} (-1)^{|\beta|} \frac{k!}{\beta!(k-|\beta|)!} \mathbf{T}^{*\beta} \mathbf{T}^\beta \mathbf{e}_\alpha = \sum_{\substack{\beta \in \mathbf{Z}_+^m \\ \beta \leq \alpha, |\beta| \leq k}} (-1)^{|\beta|} \frac{k!}{\beta!(k-|\beta|)!} \frac{\rho(\alpha-\beta)}{\rho(\alpha)} \mathbf{e}_\alpha.$$

It follows that for any $\alpha \in \mathbf{Z}_+^m$ and $1 \leq k \leq n$,

$$\sum_{\substack{\beta \in \mathbf{Z}_+^m \\ \beta \leq \alpha, |\beta| \leq k}} (-1)^{|\beta|} \frac{k!}{\beta!(k-|\beta|)!} \frac{\rho(\alpha-\beta)}{\rho(\alpha)} \geq 0. \tag{4.1}$$

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For a fixed but arbitrary $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{Z}_+^m$, suppose that the set $\{i : \alpha_i \neq 0\}$ has $l$ elements. Then we will prove that

$$\sum_{\substack{\beta \in \mathbf{Z}_+^m \\ \beta \le \alpha, |\beta| \le n}} (-1)^{|\beta|} \frac{n!}{\beta!(n-|\beta|)!} \frac{\rho(\alpha-\beta)}{\rho(\alpha)} \tag{4.2}$$
$$= 1 - \sum_{j=1}^{l} (n + x_{e_{i_j}}) \frac{\rho(\alpha - e_{i_j})}{\rho(\alpha)} + \sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \theta < \gamma < \alpha}} x_\gamma \Big[ \sum_{\substack{\xi \in \mathbf{Z}_+^m, |\xi| \le n \\ \xi \le \alpha - \gamma}} (-1)^{|\xi|} \frac{n!}{\xi!(n-|\xi|)!} \frac{\rho(\alpha-\gamma-\xi)}{\rho(\alpha-\gamma)} \Big] \frac{\rho(\alpha-\gamma)}{\rho(\alpha)},$$

where $x_\gamma = -\frac{(n-2+|\gamma|)!}{\gamma!(n-2)!} \frac{|\alpha-\gamma|}{|\alpha|}$ and $\{i_j : 1 \le j \le l\} = \{i : \alpha_i \neq 0\}$. Note that

$$\sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \theta < \gamma < \alpha}} x_\gamma \Big[ \sum_{\substack{\xi \in \mathbf{Z}_+^m, |\xi| \le n \\ \xi \le \alpha - \gamma}} (-1)^{|\xi|} \frac{n!}{\xi!(n-|\xi|)!} \frac{\rho(\alpha-\gamma-\xi)}{\rho(\alpha-\gamma)} \Big] \frac{\rho(\alpha-\gamma)}{\rho(\alpha)}$$
$$= \sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \theta < \gamma < \alpha}} \sum_{\substack{\xi \in \mathbf{Z}_+^m, |\xi| \le n \\ \gamma \le \xi + \gamma \le \alpha}} (-1)^{|\xi|} x_\gamma \frac{n!}{\xi!(n-|\xi|)!} \frac{\rho(\alpha-(\gamma+\xi))}{\rho(\alpha)} \tag{4.3}$$
$$= \sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \theta < \gamma < \alpha}} \sum_{\substack{\beta \in \mathbf{Z}_+^m \\ \gamma \le \beta \le \alpha \\ |\beta-\gamma| \le n}} (-1)^{|\beta|-|\gamma|+1} \frac{(n-2+|\gamma|)!}{\gamma!(n-2)!} \frac{|\alpha-\gamma|}{|\alpha|} \frac{n!}{(\beta-\gamma)!(n-|\beta-\gamma|)!} \frac{\rho(\alpha-\beta)}{\rho(\alpha)},$$

then we will prove (4.2) in the following three cases.

**Case 1:** When $|\beta| = 0, 1$. It is easy to see that (4.2) holds.

**Case 2:** When $2 \le |\beta| \le n$, from Lemma 3.2 and Lemma 4.1, we have that

$$\sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \theta < \gamma \le \beta}} (-1)^{|\beta|-|\gamma|+1} \frac{(n-2+|\gamma|)!}{\gamma!(n-2)!} \frac{|\alpha-\gamma|}{|\alpha|} \frac{n!}{(\beta-\gamma)!(n-|\beta-\gamma|)!} - (-1)^{|\beta|} \frac{n!}{\beta!(n-|\beta|)!}$$
$$= \sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \gamma \le \beta}} (-1)^{|\beta|-|\gamma|+1} \frac{(n-2+|\gamma|)!}{\gamma!(n-2)!} \frac{|\alpha-\gamma|}{|\alpha|} \frac{n!}{(\beta-\gamma)!(n-|\beta-\gamma|)!}$$
$$= \sum_{i=0}^{|\beta|} (-1)^{|\beta|-i+1} \frac{(n-2+i)!}{(n-2)!} \frac{|\alpha|-i}{|\alpha|} \frac{n!}{(n-|\beta|+i)!} \Big( \sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \gamma \le \beta \\ |\gamma| = i}} \frac{1}{\gamma!(\beta-\gamma)!} \Big)$$
$$= (-1)^{|\beta|+1} \frac{|\beta|!}{\beta!} \sum_{i=0}^{|\beta|} (-1)^{-i} \frac{(n-2+i)!}{i!(n-2)!} \frac{n!}{(|\beta|-i)!(n-|\beta|+i)!} \frac{|\alpha|-i}{|\alpha|}$$
$$= (-1)^{|\beta|+1} \frac{|\beta|!}{\beta!} \left\{ \Big[ \sum_{i=0}^{|\beta|} (-1)^{-i} \binom{n-2+i}{i} \binom{n}{|\beta|-i} \Big] - \frac{1}{|\alpha|} \Big[ \sum_{i=1}^{|\beta|} (-1)^{-i} \binom{n-2+i}{i} \binom{n}{|\beta|-i} i \Big] \right\}$$
$$= 0.$$

This means that (4.2) is valid in this case.

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**Case 3:** When $n < |\beta| \le |\alpha|$, from Lemma 3.2 and Lemma 4.1 again, we have that

$$\begin{aligned}
&\sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \theta < \gamma \le \beta \\ |\beta-\gamma| \le n}} (-1)^{|\beta|-|\gamma|+1} \frac{(n-2+|\gamma|)!}{\gamma!(n-2)!} \frac{|\alpha-\gamma|}{|\alpha|} \frac{n!}{(\beta-\gamma)!(n-|\beta-\gamma|)!} \\
&= \sum_{i=|\beta|-n}^{|\beta|} (-1)^{|\beta|-i+1} \frac{(n-2+i)!}{(n-2)!} \frac{|\alpha|-i}{|\alpha|} \frac{n!}{(n-|\beta|+i)!} \Big( \sum_{\substack{\gamma \in \mathbf{Z}_+^m \\ \gamma \le \beta, |\gamma| = i}} \frac{1}{\gamma!(\beta-\gamma)!} \Big) \\
&= (-1)^{|\beta|+1} \frac{|\beta|!}{\beta!} \sum_{i=|\beta|-n}^{|\beta|} (-1)^{-i} \frac{(n-2+i)!}{i!(n-2)!} \frac{n!}{(|\beta|-i)!(n-|\beta|+i)!} \frac{|\alpha|-i}{|\alpha|} \\
&= (-1)^{|\beta|+1} \frac{|\beta|!}{\beta!} \Big\{ \Big[ \sum_{i=|\beta|-n}^{|\beta|} (-1)^{-i} \binom{n-2+i}{i} \binom{n}{|\beta|-i} \Big] - \frac{1}{|\alpha|} \Big[ \sum_{i=|\beta|-n}^{|\beta|} (-1)^{-i} \binom{n-2+i}{i} \binom{n}{|\beta|-i} i \Big] \Big\} \\
&= 0.
\end{aligned}$$

This shows that (4.2) is also valid in Case 3.

From (4.1), (4.2) and $x_\gamma = -\frac{(n-2+|\gamma|)!}{\gamma!(n-2)!} \frac{|\alpha-\gamma|}{|\alpha|} < 0$ for $\theta < \gamma < \alpha$, we know that $1 - \sum_{j=1}^{l} (n + x_{e_{i_j}}) \frac{\rho(\alpha-e_{i_j})}{\rho(\alpha)} = 1 - \frac{|\alpha|+n-1}{|\alpha|} \sum_{j=1}^{l} \frac{\rho(\alpha-e_{i_j})}{\rho(\alpha)} > 0$, then $\sum_{j=1}^{l} \frac{\rho(\alpha-e_{i_j})}{\rho(\alpha)} \le \frac{|\alpha|}{|\alpha|+n-1}$, where $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{Z}_+^m$ and $\alpha_{i_j} \neq 0$ for $1 \le j \le l$. That is, $\sum_{\substack{\beta \in \mathbf{Z}_+^m, \beta \le \alpha \\ |\alpha-\beta| = 1}} \frac{\rho(\beta)}{\rho(\alpha)} \le \frac{|\alpha|}{|\alpha|+n-1}$. $\square$

**Corollary 4.3.** *Let $\mathbf{T} = (T_1, \cdots, T_m)$ be the adjoint of multiplication tuple on the reproducing kernel Hilbert space determined by the kernel function $K(z,w) = \sum_{i=0}^{\infty} a(i)(z_1\overline{w}_1 + \cdots + z_m\overline{w}_m)^i$, $a(i) > 0$. If $\mathbf{T}$ is $n$-hypercontractive, then $\frac{a(i-1)}{a(i)} \le \frac{i}{i+n-1}$ for any positive integer $i$.*

Recall the definition that a commuting $m$-tuple of operators is subnormal. Let $\mathbf{T} = (T_1, \cdots, T_m) \in \mathcal{L}(\mathcal{H})^m$. If there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a commuting $m$-tuple of normal operators $N_i, 1 \le i \le m$, in $\mathcal{L}(\mathcal{K})$ such that $\mathbf{T}$ extends to $\mathbf{N} = (N_1, \cdots, N_m)$, then $\mathbf{T}$ is said to be subnormal and $\mathbf{N}$ is said to be a commuting normal extension of $\mathbf{T}$, see Definition 5.1 in [6]. For a commuting $m$-tuple $\widehat{\mathbf{T}} = (\widehat{T}_1, \cdots, \widehat{T}_m)$, where $\widehat{T}_i, 1 \le i \le m$, are subnormal, the lifting problem asks whether $\widehat{\mathbf{T}}$ has a commuting normal extension, that is, $\widehat{\mathbf{T}}$ is subnormal. In [18] and [27], a negative answer to the above problem is given with the help of the contractive commuting $m$-tuple of unilateral weighted shifts with weight sequence $\left\{ \lambda_\alpha^{(i)} = \sqrt{\frac{\alpha_i+1}{|\alpha|+1}} : 1 \le i \le m \right\}_{\alpha \in \mathbf{Z}_+^m}$. In the following, we prove that the commuting $m$-tuple of unilateral weighted shifts whose spectrum is contained in $\overline{\mathbb{B}^m}$ is not subnormal.

**Corollary 4.4.** *The adjoint of commuting $m$-tuples of unilateral weighted shifts whose spectrum is contained in $\overline{\mathbb{B}^m}$ is not subnormal.*

*Proof.* By Theorem 5.2 in [6], we know that for a commuting $m$-tuple whose spectrum is contained in $\overline{\mathbb{B}^m}$, it is $n$-hypercontractive for all positive integers $n$ if and only if it is subnormal. Let $\mathbf{T} = (T_1, \cdots, T_m)$ be the adjoint of commuting $m$-tuple of unilateral weighted shifts on Hilbert space $\mathcal{H}$ with reproducing kernel $K(z,w)=\sum\limits_{\alpha\in\mathbf{Z}_+^m}\rho(\alpha)z^\alpha\overline{w}^\alpha$. Assume that $\mathbf{T}=(T_1,\cdots,T_m)$ is subnormal. It follows that $\mathbf{T}$ is $n$-hypercontractive for all integers $n>0$. By Theorem 4.2, for any $\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbf{Z}_+^m$, without losing generality, we assume that $\alpha_i\neq 0$, $1\leq i\leq m$, then

$$\frac{\rho(\alpha-e_i)}{\rho(\alpha)}\leq\sum_{\substack{\beta\in\mathbf{Z}_+^m,\,\beta\leq\alpha\\|\alpha-\beta|=1}}\frac{\rho(\beta)}{\rho(\alpha)}\leq\frac{|\alpha|}{|\alpha|+n-1}.$$

Since $\lim\limits_{n\to\infty}\frac{|\alpha|}{|\alpha|+n-1}=0$, we have $\frac{\rho(\alpha-e_i)}{\rho(\alpha)}=0$. This is a contradiction. $\square$

The following example shows that the $n$-hypercontractivity condition in Theorem 3.3 is necessary.

**Example 4.5.** For the adjoint $\mathbf{S}^*=(S_1^*,\cdots,S_m^*)$ of the multiplication tuple on $H_n^2$, there exists an $m$-tuple $\mathbf{T}=(T_1,\cdots,T_m)$ that is not an $n$-hypercontraction and a positive, bounded, real-analytic function $\varphi$ defined on $\mathbb{B}^m$ such that

$$\mathcal{K}_{\mathbf{S}^*}(w)-\mathcal{K}_{\mathbf{T}}(w)=\sum_{i,j=1}^m\frac{\partial^2\varphi(w)}{\partial w_i\partial\overline{w}_j}dw_i\wedge d\overline{w}_j,\quad w\in\mathbb{B}^m.$$

Moreover, $\mathbf{T}$ is not similar to $\mathbf{S}^*$.

*Proof.* Let $\mathbf{T}$ be the adjoint of multiplication tuple on some Hilbert space with reproducing kernel $K(z,w)=\sum\limits_{\alpha\in\mathbf{Z}_+^m}\widetilde{\rho}(\alpha)z^\alpha\overline{w}^\alpha$. Then $\mathcal{K}_{\mathbf{S}^*}(w)-\mathcal{K}_{\mathbf{T}}(w)=\sum\limits_{i,j=1}^m\frac{\partial^2}{\partial w_i\partial\overline{w}_j}\log[K(w,w)(1-|w|^2)^n]dw_i\wedge d\overline{w}_j$. Note that $\{\mathbf{e}_\alpha:\mathbf{e}_\alpha(z)=\sqrt{\rho(\alpha)}z^\alpha\}_{\alpha\in\mathbf{Z}_+^m}$ is an orthonormal basis of $H_n^2$ and $S_i^*\mathbf{e}_\alpha=\sqrt{\frac{\rho(\alpha-e_i)}{\rho(\alpha)}}\mathbf{e}_{\alpha-e_i}$, where $\rho(\alpha)=\frac{(n+|\alpha|-1)!}{\alpha!(n-1)!}$. Letting

$$\widetilde{\rho}(\alpha)=\begin{cases}\frac{\rho(\alpha)}{k}, & \alpha=\beta^l+ke_1,\ 1\leq k\leq l,\\ \frac{\rho(\alpha)}{k}, & \alpha=\beta^l+(2l-k)e_1,\ 1\leq k\leq l,\\ \rho(\alpha), & otherwise,\end{cases}$$

where $\{\beta^l=(\beta_1^l,\cdots,\beta_m^l)\}_{l=1}^\infty\subseteq\mathbf{Z}_+^m$ satisfying $|\beta^l|>\max\left\{\frac{n^n2^{3l+1}}{(n-1)!}-n,n-2\right\}$ and $|\beta^l|+2l<|\beta^{l+1}|$, then $[\widetilde{\rho}(\alpha)-\rho(\alpha)]\frac{\alpha!}{|\alpha|!}$ is only related to $|\alpha|$ (not $\alpha$), and

$$\begin{aligned}K(w,w)&=\frac{1}{(1-|w|^2)^n}+\sum_{l=0}^\infty\Big[\sum_{\substack{\alpha\in\mathbf{Z}_+^m\\|\alpha|=l}}\left(\widetilde{\rho}(\alpha)-\rho(\alpha)\right)w^\alpha\overline{w}^\alpha\Big]\\&=\frac{1}{(1-|w|^2)^n}+\sum_{l=2}^\infty\Big\{\sum_{k=2}^l\Big(\widetilde{\rho}(\beta^l+ke_1)-\rho(\beta^l+ke_1)\Big)\frac{(\beta^l+ke_1)!}{|\beta^l+ke_1|!}|w|^{2|\beta^l+ke_1|}\\&\quad+\sum_{k=2}^{l-1}\Big(\widetilde{\rho}(\beta^l+(2l-k)e_1)-\rho(\beta^l+(2l-k)e_1)\Big)\frac{(\beta^l+(2l-k)e_1)!}{|\beta^l+(2l-k)e_1|!}|w|^{2|\beta^l+(2l-k)e_1|}\Big\}\\&:=\frac{1}{(1-|w|^2)^n}+\sum_{l=2}^\infty g_l(w).\end{aligned}$$

Since $|\beta^l| > n-2$ for all $l \geq 1$, we have

$$\begin{aligned}
&|g_l(w)| \\
&= \tfrac{(n+|\beta^l|-1)!}{|\beta^l|!(n-1)!}|w|^{2|\beta^l|}\left|\sum_{k=2}^{l} \tfrac{(n+|\beta^l+ke_1|-1)!|\beta^l|!}{(n+|\beta^l|-1)!|\beta^l+ke_1|!}\left(\tfrac{1}{k}-1\right)|w|^{2k} + \sum_{k=2}^{l-1} \tfrac{(n+|\beta^l+(2l-k)e_1|-1)!|\beta^l|!}{(n+|\beta^l|-1)!|\beta^l+(2l-k)e_1|!}\left(\tfrac{1}{k}-1\right)|w|^{4l-2k}\right| \\
&\leq \tfrac{(n+|\beta^l|-1)!}{|\beta^l|!(n-1)!}|w|^{2|\beta^l|}\left|\sum_{k=2}^{l} 2^k\left(\tfrac{1}{k}-1\right)|w|^{2k} + \sum_{k=2}^{l-1} 2^{2l-k}\left(\tfrac{1}{k}-1\right)|w|^{4l-2k}\right|
\end{aligned}$$

and $M_l = \sup\limits_{|w|<1}\left|\sum_{k=2}^{l} 2^k\left(\frac{1}{k}-1\right)|w|^{2k} + \sum_{k=2}^{l-1} 2^{2l-k}\left(\frac{1}{k}-1\right)|w|^{4l-2k}\right| < 2^{2l-1}$. Setting $f(x) = x^{|\beta^l|}(1-x)^n$ for $0 \leq x \leq 1$. Then $f(x)$ attains a maximum of $\left(\frac{|\beta^l|}{|\beta^l|+n}\right)^{|\beta^l|}\left(\frac{n}{|\beta^l|+n}\right)^n$ at $x = \frac{|\beta^l|}{|\beta^l|+n}$. Therefore, from $|\beta^l| > \frac{n^n 2^{3l+1}}{(n-1)!} - n$,

$$|g_l(w)|(1-|w|^2)^n \leq M_l \frac{(n+|\beta^l|-1)!}{|\beta^l|!(n-1)!}|w|^{2|\beta^l|}(1-|w|^2)^n \leq 2^{2l-1}\frac{(n+|\beta^l|-1)!}{|\beta^l|!(n-1)!}\frac{|\beta^l|^{|\beta^l|}n^n}{(|\beta^l|+n)^{|\beta^l|+n}} \leq \frac{2^{2l-1}n^n}{(n-1)!(|\beta^l|+n)} \leq \frac{1}{2^{l+2}}$$

for any $w \in \mathbb{B}^m$. Since $K(w,w)(1-|w|^2)^n = 1 + \sum_{l=2}^{\infty} g_l(w)(1-|w|^2)^n$, we know that $\frac{7}{8} < K(w,w)(1-|w|^2)^n < \frac{9}{8}$. So $K(w,w)(1-|w|^2)^n$ is bounded and positive. For any $l > 2$ and $\alpha = \beta^l + (l-1)e_1 \in \mathbf{Z}_+^m$, we suppose that $\alpha_1 \neq 0$ and $\alpha_i = 0$ for $2 \leq i \leq m$, then

$$\frac{\widetilde{\rho}(\alpha - e_1)}{\widetilde{\rho}(\alpha)} = \frac{\widetilde{\rho}(\beta^l + (l-2)e_1)}{\widetilde{\rho}(\beta^l + (l-1)e_1)} = \frac{l-1}{l-2}\frac{|\alpha|}{|\alpha|+n-1} > \frac{|\alpha|}{|\alpha|+n-1}$$

and $\dfrac{\prod_{k=1}^{l-1}\frac{\widetilde{\rho}(\beta^l+ke_1)}{\widetilde{\rho}(\beta^l+(k+1)e_1)}}{\prod_{k=1}^{l-1}\frac{\rho(\beta^l+ke_1)}{\rho(\beta^l+(k+1)e_1)}} = \frac{\rho(\beta^l+le_1)}{\widetilde{\rho}(\beta^l+le_1)} = l \to +\infty$ as $l \to +\infty$. Thus, from Theorem 3.1 and Theorem 4.2 , $\mathbf{T}$ is not an $n$-hypercontraction and $\mathbf{T}$ is not similar to $\mathbf{S}^*$. $\square$

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