# SOME ASPECTS OF INVARIANT MEANS IN THE SPACES OF WEAKLY ALMOST PERIODIC FUNCTIONS AND LEFT UNIFORMLY CONTINUOUS FUNCTIONS

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ABSTRACT. The main aim of this paper is to study the F.P property for semitopological semigroups $(\Sigma)$ of non-expansive mappings on a subset of a Banach space which is nonempty closed and convex or, more generally, a L.C.S and to establish the F.P property of the space of all left uniformly continuous functions by using its amenability for $\Sigma$ and to establish the F.P property for the class of weakly almost periodic functions by using its amenability for $\Sigma$.

## 1. INTRODUCTION

For many years, there have been appreciable attentiveness in the study of fixed point property (F.P.P) for $\Sigma$ of nonexpansive (N.E) mappings on a compact convex subset of a Banach space. In 1948, Bruck [5] proved the result that "if $S$ is a Banach space having weak F.P.P, then for any commutative semigroup acting on weakly compact convex subset $M$ of $S$ has the common fixed point (C.F.P) property". In 1965, Browder's result [4] declares that if we have a uniformly convex Banach space $S$, then there exists a weak F.P.P in $S$. The extension of this result is proved by Kirk [12] which states that if $S$ has a weakly compact subset $M$ having normal structure, then there exists a F.P.P in $M$.
In 1973, Lau [13] established the necessary and sufficient relation between existence of $L.I.M$ in $A.P(\Sigma)$ and F.P.P $(D)$. In 1985, Hsu [10] proved that $\Sigma$ has F.P.P $(G)$ where $\Sigma$ is discrete and left-amenable. In 2008, Lau [16] proved that when $\Sigma$ is separable then there is iff relation between amenability of $W.A.P(\Sigma)$ and F.P.P $(F)$. For $A.P(\Sigma)$ to have a $L.I.M$, Lau [16] proved two F.P properties $(E^{'})$ and $(E)$ and also he showed that the F.P.P $(E^{'})$ and the F.P.P $(D)$ are equivalent. In 2012, Lau [17] proved that "if $\Sigma$ is a separately continuous and quasi-equicontinuous action on a compact Hausdorff space $X$, then $W.A.P(\Sigma)$ has a multiplicative $L.I.M$ iff $X$ has a C.F.P for $\Sigma$".
In 2020, related to F.P properties of semigroup actions on a non-empty closed convex subset of a Banach space, or more generally, a L.C.S, an updated survey on multiple analytic and algebraic properties of semigroups is done by Lau [18]. In 1976, Lau gave some open problems. In this article, we will deal with some of the open problems posed by Lau. These problems are about whether in the space of left uniformly continuous functions of $\Sigma$, the left amenability property implies the existence of a C.F.P for every jointly $weak^*$ continuous norm N.E which acts on a $weak^*$ compact convex subset (non-empty subset) of a dual Banach space and whether in the space of weakly almost periodic functions of a $\Sigma$, the left

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amenability property implies the existence the of a C.F.P for every jointly $weak^*$ continuous norm N.E which acts on a norm separable $weak^*$ compact convex subset (non-empty subset) of a dual Banach space.
The article is arranged in the following manner. In *Section 2* some basic definition related to F.P properties are defined. In *Section 3* some F.P properties of $\Sigma$ for $A.P(\Sigma)$ and $W.A.P(\Sigma)$ are discussed. In *Section 4* we are trying to give compact answers to the open problems posed by Lau [14].

## 2. Preliminaries

Let us recall some fundamental concepts which will be used in our work and which can be found in [16–18].

**Definition 1 (Fixed Point Property).** *Consider a Banach space $S$ and a closed bounded subset $\emptyset \neq M \subseteq S$. If a F.P for any N.E mapping $\mathcal{H} : M \to M$ are in $M$, then it possesses the F.P.P.*

**Definition 2 (Weak Fixed Point Property).** *Let $S$ be the Banach space. If any weakly compact convex subset $M \subseteq S$ has the F.P.P, then $M$ has weak F.P.P.*

**Definition 3 (Left and Right Translation).** *Let us suppose that a set of bounded complex-valued functions $C^*$-algebra $l^\infty(\Sigma)$ which have supremum norm and in which multiplication is pointwise and suppose that $\Sigma$ be a semigroup. For every element $b \in \Sigma$, $h \in l^\infty(\Sigma)$, if*

$$l_b h(s) = h(bs) \text{ where } s \in \Sigma$$

*then $l_b h$ is said to be the left translates of $h$ by $b$. Similarly if*

$$r_b h(s) = h(sb) \text{ where } s \in \Sigma$$

*then $r_b h$ are said to be the right translate of $h$ by $b$.*

**Definition 4 (Mean).** *Suppose a closed subspace $\mathcal{X}$ of $l^\infty(\Sigma)$ which is invariant under translations and containing constants . Then a mean is a linear functional $w \in \mathcal{X}^*$ if $\|w\| = w(e) = 1$, where $e(s) = 1$ for all $s \in \Sigma$.*
*$w$ is called a left invariant mean if $w(l_a h) = w(h)$ for all $a \in \Sigma, h \in \mathcal{X}$, similarly $w$ is called right invariant mean if $w(r_a h) = w(h)$ for all $a \in \Sigma, h \in \mathcal{X}$. And denoted by L.I.M and R.I.M respectively.*

**Definition 5 (Semitopological Semigroup).** *If $\Sigma$ is a semigroup with Haudorff topology is said to be a semitopological semigroup $\Sigma$ if for each $u \in \Sigma$, the mappings $t \mapsto ut$ and $t \mapsto tu$ from $\Sigma$ into itself are continuous.*

Now, we are going to list some spaces as follows:
$\boldsymbol{C_b(\Sigma)}$ is the space of all bounded and continuous complex-valued functions defined on $\Sigma$.
$\boldsymbol{L.U.C(\Sigma)}$ is the space of all $h \in C_b(\Sigma)$ in such a way that the mapping $s \mapsto l_s h : \Sigma \mapsto C_b(\Sigma)$ is continuous. i.e., the space of all left uniformly continuous functions on $\Sigma$.
$\boldsymbol{A.P(\Sigma)}$ is said to be space of all $h \in C_b(\Sigma)$ in such a way that $\mathcal{LO}(h) = \{l_s h : s \in \Sigma\}$ is relatively compact in the norm topology of $C_b(\Sigma)$.
$\boldsymbol{W.A.P(\Sigma)}$ is said to be the space of all $h \in C_b(\Sigma)$ in such a way that $\mathcal{LO}(h) = \{l_s h : s \in \Sigma\}$ is relatively compact in the weak topology of $C_b(\Sigma)$.

**Definition 6 (Left-Amenable).** • *A $\Sigma$ is said to be left-amenable if there exists a $L.I.M$ in $l^{\infty}(\Sigma)$.*
• *The $\Sigma$ is said to be left-amenable if $L.U.C(\Sigma)$ has a $L.I.M$.*

In general, the following inclusion hold:

$$A.P(\Sigma) \subseteq L.U.C(\Sigma) \subseteq C_b(\Sigma) \quad and \quad A.P(\Sigma) \subseteq W.A.P(\Sigma) \subseteq C_b(\Sigma).$$

If $\Sigma$ is discrete, then

$$A.P(\Sigma) \subseteq W.A.P(\Sigma) \subseteq L.U.C(\Sigma) = l^{\infty}(\Sigma).$$

If $\Sigma$ is compact, then

$$A.P(\Sigma) = L.U.C(\Sigma) \subseteq W.A.P(\Sigma) = C_b(\Sigma).$$

Hence both the definitions of left-reversible coincide when $\Sigma$ is discrete.

**Definition 7 (Left-Reversible).** *If any two closed right ideals of $\Sigma$ have non-empty intersection, then $\Sigma$ is said to be left-reversible, i.e.*

$$\overline{c\Sigma} \bigcap \overline{d\Sigma} \neq \varnothing \text{ for any } c, d \in \Sigma.$$

**Definition 8 ( Semitopological Semigroup Action).** *A semitopological semigroup action is a mapping $\varphi : \Sigma \times \daleth \to \daleth$ in such a way that*

$$\mathcal{H}_{v_1 v_2} s = \mathcal{H}_{v_1}(\mathcal{H}_{v_2} s), \;\; v_1, v_2 \in \Sigma, \; s \in \daleth.$$

*where $\mathcal{H}_v s$ is defined as*

$$\mathcal{H}_v s = \varphi(v, s), \text{ where } v \in \Sigma \text{ and } s \in \daleth$$

If the mapping $\varphi$ is separately continuous, then the action defined on $\varphi$ is also separately continuous. Similarly, if the mapping $\varphi$ is jointly continuous, then action defined on $\varphi$ is also jointly continuous. For simplicity, $\mathcal{H}_v s = vs$ and we say $\prod = \{\mathcal{H}_s : s \in \Sigma\}$ a representation of $\Sigma$ on $\daleth$.

**Definition 9 (Affine Action).** *Consider $\mathfrak{M}$ be the convex subset of a linear topological space and if for each $t \in \Sigma$, the mapping $u \mapsto tu : \mathfrak{M} \to \mathfrak{M}$ satisfies*

$$t(\lambda u + (1-\lambda)v) = \lambda tu + (1-\lambda)tv \text{ for } t \in \Sigma, \; u, v \in \mathfrak{M} \text{ and } 0 \leq \lambda \leq 1.$$

*then the action of $\Sigma$ on $\mathfrak{M}$ is known as affine.*

**Definition 10 (Q-N.E Action).** *Consider a collection of continuous semi-norms $Q$ on a separated L.C.S $S$ which determines the topology of $S$. Then an action of $\Sigma$ on a subset $M \subseteq S$ is Q-N.E if $\wp(t \cdot u - t \cdot v) \leq \wp(u - v)$ for all $t \in \Sigma, \; u, v \in M$ and $\wp \in Q$.*

**Definition 11 (Equicontinuous action).** *Assume that $\mathcal{U}$ be the unique uniformity which determines the topology of $\daleth$ (see [11], p. 197). An action of $\Sigma$ on a compact HS $\daleth$ is equicontinuous if, for each element in $\daleth$ and $A \in \mathcal{U}$, there is a $B$ in $\mathcal{U}$ in such a way that $(tx, ty) \in A$ for all $t \in \Sigma$ whenever $(x, y) \in B$.*

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## 3. Fixed point property of semigroup of N.E mappings

In this section, we will study the relation between the existence of $L.I.M$ for $A.P(\Sigma)$, $W.A.P(\Sigma)$ and F.P properties of $\Sigma$ acting on certain subsets of a L.C.S. Also, we will discuss the results which will be used in our main result. The following F.P.P was proved by Lau [13].

**Theorem 1.** *Then $A.P(\Sigma)$ has a $L.I.M$ iff $\Sigma$ satisfies the following F.P.P:*

(D) *Let $(S, Q)$ be a separated L.C.S and $M$ be a convex complex subset of $S$. The action of $\Sigma$ on $M$ is separately continuous and Q-N.E, then $M$ has a C.F.P for $\Sigma$.*

For a long time it has been an open question that if $W.A.P(\Sigma)$ has $L.I.M$ then the F.P.P for N.E actions of $\Sigma$ on a weakly compact convex set holds.

Hsu [10] proved that "if $\Sigma$ is left-reversible and discrete, then the following fixed point property on $\Sigma$ holds:

(G) Consider a weakly compact subset $M$ of a separated compact L.C.S $(S, Q)$ and the action of $\Sigma$ on $M$ is weakly separately continuous and Q-N.E, then $M$ must contains a C.F.P for $\Sigma$."

Since the F.P.P $(G)$ implies that $W.A.P(\Sigma)$ has $L.I.M$, if $\Sigma$ is discrete and left-reversible, $W.A.P(\Sigma)$ must have a $L.I.M$. If $\Sigma$ is left-amenable and discrete, then $\Sigma$ is left-reversible. Even $C_b(\Sigma)$ has a $L.I.M$, a general $\Sigma$ does not have to be left-reversible unless $\Sigma$ is normal (see [9]).

The above-mentioned known relations for discrete semigroup $\Sigma$ are given below in the form of implication diagram.

$$\begin{gathered}\Sigma \text{ left-amenable}\\ \Downarrow\\ \Sigma \text{ left-reversible}\\ \Downarrow\\ A.P(\Sigma) \text{ has } L.I.M\end{gathered}$$

The implication$\Sigma$ is left-reversible $\Rightarrow$ $A.P(\Sigma)$ has a $L.I.M$" for any set $\Sigma$ was proved in [13]. T. Mitchell gave two examples in conference on analysis of semigroups which was held in the Richmond, Virginia 1984, to show that $A.P(\Sigma)$ has $L.I.M \nRightarrow \Sigma$ is left-reversible for a discrete semigroup $\Sigma$, (see [15]). Later in [16] for discrete semigroup $\Sigma$, the above diagram completed as

$\Sigma$ left-amenable $\Rightarrow$ $\Sigma$ left-reversible $\Rightarrow$ $W.A.P(\Sigma)$ has $L.I.M$ $\Rightarrow$ $A.P(\Sigma)$ has $L.I.M$

**Definition 12 (Quasi-equicontinuous Action).** *Assume that $\Sigma$ is a semitopological semigroup on a Haudorff space $\daleth$. If $\overline{\Sigma}^p$, the closure of $\Sigma$ in the product space $\daleth^\daleth$, consists of only continuous mappings, then this action is called a quasi-equicontinuous action.*

**Remark 1.** *An equicontinuos action on a closed subset of a topological vector space is quasi-equicontinuous but converse may or may not be true ( [16], Example 4.14).*

The following theorem related to the existence of $L.I.M$ on $W.A.P(S)$ was proved in [16].

**Theorem 2.** *Consider $\Sigma$ be a separable. Then $W.A.P(\Sigma)$ has a $L.I.M$ iff $\Sigma$ possesses the F.P.P (F) stated below:*

(F) *Assume that $M$ be a weakly compact convex subset of a separated L.C.S $(S, Q)$. The action of $\Sigma$ on $M$ is Q-N.E, weakly quasi-equicontinuous and weakly separately continuous, then $M$ must have a C.F.P for $\Sigma$.*

Consider the following F.P.P for a $\Sigma$:

(E) Consider a subset $M$ of separated L.C.S $(S, Q)$, which is weakly compact and convex. If $\Sigma$ acts on $M$ is separately continuous, equicontinuous and a Q-N.E self-mappings when $M$ is endowed with the weak topology of $(S, Q)$, then there exist F.P for $\Sigma$ in $M$.

One can easily observe that

$$(G) \Rightarrow (F) \Rightarrow (E) \Rightarrow (D).$$

It has been an open question that above implication can be reversed. To answer this question Lau and Zhang establish a result in [16].

**Theorem 3.** *Consider $\Sigma$ be a separable. Then there exist a $L.I.M$ in $A.P(\Sigma)$ if the F.P.P $(E)$ holds and vice versa.*

**Definition 13 (Normal Structure).** *Assume that $B$ be the Banach space and $J$ be the bounded convex subset of $B$. $J$ has normal structure if for each convex subset $K \subseteq J$, containing more than one point, there is a point $q_0$ in $K$ in such a way that* $\sup\{\|q_0 - p\| : p \in K\} < \sup\{\|q - p\| : p, q \in K\}$.

**Definition 14 ($Q$-normal Structure).** *Consider a separated L.C.S $(S, Q)$ and then the $Q$-normal structure for subset $M$ of $S$ is defined as if for each subset $B$ of $M$ which is $Q$-bounded having at least two points, there exist a $t_0 \in coB$ and $q \in Q$ in such a way that*

$$\sup\{q(t - t_0) : t \in B\} < \sup\{q(t - u) : t, u \in B\}.$$

*Here the word "Q-boundedness" of $B$, means that for every element $q \in Q$, $\exists\, \delta > 0$ in such a way that $q(t) \leq \delta$, for all $t \in B$.*

**Theorem 4.** *$A.P(\Sigma)$ has $L.I.M$ iff a $\Sigma$ possesses the following F.P.P.*

$(E')$ *For a subset $M$ of a separated L.C.S $(S, Q)$ which is weakly compact and convex , if $\Sigma$ acts on an $M$ and this action is Q-N.E, separately continuous and equicontinuous then $M$ contains a C.F.P for $\Sigma$, where $M$ is equipped with the weak topology of $(S, Q)$ and has Q-normal structure.*

*Thus F.P.P $(E')$ is equivalent to the F.P.P $(D)$.*

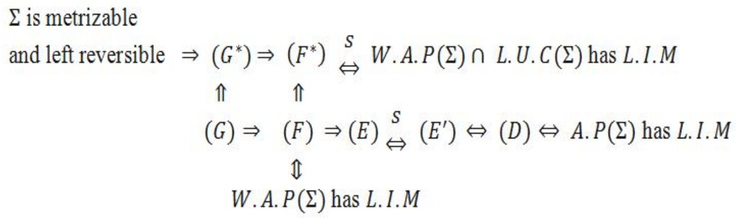
When the semigroup $\Sigma$ becomes separable(i.e. if there exists countable dense subset in $\Sigma$), then theorem 3 and 4 shows that F.P.P $(E)$ is identical to F.P.P $(E')$.

In the F.P properties (G) and (F), if we replace the term separately continuous by jointly continuous, then we have two more F.P properties (F*) and (G*) in such a way that (G*) implies (F*).

**Theorem 5.** *[16] Consider $\Sigma$ be a separable. Then F.P.P (F*) holds iff the space $W.A.P(\Sigma) \cap L.U.C(\Sigma)$ has a $L.I.M$.*

**Theorem 6.** *[16] Let $\Sigma$ be a metrizable and left-reversible. Then $\Sigma$ has the F.P.P (G*). Particularly, $W.A.P(\Sigma) \cap L.U.C(\Sigma)$ has a $L.I.M$.*

The diagram given below describe the relations between the F.P properties explained in [16].




Here "$s$" written on implication symbol represents that semigroup is separable.

**Theorem 7.** *[17, 18] Let $LUC(\Sigma)$ has a L.I.M then the following F.P.P $F_*$ hold.*

($F_*$) *Consider a weak* compact convex subset $M \neq \emptyset$ of the dual space of Banach space $S$. Let $\prod = \{\mathcal{H}_s : s \in \Sigma\}$ be a representation of $\Sigma$ on $M$ and this representation be norm N.E and jointly continuous, where $M$ is endowed with the Weak* topology of $S^*$. Then $M$ contains a C.F.P for $\Sigma$.*

**Theorem 8.** *[17, 18] If $\Sigma$ is a left-amenable or left-reversible, then the F.P.P $(F_{*s})$ holds.*

($F_{*s}$) *Assume that $S$ be the Banach space and $M \neq \emptyset$ be the norm separable $weak^*$ compact convex set of dual space of $S$. Then whenever $\prod = \{\mathcal{H}_s : s \in S\}$ is a norm N.E representation of $\Sigma$ on $M$ and the mapping $(s, x) \mapsto \mathcal{H}_s(x)$ from $\Sigma \times CM \to M$ is jointly continuous when $M$ is endowed with the $weak^*$-topology of $S^*$, then there is a C.F.P for $\Sigma$ in $M$.*

## 4. Main Results

In this section, we are trying to give a compact answers to open problems posed by Lau in 1976. These open problems are stated as follows:
**1.** Does a $\Sigma$ have the F.P.P $(F_*)$ if $L.U.C(\Sigma)$ has a $L.I.M$?
A weak version of property $(F_*)$ holds if $L.U.C(\Sigma)$ has a $L.I.M$ ( [17], Proposition 6.1).
**2**. Let us suppose that a semigroup $\Sigma$ which is discrete. If the F.P.P $(F_{*s})$ holds, does $W.A.P(\Sigma)$ have a $L.I.M$? We also do not know whether the existence of a $L.I.M$ on $W.A.P(\Sigma)$ is sufficient to ensure the F.P.P $(F_{*s})$.
In [17] (Proposition 6.5), a partial affirmative answer to Problem was given, which is stated as follows:

**Theorem 9.** *Suppose that $\Sigma$ has the F.P.P $(F_{*s})$. Then*

(a) *$A.P.(\Sigma)$ has a $L.I.M$*
(b) *$W.A.P(\Sigma)$ has a $L.I.M$ if $\Sigma$ has a countable left ideal.*

Before going to the main theorem, we first state and prove some results which will be used in our main result.

**Theorem 10** (***Riesz representation theorem***). *Let $Y$ be a locally compact Hausdorff space and $C_c(Y)$ be the space of continuous compactly supported complex valued functions defined on $Y$. For any positive linear functional $\psi$ on $C_c(Y)$, there is a unique radon measure $\mu$ on $Y$ such that*

$$\psi(f) = \int_Y f(y)d\mu(y), \; for \; all \; f \in C_c(Y).$$

**Lemma 1.** *Let $L.U.C(\Sigma)$ has a left invariant meant and $\prod = \{\mathcal{H}_s : s \in \Sigma\}$ is a representation of $\Sigma$ on non-empty subset $M$ of the dual space of a complete normed space $S$. This subset is also weak* compact and convex. Suppose $L$ is weak* convex closed $\Sigma$-invariant and minimal subset of $M$ and $K$ be a non-empty minimal weak* closed $\Sigma$-invariant subset of $L$, then $\mathcal{H}_s K = K$ for each $s \in \Sigma$.*

*Proof.* Let $y \in K$. Using Theorem 10, the functional $\phi$ defines a (regular) probability meassure $\eta$ on $K$ in such a way that

$$\eta(A) = \eta(\mathcal{H}_a^{-1} A)$$

for all $\mathcal{H}_a^{-1} A \in \prod$ and for each Borel subset $A$ of $K$.
Suppose the family of all closed subset $A$ of $K$ i.e.

$$\Im = \{A \in K : \eta(A) = 1\},$$

and let

$$K_0 = \cap \Im,$$

which is nonempty, indeed let $K_0 = \cap \Im = \emptyset$ which is closed and subset of $K$ but $K$ is minimal so this is not possible. Hence

$$K_0 \neq \emptyset.$$

If $A \in \Im$ and $\mathcal{H}_s^{-1} \in \Sigma$

$$\begin{aligned} \eta(A) &= 1 \\ \eta(\mathcal{H}_s^{-1} A) &= 1 \quad (\because \eta(A) = \eta(\mathcal{H}_s^{-1} A)) \end{aligned}$$

This implies $\mathcal{H}_s^{-1} A \in \Im$. Hence $K_0 \subset \mathcal{H}_s^{-1} K_0$ or $\mathcal{H}_s K_0 \subset K_0$. By minimality of $K$, $K_0 = K$.
Since

$$\begin{aligned} \eta(\mathcal{H}_s K) &= \eta(\mathcal{H}_s^{-1} \mathcal{H}_s K) \\ &= \eta(K) \\ &= 1. \end{aligned}$$

This implies $\mathcal{H}_s K \in \Im$, for each $\mathcal{H}_s \in \prod$. Therefore

$$\begin{aligned} K_0 &= K \\ &\subset \mathcal{H}_s K \\ &\subset K; \end{aligned}$$

hence $\mathcal{H}_s K = K$ holds for each $s \in \Sigma$. □

**Theorem 11.** *Consider a Banach space $S$. Let us suppose a Hausdorff locally convex topology $\tau$ on $S$. The topology $\tau$ is weaker than the norm topology $\mathfrak{N}$. Now suppose a $\tau$-compact norm separable subset $X$ of $S$ and suppose a N.E representation $\Pi = \{\mathcal{H}_s : s \in \Sigma\}$ of a semigroup $\Sigma$ and $\tau - \tau$-continuous self-maps of $X$ such that $\{\mathcal{H}_u x : u \in \Sigma\}$ is $\tau$-dense in $X$ for each $x \in X$. Then there exist a sequence $(u_i) \in \Sigma$ where $i = 1, 2, \ldots p$ with the condition $X = \bigcup_{j=1}^{p} \{\mathcal{H}_{s_j}^{-1}[(z + U) \cap X]\}$ where $s_j = u_j u_{j-1} ... u_1$. Also if $\{x \in S : \|x\| \leq 1\}$ is $\tau$-closed and each mapping $\mathcal{H}_s$ is onto, then both $\tau$-topology and $\mathfrak{N}$-topology on $X$ coincides. Therefore, $X$ becomes a norm-compact.*

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*Proof.* Consider a set $U$ as a $\tau$-neighborhood of 0. Now suppose $N_\delta = \{x \in S : \|x\| < \delta\ , \delta > 0\}$. Take an open neighborhood $U_1$ of 0 in $\tau$ such that $U_1 + U_1 \subseteq U$. As $U_1$ is a $\mathfrak{N}$-neighborhood of 0, for $\varepsilon > 0$ there exists $N_\varepsilon$ such that $N_\varepsilon \subseteq U_1$. Let $z \in X$ be fixed. As $X$ is $\tau$-compact, therefore, $X$ is cover by countably many sets and $x_i + N_\varepsilon; x_i \in X$ is a covering of $X$. As given that $\{\mathcal{H}_u x_1 : u \in \Sigma\}$ is $\tau$-dense in $X$, so we can take an element $u_1 \in \Sigma$ such that $\mathcal{H}_{u_1} x_1 \in (z + U_1) \cap X$. By induction, we can take a sequence $(u_j), j = 1, 2, ...$ in $\Sigma$ such that $T_{s_j} x_j \in (z + V_1) \cap K$ where $s_j = t_j t_{j-1} ... t_1$.
As given that each $\mathcal{H}_s$ is N.E, so we have

$$\mathcal{H}_{s_j}[(x_j + N_\varepsilon) \cap X \subseteq (z + N_\varepsilon + U_1) \cap X.$$

Also

$$(z + N_\varepsilon + U_1) \cap X \subseteq (z + U) \cap X$$

Therefore

$$\mathcal{H}_{s_j}[(x_j + N_\varepsilon) \cap X \subseteq (z + U) \cap X.$$

Now by applying $\{\mathcal{H}_{s_j}^{-1}$ on both sides, we get

$$(x_j + N_\varepsilon) \cap X \subseteq \mathcal{H}_{s_j}^{-1}[(z + U) \cap X].$$

Consequently, $\{\mathcal{H}_{s_j}^{-1}[(z+U) \cap X]\}_{j=1}^{\infty}$ is a $\tau$-open covering of $X$. Since $X$ is $\tau$-compact, there exist $p$ such that

$$X = \bigcup_{j=1}^{p} \mathcal{H}_{s_j}^{-1}[(z + U) \cap X].$$

Now if each $\mathcal{H}_s$ is onto and $\{x \in S : \|x\| \leq 1\}$ is $\tau$-closed, let $\delta > 0$ be fixed. Now again, $y_i + \frac{1}{2} N_\delta; y_i \in X$ is the covering of $X$, as $X$ is $\tau$-compact Hausdorff locally convex topology, therefore $X$ is second category in itself, so there exist a point $y \in X$ and an open set $V \in \tau$ such that

$$X \cap (\frac{1}{2} N_\delta + y) \supseteq V \cap X \neq \phi$$

Consider an open neighborhood $U$ of 0 in $\tau$-of such that $z + U \subseteq V$ where $z \in V \cap X$ . So we have $(z + U) \cap X \neq \phi$ and

$$(z + U) \cap X \subseteq (y + \frac{1}{2} N_\delta) \cap X.$$

And

$$(y + \frac{1}{2} N_\delta) \cap X \subseteq (z + N_\delta) \cap X.$$

Therefore

$$(z + U) \cap X \subseteq (z + N_\delta) \cap X. \tag{4.1}$$

Now we can find a sequence $(u_i) \in \Sigma$ where $i = 1, 2 \ldots p$ such that

$$X = \bigcup_{j=1}^{p} \mathcal{H}_{s_j}^{-1}[(z + U) \cap X] \quad \text{where} s_j = u_j t_{j-1} \ldots u_1.$$

Since each $\mathcal{H}_s$ is onto, we have

$$\begin{aligned} K &= \mathcal{H}_{s_p} X \\ &= \mathcal{H}_{s_p}\{\bigcup_{j=1}^{p} \mathcal{H}_{s_j}^{-1}[(z+U)\cap X]\} \\ &\subseteq \bigcup_{j=1}^{p}\{\mathcal{H}_{u_p u_{p-1}\dots u_{j+1}}[(z+U)\cap X]\} \\ &\subseteq \bigcup_{j=1}^{p}\{\mathcal{H}_{u_p u_{p-1}\dots u_{j+1}}[(z+N_\delta)\cap X]\} \quad (by 4.1) \\ &\subseteq \bigcup_{j=1}^{p}\{\mathcal{H}_{u_p u_{p-1}\dots u_{j+1}}(z)+N_\delta)\} \end{aligned}$$

by non-expansiveness of $\mathcal{H}_s, s \in \Sigma$. Hence $X$ is totally bounded, therefore $X$ becomes a compact in $\mathfrak{N}$. As given that $\tau$ on $X$ is weaker than the norm topology $\mathcal{N}$, therefore, both topologies on $X$ coincides. Hence $X$ is a norm compact. □

In 2012, Lau and Zhang prove a weak result regarding this open problem by imposing an extra condition of normal structure on F.P.P $(F_*)$. For proof of our main result we use a Lemma 2 in [6] to make a condition similar to normal structure, without taking the assumption of normal structure in fixed point property $(F_*)$.

**Theorem 12.** *Let $L.U.C(\Sigma)$ has a left invariant mean, then F.P.P $(F_*)$ hold.*

*Proof.* Let a weak* convex closed $\Sigma-$invariant minimal subset $L$ of $M$ and $Z \neq \emptyset$ be weak* closed minimal $\Sigma-$invariant subset of $L$. Take a $y \in Z$. Indeed, the action of $\Sigma$ on $M$ is jointly weak* continuous, for each $f \in C(Z)$ (i.e. $f : Z \to \mathbb{C}$ is continuous) the mapping $s \mapsto f(sy)$ is continuous. This implies the mapping $s \mapsto l_s f$, for each $s \in \Sigma \in L.U.C(\Sigma)$; $Z$ is endowed with the $W * T$ of $S^*$.
By Lemma 4.1 we get $\mathcal{H}_s Z = Z$ holds for each $s \in \Sigma$. Now by using the standard argument of ( [13], Theorem 4.1) and norm nonexpansiveness we show that $Z$ contains only one point. For this we are following a concept similar to the that of ( [6],Lemma 2). If $Z$ consists of only one point then according to that lemma in [6],
$x \in Z,\ f(x) \in Z \Rightarrow\ x = f(x)$, we are done. Otherwise, there exists a continuous norm in such a way that

$$d = \sup\{\|x-y\|; x, y \in Z\} > 0.$$

Then, from Lemma 1 in [6], there exist $\eta \in \overline{co}Z$ in such a way that

$$d_0 = \{\sup \|\eta - x\|; x \in Z\} < d.$$

Let $L_0 = L \cap (\bigcap\{B[x, d_0]; x \in Z\})$, where

$$B[x, d_0] = \{y \in Z; \|y - x\| \leqq d_0\}.$$

Then $\eta \in L_0$ and $L_0$ is nonempty closed convex proper subset of $L$. Furthermore, if $x \in L_0$ then $x \in L\ \because L_0 \subset L$ and $z \subseteq B[x, d_0]$. Hence for any $\mathcal{H}_s \in \prod$, by nonexpansiveness of $\Sigma$ on $L$ we have $Z = \mathcal{H}_s Z \subseteq B[\mathcal{H}_s x, d_0]$. It follows that $sL_0 \subseteq L_0$ for all $s \in \Sigma$, contracting the minimality of $L$. Consequently, $Z$ must consist of a single point. So the single point in $Z$ is a C.F.P for $\Sigma$. □

Now here, we are going to give the compact answer to the converse part of the open problem posed by Lau in 1976.

**Theorem 13.** *Let $\Sigma$ be a semitopological semigroup. If $W.A.P(\Sigma)$ has $L.I.M$ then the following F.P.P $F_{*s}$ holds:*

*Proof.* Assume that $W.A.P(\Sigma)$ has $L.I.M$. By Zorn's lemma, there exist $weak^*$-compact convex subset $\emptyset \neq L \subseteq M$ which is minimal with respect to being $weak^*$ invariant and closed convex under each element of $\Sigma$.
A second application of Zorn's lemma shows that there exist a subset $\emptyset \neq Z \subseteq L$ which is minimal with respect to being $weak^*$-closed and invariant under each element of $\Sigma$.
By Theorem 11, $Z$ is norm compact. If $Z$ consists of only one point according to that lemma in [6],
$x \in Z,\ f(x) \in Z \Rightarrow\ x = f(x)$, we are done. Otherwise, let $d = diam(Z)$. Then by ( [6], Lemma 1), there is $\eta \in \overline{co}Z \subseteq L$ such that

$$d_\circ = \sup\{\|\eta - x\| : x \in Z\} < d.$$

Let

$$L_\circ = L \cap (\cap_{x\in Z} B[x, d_\circ]).$$

Where

$$B[x, d\circ] = \{y \in Z : \|x - y\| \leq d_\circ\}$$

which is $weak^*$-closed. Then $\eta \in L_\circ$ and $L_\circ$ is a non-empty $weak^*$-closed convex proper subset of $L$.
If $x \in L_\circ$, then $x \in L$ and $Z \subseteq B[x, d_\circ]$. Hence for any $a \in \Sigma$

$$Z = a \cdot Z \subseteq B[a \cdot x, d_\circ]$$

by non-expansiveness of $\Sigma$ on $L$.
This implies that $aL_\circ \subseteq L_\circ$. But according to our supposition $L$ is minimal. Hence our supposition is wrong. Consequently, $Z$ must consists of a single point. So the single point in $Z$ is a C.F.P for $\Sigma$. □

## 5. OPEN PROBLEM

This paper is concluded by the following open questions.
**Open Problem 1** Does a $\Sigma$ have the F.P.P $(F_*)$ by omitting the convexity assumption if $L.U.C(\Sigma)$ has $L.I.M$.
Related to Problem 1, the following theorem were proved in [7].

**Theorem 14.** *If $\Sigma$ is compact, then the following statements are equivalent.*

1. *There exists a compact subset $K \subseteq \Delta(A.P(\Sigma))$ such that $L_s^* K = K$*
2. *Whenever $\Sigma$ acts on a compact Hausdorff space X, where the action is jointly continuous, there exist a non-empty compact subset $K^{,}$ of $X$ such that $s \cdot K^{,} = K^{,}$ $\forall s \in \Sigma$.*

Note that for a $\Sigma$ to be compact, $A.P(\Sigma) = L.U.C(\Sigma)$.
**Open Problem 2** Does F.P.P $(F_{*s})$ holds for a $\Sigma$ if $A.P(\Sigma)$ has $L.I.M$.
Related to Problem 2, the theorem 8 were proved in [17].

**Remark 2.** *If $\Sigma$ is compact then $A.P(\Sigma) = L.U.C(\Sigma)$ then one can easily see that the theorem 8 gives the answer to the open Problem 2.*

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