ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

MAPS PRESERVING THE TRUNCATION OF OPERATORS ON POSITIVE CONES

YADI SONG AND GUOXING JI

ABSTRACT. Let $\mathscr{H}$ be a complex Hilbert space with $\dim\mathscr{H} \geq 2$ and $\mathscr{B}(\mathscr{H})^+$ the positive cone of the algebra of all bounded linear operators on $\mathscr{H}$. For $A, B \in \mathscr{B}(\mathscr{H})^+$, A is called a positive truncation of B if $A = P_A B P_A$, where P_A denotes the orthogonal projection onto the closure of $R(A)$. In this paper, we determine the structures of all bijections preserving the positive truncations of operators in both directions on the positive cone $\mathscr{B}(\mathscr{H})^+$.

1. Introduction

The preserver problem is one of the important research content of operator algebra, which has attracted extensive attention in academic circles in recent years and has achieved many remarkable results(cf.[1, 4]). The aim is to find certain rigid characterizations of isomorphisms of operator algebras. Meanwhile, to describe algebraic or geometric characterizations of operator algebras, many authors consider maps preserving certain properties on some important operator classes. For example, those maps on positive cones of operator algebras preserving certain operator means have been studied recently in [2, 5, 6, 7]. Among those topics, it is an important object to characterize those maps which preserve some operator relations. Very recently, preserver problems involving truncation of operators have been considered(cf.[3, 10]). Let $\mathscr{H}$ be a complex Hilbert space with $\dim\mathscr{H} \geq 2$ and let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. For $A, B \in \mathscr{B}(\mathscr{H})$, A is said to be a truncation of B if $A = P_A B P_{A^*}$, where P_A and P_{A^*} denote the orthogonal projections from $\mathscr{H}$ onto the closures of the range of A and A^* respectively. Note that the truncation is an elementary relation between two operators. It reveals a certain "size" relation between A and B. How this relation effects the isomorphisms of operator algebras? Authors in [10] gave the forms of all additive surjective maps preserving the truncation of operators in both directions on $\mathscr{B}(\mathscr{H})$. We also characterized the forms of bijective maps preserving the truncation of products of operators on $\mathscr{B}(\mathscr{H})$ in [3]. If we consider A and B are both positive operators, then A is a truncation of B if and only if $A = PBP$ for some projection P. Thus, it is an interesting problem to consider those maps which preserving the truncation of operators on the positive cone of operators. We will consider this problem in this paper.

We denote by $\mathscr{B}(\mathscr{H})^+ = \{A \in \mathscr{B}(\mathscr{H}) : A \geq 0\}$ the positive operator cone of $\mathscr{B}(\mathscr{H})$. For nonzero vectors $x, y \in \mathscr{H}$, the symbol $x \otimes y$ stands for the rank-1 bounded linear operator defined by $(x \otimes y)z = \langle z, y\rangle x$ for all $z \in \mathscr{H}$, where $\langle z, y\rangle$ is the inner product of z and y. Note that every operator of rank-1 can be written in this form. Then $P_x = x \otimes x$ is a rank-1 projection for any unit vector x. For a subset

This research was supported by the National Natural Science Foundation of China(No. 12271323) and the Fundamental Research Funds for the Central Universities (Grant No. GK202107014).

2020 *Mathematics Subject Classification.* 47B49; 47B48.

Key words and phrases. positive operator; positive cone; truncation of operator; preserver.

$S \subseteq \mathscr{H}$, we denote by $\vee S$ and $S^{\perp}$ the closed subspace generated by S and the orthogonal complement subspace of S in $\mathscr{H}$ respectively. Let $\mathscr{P}_1(\mathscr{H}) = \{P_x : x \in \mathscr{H}, \|x\| = 1\}$ and let $\mathscr{B}(\mathscr{H})_p$ be the set of all projections on $\mathscr{H}$. For $P, Q \in \mathscr{B}(\mathscr{H})_p$, we say $P \leq Q$ if $PQ = QP = P$. Two projections P and Q are said to be orthogonal (in symbol $P \perp Q$) if $PQ = 0$. For $A \in \mathscr{B}(\mathscr{H})^+$, let $\mathscr{T}(A)$ denote the set of all positive truncations of A, that is, $\mathscr{T}(A) = \{PAP : P \in \mathscr{B}(\mathscr{H})_p\}$. In this paper, we denote by $\mathbb{R}^+$ and $\mathbb{Q}^+$ the set of all nonnegative real numbers and the set of all nonnegative rational numbers.

2. Main results

Let $A, B \in \mathscr{B}(\mathscr{H})^+$. It is known that

$$A = P_A B P_A \Longleftrightarrow A^3 = ABA \tag{2.1}$$

from [10, Lemma 2.1]. Note that the relationship of truncations of operators is not an order in general. In fact, when A is a truncation of B and B is that of C, it is not true that A is that of C. However we may define a relationship " $\prec$ " in $\mathscr{T}(A)$ for any $A \in \mathscr{B}(\mathscr{H})^+$, which is useful in our proofs. For $A_1, A_2 \in \mathscr{T}(A)$, if A_1 is also a truncation of A_2, then we say that $A_1 \prec A_2$. We say a nonzero $A_0 \in \mathscr{T}(A)$ is minimal if there is not any nonzero $A_1 \in \mathscr{T}(A)$ with $A_1 \neq A_0$ such that $A_1 \prec A_0$. The following lemma is elementary.

Lemma 2.1. *If $A_0 \in \mathscr{T}(A)$ is minimal, then A_0 is a rank-1 operator.*

Proof. Suppose that $\operatorname{rank}(A_0) \geq 2$. Then $A_0 = P_{A_0} A P_{A_0}$ with $\operatorname{rank}(P_{A_0}) \geq 2$. Take any unit vector $x \in R(A_0)$ and put $A_x = P_x A_0 P_x = \langle A_0 x, x\rangle P_x$. It is trivial that $A_x \neq A_0$. Since $P_x \leq P_{A_0}$, $A_x = P_x A_0 P_x = P_x P_{A_0} A P_{A_0} P_x = P_x A P_x$. Thus $A_x \in \mathscr{T}(A)$ and $A_x \prec A_0$, which implies that A_0 is not minimal in $\mathscr{T}(A)$, a contradiction. Hence A_0 is a rank-1 operator. □

It is known that if $A_0 \in \mathscr{T}(A)$ is minimal, then $A_0 = \langle Ax, x\rangle P_x$ for some unit vector x with $Ax \neq 0$. However the converse is false in general. The next lemma gives a characterization of minimal truncations in $\mathscr{T}(A)$. We next put $A_x = P_x A P_x = \langle Ax, x\rangle P_x$ for any unit vector x with $Ax \neq 0$.

Lemma 2.2. *Let $A_x \in \mathscr{T}(A)$. Then A_x is minimal if and only if there exists some $a > 0$ such that $Ax = ax$ and $A|_{\{x\}^{\perp}}$ is injective.*

Proof. $\Longrightarrow$ Assume that $A_x = P_x A P_x$ is minimal in $\mathscr{T}(A)$. Put $M = \{x\}^{\perp}$. Then M is the orthogonal complement subspace of the one dimensional subspace $\vee\{x\}$. Thus $\mathscr{H} = \vee\{x\} \oplus M$. Assume by way of contradiction that there exists a unit vector $y \in M$ such that $\langle Ax, y\rangle \neq 0$. Let $\mathscr{H}_2 = \vee\{x, y\}$. Denote by $P_{\mathscr{H}_2}$ the orthogonal projection onto $\mathscr{H}_2$. Then $P_{\mathscr{H}_2} A P_{\mathscr{H}_2} \in \mathscr{T}(A)$. Since $\langle Ax, y\rangle \neq 0$, there exists a scalar $\theta \in [0, 2\pi]$ such that $\langle Ax, y\rangle = e^{i\theta}|\langle Ax, y\rangle|$. Let $\tilde{y} = -e^{i\theta} y$, then $\langle Ax, \tilde{y}\rangle = -|\langle Ax, y\rangle| < 0$. We may assume that $\langle Ax, y\rangle < 0$ without loss of generality. Under the decomposition $\mathscr{H} = \mathscr{H}_2 \oplus \mathscr{H}_2{}^{\perp}$,

$$A = \begin{pmatrix} a_{11} & a_{12} & A_{13} \\ a_{12} & a_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{pmatrix},$$

where $a_{11} = \langle Ax, x\rangle > 0$, $a_{22} = \langle Ay, y\rangle > 0$ and $a_{12} = \langle Ax, y\rangle < 0$. Thus

$$P_{\mathscr{H}_2} A P_{\mathscr{H}_2} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } P_x A P_x = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For any $a, b \in (0,1)$ with $a^2 + b^2 = 1$, it is easy to see that

$$P = \begin{pmatrix} a^2 & ab & 0 \\ ab & b^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

is a rank-1 projection such that $P \le P_{\mathscr{H}_2}$. By an elementary calculation, we have

$$\begin{aligned} PAP &= PP_{\mathscr{H}_2} A P_{\mathscr{H}_2} P \\ &= \begin{pmatrix} a^4 a_{11} + 2a^3 b a_{12} + a^2 b^2 a_{22} & a^3 b a_{11} + 2a^2 b^2 a_{12} + ab^3 a_{22} & 0 \\ 2a^2 b^2 a_{12} + a^3 b a_{11} + ab^3 a_{22} & b^4 a_{22} + 2ab^3 a_{12} + a^2 b^2 a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$PP_x A P_x P = \begin{pmatrix} a^4 a_{11} & a^3 b a_{11} & 0 \\ a^3 b a_{11} & a^2 b^2 a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume that $PAP = PP_x A P_x P$. Then we have

$$2a a_{12} + b a_{22} = 0. \tag{2.3}$$

Since $a_{12} < 0$ and $b = (1 - a^2)^{\frac{1}{2}}$, the equation (2.3) has a positive solution

$$a = \sqrt{\frac{{a_{22}}^2}{4{a_{12}}^2 + {a_{22}}^2}}. \tag{2.4}$$

That is, if we take a as (2.4) and $b = (1 - a^2)^{\frac{1}{2}}$, then the rank-1 projection P defined in (2.2) satisfies $PAP = PP_x A P_x P$, which shows that $PAP \prec P_x A P_x$. It contradicts with the minimality of $P_x A P_x$ since $PAP \ne P_x A P_x$. It follows that $\langle Ax, y\rangle = 0$ for any $y \in M$. Hence $Ax = a_{11} x$ and A has the matrix representation

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_2 = A|_M$.

Suppose on the contrary that there exists a nonzero $y \in M$ such that $A_2 y = Ay = 0$. Take any $P \in \mathscr{P}_1(\mathscr{H})$ defined as (2.2), then we have $PAP = PP_x A P_x P \ne P_x A P_x$. It follows that $PAP \in \mathscr{T}(A)$ and $PAP \prec P_x A P_x$. We again obtain that $P_x A P_x$ is not minimal, a contradiction. Hence A_2 is injective.

$\Longleftarrow$ Assume that A has the matrix representation $A = \begin{pmatrix} a_{11} & 0 \\ 0 & A_2 \end{pmatrix}$ under the decomposition $\mathscr{H} = \vee\{x\} \oplus M$, where $M = \{x\}^{\perp}$ and $A_2 = A|_M$ is injective.

Take any unit vector $y \notin \vee\{x\}$. Then there exists a unit vector $\xi \in M$ such that $\vee\{x, y\} = \vee\{x, \xi\}$. Let $\mathscr{H}_2 = \vee\{x, y\} = \vee\{x, \xi\}$ be the 2-dimensional subspace of $\mathscr{H}$ generated by x and y. Then

$P_{\mathscr{H}_2}AP_{\mathscr{H}_2} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $a_{22} = \langle A\xi, \xi \rangle > 0$. Then there exist an $\alpha \in [0,1)$ and a complex number λ with $|\lambda| = 1$ such that

$$P_y = \begin{pmatrix} \alpha & \lambda\sqrt{\alpha(1-\alpha)} & 0 \\ \bar{\lambda}\sqrt{\alpha(1-\alpha)} & 1-\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is an elementary exercise to check that

$$P_xAP_x = P_xP_{\mathscr{H}_2}AP_{\mathscr{H}_2}P_x = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} P_yAP_y &= P_yP_{\mathscr{H}_2}AP_{\mathscr{H}_2}P_y \\ &= \begin{pmatrix} a_{11}\alpha^2 + a_{22}\sqrt{\alpha(1-\alpha)} & \lambda\sqrt{\alpha(1-\alpha)}\,(a_{11}\alpha + a_{22}(1-\alpha)) & 0 \\ \bar{\lambda}\sqrt{\alpha(1-\alpha)}\,(a_{11}\alpha + a_{22}(1-\alpha)) & a_{11}\alpha(1-\alpha) + a_{22}(1-\alpha)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$P_yP_xAP_xP_y = \begin{pmatrix} a_{11}\alpha^2 & \lambda a_{11}\alpha\sqrt{\alpha(1-\alpha)} & 0 \\ \bar{\lambda}a_{11}\alpha\sqrt{\alpha(1-\alpha)} & a_{11}\alpha(1-\alpha) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $\alpha \in [0,1)$. It follows that $P_yAP_y \neq P_yP_xAP_xP_y$. From the arbitrariness of y we obtain that P_xAP_x is minimal. $\square$

Let φ be a bijection on $\mathscr{B}(\mathscr{H})^+$. We say that φ preserves the truncation of operators if $\varphi(A)$ is the positive truncation of $\varphi(B)$ whenever A is that of B for any $A, B \in \mathscr{B}(\mathscr{H})^+$. φ is said to preserve the truncation of operators in both directions if $\varphi(A)$ is the truncation of $\varphi(B)$ if and only if A is that of B for any $A, B \in \mathscr{B}(\mathscr{H})^+$. Before giving the main theorem, we firstly show the following lemma.

Lemma 2.3. *Let φ be a bijiection on $\mathscr{B}(\mathscr{H})^+$ preserving the truncation of positive operators in both directions. Then $\varphi(\mathbb{R}^+I) = \mathbb{R}^+I = \{aI : a \in \mathbb{R}^+\}$.*

Proof. It is trivial that $\varphi(0) = 0$. Fix an $a > 0$. Define a map $\psi : \mathscr{B}(\mathscr{H})^+ \to \mathscr{B}(\mathscr{H})^+$ by $\psi(A) = \varphi(aA)$ for any $A \in \mathscr{B}(\mathscr{H})^+$. Then ψ has the same preserver properties with φ and $\psi(I) = \varphi(aI)$. So in the following we only need to show that there exists a scalar $b > 0$ such that $\varphi(I) = bI$. Put $B = \varphi(I)$. Since B has a minimal truncation, it follows from Lemma 2.2 that the range of B is dense and hence B is necessarily injective. Since $\mathscr{T}(I) = \mathscr{B}(\mathscr{H})_p$ and φ preserves the truncation of positive operators in both directions, it is trivial that $\varphi(\mathscr{T}(I)) = \mathscr{T}(\varphi(I)) = \mathscr{T}(B)$ and φ preserves the minimal truncations in both directions.

If P_xBP_x and P_yBP_y are two minimal truncations in $\mathscr{T}(B)$ such that $Bx = bx$ and $By = by$ for some $b > 0$, then for any unit vector $z \in \vee\{x, y\}$, $P_zBP_z \in \mathscr{T}(B)$ is also a minimal truncation with $Bz = bz$. Now take a maximal orthogonal family of maximal subspaces $\{M_i \subseteq \mathscr{H} : i \in \Lambda\}$ such that $P_{x_i}BP_{x_i}$ is minimal in $\mathscr{T}(B)$ for any unit vector $x_i \in M_i$, $\forall i \in \Lambda$. Then for any $i \in \Lambda$ there exists a scalar $b_i > 0$ such that $BP_{M_i} = P_{M_i}BP_{M_i} = b_i P_{M_i}$.

Put $M = \bigvee\{M_i : i \in \Lambda\}$ and Q be the projection onto $M^{\perp}$. If $Q \neq 0$, then $BQ = QBQ \in \mathscr{T}(B)$ and is nonzero because of the injectivity of B. Thus there exists a nonzero projection $P \in \mathscr{B}(\mathscr{H})_p$ such that $\varphi(P) = QBQ$. It is known that P_x is minimal in $\mathscr{T}(I)$ for any unit vector $x \in P(\mathscr{H})$ by Lemma 2.1. Fix a unit vector $x_0 \in P(\mathscr{H})$, then $\varphi(P_{x_0}) \in \mathscr{T}(B)$ is also minimal. Thus there exist a unit vector $e_0 \in \mathscr{H}$ and a scalar $b_0 > 0$ such that $\varphi(P_{x_0}) = b_0 P_{e_0}$. By Lemma 2.2 again, we have $Be_0 = b_0 e_0$. Let M_0 be the maximal subspace of $\mathscr{H}$ such that $Bx = b_0 x$ for any unit $x \in M_0$. If $b_0 \neq b_i$ for any $i \in \Lambda$, then $\{M_i : i \in \Lambda\} \cup \{M_0\}$ is an orthogonal family of subspaces in $\mathscr{H}$ such that $P_{M_i}BP_{M_i} = b_i P_{M_i}$ for any $i \in \Lambda \cup \{0\}$. This contradicts to the maximality of $\{M_i : i \in \Lambda\}$. Hence there exists some $i \in \Lambda$ such that $b_0 = b_i$ and so $M_0 = M_i$. Note that $b_i P_{e_0} = \varphi(P_{x_0}) \prec \varphi(P) = QBQ = BQ$ in $\mathscr{T}(B)$. It follows that $b_i = \langle QBQe_0, e_0 \rangle = 0$ since $e_0 \in M_i \subseteq M$. This is a contradiction. Hence $Q = 0$ and $\mathscr{H} = \oplus_{i \in \Lambda} M_i$.

Suppose that $b_k \neq b_l$ for some $k, l \in \Lambda$ and we take two unit vectors $e_i \in M_i$ for $i = k, l$. Denote by Q_2 the projection onto $\vee\{e_k, e_l\}$. Then $Q_2BQ_2 = Q_2B \in \mathscr{T}(B)$. It is known that $b_l P_{e_l}$ and $b_k P_{e_k}$ are the only two minimal truncations in $\mathscr{T}(B)$ such that $b_i P_{e_i} \prec Q_2BQ_2$, $i = l, k$. Put $P_2 = \varphi^{-1}(Q_2B) \in \mathscr{T}(I)$. Then $\operatorname{rank}(P_2) \geq 2$ and thus there are innumerable minimal truncations $P_x \in \mathscr{T}(I)$ such that $P_x \prec P_2$. This means that there are also innumerable minimal truncations $P_y \in \mathscr{T}(B)$ such that $P_y \prec Q_2B$, a contradiction. Hence $b_i = b_j$ for any $i, j \in \Lambda$ and we denote it by b. Therefore, $B = bI$. □

Theorem 2.1. *Let $\varphi : \mathscr{B}(\mathscr{H})^+ \to \mathscr{B}(\mathscr{H})^+$ be a bijection preserving the truncation of positive operators in both directions. Then there exist $\alpha > 0$ and a unitary or an anti-unitary operator U on $\mathscr{H}$ such that $\varphi(A) = \alpha UAU^*$ for any $A \in \mathscr{B}(\mathscr{H})^+$.*

Proof. By Lemma 2.3, there exists a scalar $\alpha > 0$ such that $\varphi(I) = \alpha I$. Define $\psi(A) = \alpha^{-1}\varphi(A)$ for any $A \in \mathscr{B}(\mathscr{H})^+$, then ψ and φ have the same preserver properties such that $\psi(I) = I$. In this case, there exists a bijective function f on $\mathbb{R}^+$ such that $\psi(aI) = f(a)I$ for any $a \in \mathbb{R}^+$. It follows that $f(0) = 0$ and $f(1) = 1$. It is known that ψ preserves the projections and the order of projections in both directions from Lemma 2.2. Furthermore, ψ preserves rank-1 projections in both directions. We next show that $\psi(aP) = f(a)\psi(P)$ for any $a \in \mathbb{R}^+$ and any $P \in \mathscr{P}_1(\mathscr{H})$. It holds obviously when $a = 0$ and $a = 1$, so we need only to prove the corrections for $a \neq 0$ and $a \neq 1$. It is elementary that $\psi(aP) = f(a)Q_{aP}$ for some $Q_{aP} \in \mathscr{P}_1(\mathscr{H})$ since aP is minimal in $\mathscr{T}(aI)$. Note that $I - P$ is one co-dimensional. If $\operatorname{rank}(I - \psi(I - P)) \geq 2$, then there exists a $Q \in \mathscr{B}(\mathscr{H})_p$ such that $\psi(I - P) \precneqq Q \precneqq I$. Thus $I - P \precneqq \psi^{-1}(Q) \precneqq I$, a contradiction. Hence $\psi(I - P)$ is also a one co-dimensional projection. That is, $\psi(I - P) = I - Q$ for some $Q \in \mathscr{P}_1(\mathscr{H})$. Put $A = aP + (I - P)$ and $B = \psi(A)$. Note that $aP \in \mathscr{T}(A)$ is minimal. So is $\psi(aP) = f(a)Q_{aP} \in \mathscr{T}(B)$. Thus from Lemma 2.2 we get that $B = f(a)Q_{aP} \oplus B_2$ for an injective operator B_2 on $R(I - Q_{aP})$. We also have that $I - Q \in \mathscr{T}(B)$ since $I - P \in \mathscr{T}(A)$.

On the other hand, for any unit $x \in R(I - P)$, it is trivial that $P_x \prec I - P$ and P_x is minimal in $\mathscr{T}(A)$, then $P_y = \psi(P_x) \prec \psi(I - P) = I - Q$ and P_y is minimal in $\mathscr{T}(B)$. Conversely, for any unit vector $y \in R(I - Q)$, we have $P_y = P_y(I - Q)P_y = P_y(I - Q)B(I - Q)P_y = P_yBP_y$. This means that $P_y \in \mathscr{T}(B)$. Thus $\psi^{-1}(P_y) \in \mathscr{T}(A)$. We again have $\psi^{-1}(P_y) \prec I - P$ and $\psi^{-1}(P_y) \in \mathscr{T}(A)$ is minimal. This implies that $P_y \in \mathscr{T}(B)$ is minimal. Thus $I - Q = (I - Q)B$ by Lemma 2.2. Note that $f(a) \neq 1$. Then $Q_{aP}(I - Q) = 0$ so we have $Q_{aP} = Q$. Hence $\psi(aP) = f(a)Q$.

16 Feb 2023 22:13:56 PST
220330-GuoxingJi Version 2 - Submitted to Rocky Mountain J. Math.

Next we prove that $\psi(P) = Q$. Note that $Q\psi(P)Q$ is a truncation of $\psi(P)$. Then there exists a $\mu \in [0,1]$ such that $Q\psi(P)Q = \mu Q$. If $\mu = 0$, then $Q\psi(P)Q = 0$. Therefore $\psi(P) \leq I - Q = \psi(I-P)$, a contradiction. Hence $\mu \in (0,1]$. Then there exists a $\lambda \in \mathbb{R}^+$ such that $f(\lambda) = \mu$. That is, $Q\psi(P)Q = \mu Q = f(\lambda)Q$. Now $\psi^{-1}(\mu Q) = \psi^{-1}(f(\lambda)Q) = \lambda P$. Since $\mu Q = Q\psi(P)Q$ is a truncation of $\psi(P)$, $\lambda P = \psi^{-1}(\mu Q)$ is a truncation of P. We thus have $\lambda = 1$ and $\mu = f(1) = 1$. Hence $Q\psi(P)Q = Q$, that is, $\psi(P) = Q$. Thus $\psi(aP) = f(a)\psi(P)$ and $\psi(aP \oplus (I-P)) = f(a)\psi(P) \oplus (I - \psi(P))$ for any $P \in \mathscr{P}_1(\mathscr{H})$.

Take any $P, Q \in \mathscr{P}_1(\mathscr{H})$ with $PQ = 0$. Then $Q \leq I - P$ and hence $\psi(Q) \leq \psi(I-P)$. This implies that $\psi(P)\psi(Q) = 0$. It follows that ψ preserves the orthogonality of rank-1 projections in both directions. Moreover, if we assume that $P = P_x$ for some unit vector $x \in \mathscr{H}$, then we have $PQP = \langle Qx, x\rangle P$, that is, $\langle Qx, x\rangle P$ is a truncation of Q. It follows that $\psi(PQP) = f(\langle Qx, x\rangle)\psi(P)$ is a truncation of $\psi(Q)$ and so $\psi(PQP) = \psi(P)\psi(Q)\psi(P)$. We will show the conclusion from the following two cases.

Case 1 $\dim \mathscr{H} = 2$.

Let $\{e_1, e_2\}$ be an arbitrary orthonormal basis of $\mathscr{H}$. Take any pair of rank-1 projections E_1, E_2 with $E_1 \perp E_2$. Without loss of generality, we may assume that

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

under this basis.

Since ψ preserves the rank-1 projections as well as the orthogonality of rank-1 projections in both directions, we may assume that $\psi(E_1) = E_1$ and $\psi(E_2) = E_2$ by considering a unitary transformation if necessary. In this case, for any $P \in \mathscr{P}_1(\mathscr{H})$, there are $a, b, c, d \in [0,1]$ and $\lambda, \mu \in \mathbb{C}$ with $a^2 + b^2 = c^2 + d^2 = 1$ and $|\lambda| = |\mu| = 1$ such that

$$P = \begin{pmatrix} a^2 & \lambda ab \\ \bar{\lambda} ab & b^2 \end{pmatrix} \text{ and } \psi(P) = \begin{pmatrix} c^2 & \mu cd \\ \bar{\mu} cd & d^2 \end{pmatrix}. \tag{2.5}$$

Thus

$$E_1 P E_1 = \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} = a^2 E_1 \text{ and } E_1 \psi(P) E_1 = \begin{pmatrix} c^2 & 0 \\ 0 & 0 \end{pmatrix} = c^2 E_1$$

and $\psi(E_1)\psi(P)\psi(E_1) = \psi(E_1 P E_1) = \psi(a^2 E_1) = f(a^2) E_1 = c^2 E_1$. This implies that $f(a^2) = c^2$. We also have $f(b^2) = d^2$ in the same way. It follows that

$$f(1 - a^2) = 1 - c^2 = 1 - f(a^2). \tag{2.6}$$

Now for any $t > 0$,

$$P(tE_1)P = \begin{pmatrix} a^4 t & \lambda a^3 bt \\ \bar{\lambda} a^3 bt & a^2 b^2 t \end{pmatrix} = a^2 t P \tag{2.7}$$

and

$$\psi(P)\psi(tE_1)\psi(P) = \begin{pmatrix} c^4 f(t) & \mu c^3 d f(t) \\ \bar{\mu} c^3 d f(t) & c^2 d^2 f(t) \end{pmatrix} = c^2 f(t)\psi(P). \tag{2.8}$$

It follows that $\psi(P(tE_1)P) = \psi\left(a^2tP\right) = f\left(a^2t\right)\psi(P)$. Since $P(tE_1)P$ is a truncation of tE_1, $\psi(P(tE_1)P)$ is also a truncation of $\psi(tE_1) = f(t)E_1$. By using of (2.7) and (2.8) we have

$$f\left(a^2t\right)\psi(P) = \psi(P(tE_1)P) = \psi(P)\psi(tE_1)\psi(P) = c^2f(t)\psi(P) = f\left(a^2\right)f(t)\psi(P).$$

Therefore for any $0 < a < 1$ and $t > 0$, we have $f\left(a^2t\right) = f\left(a^2\right)f(t)$. This means that

$$f(xy) = f(x)f(y), \quad \forall 0 < x \le 1,\ y > 0.$$

It follows that $f(x^{-1}) = f(x)^{-1}$ and $f(x^2) = f(x)^2$ for any $x > 0$. Now for any $1 < x < y$,

$$f(xy) = f(\frac{x}{y}y^2) = f(\frac{x}{y})f(y^2) = f(x)f(y^{-1})f(y)^2 = f(x)f(y)^{-1}f(y)^2 = f(x)f(y)$$

since $y^{-1} < 1$. Thus

$$f(xy) = f(x)f(y), \forall x,y > 0.$$

Moreover, $f(\frac{1}{2}) = \frac{1}{2}$ and $f\left(1-a^2\right) = f((1-a)(1+a)) = f(1-a)f(1+a) = 1-f(a)^2 = (1-f(a))(1+f(a))$ for and $0 < a < 1$ by (2.6). We again have $f(2) = 2$ and $f(1+a) = 1+f(a)$ for any $0 < a < 1$. Let x,y be real numbers with $x > y > 0$. Then $x^{-1}y < 1$ and $f(x+y) = f(x)(1+f(x^{-1}y)) = f(x)+f(y)$ and $f(2x) = f(2)f(x) = 2f(x)$. It follows that $f(r) = r$ for any $r \in \mathbb{Q}^+$. Assume $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}^+$ with $x_n < x < y_n$ and $x_n \to x, y_n \to x (n \to \infty)$. Note that $x_n = f(x_n) < f(x) < y_n = f(y_n)$. Thus $f(x) = x, \forall x \in \mathbb{R}^+$.

For any $P \in \mathscr{P}_1(\mathscr{H})$, P and $\psi(P)$ can be represented as (2.5), then $\operatorname{Tr}(E_1P) = \operatorname{Tr}(\psi(E_1)\psi(P)) = a^2$. Now for any rank-1 projection $E = e \otimes e \in \mathscr{P}_1(\mathscr{H})$, we may take an orthononal basis $\{e_1, e_2\}$ of $\mathscr{H}$ with $e_1 = e$. Then $E_1 = E$. Thus for any projections $E, P \in \mathscr{P}_1(\mathscr{H})$, we have $\operatorname{Tr}(EP) = \operatorname{Tr}(\psi(E)\psi(P))$. By the Wigner's theorem in [9], there exists a unitary or an anti-unitary operator U such that

$$\psi(P) = UPU^*, \quad \forall P \in \mathscr{P}_1(\mathscr{H}).$$

Case 2 $\dim \mathscr{H} \ge 3$.

Since ψ preserves the orthogonality of rank-1 projections in both directions, there exists a unitary or an anti-unitary operator U such that

$$\psi(P) = UPU^* \text{ for any } P \in \mathscr{P}_1(\mathscr{H})$$

by the Uhlhorn's theorem in [8].

Now in both cases, we define $\phi(A) = U^*\psi(A)U$ for any $A \in \mathscr{B}(\mathscr{H})^+$. Then ϕ and ψ have the same properties such that $\phi(P) = P$ for any $P \in \mathscr{P}_1(\mathscr{H})$. We note that $f(a) = a$ for all $a \in \mathbb{R}^+$. In fact, we similarly obtain this fact by taking $E_1, E_2 \in \mathscr{P}_1(\mathscr{H})$ with $E_1 \perp E_2$ when $\dim \mathscr{H} > 2$ as in the proof of Case 1.

Let $A \in \mathscr{B}(\mathscr{H})^+$. Since $P_xAP_x = \langle Ax, x\rangle P_x$ is the truncation of A for any unit vector $x \in \mathscr{H}$, $\phi(P_xAP_x)$ is the truncation of $\phi(A)$ and $\phi(P_xAP_x) = f(\langle Ax, x\rangle)\phi(P_x) = f(\langle Ax, x\rangle)P_x = \langle Ax, x\rangle P_x = \langle \phi(A)x, x\rangle P_x$. Hence

$$\langle Ax, x\rangle = \langle \phi(A)x, x\rangle$$

for any unit vector $x \in \mathscr{H}$. Thus $\phi(A) = A$ and $\psi(A) = UAU^*$. Consequently,

$$\varphi(A) = \alpha UAU^* \text{ for any } A \in \mathscr{B}(\mathscr{H})^+.$$

□

Acknowledgment The authors are deeply grateful to the referees for their valuable comments which helped to improve the manuscript.

References

[1] M. Brešar, P. Šemrl, *Linear preservers on B(X)*, Banach Center Publ., **38**(1997), 49-58.

[2] Y. Dong, L. Li, L. Molnár, N-C. Wong, *Transformations preserving the norm of means between positive cones of general and commutative C^*-algebras*, J. Operator Theory, **88**(2022), 365-406.

[3] X. Jia, W. Shi, G. Ji, *Maps preserving the truncation of products of operators*, Ann. Funct. Anal., **13**(2022), Article number: 40.

[4] C. Li, S. Pierce, *Linear preserver problems*, Amer. Math. Monthly, **108**(2001), 591-605.

[5] L. Li, L. Molnár, L. Wang, *On preservers related to the spectral geometric mean*, Linear Algebra Appl., **610**(2021), 647-672.

[6] L. Molnár, *Maps on positive cones in operator algebras preserving power means*, Aequationes Math., **94**(2020), 703-722.

[7] L. Molnár, *Maps on positive definite cones of C^*-algebras preserving the Wasserstein mean*, Proc. Amer. Math. Soc., **150**(2022), 1209-1221.

[8] U. Uhlhorn, *Representation of symmetry transformations in quantum mechanics*, Ark. Fys., **23**(1963), 307-340.

[9] E. Wigner, *Group theory and its application to the quantum theory of atomic spectra*, Academic Press Inc., New York, (1959).

[10] J. Yao, G. Ji, *Additive maps preserving the truncation of operators*, J. Math. Res. Appl., **42**(2022), 89-94.

School of Mathematics and Statistics, Shaanxi Normal University, Xian , 710119, People's Republic of China

E-mail address: 15651783951@163.com

School of Mathematics and Statistics, Shaanxi Normal University, Xian , 710119, People's Republic of China

E-mail address: gxji@snnu.edu.cn