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ISOMETRIC ACTIONS ARE QUASIDIAGONAL

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ABSTRACT. We show every isometric action of a discrete group on a compact space is quasidiagonal in a strong sense. This shows that reduced crossed products by such actions are quasidiagonal or MF whenever the reduced group C^*-algebra of the acting group is quasidiagonal or MF. We use this to show new examples of group actions whose crossed products are MF.

We begin with the definition of quasidiagonality for group actions, which describes actions which are approximated by actions on finite dimensional C^*-algebras.

Definition 1. *An action $\Gamma \curvearrowright A$ of a group on a unital C^*-algebra is quasidiagonal if for all $F \subset A$, $S \subset \Gamma$ finite, and $\varepsilon > 0$ there exists $k > 0$, a unital completely positive map $\phi : A \to M_k(\mathbb{C})$, and an action $\Gamma \curvearrowright M_k(\mathbb{C})$ such that*

(1): $||\phi(ab) - \phi(a)\phi(b)|| < \varepsilon$ *for all* $a, b \in F$

(2): $||\phi(\gamma \cdot a) - \gamma \cdot \phi(a)|| < \varepsilon$ *for all* $a \in F$, $\gamma \in S$.

Quasidiagonality of group actions was first introduced in [3, 3.2] where it was shown to relate to quasidiagonality and the MF property when passing to the crossed product C^*-algebra. Our reason for studying this property is to show general conditions on a group action which imply the crossed product is MF. Speaking loosely, the MF property means that a C^*-algebra is a generalized inductive limit of finite dimensional C^*-algebras [1, 3.2.1]. More concretely, this means in the separable case that such a C^*-algebra embeds inside $\prod_n M_n(\mathbb{C}) / \bigoplus_n M_n(\mathbb{C})$ [1, 3.2.2]. One can see this is a weaker condition than quasidiagonality of C^*-algebras since the latter is equivalent to the existence of such an embedding with a completely positive lift. While there are general theorems about when a crossed product C^*-algebra is quasidiagonal, somewhat less is known in this regard about the MF property. We will show that all isometric actions are quasidiagonal and therefore that their reduced crossed products are MF whenever the reduced C^*-algebra of the acting group is MF. This can be thought of as a spacial analog of the Peter-Weyl theorem where we have replaced an isometric representation by an isometric action.

The proof of the main theorem relies on convolution with an approximate identity having certain properties. We begin with a construction which applies the Peter-Weyl theorem to produce such kernels.

Proposition 2. *Suppose G is a compact group and $G \curvearrowright C(G)$ the action induced by left multiplication, that is, the action defined by $g \cdot f(h) := f(g^{-1}h)$. Then there exists a net of functions $k_i \in C(G)$ such that the following hold:*

(1): $k_i \geq 0$ *for all* i.

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(2): $\int_G k_i d\mu = 1$ *for all* i.
(3): $k_i * f$ *converges uniformly to* f *for all* $f \in C(G)$.
(4): k_i *lies in a finite-dimensional* G*-invariant subspace of* $C(G)$ *for all* i.

Proof. First, consider a neighborhood basis (U_i) for the identity in G and let (h_i) be a net of non-negative, $L^1(G)$-normalized continuous functions with h_i supported inside U_i. Then (h_i) is an approximate identity in the sense that $f * h_i \to f$ uniformly for $f \in C(G)$. So (h_i) satisfies (1), (2), and (3).

Let $C_{\text{fin}}(G)$ be the collection of functions in $C(G)$ whose G-orbits are contained in a finite-dimensional subspace. Then $C_{\text{fin}}(G)$ is dense in $C(G)$ by the Peter-Weyl theorem (as it contains the matrix coefficients of all irreducible unitary representations of G). Also, $C_{\text{fin}}(G)$ is a sub $*$-algebra: if f and g are linear combinations of $\{f_i\}_{i=1}^n$ and $\{g_j\}_{j=1}^m$ respectively, then fg is a linear combination of $\{f_i g_j\}_{i,j}$, and the action $G \curvearrowright C(G)$ respects products so that a product of functions with finite orbits will also have a finite orbit.

Now, let $\sqrt{h_i}$ be the pointwise square root of h_i. Let $l_i \in C_{\text{fin}}(G)$ be such that $\|l_i - \sqrt{h_i}\|_{C(G)} \leq \frac{\varepsilon_i}{\|\sqrt{h_i}\|_{C(G)}}$ with $\varepsilon_i \to 0$. Put $k_i = \frac{l_i l_i^*}{\|l_i l_i^*\|_1}$.

Note that k_i satisfies (1) by construction. Moreover, $\|l_i l_i^* - h_i\|_{C(G)} \leq \frac{\varepsilon_i}{\|\sqrt{h_i}\|_{C(G)}} \cdot 2\|\sqrt{h_i}\|_{C(G)} + \frac{\varepsilon_i^2}{\|\sqrt{h_i}\|_{C(G)}^2} < 3\varepsilon_i \to 0$ as $i \to \infty$, so $\|k_i - h_i\|_{L^1(G)} \to 0$ and so $\left| \int l_i l_i^* d\mu - 1 \right| < 3\varepsilon_i$. But that means normalizing $l_i l_i^*$ to have integral equal to 1 has the effect of scaling by at most $\frac{1}{1-3\varepsilon_i}$ for all i. Thus, (k_i) satisfies (3) since $\varepsilon_i \to 0$. It also satisfies (4) since each l_i satisfies (4) and $C_{\text{fin}}(G)$ is a $*$-subalgebra. $\square$

We are now ready to prove the main result. The perturbation argument found on the next page is the basic idea behind [2, Corollary B.9], although we do not use that directly.

Theorem 3. *Suppose* G *is a compact group. Then the action* $G \curvearrowright C(G)$ *induced by left multiplication is quasidiagonal. In fact, the u.c.p. maps can be taken to be approximately equivariant for all* $\gamma \in G$.

Proof. Fix $F \subset C(G)$ a finite subset and $\varepsilon > 0$. We can assume each $f \in F$ has $\|f\|_{C(G)} \leq 1$. Let (k_i) be as in the previous proposition. Let W_i be a finite dimensional, G-invariant subspace of $C(G)$ containing k_i.

Denote by Φ_i the map $C(G) \to C(G)$ given by $f \mapsto f * k_i$. This is equivariant for the action $G \curvearrowright C(G)$. Also the image of Φ_i lies in W_i. To see this, observe that $(f * k_i)(g) = \int_G f(h) k_i(h^{-1}g) dh = \int_G f(h)(\lambda_h k_i)(g) dh = \left(\int_G f(h)(\lambda_h k_i) dh \right)(g)$ where λ is the aforementioned action $G \curvearrowright C(G)$ and the last integral in parentheses is $C(G)$-valued. This integral is then a limit of linear combinations of $(\lambda_h k_i)$ for different $h \in G$, and so is contained in W_i since W_i is a closed, invariant subspace. Moreover, since $k_i \geq 0$, Φ_i is positive, hence completely positive as a map $C(G) \to C(G)$, since any positive map of commutative C^*-algebras is completely positive.

Consider W_i now with the $L^2(G, \mu)$-inner product where μ is normalized Haar measure. This is a finite dimensional Hilbert space on which G acts unitarily. Finite-dimensionality implies we can find a sufficiently large $E_i \subset G$ such that the map $\psi_i : W_i \to \mathbb{C}^{E_i}$ given by taking evaluations is an isomorphism of W_i onto its image and preserves the suprema of functions in $(F \cup F^2) * k_i \pm [(F \cup F^2) * k_i]^2$ (and we assume $0 \in F$), since continuous functions on G achieve their suprema. Then we can conjugate over the action on W_i to an action on the subspace $\psi_i(W_i) \subset \mathbb{C}^{E_i}$.

Partition G into measurable sets, P_e each containing exactly one $e \in E_i$. For a suitable choice of E_i we can ensure, for any neighborhood N of the diagonal in $G \times G$, that $P_e \times \{g\}$ is contained in N for all $e \in E_i$ and all $g \in G$ (that is, in the metrizable case, we can ensure the diameters of the P_e are as small as we wish). This implies elements of F will stay very close to their value at e within P_e. Equip $\mathbb{C}^{E_i}$ with the inner product $\langle v,w\rangle_i = \sum_{e\in E_i} v(e)\overline{w(e)}\mu(P_e)$.

Extend the action $G \curvearrowright \psi_i(W_i)$ to an action (representation) on all of $\mathbb{C}^{E_i}$ by defining the new action to be trivial on the orthogonal complement of $\psi_i(W_i)$ in $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i)$. Since all of $C(G)$ is mapped by $\psi_i \circ \Phi_i$ into $\psi_i(W_i)$, extending this way won't affect our estimates later.

Recall the definition of the inner product $\langle\cdot,\cdot\rangle_i$ and notice that the canonical basis of $\mathbb{C}^{E_i}$ is still orthogonal with respect to this inner product, so functions on E_i, thought of as multiplication operators, are still represented by the same diagonal matrices as they would be with respect to the canonical basis if we simply choose the appropriate renormalization of the canonical basis as a new orthonormal basis.

Since ψ_i is not unitary as a map $(C(G), \langle\cdot,\cdot\rangle_{L^2G}) \to (\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i)$, the representation of G on $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i)$ given above is not unitary. However, since G is compact, we can make this representation unitary by replacing the original inner product with its average, $\langle\cdot,\cdot\rangle_i^*$, over G.

We now have a unitary representation of G on $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$, hence an action of G on $B(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$. We can also represent elements of $\mathbb{C}^{E_i}$ as multiplication operators on $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i)$ (diagonal matrices) by the representation $\mathbb{C}^{E_i} \to B(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i)$ mapping $v \mapsto M_v$, using the orthonormal basis of $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i)$ mentioned earlier. For V an operator which takes this basis to an orthonormal basis of $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$, we define $\tilde{\psi}_i : W_i \to B(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$ by $f \mapsto VM_{\psi_i(f)}V^{-1}$. The unitary representation of G on $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$ also gives an action on $B(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$ by conjugation.

Some care is needed here since, although the averaged inner product is equivalent to the original, a priori the constants could increase as i increases so that our errors are amplified. However, if we assume E_i is sufficiently large and that diameters of the P_e are sufficiently small, we have that $\langle f,g\rangle \approx_\varepsilon \langle\psi_i(f),\psi_i(g)\rangle_i$ for all $f,g \in B_2(W_i)$ (the ball of radius 2 centered at the origin in W_i). In other words, ψ_i, and hence the action on $(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle)$, is 'almost unitary'. Assume $\varepsilon < 1$ so that the pullback of the unit ball by ψ_i is contained in $B_2(W_i)$. Then for $\gamma \in G$ and $\psi_i(f), \psi_i(g) \in B_1(\psi_i(W_i))$, $\langle\gamma\cdot\psi_i(f), \gamma\cdot\psi_i(g)\rangle_i = \langle\psi_i(\gamma\cdot f), \psi_i(\gamma\cdot g)\rangle_i \approx_\varepsilon \langle\gamma\cdot f, \gamma\cdot g\rangle_i = \langle f,g\rangle_i \approx_\varepsilon \langle\psi_i(f),\psi_i(g)\rangle$. This shows $\langle v,w\rangle_i^* \approx_{2\varepsilon} \langle v,w\rangle_i$ for $v,w \in B_1(\mathbb{C}^{E_i})$ (since the two are the same on the orthogonal complement of $\psi_i(W_i) \subset (\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i)$) and so errors are only amplified by a constant multiple which tends to 1 as E_i becomes larger.

A second issue arises since the canonical basis of $\mathbb{C}^{E_i}$ is no longer orthogonal with respect to $\langle\cdot,\cdot\rangle_i^*$. We therefore have to choose a new basis, which means the multiplication operators coming from $\mathbb{C}^{E_i}$ are represented by different matrices. This means the map $\tilde{\psi}_i : W_i \to B(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$ may not be $*$-linear, since $\tilde{\psi}_i(f) = VM_{\psi_i(f)}V^{-1}$ where V is invertible, but not unitary. Notice that, for an appropriate choice of E_i, V takes a basis which is orthonormal (with respect to $\langle\cdot,\cdot\rangle_i$) to a basis which is orthonormal with respect to $\langle\cdot,\cdot\rangle_i^*$ and hence has $\langle e_j,e_k\rangle_i^* \approx_{2\varepsilon} 0$ and $\langle e_j,e_j\rangle_i^* \approx_{2\varepsilon} 1$. This implies V is 'almost unitary' in the sense that $\langle Vv,Vu\rangle_i^* \approx_{4\varepsilon} \langle v,u\rangle_i^*$. Recall that we can write $V = AU$ in a unique way where A is a positive matrix and U is a unitary. Since V is invertible and bounded below and above by $1-4\varepsilon$ and $1+4\varepsilon$ respectively, the same is true for A. Thus, A is within 4ε of the identity. The matrix U is

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therefore a unitary such that $\|V-U\| < 4\varepsilon$. To simplify things later, we can go back and choose E_i so that $\|V-U\|$ is sufficiently small so that $\|V\psi_i(f)V^{-1} - U\psi_i(f)U^*\| < \varepsilon$ for all f with $\|f\|_{C(G)} \leq 1$.

Defining $\Psi_i(f) := U\psi_i(f*k_i)U^*$ now gives a map $C(G) \to B(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$ which is a composition of a unital, completely positive map and a unital $*$-homomorphism and hence u.c.p.. Since Ψ_i is an ε-perturbation of an equivariant map, $\|\Psi_i(\gamma\cdot f) - \gamma\cdot\Psi_i(f)\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle^*)} < \varepsilon$ for all $f \in F$ and $\gamma \in G$. Notice that this approximation works for all of G rather than just a finite subset.

Moreover, for $f \in F$, we can choose i and E_i so that

$$\|f\|_{C(G)} \approx_\varepsilon \|\psi_i(f*k_i)\|_{C(G)} = \|\psi_i(f*k_i)\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle)} \approx_\varepsilon \|V\psi_i(f*k_i)V^{-1}\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle^*)}$$

$$\approx_\varepsilon \|U\psi_i(f*k_i)U^*\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle^*)} = \|\Psi_i(f*k_i)\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle^*)}$$

for all $f \in F$.

Also, if we choose i large enough that f, g, and fg are all approximated to within ε by their convolutions with k_i, we have, using that ψ_i and conjugation by U are both multiplicative and that the moduli of f and g are bounded above by 1,

$$\begin{aligned}
&\|U\psi_i((fg)*k_i)U^* - U\psi_i(f*k_i)U^*U\psi_i(f*k_i)U^*\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle_i^*)} \\
&= \|U\psi_i((fg)*k_i) - \psi_i(f*k_i)\psi_i(g*k_i)U^*\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle_i^*)} \\
&= \|U\psi_i((fg)*k_i - (f*k_i)(g*k_i))U^*\|_{B(\mathbb{C}^{E_i},\langle\cdot,\cdot\rangle_i^*)} \\
&= \|(fg)*k_i - (f*k_i)(g*k_i)\|_{C(G)} \\
&= \|(((fg)*k_i - fg) + fg) - ((f*k_i - f) + f)((g*k_i - g) + g)\|_{C(G)} \\
&\leq \varepsilon + \|fg - ((f*k_i - f)(g*k_i - g) + (f*k_i - f)g + (g*k_i - g)f + fg)\|_{C(G)} \\
&\leq \varepsilon + \varepsilon^2 + \varepsilon + \varepsilon < 4\varepsilon
\end{aligned}$$

Observing that $B(\mathbb{C}^{E_i}, \langle\cdot,\cdot\rangle_i^*)$ identifies with $M_{|E_i|}(\mathbb{C})$ now completes the proof. □

It is now more or less straightforward to extend the previous theorem and obtain our main result.

Theorem 4. *Suppose $\Gamma \curvearrowright X$ is an isometric action by a countable discrete group on a compact space. Then $\Gamma \curvearrowright X$ is quasidiagonal. In fact, the u.c.p. maps can be taken to be approximately equivariant for all $\gamma \in \Gamma$.*

Proof. Consider first a minimal action $\Gamma \curvearrowright X$. Let G be the closure of $\Gamma \subset \mathrm{Isom}(X)$. Then picking any $x \in X$ and taking the orbit map $G \to Gx$ gives a continuous, equivariant map $h: G \to X$. Then the map which sends $f \in C(X)$ to $f \circ h$ is an equivariant homomorphism $C(X) \to C(G)$, and minimality and the definition of h imply this map preserves the supremum norm. The previous theorem now shows $\Gamma \curvearrowright X$ is quasidiagonal with completely positive maps as in the previous theorem, as any such map can be pulled back to such a map from $C(X)$.

If $\Gamma \curvearrowright X$ is not minimal, for any fixed $\delta > 0$ we can find a finite collection $\{O_i\}$ of orbits so that the union of all their closures, Y, is δ-dense in X. Then the map $\psi: C(X) \to C(Y)$ given by restriction is an equivariant homomorphism, and for any finite subset $F \subset C(X)$, δ can be chosen small enough so that

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ψ decreases the norms of elements of F by less than any $\varepsilon > 0$. Composing with $\psi : C(Y) \to \oplus_i C(O_i)$ and using the previous paragraph now finishes the proof. □

The theorem above essentially implies that crossed products by isometric actions have finite dimensional approximations as nice as those of the acting group's C^*-algebra. More precisely, we have the following corollary.

Corollary 5. *Suppose $\Gamma \curvearrowright X$ is an isometric action by a countable discrete group. Then $C(X) \rtimes_r \Gamma$ is quasidiagonal or MF exactly when $C_r^*(\Gamma)$ is.* □

This can be combined with [5, 3.9] to show that many group actions (by non-amenable groups) have MF crossed products.

Corollary 6. *Let Γ_i, $i \in I$ be a finite or countable collection of discrete abelian groups with a common subgroup H and let $\Gamma = \star_H \Gamma_i$ be their amalgamated product. Suppose $\Gamma \curvearrowright X$ is an isometric action. Then $C(X) \rtimes_r \Gamma$ is MF.* □

Corollary 7. *Suppose Γ is virtually free and $\Gamma \curvearrowright X$ is an isometric action. Then $C(X) \rtimes_r \Gamma$ is MF.*

Proof. We will show $C_r^*(\Gamma)$ is MF. If $\Lambda < \Gamma$ is a free normal subgroup and $F := \Gamma/\Lambda$ is finite, we have $B(l^2\Gamma) \cong B(l^2(\Lambda \times F)) \cong B(l^2\Lambda \otimes l^2F) \cong B(l^2\Lambda) \otimes B(l^2F)$, so $C_r^*(\Gamma)$ can be realized as an algebra of $|F| \times |F|$ matrices with entries in $C_r^*(\Lambda)$. Thus, $C_r^*(\Gamma)$ is MF since $C_r^*(\Lambda)$ is MF by [5, 3.9] and since it is straightforward to show using [3, 3.1] that matrix algebras over MF algebras are themselves MF. □

To show more concretely that these results advance our understanding of the MF property, we finish with an example.

Example 8. *Consider the amalgamated product $\Gamma = \star_{i,\mathbb{Z}} \mathbb{Z}$ for $i \in I$ and I finite or countable where $\mathbb{Z}$ is embedded into the i-th copy of $\mathbb{Z}$ via multiplication by $k_i \in \mathbb{Z}$. This is the group $\langle z_1, z_2, \ldots | z^{k_1} = z^{k_2} = \cdots \rangle$. Choose any $n \geq 1$ and let $v_i \in U(n)$ be unitaries with order dividing k_i (e.g. permutation matrices or diagonal matrices with ± 1 in each diagonal entry). Let $u \in U(n)$ be any other unitary. Functional calculus allows us to find u^{1/k_i} for all k_i, so we can define $z_i \cdot x = (v_i \otimes u^{1/k_i})x$ for $x \in U(M_n(\mathbb{C}) \otimes M_n(\mathbb{C})) \cong U(n^2)$. This defines an action $\Gamma \curvearrowright U(n^2)$ which is implemented by translations and therefore isometric. The crossed products $C(U(n^2)) \rtimes_r \Gamma$ are therefore MF by the previous corollary.*

When $|I| \geq 2$, $k_1 \geq 2$, and $k_2 \geq 3$, the subgroup $\mathbb{Z} \star_{\mathbb{Z}} \mathbb{Z}$ contains a free group on two generators by [4, 2.8] *and is therefore not amenable. We can also construct similar examples where Γ is not virtually free taking a free product of Γ with $\mathbb{Z} \oplus \mathbb{Z}$ and defining each factor of $\mathbb{Z} \oplus \mathbb{Z}$ to act via multiplication by one of two commuting unitaries.*

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