Several sufficient conditions for the log-balancedness of the difference sequence of a log-convex sequence

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Abstract. Let $\{z_n\}_{n\geq 0}$ be a log-convex sequence with $z_{n+1}-z_n>0$ for $n\geq 0$. In this paper, we mainly give several sufficient conditions for the log-balancedness of $\{z_{n+1}-z_n\}_{n\geq 0}$, where $\{z_n\}_{n\geq 0}$ satisfies a three-term (four-term) recurrence. Then, we apply these results to a series of combinatorial numbers such as Motzkin numbers, middle trinomial coefficients numbers, the Fine numbers, and so on.

Key words. log-convexity, log-concavity, log-balancedness.

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1 Introduction

We first recall some definitions related to the log-behavior of positive sequences. A positive sequence $\{z_n\}_{n\geq 0}$ is said to be *log-convex* (*log-concave*) if $z_n^2\leq z_{n-1}z_{n+1}$ ($z_n^2\geq z_{n-1}z_{n+1}$) for all $n\geq 1$. A log-convex sequence $\{z_n\}_{n\geq 0}$ is said to be *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave (Došlić [2] gave this definition). It is clear that a sequence $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (nonincreasing) and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $\frac{z_{n+1}}{(n+1)z_n}\geq\frac{z_{n+2}}{(n+2)z_{n+1}}$ for each $n\geq 0$. Log-behavior is an important source of inequalities. In particular, since log-balancedness involves log-convexity and log-concavity, it can help us find more inequalities. In addition, log-balanced sequences can provide important examples in white noise distribution theory (see Asai et al. [1] for more details). Hence the log-balancedness of sequences deserves to be studied. For the investigation of log-balancedness, see Došlić [2], Došlić [3], Zhang and Zhao [9], and Liu and Zhao [7] for instance. For a log-convex sequence $\{z_n\}_{n\geq 0}$, where $z_{n+1}-z_n>0$ for each $n\geq 0$, Zhao [12] investigated the log-convexity of $\{z_{n+1}-z_n\}_{n\geq 0}$,

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where $\{z_n\}_{n\geq 0}$ satisfies a three-term recurrence. In this paper, we are interested in the log-balancedness of $\{z_{n+1}-z_n\}_{n\geq 0}$. In Section 2, we mainly give several sufficient conditions for the log-balancedness of $\{z_{n+1}-z_n\}_{n\geq 0}$, where $\{z_n\}_{n\geq 0}$ satisfies a three-term (four-term) recurrence. In Section 3, we apply these results to a series of combinatorial numbers.

2 Several sufficient conditions for the log-balancedness of the difference sequence of a log-convex sequence

The following lemma will be used.

Lemma 2.1 *[11] If the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are both log-balanced, then so is their binomial convolution*

$$z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}, \quad n = 0, 1, 2, \cdots .$$

Now we give several sufficient conditions for the log-balancedness of the difference sequence of a log-convex sequence.

Theorem 2.1 *Suppose that $\{z_n\}_{n\geq 0}$ is a log-balanced sequence. Let $u_n = \sum_{k=0}^{n} \binom{n}{k} z_k$. Then $\{u_{n+1}-u_n\}_{n\geq 0}$ is also log-balanced.*

Proof. It is clear that $u_1 - u_0 = z_1$. For $n \geq 1$, by using $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ $(k \geq 1)$, we have

$$u_{n+1} - u_n = \sum_{k=1}^{n} \binom{n}{k-1} z_k + z_{n+1} = \sum_{k=0}^{n} \binom{n}{k} z_{k+1}.$$

In order to prove that the sequence $\{z_{k+1}\}_{k\geq 0}$ is log-balanced, we only need to show that $\{\frac{z_{k+2}}{(k+1)z_{k+1}}\}_{k\geq 0}$ is decreasing. It is evident that

$$\frac{z_{k+2}}{(k+1)z_{k+1}} = \frac{z_{k+2}}{(k+2)z_{k+1}} \cdot \frac{k+2}{k+1}.$$

Since $\{z_k\}_{k\geq 0}$ is log-balanced, $\{\frac{z_{k+2}}{(k+2)z_{k+1}}\}_{k\geq 0}$ is decreasing. On the other hand, $\{\frac{k+2}{k+1}\}_{k\geq 0}$ is decreasing. Then $\{\frac{z_{k+2}}{(k+1)z_{k+1}}\}_{k\geq 0}$ is decreasing. It follows from Lemma 2.1 that $\{u_{n+1}-u_n\}_{n\geq 0}$ is log-balanced.

■

Theorem 2.2 *Assume that $\{z_n\}_{n\geq 0}$ is a log-balanced sequence. For $n \geq 0$, let $x_n = \frac{z_{n+1}}{z_n}$. If $x_n > n+1$ for $n \geq 0$ and $\{z_{n+1}-z_n\}_{n\geq 0}$ is log-convex, $\{z_{n+1}-z_n\}_{n\geq 0}$ is log-balanced.*

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Proof. For $n \geq 0$, let $y_n = \frac{z_{n+2}-z_{n+1}}{z_{n+1}-z_n}$. Then we have

$$y_n = \frac{x_n(x_{n+1}-1)}{x_n - 1} \quad \text{and} \quad \frac{y_n}{n+1} = \frac{x_n(x_{n+1}-1)}{(x_n-1)(n+1)}.$$

In order to prove that $\{z_{n+1} - z_n\}_{n\geq 0}$ is log-balanced, we need to show that $\{\frac{y_n}{n+1}\}_{n\geq 0}$ is decreasing. It is obvious that

$$\frac{x_{n+1}-1}{n+1} - \frac{x_{n+2}-1}{n+2} = \frac{(n+2)x_{n+1} - (n+1)x_{n+2} - 1}{(n+1)(n+2)}.$$

Since $\{z_n\}_{n\geq 0}$ is log-balanced, $\frac{x_{n+1}}{n+2} \geq \frac{x_{n+2}}{n+3}$. Then we get

$$\begin{aligned}(n+2)x_{n+1} - (n+1)x_{n+2} - 1 &\geq \frac{(n+2)^2}{n+3}x_{n+2} - (n+1)x_{n+2} - 1 \\ &= \frac{x_{n+2} - n - 3}{n+3}.\end{aligned}$$

Due to $x_n > n+1$ for $n \geq 0$, we obtain $(n+2)x_{n+1} - (n+1)x_{n+2} - 1 > 0$. Then $\{\frac{x_{n+1}-1}{n+1}\}_{n\geq 0}$ is decreasing. On the other hand, $\{\frac{x_n}{x_n-1}\}_{n\geq 0}$ is decreasing. Hence $\{\frac{y_n}{n+1}\}_{n\geq 0}$ is decreasing. ■

Theorem 2.3 *Assume that $\{z_n\}_{n\geq 0}$ is a log-convex sequence and satisfies the recurrence*

$$z_{n+1} = R_n z_n - S_n z_{n-1}, \quad n \geq 1, \tag{2.1}$$

where $R_n \geq 1, S_n \geq 0$, and $R_n - S_n - 1 \geq 0$ for each $n \geq 1$. For $n \geq 0$, let $x_n = \frac{z_{n+1}}{z_n}$. If there exists a nonnegative integer n_0 such that $x_{n_0} > 1$, the sequences $\{\frac{R_{n+1}-1}{n+1}\}_{n\geq n_0}$ and $\{\frac{R_{n+1}-1-S_{n+1}}{n+1}\}_{n\geq n_0}$ are decreasing, and $\{z_{n+1} - z_n\}_{n\geq n_0}$ is log-convex, $\{z_{n+1} - z_n\}_{n\geq n_0}$ is log-balanced.

Proof. For $n \geq 0$, let $y_n = \frac{z_{n+2}-z_{n+1}}{z_{n+1}-z_n}$. It follows from (2.1) that

$$z_{n+1} - z_n = (R_n - 1)(z_n - z_{n-1}) + (R_n - 1 - S_n)z_{n-1}, \quad n \geq 1.$$

For $n \geq n_0$, we have

$$\begin{aligned}y_n &= R_{n+1} - 1 + \frac{R_{n+1} - 1 - S_{n+1}}{x_n - 1}, \\ \frac{y_n}{n+1} &= \frac{R_{n+1}-1}{n+1} + \frac{R_{n+1} - 1 - S_{n+1}}{n+1} \cdot \frac{1}{x_n - 1}.\end{aligned}$$

Since $\{z_n\}_{n\geq 0}$ is log-convex and $x_{n_0} > 1$, $\{\frac{1}{x_n-1}\}_{n\geq n_0}$ is decreasing. Noting that $\{\frac{1}{x_n-1}\}_{n\geq n_0}$, $\{\frac{R_{n+1}-1}{n+1}\}_{n\geq n_0}$, and $\{\frac{R_{n+1}-1-S_{n+1}}{n+1}\}_{n\geq n_0}$ are decreasing, we have that $\{\frac{y_n}{n+1}\}_{n\geq n_0}$ is decreasing. On the other hand, the sequence $\{z_{n+1} - z_n\}_{n\geq n_0}$ is log-convex. Thus $\{z_{n+1} - z_n\}_{n\geq n_0}$ is log-balanced. ■

Theorem 2.4 *Suppose that* $\{z_n\}_{n\geq 0}$ *is a log-convex sequence and satisfies the recurrence*

$$z_{n+1} = R_n z_n + S_n z_{n-1}, \quad n \geq 1,$$

where $R_n > 1$ *and* $S_n > 0$ *for each* $n \geq 1$. *For* $n \geq 0$, *let* $x_n = \frac{z_{n+1}}{z_n}$. *If there exists a nonnegative integer* n_0 *such that* $x_{n_0} > 1$, $\{\frac{R_{n+1}-1}{n+1}\}_{n\geq n_0}$ *and* $\{\frac{S_{n+1}}{n+1}\}_{n\geq n_0}$ *are decreasing, and* $\{z_{n+1} - z_n\}_{n\geq n_0}$ *is log-convex,* $\{z_{n+1} - z_n\}_{n\geq n_0}$ *is log-balanced.*

The proof of Theorem 2.4 is similar to that of Theorem 2.3 and is omitted here.

Theorem 2.5 *Suppose that* $\{z_n\}_{n\geq 0}$ *is a log-convex sequence and satisfies the recurrence*

$$z_{n+1} = R_n z_n - S_n z_{n-1}, \quad n \geq 1,$$

where $R_n > 1$ *and* $S_n > R_n - 1$ *for* $n \geq 1$. *For* $n \geq 0$, *let* $x_n = \frac{z_{n+1}}{z_n}$. *Assume that* N *is a nonnegative integer. For* $n \geq N$, $\varphi_n \geq x_n \geq \psi_n > 1$. *For* $n \geq N$, *put*

$$\begin{aligned}\Omega_n &= (n+2)R_{n+1} - (n+1)R_{n+2} - 1 + \frac{(n+1)(S_{n+2} - R_{n+2} + 1)}{\varphi_{n+1} - 1} \\ &\quad - \frac{(n+2)(S_{n+1} - R_{n+1} + 1)}{\psi_n - 1}.\end{aligned}$$

If there exists a nonnegative integer $N_1 \geq N$ *such that* $\{z_{n+1} - z_n\}_{n\geq N_1}$ *is log-convex and* $\Omega_n \geq 0$ *for* $n \geq N_1$, $\{z_{n+1} - z_n\}_{n\geq N_1}$ *is log-balanced.*

Proof. For $n \geq N$, let $y_n = \frac{z_{n+2}-z_{n+1}}{z_{n+1}-z_n}$. Then $y_n = R_{n+1} - 1 - \frac{S_{n+1}-R_{n+1}+1}{x_n-1}$ $(n \geq N)$. Now we prove that $\{\frac{y_n}{n+1}\}_{n\geq N_1}$ is decreasing. It is evident that

$$\begin{aligned}(n+2)y_n - (n+1)y_{n+1} &= (n+2)R_{n+1} - (n+1)R_{n+2} - 1 - \frac{(n+2)(S_{n+1} - R_{n+1} + 1)}{x_n - 1} \\ &\quad + \frac{(n+1)(S_{n+2} - R_{n+2} + 1)}{x_{n+1} - 1} \\ &\geq (n+2)R_{n+1} - (n+1)R_{n+2} - 1 + \frac{(n+1)(S_{n+2} - R_{n+2} + 1)}{\varphi_{n+1} - 1} \\ &\quad - \frac{(n+2)(S_{n+1} - R_{n+1} + 1)}{\psi_n - 1} \\ &= \Omega_n \quad (n \geq N_1) \\ &> 0.\end{aligned}$$

This implies that $\{\frac{y_n}{n+1}\}_{n\geq N_1}$ is decreasing. Hence the sequence $\{z_{n+1} - z_n\}_{n\geq N_1}$ is log-balanced. ■

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Theorem 2.6 *Assume that* $\{z_n\}_{n\geq 0}$ *is a log-convex sequence satisfying the recurrence*

$$z_{n+1} = Q_n z_n + R_n z_{n-1} + S_n z_{n-2}, \quad n \geq 2, \tag{2.2}$$

where $Q_n > 0$, $R_n \geq 0$, $S_n \geq 0$, *and* $Q_{n+1}+R_{n+1}-1 \geq 0$ *for* $n \geq 2$. *For* $n \geq 0$, *let* $x_n = \frac{z_{n+1}}{z_n}$. *Suppose that*

$$\phi_n \leq x_n \leq \varphi_n, \quad n \geq N_1,$$

where N_1 *is an integer with* $N_1 \geq 0$ *and* $\phi_n > 1$ *for* $n \geq N_1$. *For* $n \geq N_1 + 1$, *define*

$$\begin{aligned}
\Delta_n &= Q_{n+2} - Q_{n+1} + \frac{Q_{n+2} + R_{n+2} - 1}{\varphi_{n+1} - 1} + \frac{S_{n+2}}{\varphi_n(\varphi_{n+1} - 1)} - \frac{Q_{n+1} + R_{n+1} - 1}{\phi_n - 1} \\
&\quad - \frac{S_{n+1}}{\phi_{n-1}(\phi_n - 1)}, \\
\Upsilon_n &= (n+2)Q_{n+1} - (n+1)Q_{n+2} - 1 + \frac{(n+2)(Q_{n+1} + R_{n+1} - 1)}{\varphi_n - 1} + \frac{(n+2)S_{n+1}}{\varphi_{n-1}(\varphi_n - 1)} \\
&\quad - \frac{(n+1)(Q_{n+2} + R_{n+2} - 1)}{\phi_{n+1} - 1} - \frac{(n+1)S_{n+2}}{\phi_n(\phi_{n+1} - 1)}.
\end{aligned}$$

(i) If there exists an integer $N_2 \geq N_1 + 1$ *such that* $\Delta_n \geq 0$ *for* $n \geq N_2$, *the sequence* $\{z_{n+1} - z_n\}_{n\geq N_2}$ *is log-convex.*

(ii) If there exists an integer $N_3 \geq N_1 + 1$ *such that* $\Delta_n \geq 0$ *and* $\Upsilon_n \geq 0$ *for* $n \geq N_3$, *the sequence* $\{z_{n+1} - z_n\}_{n\geq N_3}$ *is log-balanced.*

Proof. For $n \geq N_1$, let $y_n = \frac{z_{n+2}-z_{n+1}}{z_{n+1}-z_n}$. It is clear that $y_n = \frac{x_n(x_{n+1}-1)}{x_n-1}$. We first give the proof of (i). In order to prove that the sequence $\{z_{n+1} - z_n\}_{n\geq N_2}$ is log-convex, we only need to show that $\{y_n\}_{n\geq N_2}$ is increasing. It follows from (2.2) that

$$x_n = Q_n + \frac{R_n}{x_{n-1}} + \frac{S_n}{x_{n-1}x_{n-2}}, \quad n \geq 2. \tag{2.3}$$

By applying (2.3), we derive

$$y_n = Q_{n+1} - 1 + \frac{Q_{n+1} + R_{n+1} - 1}{x_n - 1} + \frac{S_{n+1}}{x_{n-1}(x_n - 1)}, \quad n \geq N_1. \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned}
y_{n+1} - y_n &= Q_{n+2} - Q_{n+1} + \frac{Q_{n+2} + R_{n+2} - 1}{x_{n+1} - 1} + \frac{S_{n+2}}{x_n(x_{n+1} - 1)} \\
&\quad - \frac{Q_{n+1} + R_{n+1} - 1}{x_n - 1} - \frac{S_{n+1}}{x_{n-1}(x_n - 1)}.
\end{aligned}$$

Since $\phi_n \leq x_n \leq \varphi_n$ for $n \geq N_1$,

$$\begin{aligned}
y_{n+1} - y_n &\geq Q_{n+2} - Q_{n+1} + \frac{Q_{n+2} + R_{n+2} - 1}{\varphi_{n+1} - 1} + \frac{S_{n+2}}{\varphi_n(\varphi_{n+1} - 1)} \\
&\quad - \frac{Q_{n+1} + R_{n+1} - 1}{\phi_n - 1} - \frac{S_{n+1}}{\phi_{n-1}(\phi_n - 1)} \quad (n \geq N_1 + 1) \\
&= \Delta_n.
\end{aligned}$$

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

It follows from $\Delta_n \geq 0$ $(n \geq N_2)$ that $\{y_n\}_{n\geq N_2}$ is increasing. Now we complete the proof of (ii). It is evident that the log-convex sequence $\{z_{n+1}-z_n\}_{n\geq N_3}$ is log-balanced if and only if

$$(n+2)y_n - (n+1)y_{n+1} \geq 0 \quad (n \geq N_3).$$

By means of (2.4) and $\phi_n \leq x_n \leq \varphi_n$ $(n \geq N_1)$, we get

$$(n+2)y_n - (n+1)y_{n+1} \geq \Upsilon_n, \quad n \geq N_1+1.$$

Noting that $\Delta_n \geq 0$ and $\Upsilon_n \geq 0$ for $n \geq N_3$, we have that $(n+2)y_n - (n+1)y_{n+1} \geq 0$ for $n \geq N_3$. ■

Theorem 2.7 *Assume that $\{z_n\}_{n\geq 0}$ is a log-convex sequence satisfying the recurrence*

$$z_{n+1} = Q_n z_n + R_n z_{n-1} - S_n z_{n-2}, \quad n \geq 2,$$

where $Q_n > 0$, $R_n > 0$, $S_n > 0$, and $Q_{n+1}+R_{n+1}-1 \geq 0$ for $n \geq 2$. For $n \geq 0$, let $x_n = \frac{z_{n+1}}{z_n}$. Suppose that

$$\phi_n \leq x_n \leq \varphi_n, \quad n \geq N_1,$$

where N_1 is an integer with $N_1 \geq 0$ and $\phi_n > 1$ for $n \geq N_1$. For $n \geq N_1+1$, define

$$\begin{aligned}
\Omega_n &= Q_{n+2} - Q_{n+1} + \frac{Q_{n+2}+R_{n+2}-1}{\varphi_{n+1}-1} - \frac{S_{n+2}}{(\phi_{n+1}-1)\phi_n} - \frac{Q_{n+1}+R_{n+1}-1}{\phi_n - 1} \\
&\quad + \frac{S_{n+1}}{(\varphi_n - 1)\varphi_{n-1}}, \\
\Phi_n &= (n+2)Q_{n+1} - (n+1)Q_{n+2} - 1 + \frac{(n+2)(Q_{n+1}+R_{n+1}-1)}{\varphi_n - 1} - \frac{(n+2)S_{n+1}}{(\phi_n-1)\phi_{n-1}} \\
&\quad - \frac{(n+1)(Q_{n+2}+R_{n+2}-1)}{\phi_{n+1}-1} + \frac{(n+1)S_{n+2}}{(\varphi_{n+1}-1)\varphi_n}.
\end{aligned}$$

(i) If there exists an integer $N_2 \geq N_1+1$ such that $\Omega_n \geq 0$ for $n \geq N_2$, the sequence $\{z_{n+1}-z_n\}_{n\geq N_2}$ is log-convex.
(ii) If there exists an integer $N_3 \geq N_1+1$ such that $\Omega_n \geq 0$ and $\Phi_n \geq 0$ for $n \geq N_3$, the sequence $\{z_{n+1}-z_n\}_{n\geq N_3}$ is log-balanced.

The proof of Theorem 2.7 is similar to that of Theorem 2.6 and is omitted here.

3 Log-balancedness for the difference sequence of a log-convex sequence

We apply the results of Theorems 2.2–2.7 to a series of sequences in this section.

Example 3.1 *Let $\{a_n\}_{n\geq 1}$ denote the sequence of counting directed column-convex polyominoes of height n. The value of a_n is equal to the number of outcomes to a race with n contestants in which there is at most one tie (of at least two contestants). The sequence $\{a_n\}_{n\geq 1}$ is Sloane's A007808 and satisfies the recurrence*

$$a_n = (n+2)a_{n-1} - (n-1)a_{n-2}, \quad n \geq 3, \tag{3.1}$$

where $a_1 = 1$ and $a_2 = 3$. Some values of $\{a_n\}_{n\geq 1}$ are as follows:

n	1	2	3	4	5	6	7	8	9
a_n	1	3	13	69	431	3103	25341	231689	2345851

Došlić [2] proved that $\{a_n\}_{n\geq 1}$ is log-balanced. Now we discuss the log-balancedness of $\{a_{n+1} - a_n\}_{n\geq 1}$.

Corollary 3.1 *For the sequence $\{a_n\}_{n\geq 1}$ of satisfying (3.1), $\{a_{n+1} - a_n\}_{n\geq 1}$ is log-balanced.*

Proof. For $n \geq 1$, put $x_n = \frac{a_{n+1}}{a_n}$. Došlić [2] showed that $n+2 \leq x_n \leq n+3$ for $n \geq 1$. Zhao [12] showed that $\{a_{n+1} - a_n\}_{n\geq 1}$ is log-convex. It follows from Theorem 2.2 that $\{a_{n+1} - a_n\}_{n\geq 1}$ is log-balanced.

■

Example 3.2 *Let h_n denote the number of the set of all tree-like polyhexes with $n+1$ hexagons (see Harary and Read [5]). The value of h_n is also the number of lattice paths, from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(1,-1)$ and $(2,0)$, never falling below the x-axis and with no peaks at odd level. The sequence $\{h_n\}_{n\geq 0}$ is Sloane's A002212 and satisfies the following recurrence*

$$(n+1)h_n = 3(2n-1)h_{n-1} - 5(n-2)h_{n-2}, \quad n \geq 2,$$

where $h_0 = h_1 = 1$ and $h_2 = 3$. Some values of $\{h_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9
h_n	1	1	3	10	36	137	543	2219	9285	39587

Now we discuss the log-balancedness of $\{h_{n+1} - h_n\}_{n\geq 1}$.

Corollary 3.2 *For the sequence $\{h_n\}_{n\geq 0}$ counting tree-like polyhexes, $\{h_{n+1} - h_n\}_{n\geq 1}$ is log-balanced.*

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Proof. For $n \geq 0$, set $x_n = \frac{h_{n+1}}{h_n}$. It is obvious that

$$x_1 = 3, \quad R_n = \frac{3(2n+1)}{n+2}, \quad S_n = \frac{5(n-1)}{n+2}, \quad R_n - S_n - 1 = \frac{6}{n+2}, \quad R_{n+1} - 1 = \frac{5n+6}{n+3}.$$

We note that $\{\frac{R_{n+1}-1}{n+1}\}_{n\geq 1} = \{\frac{5n+6}{(n+1)(n+3)}\}_{n\geq 1}$ and $\{\frac{R_{n+1}-S_{n+1}-1}{n+1}\}_{n\geq 1} = \{\frac{6}{(n+1)(n+3)}\}_{n\geq 1}$ are both decreasing. Liu and Wang [6] proved that the sequence $\{h_n\}_{n\geq 0}$ is log-convex and Zhao [12] showed that $\{h_{n+1} - h_n\}_{n\geq 1}$ is also log-convex. It follows from Theorem 2.3 that the sequence $\{h_{n+1} - h_n\}_{n\geq 1}$ is log-balanced.

■

Example 3.3 *Let $\{M_n\}_{n\geq 0}$ denote the Motzkin sequence. The value of M_n is the number of lattice paths from $(0,0)$ to (n,n), with steps $(0,2)$, $(2,0)$ and $(1,1)$, never rising above the line $y = x$. The Motzkin sequence $\{M_n\}_{n\geq 0}$ is Sloane's A001006 and satisfies the following recurrence*

$$M_{n+1} = \frac{2n+3}{n+3} M_n + \frac{3n}{n+3} M_{n-1}, \quad n \geq 1,$$

where $M_0 = M_1 = 1$ and $M_2 = 2$. Some values of $\{M_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9
M_n	1	1	2	4	9	21	51	127	323	835

Now we investigate the log-balancedness of $\{M_{n+1} - M_n\}_{n\geq 3}$.

Corollary 3.3 *For the Motzkin sequence $\{M_n\}_{n\geq 0}$, $\{M_{n+1} - M_n\}_{n\geq 3}$ is log-balanced.*

Proof. For $n \geq 0$, let $x_n = \frac{M_{n+1}}{M_n}$. It is evident that

$$R_n = \frac{2n+3}{n+3}, \quad S_n = \frac{3n}{n+3}, \quad \frac{R_{n+1}-1}{n+1} = \frac{1}{n+4}, \quad \frac{S_{n+1}}{n+1} = \frac{3}{n+4}.$$

We note that $\{\frac{R_{n+1}-1}{n+1}\}_{n\geq 3}$ and $\{\frac{S_{n+1}}{n+1}\}_{n\geq 3}$ are monotonic decreasing and $x_3 > 1$. Došlić [2] proved that $\{M_n\}_{n\geq 0}$ is log-balanced and Zhao [12] showed that $\{M_{n+1} - M_n\}_{n\geq 3}$ is log-convex. It follows from Theorem 2.4 that the sequence $\{M_{n+1} - M_n\}_{n\geq 3}$ is log-balanced. ■

Example 3.4 *Let $\{T_n\}_{n\geq 0}$ denote the sequence of middle trinomial coefficients. The coefficient of x^n in $(1 + x + x^2)^n$ is T_n. The sequence $\{T_n\}_{n\geq 0}$ is Sloane's A002426 and satisfies the following recurrence*

$$(n+1)T_{n+1} = (2n+1)T_n + 3nT_{n-1}, \quad n \geq 1, \tag{3.2}$$

where $T_0 = 1$ and $T_1 = 1$. The value of T_n is equal to the number of lattice paths from $(0,0)$ to $(n,0)$ using steps $(1,0)$, $(1,1)$, and $(1,-1)$. One can find more information of $\{T_n\}_{n\geq 0}$ in [8]. Some values of $\{T_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9	10
T_n	1	1	3	7	19	51	141	393	1107	3139	8953

Došlić [4] proved that the sequence $\{T_n\}_{n\geq 4}$ is log-convex. The sequence $\{T_{n+1} - T_n\}_{n\geq 1}$ is Sloane's A025178. One can find more properties of $\{T_{n+1} - T_n\}_{n\geq 1}$ in [8]. Now we discuss the log-balancedness of $\{T_{n+1} - T_n\}_{n\geq 1}$.

Corollary 3.4 *For the sequence of middle trinomial coefficients $\{T_n\}_{n\geq 0}$, $\{T_{n+1} - T_n\}_{n\geq 5}$ is log-balanced.*

Proof. For $n \geq 0$, let $x_n = \frac{T_{n+1}}{T_n}$ and $y_n = \frac{T_{n+2}-T_{n+1}}{T_{n+1}-T_n}$ $(n \geq 1)$. It follows from (3.2) that

$$x_n = \frac{2n+1}{n+1} + \frac{3n}{(n+1)x_{n-1}}, \quad n \geq 1. \tag{3.3}$$

We first prove by induction that

$$\theta_{n-1} \leq x_n \leq \theta_n, \quad n \geq 5,$$

where $\theta_n = \frac{2n+3+\sqrt{16n^2+48n+33}}{2(n+2)}$. By computation, we have $\theta_{k-1} \leq x_k \leq \theta_k$ for $5 \leq k \leq 8$. Assume that $\theta_{k-1} \leq x_k \leq \theta_k$ for $k \geq 8$. It follows from (3.3) and $\theta_{k-1} \leq x_k \leq \theta_k$ $(k \geq 8)$ that

$$\begin{aligned} x_{k+1} - \theta_k &= \frac{2k+3}{k+2} + \frac{3(k+1)}{(k+2)x_k} - \theta_k \\ &\geq \frac{2k+3}{k+2} + \frac{3(k+1)}{(k+2)\theta_k} - \theta_k \\ &= 0 \quad (k \geq 8) \end{aligned}$$

and

$$\begin{aligned} x_{k+1} - \theta_{k+1} &= \frac{2k+3}{k+2} + \frac{3(k+1)}{(k+2)x_k} - \theta_{k+1} \\ &\leq \frac{2k+3}{k+2} + \frac{3(k+1)}{(k+2)\theta_{k-1}} - \theta_{k+1} \\ &= \frac{2k+3}{k+2} + \frac{(k+1)(\sqrt{16k^2+16k+1} - 2k - 1)}{2k(k+2)} - \frac{2k+5+\sqrt{16k^2+80k+97}}{2(k+3)} \\ &= \frac{1}{2k(k+2)(k+3)}\Bigg[- 2k - 3 + (2k+3)\sqrt{16k^2+16k+1} \\ &\quad +(k^2+2k)\left(\sqrt{16k^2+16k+1} - \sqrt{16k^2+80k+97}\right)\Bigg]. \end{aligned}$$

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Since

$$\begin{aligned}\sqrt{16k^2+16k+1}-\sqrt{16k^2+80k+97} &\leq -8,\\ (2k+3)\sqrt{16k^2+16k+1}-8k^2-18k-3 &< 0,\end{aligned}$$

$x_{k+1}-\theta_{k+1}<0$ holds for $k\geq 8$. Hence we have $\theta_{n-1}\leq x_n\leq \theta_n$ for $n\geq 5$. It is clear that

$$y_n=\frac{n+1}{n+2}+\frac{4(n+1)}{(n+2)(x_n-1)},\quad n\geq 1.$$

In order to prove that $\{T_{n+1}-T_n\}_{n\geq 5}$ is log-convex, we need to show that $\{y_n\}_{n\geq 5}$ is increasing. We note that

$$\begin{aligned}&(n+2)^2(x_n-1)-(n+1)(n+3)(x_{n+1}-1)\\ &=(n+2)^2(x_n-1)-(n+1)(n+3)\left[\frac{n+1}{n+2}+\frac{3(n+1)}{(n+2)x_n}\right]\\ &\geq (n+2)^2(\theta_{n-1}-1)-(n+1)(n+3)\left[\frac{n+1}{n+2}+\frac{3(n+1)}{(n+2)\theta_{n-1}}\right]\\ &=-\frac{(2n+3)(\sqrt{16n^2+16n+1}-1)}{2n(n+1)(n+2)}\quad (n\geq 5).\end{aligned}$$

Then we derive

$$\begin{aligned}&(x_n-1)(x_{n+1}-1)+4[(n+2)^2(x_n-1)-(n+1)(n+3)(x_{n+1}-1)]\\ &\geq(\theta_{n-1}-1)(\theta_n-1)-\frac{2(2n+3)(\sqrt{16n^2+16n+1}-1)}{n(n+1)(n+2)}\\ &=\frac{(\sqrt{16n^2+16n+1}-1)[n(\sqrt{16n^2+48n+33}-1)-8(2n+3)]}{4n(n+1)(n+2)}\\ &>0\quad (n\geq 5).\end{aligned}$$

On the other hand,

$$\begin{aligned}&y_{n+1}-y_n\\ &=\frac{(x_n-1)(x_{n+1}-1)+4[(n+2)^2(x_n-1)-(n+1)(n+3)(x_{n+1}-1)]}{(n+2)(n+3)(x_n-1)(x_{n+1}-1)}.\end{aligned}$$

This implies that $y_{n+1}-y_n>0$ for $n\geq 5$. Hence $\{y_n\}_{n\geq 5}$ is increasing. It follows from Theorem 2.4 that the sequence $\{T_{n+1}-T_n\}_{n\geq 5}$ is log-balanced. ■

For the sequence of middle trinomial coefficients $\{T_n\}_{n\geq 0}$, we can obtain the following results:
(i) For $n\geq 0$, let $x_n=\frac{T_{n+1}}{T_n}$. It follows from (3.3) that

$$\frac{x_n}{n+1}=\frac{2n+1}{(n+1)^2}+\frac{3n}{(n+1)^2x_{n-1}}.$$

Since $\{\frac{2n+1}{(n+1)^2}\}_{n\geq 1}$, $\{\frac{3n}{(n+1)^2}\}_{n\geq 1}$, and $\{\frac{1}{x_{n-1}}\}_{n\geq 5}$ are decreasing, $\{\frac{x_n}{n+1}\}_{n\geq 5}$ is decreasing. This implies that $\{T_n\}_{n\geq 5}$ is log-balanced. We note that $\{T_4, T_5, T_6\}$ is log-balanced. Hence the sequence $\{T_n\}_{n\geq 4}$ is log-balanced.

(ii) Let $\{\mu_n\}_{n\geq 0}$ be Sloane's A007971. For the Motzkin sequence $\{M_n\}_{n\geq 0}$, $\mu_{n+2} = 2M_n$. For the sequence of middle trinomial coefficients $\{T_n\}_{n\geq 0}$, $\frac{T_{n+1}-T_n}{n} = \mu_{n+1}$ $(n \geq 3)$. See [8] for more properties of $\{\mu_n\}_{n\geq 0}$. It is clear that $T_{n+1} - T_n = 2nM_{n-1}$ $(n \geq 3)$. We have that $\{nM_{n-1}\}_{n\geq 5}$ is log-balanced from Corollary 3.4.

Example 3.5 *Let $\{w_n\}_{n\geq 0}$ denote the sequence counting walks on cubic lattice. The sequence $\{w_n\}_{n\geq 0}$ is Sloane's A005572 and satisfies the following recurrence*

$$(n+2)w_n = 4(2n+1)w_{n-1} - 12(n-1)w_{n-2}, \quad n \geq 2,$$

where $w_0 = 1$, $w_1 = 4$ and $w_2 = 17$. Some values of $\{w_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9
w_n	1	4	17	76	354	1704	8421	42508	218318	1137400

Liu and Wang [6] proved that the sequence $\{w_n\}_{n\geq 0}$ counting walks on cubic lattice is log-convex. In particular, Zhao [12] showed that $\{w_{n+1} - w_n\}_{n\geq 0}$ is also log-convex. Now we discuss the log-balancedness of $\{w_{n+1} - w_n\}_{n\geq 0}$.

Corollary 3.5 *For the sequence $\{w_n\}_{n\geq 0}$ counting walks on cubic lattice, $\{w_{n+1} - w_n\}_{n\geq 0}$ is log-balanced.*

Proof. For $n \geq 0$, let $x_n = \frac{w_{n+1}}{w_n}$. Zhao [12] proved that

$$\lambda_n \leq x_n \leq \lambda_{n+1}, \quad n \geq 3,$$

where $\lambda_n = \frac{6n+9}{n+3}$. It is evident that $R_n = \frac{4(2n+3)}{n+3}$ and $S_n = \frac{12n}{n+3}$. We note that

$$\begin{aligned}
\Omega_n &= (n+2)R_{n+1} - (n+1)R_{n+2} - 1 + \frac{(n+1)(S_{n+2} - R_{n+2} + 1)}{\lambda_{n+2} - 1} \\
&\quad - \frac{(n+2)(S_{n+1} - R_{n+1} + 1)}{\lambda_n - 1} \\
&= \frac{7n^2 + 39n + 68}{(n+4)(n+5)} - \frac{25n^3 + 105n^2 - 130n - 408}{(n+4)(5n+6)(5n+16)} \\
&= \frac{3(50n^4 + 505n^3 + 2089n^2 + 4094n + 2856)}{(n+4)(n+5)(5n+6)(5n+16)} > 0 \quad (n \geq 3).
\end{aligned}$$

It follows from Theorem 2.5 that the sequence $\{w_{n+1} - w_n\}_{n\geq 3}$ is log-balanced. On the other hand, $\{w_{k+1} - w_k\}_{0\leq k\leq 4}$ is log-balanced. Hence $\{w_{n+1} - w_n\}_{n\geq 0}$ is log-balanced. ■

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Example 3.6 *Let g_n count the number of undirected 2-regular labeled graphs. The sequence $\{g_n\}_{n\geq 0}$ is Sloane's A001205 and satisfies the following recurrence*

$$g_{n+1} = ng_n + \binom{n}{2} g_{n-2}, \quad n \geq 2, \tag{3.4}$$

where $g_0 = 1$ and $g_1 = g_2 = 0$. Some values of $\{g_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11
g_n	1	0	0	1	3	12	70	465	3507	30016	286884	3026655

Zhao [10] proved that $\{g_n\}_{n\geq 3}$ is log-convex. Now we discuss the log-behavior of $\{g_{n+1} - g_n\}_{n\geq 2}$.

Corollary 3.6 *For the sequence of counting the number of undirected 2-regular labeled graphs $\{g_n\}_{n\geq 3}$, $\{g_{n+1} - g_n\}_{n\geq 2}$ is log-convex and $\{g_{n+1} - g_n\}_{n\geq 4}$ is log-balanced.*

Proof. For $n \geq 3$, let $x_n = \frac{g_{n+1}}{g_n}$. It follows from (3.4) that

$$x_n = n + \binom{n}{2} \frac{1}{x_{n-1}x_{n-2}}, \quad n \geq 5. \tag{3.5}$$

We first prove by induction that

$$n + \frac{1}{2} \leq x_n \leq n + \frac{1}{2} + \frac{1}{n}, \quad n \geq 6. \tag{3.6}$$

We observe that $k + \frac{1}{2} \leq x_k \leq k + \frac{1}{2} + \frac{1}{k}$ for $6 \leq k \leq 10$. Assume that $k + \frac{1}{2} \leq x_k \leq k + \frac{1}{2} + \frac{1}{k}$ for $k \geq 10$. By applying (3.5), we have

$$\begin{aligned} x_{k+1} - k - \frac{3}{2} &= -\frac{1}{2} + \binom{k+1}{2} \frac{1}{x_k x_{k-1}}, \\ x_{k+1} - k - \frac{3}{2} - \frac{1}{k+1} &= -\frac{1}{2} - \frac{1}{k+1} + \binom{k+1}{2} \frac{1}{x_k x_{k-1}}. \end{aligned}$$

Since $k - \frac{1}{2} \leq x_{k-1} \leq k - \frac{1}{2} + \frac{1}{k-1} < k$ and $k + \frac{1}{2} \leq x_k \leq k + \frac{1}{2} + \frac{1}{k} < k+1$,

$$\begin{aligned} x_{k+1} - k - \frac{3}{2} &\geq -\frac{1}{2} + \binom{k+1}{2} \frac{1}{(k+1)k} \\ &= 0 \quad (k \geq 10), \\ x_{k+1} - k - \frac{3}{2} - \frac{1}{k+1} &\leq -\frac{1}{2} - \frac{1}{k+1} + \binom{k+1}{2} \frac{1}{(k+\frac{1}{2})(k-\frac{1}{2})} \\ &= -\frac{k^2 - \frac{5k}{4} - \frac{3}{4}}{2(k+1)(k+\frac{1}{2})(k-\frac{1}{2})} \\ &< 0 \quad (k \geq 10). \end{aligned}$$

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Hence we have (3.6). Now we discuss the log-convexity of $\{g_{n+1}-g_n\}_{n\geq 2}$ by Theorem 2.6. It is clear that

$$Q_n = n, \quad R_n = 0, \quad S_n = \binom{n}{2}, \quad \varphi_n = n + \frac{1}{2} + \frac{1}{n}, \quad \phi_n = n + \frac{1}{2},$$

$$\begin{aligned}\Delta_n &= 1 + \frac{n+1}{n+\frac{1}{2}+\frac{1}{n+1}} + \frac{\binom{n+2}{2}}{(n+\frac{1}{2}+\frac{1}{n})(n+\frac{1}{2}+\frac{1}{n+1})} - \frac{n}{n-\frac{1}{2}} - \frac{\binom{n+1}{2}}{(n-\frac{1}{2})^2} \\ &\geq 2 + \frac{n+2}{2(n+1)} - \frac{n}{n-1} - \frac{n(n+1)}{2(n-1)^2} = \frac{n^3-3n^2-3n+3}{(n+1)(n-1)^2} > 0 \quad (n \geq 7)\end{aligned}$$

and

$$\begin{aligned}\Upsilon_n &= -1 + \frac{(n+2)n}{n-\frac{1}{2}+\frac{1}{n}} + \frac{(n+2)\binom{n+1}{2}}{(n-\frac{1}{2}+\frac{1}{n-1})(n-\frac{1}{2}+\frac{1}{n})} - \frac{(n+1)^2}{n+\frac{1}{2}} - \frac{(n+1)\binom{n+2}{2}}{(n+\frac{1}{2})^2} \\ &= \frac{2n^3+2n^2+n-2}{2n^2-n+2} + \frac{2(n-1)n^2(n+1)(n+2)}{(2n^2-3n+3)(2n^2-n+2)} - \frac{6(n+1)^3}{(2n+1)^2} \\ &> \frac{2n^3+2n^2+n-2}{2n^2-n+2} + \frac{n(n+1)(n+2)}{2n^2-n+2} - \frac{6(n+1)^3}{(2n+1)^2} \quad (n \geq 8) \\ &= \frac{2n^4+5n^3-21n^2-35n-14}{(2n^2-n+2)(2n+1)^2} > 0.\end{aligned}$$

It follows from Theorem 2.6 that the sequence $\{g_{n+1}-g_n\}_{n\geq 8}$ is log-balanced. On the other hand, we find that $\{g_{k+1}-g_k\}_{2\leq k\leq 9}$ is log-convex and $\{g_{k+1}-g_k\}_{4\leq k\leq 9}$ is log-balanced. Hence $\{g_{n+1}-g_n\}_{n\geq 2}$ is log-convex and $\{g_{n+1}-g_n\}_{n\geq 4}$ is log-balanced. ■

Example 3.7 *Let G_n stand for the number of graphs on the vertex set $[n] = \{1, 2, \cdots, n\}$, whose every component is a cycle. The sequence $\{G_n\}_{n\geq 0}$ is Sloane's A002135 and satisfies the following recurrence*

$$G_{n+1} = (n+1)G_n - \binom{n}{2}G_{n-2}, \quad n \geq 2, \tag{3.7}$$

where $G_0 = 1$, $G_1 = 1$, and $G_2 = 2$. Some values of $\{G_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9	10
G_n	1	1	2	5	17	73	388	2461	18155	152531	1436714

Došlić and D. Veljan [3] proved that $\{G_n\}_{n\geq 0}$ is log-convex. Now we investigate the log-behavior of $\{G_{n+1}-G_n\}_{n\geq 1}$.

Corollary 3.7 *For the sequence $\{G_n\}_{n\geq 0}$ defined by (3.7), $\{G_{n+1}-G_n\}_{n\geq 1}$ is log-balanced.*

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Proof. For $n \geq 0$, let $x_n = \frac{G_{n+1}}{G_n}$. It follows from (3.7) that

$$x_n = n + 1 - \frac{\binom{n}{2}}{x_{n-1}x_{n-2}}, \quad n \geq 2.$$

We can prove by induction that

$$n + \frac{1}{2} - \frac{1}{n} \leq x_n \leq n + \frac{1}{2}, \quad n \geq 2. \tag{3.8}$$

The proof of (3.8) is similar to (3.6) and is omitted here. Now we investigate the log-convexity of $\{G_{n+1} - G_n\}_{n\geq 1}$ by Theorem 2.7. It is evident

$$Q_n = n + 1, \quad R_n = 0, \quad S_n = \binom{n}{2}, \quad \varphi_n = n + \frac{1}{2}, \quad \phi_n = n + \frac{1}{2} - \frac{1}{n}.$$

For $n \geq 3$, we note that

$$\begin{aligned}
\Omega_n &= 1 + \frac{n+2}{n+\frac{1}{2}} - \frac{\binom{n+2}{2}}{(n+\frac{1}{2}-\frac{1}{n+1})(n+\frac{1}{2}-\frac{1}{n})} - \frac{n+1}{n-\frac{1}{2}-\frac{1}{n}} + \frac{\binom{n+1}{2}}{(n-\frac{1}{2})^2} \\
&\geq 2 + \frac{3}{2n+1} - \frac{2(n+1)(n+2)}{(2n-1)^2} - \frac{2n(n+1)}{2n^2-n-2} + \frac{2n(n+1)}{(2n-1)^2} \\
&= \frac{2(n^2-2n-2)}{2n^2-n-2} + \frac{3}{2n+1} - \frac{4(n+1)}{(2n-1)^2} \\
&> \frac{1}{2} + \frac{3}{2n+1} - \frac{4(n+1)}{(2n-1)^2} > 0 \quad (n \geq 5)
\end{aligned}$$

and

$$\begin{aligned}
\Phi_n &= \frac{(n+1)(n+2)}{n-\frac{1}{2}} - \frac{(n+2)\binom{n+1}{2}}{(n-\frac{1}{2}-\frac{1}{n})(n-\frac{1}{2}-\frac{1}{n-1})} - \frac{(n+2)(n+1)}{n+\frac{1}{2}-\frac{1}{n+1}} \\
&\quad + \frac{(n+1)\binom{n+2}{2}}{(n+\frac{1}{2})^2} \\
&= \frac{2(n+1)(n+2)}{2n-1} - \frac{2(n-1)n^2(n+1)(n+2)}{(2n^2-n-2)(2n^2-3n-1)} - \frac{2(n+1)^2(n+2)}{2n^2+3n-1} \\
&\quad + \frac{2(n+1)^2(n+2)}{(2n+1)^2} \\
&= \frac{4n(n+1)(n+2)}{(2n-1)(2n^2+3n-1)} - \frac{2(n+1)(n+2)(4n^4+8n^3-5n^2-9n-2)}{(2n+1)^2(2n^2-n-2)(2n^2-3n-1)} \\
&= \frac{2(n+1)(n+2)(16n^7-80n^6-72n^5+108n^4+77n^3-2n^2+3n+2)}{(2n-1)(2n^2+3n-1)(2n+1)^2(2n^2-n-2)(2n^2-3n-1)} \\
&> 0 \quad (n \geq 6).
\end{aligned}$$

It follows from Theorem 2.7 that $\{G_{n+1} - G_n\}_{n\geq 6}$ is log-balanced. We observe that $\{G_{k+1} - G_k\}_{1\leq k\leq 7}$ is log-balanced. Hence $\{G_{n+1} - G_n\}_{n\geq 1}$ is log-balanced. ■

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

Example 3.8 *Let Γ_n be the number of $n\times n$ symmetric matrices with nonnegative integer entries, trace 0 and all row sum 2. The sequence $\{\Gamma_n\}_{n\geq 0}$ is Sloane's A002137 and satisfies the following recurrence*

$$\Gamma_{n+1}=n\Gamma_n+n\Gamma_{n-1}-\binom{n}{2}\Gamma_{n-2},\quad n\geq 2, \tag{3.9}$$

where $\Gamma_0=1,\Gamma_1=0$, and $\Gamma_2=1$. Some values of $\{\Gamma_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
Γ_n	1	0	1	1	6	22	130	822	6202	52552	499194	5238370	60222844

Došlić and D. Veljan [3] showed that $\{\Gamma_n\}_{n\geq 6}$ is log-convex. Now we discuss the log-behavior of $\{\Gamma_{n+1}-\Gamma_n\}_{n\geq 1}$.

Corollary 3.8 *For the sequence $\{\Gamma_n\}_{n\geq 0}$ defined by (3.9), $\{\Gamma_{n+1}-\Gamma_n\}_{n\geq 5}$ is log-convex and $\{\Gamma_{n+1}-\Gamma_n\}_{n\geq 6}$ is log-balanced.*

Proof. For $n\geq 2$, let $x_n=\frac{\Gamma_{n+1}}{\Gamma_n}$. It follows from (3.9) that

$$x_n=n+\frac{n}{x_{n-1}}-\frac{\binom{n}{2}}{x_{n-1}x_{n-2}},\quad n\geq 4. \tag{3.10}$$

We first prove by induction that

$$n+\frac{1}{2}-\frac{1}{2n}\leq x_n\leq n+\frac{1}{2},\quad n\geq 8. \tag{3.11}$$

We observe that $k+\frac{1}{2}-\frac{1}{2k}\leq x_k\leq k+\frac{1}{2}$ for $8\leq k\leq 10$. Assume that $k+\frac{1}{2}-\frac{1}{2k}\leq x_k\leq k+\frac{1}{2}$ for $k\geq 10$. By using (3.10), we have

$$\begin{aligned}
x_{k+1}-k-\frac{3}{2}+\frac{1}{2(k+1)} &= -\frac{1}{2}+\frac{1}{2(k+1)}+\frac{k+1}{x_k}-\frac{\binom{k+1}{2}}{x_kx_{k-1}},\\
x_{k+1}-k-\frac{3}{2} &= \frac{2(k+1)x_{k-1}-k(k+1)-x_kx_{k-1}}{2x_kx_{k-1}}\\
&= \frac{(k+2)x_{k-1}-k^2-2k+\frac{\binom{k}{2}}{x_{k-2}}}{2x_kx_{k-1}}.
\end{aligned}$$

Since $k+\frac{1}{2}-\frac{1}{2k}\leq x_k\leq k+\frac{1}{2}$,

$$\begin{aligned}
x_{k+1}-k-\frac{3}{2}+\frac{1}{2(k+1)} &\geq -\frac{1}{2}+\frac{1}{2(k+1)}+\frac{k+1}{k+\frac{1}{2}}-\frac{\binom{k+1}{2}}{(k+\frac{1}{2}-\frac{1}{2k})[k-\frac{1}{2}-\frac{1}{2(k-1)}]}\\
&\geq \frac{1}{2}+\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{k(k+1)}{2(k+\frac{1}{4})(k-\frac{3}{4})}\\
&\geq \frac{-\frac{3}{2}k-\frac{3}{16}}{2(k+\frac{1}{4})(k-\frac{3}{4})}+\frac{1}{k+1}\\
&= \frac{k^2-\frac{43k}{8}-\frac{9}{8}}{4(k-\frac{3}{4})(k+\frac{1}{4})(k+1)}>0\quad (k\geq 10)
\end{aligned}$$

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

and

$$\begin{aligned}(k+2)x_{k-1}-k^2-2k+\frac{\binom{k}{2}}{x_{k-2}} &\le (k+2)\left(k-\frac{1}{2}\right)-k^2-2k+\frac{\binom{k}{2}}{k-\frac{3}{2}-\frac{1}{2(k-2)}}\\ &= -\frac{3k-10}{2(2k-5)}<0 \quad (k\ge 10).\end{aligned}$$

Hence we have (3.11). Now we study the log-convexity of $\{\Gamma_{n+1}-\Gamma_n\}_{n\ge 3}$ by using Theorem 2.7. It is obvious that

$$Q_n=n,\quad R_n=n,\quad S_n=\binom{n}{2},\quad \phi_n=n+\frac{1}{2}-\frac{1}{2n},\quad \varphi_n=n+\frac{1}{2}.$$

For $n\ge 9$, we have

$$\begin{aligned}\Omega_n &= 1+\frac{2n+3}{n+\frac{1}{2}}-\frac{\binom{n+2}{2}}{[n+\frac{1}{2}-\frac{1}{2(n+1)}](n+\frac{1}{2}-\frac{1}{2n})}-\frac{2n+1}{n-\frac{1}{2}-\frac{1}{2n}}+\frac{\binom{n+1}{2}}{(n-\frac{1}{2})^2}\\ &= 1+\frac{2(2n+3)}{2n+1}-\frac{2(n+1)(n+2)}{(2n+3)(2n-1)}-\frac{2n}{n-1}+\frac{2n(n+1)}{(2n-1)^2}\\ &= 1-\frac{2(n+1)(n+2)}{(2n-1)(2n+3)}+2\left(1+\frac{2}{2n+1}\right)-\frac{2n}{n-1}+\frac{2n(n+1)}{(2n-1)^2}\\ &> \frac{4}{2n+1}-\frac{2}{n-1}+\frac{2n(n+1)}{(2n-1)^2}>\frac{4}{2n+1}>0\end{aligned}$$

and

$$\begin{aligned}\Phi_n &= -1+\frac{(n+2)(2n+1)}{n-\frac{1}{2}}-\frac{(n+2)\binom{n+1}{2}}{(n-\frac{1}{2}-\frac{1}{2n})[n-\frac{1}{2}-\frac{1}{2(n-1)}]}-\frac{(n+1)(2n+3)}{n+\frac{1}{2}-\frac{1}{2(n+1)}}\\ &\quad +\frac{(n+1)\binom{n+2}{2}}{(n+\frac{1}{2})^2}\\ &= -1+\frac{2(n+2)(2n+1)}{2n-1}-\frac{2n(n+1)(n+2)}{(2n+1)(2n-3)}-\frac{2(n+1)^2}{n}+\frac{2(n+1)^2(n+2)}{(2n+1)^2}\\ &= \frac{(n+2)(2n+1)}{n(2n-1)}-\frac{2(n+1)(n+2)(2n+3)}{(2n+1)^2(2n-3)}\\ &> (n+2)\left[\frac{2n+1}{2n^2}-\frac{2(n+1)(2n+3)}{(2n-3)4n^2}\right]=\frac{(n+2)(2n^2-9n-6)}{2n^2(2n-3)}>0.\end{aligned}$$

It follows from Theorem 2.7 that $\{\Gamma_{n+1}-\Gamma_n\}_{n\ge 9}$ is log-balanced. On the other hand, we observe that $\{\Gamma_{k+1}-\Gamma_k\}_{5\le k\le 10}$ is log-convex and $\{\Gamma_{k+1}-\Gamma_k\}_{6\le k\le 10}$ is log-balanced. Then $\{\Gamma_{n+1}-\Gamma_n\}_{n\ge 5}$ is log-convex and $\{\Gamma_{n+1}-\Gamma_n\}_{n\ge 6}$ is log-balanced. ■

4 Conclusions

For a log-convex sequence $\{z_n\}_{n\ge 0}$ with $z_{n+1}-z_n>0$ for $n\ge 0$, we have derived several sufficient conditions for the log-balancedness of $\{z_{n+1}-z_n\}_{n\ge 0}$. We have further applied

3 Apr 2023 19:36:48 PDT
220407-Zhao-3 Version 2 - Submitted to Rocky Mountain J. Math.

these new results to a series of combinatorial sequences. Our future work is to find more sufficient conditions for the log-balancedness of sequences.

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