Characterizations of inessential linear relations

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Abstract In order to study the inessential operators Tarafdar introduced the notion of improjective operators. Aiena and González characterize the class of inessential operators in more algebraic terms and they also obtain characterizations of the inessential operators among the improjective operators in terms of the complementability of some subspaces. In the present paper, we extend these characterizations to the general case of linear relations. Firstly, we characterize the inessential linear relations by the nullity and the deficiencies. Secondly, we apply such algebraic results to characterize the inessential linear relations among the improjective linear relations in terms of the complementability of some null and range spaces.

Key words inessential linear relations, improjective linear relations, null space and range space.

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1 Introduction

The concept of bounded inessential operator in a Banach space was first introduced by Kleinecke [18] in order to find the largest ideal of Riesz operators. In [20, 21] this notion is extended to operators acting between different Banach spaces, as follows: Let $B(X, Y)$ be the set of all bounded operators from X to Y. An element $T \in B(X, Y)$ is called inessential if $I - ST$ is Fredholm for every $S \in B(Y, X)$. Later, the inessential operators in Banach spaces have been studied in several papers by Aiena [1], Aiena and González [2, 3, 4] among others. There exist various characterizations of such class of operators. In [2, Lemma 1.1 and Theorem 1.4] the authors show that this class of operators can be characterized in more algebraic terms, showing that the class of inessential operators presents a perfect symmetry with respect to the nullity and the deficiency. Concretely, they proved the following result

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Theorem 1.1 *[2, Lemma 1.1 and Theorem 1.4] For an operator $T \in B(X,Y)$ the following assertions are equivalent:*
(i) T *is inessential.*
(ii) $\dim N(I-ST) < \infty$ *for every* $S \in B(Y,X)$.
(iii) $\dim N(I-TS) < \infty$ *for every* $S \in B(Y,X)$.
(iv) $\dim X/\overline{R(I-ST)} < \infty$ *for every* $S \in B(Y,X)$.
(v) $\dim Y/\overline{R(I-TS)} < \infty$ *for every* $S \in B(Y,X)$.

On the other hand, following Tarafdar [24, 25] we say that $T \in B(X,Y)$ is improjective if there is no infinite dimensional closed subspace M of X such that the restriction $T_{|M}$ is injective and there exists a closed subspace N of Y such that $TM \oplus N = Y$. Tarafdar [25] proved, for operators in $B(X)$, that inessential operators are improjective. Later, in [3, Proposition 2.4] the authors proved the validity of this property for operators in $B(X,Y)$. This fact combined with Theorem 1.1 and some auxiliary results permits to obtain some interesting characterizations of the inessential operators among the improjective operators in terms of the complementability of some subspaces. Concretely, we have the following result

Theorem 1.2 *[3, Theorem 2.6] For an operator $T \in B(X,Y)$ the following assertions are equivalent:*
(i) T *is inessential.*
(ii) T *is improjective and* $N(I-ST)$ *is topologically complemented in* X *for every* $S \in B(Y,X)$.
(iii) T *is improjective and* $N(I-TS)$ *is topologically complemented in* Y *for every* $S \in B(Y,X)$.
(iv) T *is improjective and* $\overline{R(I-ST)}$ *is topologically complemented in* X *for every* $S \in B(Y,X)$.
(v) T *is improjective and* $\overline{R(I-TS)}$ *is topologically complemented in* Y *for every* $S \in B(Y,X)$.

The main purpose of the present paper is to extend the above Theorems 1.1 and 1.2 to the general case of linear relations.
We note that the class of operators is unstable under the operations closure, inverse and adjoint. This is not the case if we consider linear relations (sometimes called multivalued linear operators). We emphasize that the linear relations made their appearance in Functional Analysis motivated by the need to consider adjoints of non densely defined Fredholm type operators which arise in physical applications (see, for instance [19] and [23]) and also by the need to consider the inverse of certain operators, used, for example in the study of some Cauchy problems associated to parabolic type equations in Banach spaces (see, for instance [14]). In the last decade, the investigation of Fredholm-type linear relations become more significance. The motivation of such an investigation is manifold: there is of course interest from a purely mathematical point of view, (see, for instance [5], [6], [7], [8] and [13] among others). Furthermore, it is of substantial help in the study of

many problems in Operator Theory, Physics and other areas of Applied Mathematical. We cite some of them

⋄ Linear bundles. Let S and T be bounded operators. The map $p(\lambda) := \lambda S - T,\ \lambda \in \mathbb{C}$ is called a linear bundle. Many problems of Mathematical Physics (for example, quantum theory, transport theory) are reduced to the study of certain reversibility conditions of $\lambda S - T$, and this study is reduced to the analysis of certain essential spectra of the linear relations TS^{-1} and ST^{-1}. (See, for instance [9] and [17]).

⋄ Pseudoresolvents. Note that any pseudoresolvent is the resolvent of a certain linear relation. A bibliography on this topic including references to other applications of the spectral theory of Fredholm-type linear relations can be found in [9].

⋄ Applications of some perturbation results for Fredholm-type linear relations to the study of degenerate elliptic-parabolic evolution equations. (See [12] and the references therein).

In view of the above remarks, the attempt to generalize the existing results for operators to the general case of multivalued linear operators appears as natural.

The structure of this paper is as follows. To make the paper easily accessible the exposition is more or less selfcontained.

In section 2 some auxiliary notions and results from the theory of linear relations in vector spaces and Banach spaces are presented. In particular, results concerning the closedness and the complementability of subspaces are described and the concepts of inessential and improjective linear relations are introduced. Section 3 is devoted to the extension of the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ of Theorems 1.1 and 1.2 for operators to the general case of linear relations. So Theorem 3.1 characterizes the inessential linear relations only by the nullity and this result will be the key to obtain Theorem 3.2 which characterizes the inessential linear relations among the improjective linear relations in terms of the complementability of some null spaces. The main results of section 4 are Theorems 4.1 and 4.2 which are the multivalued version of the corresponding results Theorems 1.1 and 1.2 $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ for operators. So Theorem 4.1 and Corollary 4.2 give purely algebraic characterizations of the inessential linear relations only by means of the deficiencies $\bar{\beta}$ and β.

Theorem 4.2 provides characterizations of the inessential linear relations among the improjective linear relations in terms of the complementability of some range spaces.

2 Basic definitions and auxiliary results

Throughout this paper we adhered to the notation and terminology of the book [10]. In this section, we collect some fundamental results and we obtain our auxiliary lemmas which are all necessary for proofs of the later main results in this paper.

We commence recalling some basic but useful notions and properties of linear relations in vector spaces. Let E, F and G be vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation or multivalued linear operator T from E to F is a mapping T having domain $D(T)$ a nonempty subspace of E and taking values in the collection of nonempty subsets of

F such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all nonzero scalars α_1, α_2 and $x_1, x_2 \in D(T)$. For $x \in X \backslash D(T)$ we define $Tx \neq \emptyset$. With this convention, we have that $D(T) := \{x \in E : \ Tx \neq \emptyset\}$. The class of all linear relations from E to F is denoted by $\mathcal{LR}(E, F)$ and $\mathcal{LR}(E) := \mathcal{LR}(E, E)$. If $T \in \mathcal{LR}(E, F)$ maps the points of its domain to singletons, then T is said to be single valued or an operator. Note that T is single valued if and only if $T(0) = \{0\}$.

Every $T \in \mathcal{LR}(E, F)$ is uniquely determined by its graph, $G(T)$, which is defined by $G(T) = \big\{(x, y) \in E \times F : x \in D(T), \ y \in Tx\big\}$.

Let $T \in \mathcal{LR}(E, F)$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) := \big\{(y, x)/ \ (x, y) \in G(T)\big\}$. The subspace $T^{-1}(0)$ denoted by $N(T)$, is called the null space of T and we say that T is injective if $N(T) = \{0\}$. The range of T is the subspace $R(T) := T(D(T))$ and T is called surjective if $R(T) = F$. We define $\alpha(T) := \dim N(T)$ and $\beta(T) := \dim F/R(T)$ called the nullity and the deficiency of T respectively. The index $K(T)$ of T is defined by $K(T) := \alpha(T) - \beta(T)$ provided $\alpha(T)$ and $\beta(T)$ are not both infinite.

For $T \in \mathcal{LR}(E, F)$ and $S \in \mathcal{LR}(F, G)$ the product ST is the linear relation given by $G(ST) := \{(x, z) \in E \times G : \ (x, y) \in G(T), \ (y, z) \in G(S) \text{ for some } y \in F\}$. We note that $D(ST) := \{x \in D(T) : \ Tx \cap D(S) \neq \emptyset\}$. If M is a subspace of E then the restriction $T_{|M}$ is defined by $G(T_{|M}) := G(T) \cap (M \times F)$. If $\lambda \in \mathbb{K}$ and $T \in \mathcal{LR}(E)$ then $\lambda - T$ is the linear relation given by $G(\lambda - T) := \{(x, \lambda x - y) : \ (x, y) \in G(T)\}$.

The following three lemmas are purely algebraic and they will be of substantial help to obtain the main results of this paper.

Lemma 2.1 *[10, Proposition I.2.8 and I.3.1] Let $T \in \mathcal{LR}(E, F)$. Then*
(i) For $x \in D(T)$, $y \in Tx$ if and only if $Tx = y + T(0)$. Hence $x \in N(T)$ if and only if $0 \in Tx$ if and only if $Tx = T(0)$ and for $x_1, \ x_2 \in D(T)$, $Tx_1 \cap Tx_2 \neq \emptyset$ if and only if $Tx_1 = Tx_2$.
(ii) $T(M_1 + M_2) = TM_1 + TM_2$, $M_1 \subset E$, $M_2 \subset D(T)$.
(iii) $T^{-1}TM = (M \cap D(T)) + T^{-1}(0)$, $M \subset E$.
(iv) $TT^{-1}N = (N \cap R(T)) + T(0)$, $N \subset F$.

Lemma 2.2 *Let $T \in \mathcal{LR}(E, F)$ and let M_1 and M_2 be subspaces of E. Then*
(i) $T(M_1 \cap M_2) \subset TM_1 \cap TM_2$ with the equality if $N(T) \subset M_1$.
(ii) Let $E = M_1 + M_2$, $M_1 \cap M_2 = \{0\}$ and suppose that $D(T) = E$ and $N(T) \subset M_1$. Then $R(T) = TM_1 + TM_2$ and $T(0) = TM_1 \cap TM_2$.

Proof: (i) The inclusion is clear. Assume that $N(T) \subset M_1$ and let $y \in Tm_1 \cap Tm_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Then by Lemma 2.1 (i) $Tm_1 = Tm_2$, so that $0 \in T(m_2 - m_1)$, that is, $m_2 - m_1 \in N(T) \subset M_1$. Hence $m_2 \in M_1 \cap M_2$.
(ii) Since T is everywhere defined, is $R(T) = TM_1 + TM_2$ by Lemma 2.1 (ii) and since $N(T) \subset M_1$ and $M_1 \cap M_2 = \{0\}$ it follows from the part (i) that $TM_1 \cap TM_2 = T(0)$. $\square$

Lemma 2.3 *[22, Lemma 3.1 and Theorem 4.1] Let $T \in \mathcal{LR}(E, F)$ and $S \in \mathcal{LR}(F, G)$. We have*

21 Mar 2022 23:00:36 PDT
220321-Keskes Version 1 - Submitted to Rocky Mountain J. Math.

(i) $\dim D(T) + \dim T(0) = \dim R(T) + \dim N(T)$.
(ii) If T *and* S *are everywhere defined then* $\dim ST(0) + \dim(T(0) \cap N(S)) = \dim S(0) + \dim T(0)$.

In the sequel X, Y and Z will denote infinite dimensional Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For a closed subspace M of X, Q_M will denote the quotient map from X onto X/M. Let $T \in \mathcal{LR}(X,Y)$. If Q_T denotes the quotient map from Y onto $Y/\overline{T(0)}$. Then $Q_T T$ is single valued and so we can define for $x \in D(T)$, $\|Tx\| := \|Q_T Tx\|$ and $\|T\| := \|Q_T T\|$ is called the norm of T. We remark that $\|.\|$ is not a true norm since a nonzero linear relation may have zero norm. But, in this paper all linear relations considered are assumed with nonzero norm. We say that $T \in \mathcal{LR}(X,Y)$ is continuous if $\|T\| < \infty$, bounded if it is continuous and everywhere defined, open if its inverse is continuous, closed if its graph is a closed subspace of $X \times Y$, $T \in \phi_+(X,Y)$ if it is closed with closed range and $\dim N(T) < \infty$, $T \in \phi_-(X,Y)$ if it is closed and its range is a closed finite codimensional subspace of Y and $T \in \phi(X,Y)$ if $T \in \phi_+(X,Y) \cap \phi_-(X,Y)$.
We list some known but useful properties of closed and continuous linear relations in Banach spaces.

Lemma 2.4 *[10, Corollary II.3.13, Definitions II.5.1, Proposition II.5.3, Theorem III.4.2 and Lemma V.2.9] Let* $T \in \mathcal{LR}(X,Y)$ *and* $S \in \mathcal{LR}(Y,Z)$. *We have*
(i) If $T(0) \subset D(S)$, *then* $\|ST\| \leq \|S\|\|T\|$.
(ii) T *is closed if and only if* T^{-1} *is closed.*
(iii) T *is closed if and only if* $Q_T T$ *is closed and* $T(0)$ *is closed.*
(iv) If T *is continuous with* $D(T)$ *and* $T(0)$ *closed subspaces, then* T *is closed.*
(v) If T *is closed then* T *is continuous if and only if* $D(T)$ *is closed and* T *is open if and only if* $R(T)$ *is closed.*
(vi) If S *is continuous and* $\overline{T(0)} \subset D(S)$, *then* $Q_{ST} ST = (Q_{ST} S Q_T^{-1})(Q_T T)$.

The following three lemmas are preliminary results from which information concerning the closedness of certain subspaces will follow.

Lemma 2.5 *[10, Lemma IV.5.2] Let* M *and* N *be subspaces of* X *such that* $N \subset M$ *and* N *is closed. Then*
(i) M *is closed in* X *if and only if* M/N *is closed in* X/N.
(ii) If M *is closed, then* $(X/N)/(M/N) = X/M$ *(where the equality is a canonical isometry) and* $Q_M = Q_{M/N} Q_N$.

Lemma 2.6 *Let* $T \in \mathcal{LR}(X,Y)$ *be closed. Then*
(i) $N(T) = N(Q_T T)$.
(ii) $R(T)$ *is closed if and only if* $R(Q_T T)$ *is closed.*
(iii) $T \in \phi_+(X,Y)$ *if and only if* $Q_T T \in \phi_+(X, Y/T(0))$ *and* $T \in \phi_-(X,Y)$ *if and only if* $Q_T T \in \phi_-(X, Y/T(0))$. *In such case* $K(T) = K(Q_T T)$.

21 Mar 2022 23:00:36 PDT
220321-Keskes Version 1 - Submitted to Rocky Mountain J. Math.

Proof: (i) Follows immediately from Lemmas 2.1 (iii) and 2.4 (iii).
The part (i) combined with Lemmas 2.4 (iii) and 2.5 ensures the validity of (ii) and (iii).
□

Lemma 2.7 *Let $T \in \mathcal{LR}(X,Y)$ be closed. We have*
(i) If $R(T)$ is closed and M is a subspace of X such that $N(T)+M$ is closed, then TM is closed.
(ii) If N is a closed subspace of Y such that $T(0) \subset N$ and $R(T)+N$ is closed, then $R(T)$ is closed.

Proof: (i) A combination of Lemma 2.6 (i), (ii) and [15, Lemma IV.2.9] leads to $Q_T TM$ closed and thus we infer from Lemma 2.5 (i) that TM is closed.
(ii) Using the hypotheses and Lemma 2.5 (i) it follows that $Q_T N$ and $R(Q_T T) + Q_T N$ are both closed subspaces of Y/N. Theses facts together with Lemma 2.6 (i), (ii) and [15, Theorem IV.1.12] prove that $R(Q_T T)$ is closed and thus $R(T)$ is closed by Lemma 2.5 (i).
□
Some useful results about the behaviour of the complementability in sums and quotients are now presented.
Recall that a closed subspace M of X is said to be topologically complemented in X if there exists a closed subspace N of X such that $X = M \oplus N$, that is, $X = M + N$ and $\{0\} = M \cap N$. In such case the subspace N is called a complement of M.

Definition 2.1 *[11] Let $P \in \mathcal{LR}(X)$. We say that P is a multivalued projection if $P^2 = P$ (that is, P is idempotent) and $R(P) \subset D(P)$. Multivalued projections can be characterized in terms of a pair of subspaces as follows: Let M and N be subspaces of X. Define $P \in \mathcal{LR}(X)$ by*

$$G(P) = \{(m+n, n);\ m \in M,\ n \in N\}. \tag{2.1}$$

Then P is a multivalued projection satisfying

$$D(P) = M + N,\ R(P) = M,\ N(P) = N \text{ and } P(0) = M \cap N. \tag{2.2}$$

Conversely, if P is a multivalued projection in X, then P determines a pair of subspaces M and N of X such that (2.1) and (2.2) hold.

Lemma 2.8 *[11, Proposition 1.2, Theorem 3.4 and Proposition 3.13] We have:*
(i) If $P \in \mathcal{LR}(X)$ then P is idempotent if and only if P^{-1} is idempotent.
Let M and N be subspaces of X and let P denote the multivalued projection in X with $D(P) = M + N$, $R(P) = M$, $N(P) = N$ and $P(0) = M \cap N$. Then
(ii) If M and N are closed, then P is continuous if and only if $M + N$ is closed.
(iii) If P is continuous, $M + N$ topologically complemented in X, $M \cap N$ is topologically complemented in $M + N$, then M and N are topologically complemented in X.

Lemma 2.9 *Let M and N be closed subspaces of X. We have*
(i) If M is topologically complemented in X and N is finite dimensional, then $M + F$ is topologically complemented in X.
(ii) Assume that $N \subset M$, N and M/N are topologically complemented in X and X/N respectively. Then M is topologically complemented in X.

Proof: (i) A proof of this assertion can be found in [10, Lemma V.16.3].
(ii) Assume that M/N is topologically complemented in X/N. Then we can find a closed subspace W of X such that $N \subset W$ and $M/N \oplus W/N = X/N$. This equality together with Lemma 2.2 (ii) gives $Q_N^{-1}Q_N X = Q_N^{-1}Q_N M + Q_N^{-1}Q_N W$ and $Q_N^{-1}Q_N(0) = Q_N^{-1}Q_N M \cap Q_N^{-1}Q_N W$. After that, we infer from Lemma 2.1 (iii) that

$$X = M + N \text{ and } N = M \cap W. \tag{2.3}$$

Let P be the multivalued projection in X with $D(P) = X$, $R(P) = M$, $N(P) = W$ and $P(0) = M \cap W$. If N is topologically complemented in X, then a combination of (2.3) and Lemma 2.8 (iii) leads to M and W are topologically complemented in X. □
The last part of section 2 is concentrated on the introduction of the notions of inessential and improjective linear relations which will be studied in this paper. To this end, the following results help to understand Definition 2.2 below

Lemma 2.10 *Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,Z)$ be closed and everywhere defined. Then*
(i) ST is bounded.
(ii) If $\dim T(0) < \infty$ then ST is closed.

Proof: (i) Note that T and S are bounded by Lemma 2.4 (iv) and clearly $D(ST) = X$. Applying Lemma 2.4 (i) we have that ST is continuous.
(ii) By the part (ii) of Lemma 2.4 it is enough to show that $(ST)^{-1} = T^{-1}S^{-1}$ is closed and this property follows immediately from [10, Proposition II.5.17] noting that $N(T^{-1}) = T(0)$ is finite dimensional by hypothesis. □
As an immediate application of Lemmas 2.3 (ii) and 2.10 combined with [10, Exercise II.5.16], we have

Corollary 2.1 *Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,Z)$ be closed and everywhere defined. If $T(0)$ and $S(0)$ are both finite dimensional subspaces then ST, $I - ST$, TS and $I - TS$ are bounded and closed linear relations with $ST(0) = (I - ST)(0)$ and $TS(0) = (I - TS)(0)$ finite dimensional subspaces.*

In the sequel we shall adopt the notation

$$CR_F(X,Y) := \{T \in \mathcal{LR}(X,Y) : \ T \text{ closed everywhere defined and } \dim T(0) < \infty\}.$$

The above properties suggest the following definitions.

Definition 2.2 *Let $T \in \mathcal{LR}(X,Y)$. We say that T is inessential if $I - ST \in \phi(X)$ for every $S \in CR_F(Y,X)$ and T is called improjective if there is no closed infinite dimensional subspace M of X such that $T_{|M}$ is injective and TM is topologically complemented in Y.*

The class of all inessential linear relations from X to Y is denoted by $IN(X,Y)$ and we denote by $IMP(X,Y)$ the class of all improjective linear relations from X to Y and we write $IN(X) := IN(X,X)$ and $IMP(X) := IMP(X,X)$.

Example 2.1 *Let K and L be the bounded operators on l_2 defined by $K(x_1,x_2,x_3,) := (x_1, \frac{x_2}{2}, \frac{x_3}{3}, .)$ and $L(x_1,x_2,x_3,.) := (x_2,x_3,.)$. Then*
(i) $L^{-1} \in CR_F(l_2)$ is not inessential.
(ii) $L^{-1}K \in CR_F(l_2) \cup IN(l_2)$.

Proof: (i) Clearly $L \in B(l_2)$ is surjective whose null space coincides with the subspace generated by $(1,0,0,0,...)$ which implies that $L^{-1} \in CR_F(l_2)$ is not single valued and $LL^{-1} = I$. So that (i) holds true.
(ii) It is obvious that K is a bounded compact operator and thus one deduces from Lemmas 2.3 and 2.10 combined with [10, Proposition V.2.10], that for every $S \in CR_F(l_2)$, $SL^{-1}K \in CR_F(l_2)$ is compact and since clearly $Q_{SL^{-1}K}$ is a bounded Fredholm operator we have that $Q_{SL^{-1}K}(I - SL^{-1}K)$ is a bounded Fredholm operator. Therefore $I - SL^{-1}K \in \phi(l_2)$ by Lemma 2.6. This completes the proof of (ii). $\square$

3 Some characterizations of inessential linear relations in terms of the nullity and others among the improjective linear relations in terms of the complementability of some null spaces

Our main objective in this section is to extend the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ of Theorems 1.1 and 1.2 mentioned in section 1 to the case of linear relations. We start our study with an entirely algebraic lemma.

Lemma 3.1 *Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,Z)$. Then*
(i) $N(I - ST) \subset SN(I - TS)$ and $N(I - TS) \subset TN(I - ST)$. If $T(0)$ and $S(0)$ are both finite dimensional subspaces then $\dim N(I - TS) = \dim N(I - TS)$.
(ii) $STN(I - ST) = N(I - ST) + ST(0)$ and $TSN(I - TS) = N(I - TS) + TS(0)$.
(iii) $N(I - ST) \cap N(ST) = N(I - ST) \cap ST(0) = N(ST) \cap ST(0)$ and $N(I - TS) \cap N(TS) = N(I - TS) \cap TS(0) = N(TS) \cap TS(0)$.

Proof: (i) Let $x \in N(I - ST)$. From Lemma 2.1 (i) we have that $x \in STx$. So that $y \in Tx$ and $x \in Sy$ for some $y \in Y$ equivalently $Tx = y + T(0)$ and $Sy = x + S(0)$ which implies that $TSy = y + TS(0)$, that is $y \in TSy$. Hence $y \in N(I - TS)$ and so

$x \in SN(I-TS)$ which shows that $N(I-ST) \subset SN(I-TS)$. In a similar way we obtain that $N(I-TS) \subset TN(I-ST)$. Now the use of Lemma 2.3 (i) applied to $T_{|N(I-ST)}$ and $S_{|N(I-TS)}$ makes us to deduce that $\dim N(I-ST) < \infty$ if and only if $\dim N(I-TS) < \infty$ whenever $T(0)$ and $S(0)$ are both finite dimensional subspaces.
(ii) Let $x \in STN(I-ST)$. Then there exists $a \in N(I-ST)$ for which $x \in STa$. It follows from Lemma 2.1 (i) that $STa = a + ST(0) = x + ST(0)$ and hence $x \in a + ST(0) \subset N(I-ST) + ST(0)$. Conversely, let $x = u + v \in N(I-ST) + ST(0)$. Then applying again Lemma 2.1 (i) we have that $x \in STu \subset STN(I-ST)$. Therefore $STN(I-ST) = N(I-ST) + ST(0)$. Proceeding exactly as above we obtain that $TSN(I-TS) = N(I-TS) + TS(0)$.
(iii) The proof is along the lines of the proof of the previous assertions, using the definitions and Lemma 2.1 (i). □

The following Lemmas are useful: they give information of ϕ_+ and ϕ_- linear relations.

Lemma 3.2 *Let $T \in \mathcal{LR}(X,Y)$ be closed. We have*
(i) If $T \in \phi_+(X,Y)$ then there exists $A \in B(Y,X)$ and a bounded finite rank projection P such that $AT = I_{D(T)} - P$.
(ii) If $D(T) = X$ and $T \notin \phi_+(X,Y)$ then there exists a compact operator K such that $D(K)$ is a closed infinite dimensional subspace of X and $\dim N(T+K) = \infty$.

Proof: It is proved in [10, Corollary V.1.7] that $T \in \phi_+(X,Y)$ if and only if there exists a finite codimensional subspace M of X which $T_{|M}$ is injective and open. This combined with [10, Theorems V.1.6 and V.10.3] proves the validity of (i) and (ii). □

Lemma 3.3 *Let $T \in CR_F(X)$. If $\lambda - T \in \phi_+(X)$ for every $\lambda \in \mathbb{C}\backslash\{0\}$ then $\lambda - T \in \phi_-(X)$ for every $\lambda \in \mathbb{C}\backslash\{0\}$.*

Proof: We claim that there exists $\eta > 0$ such that

$$\dim N(\eta - T) = \dim T(0) \text{ and } R(\eta - T) = X. \text{ In particular } \eta - T \in \phi(X). \tag{3.1}$$

Indeed, by Lemma 2.4 (v), ηQ_T and $Q_T T$ are bounded operators from X to $X/T(0)$ where $\eta = 2\|T\|$ and clearly $\|Q_T T\| < \gamma(\eta Q_T)$ where $\gamma(\eta Q_T)$ denotes the minimum modulus of the operator ηQ_T. In this situation we infer from [15, Theorem V.1.6] that $Q_T(\eta - T)$ has closed range, $\dim N(Q_T(\eta - T)) \leq \dim N(\eta Q_T) = \dim T(0) < \infty$, $\dim(X/T(0))/R(Q_T(\eta - T)) \leq \dim(X/T(0))/R(\eta Q_T) = 0$ and $K(Q_T(\eta - T)) = K(\eta Q_T) = \dim T(0)$.
These facts together with Lemma 2.6 give (3.1).
We claim that for every $\lambda \in \mathbb{C}\backslash\{0\}$

$$\text{the index } K(\lambda - T) \text{ has a constant value.} \tag{3.2}$$

Since $\lambda - T \in \phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ one has that the set $\{\mu \in \mathbb{C} : \ \mu - T \in \phi_+(X)\}$. has a unique component $\mathbb{C}\backslash\{0\}$ and thus we infer from [26, Proposition 3.10] the validity

of (3.2).
After that it follows immediately from (3.1) and (3.2) that $\lambda - T \in \phi_+(X)$ and $K(\lambda - T) = \dim T(0) < \infty$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ and hence $\lambda - T \in \phi_-(X)$, as desired. □

For bounded operators this lemma was proved by Aiena in [1, Theorem 3.111] with a different scheme of proof. He uses the notion of SVEP property which is not considered in this paper.
Now, we are ready to state the first main result of this section. It shows that the inessential linear relations may be characterized only by the nullity α. Furthermore, it will be crucial to obtain the second main result of this section, Theorem 3.2 below.

Theorem 3.1 *Let $T \in CR_F(X,Y)$. The following statements are equivalent:*
(i) $T \in IN(X,Y)$.
(ii) $\dim N(I - ST) < \infty$ *for every* $S \in CR_F(Y,X)$.
(iii) $\dim N(I - TS) < \infty$ *for every* $S \in CR_F(Y,X)$.

Proof: Firstly we note that by virtue of Corollary 2.1 $I - ST \in CR_F(X)$ and $I - TS \in CR_F(Y)$.
The implication $(i) \Rightarrow (ii)$ is obvious, while the equivalence of (ii) and (iii) is covered by Lemma 3.1 (i). Hence it only remains to verify the implication $(ii) \Rightarrow (i)$.
Assume that the assertion (ii) holds and let us consider two cases:
Case 1: $I - ST \in \phi_+(X)$ for all $S \in CR_F(Y,X)$. In such case, according to Lemma 3.3 one has that $I - ST \in \phi_-(X)$ for all $S \in CR_F(Y,X)$. Hence T is inessential.
Case 2: There is $S_1 \in CR_F(Y,X)$ such that $I - S_1T \notin \phi_+(X)$. Thus applying Lemma 3.2 (ii) we can find a compact operator K such that $D(K)$ is a closed infinite dimensional subspace of X and $\alpha(I - S_1T - K) = \infty$. Define $M := N(I - S_1T - K)$.
We claim that there exists $A \in B(X)$ and a bounded finite rank projection P such that

$$A(I - K) = I_{D(K)} - P. \tag{3.3}$$

The required property follows immediately from Lemma 3.2 (i) observing that $I - K \in \phi(X)$.
We claim that

$$\dim(I - AS_1T)M < \infty. \tag{3.4}$$

Since $M := N(I - S_1T - K)$ we have that $(I - K)M - S_1TM = S_1T(0)$ and multiplying the left side by A we obtain that $A(I - K)M - AS_1TM = AS_1T(0)$ and applying (3.3) we get $(I - AS_1T)M = PM + AS_1T(0)$ which is finite dimensional. Hence (3.4) holds true.
We claim that

$$\dim N(I - AS_1T) = \infty. \tag{3.5}$$

Immediate from (3.4) combined with Lemma 2.3 (i) applied to $I - AS_1T_{|M}$, observing that $dimM = \infty$ and $\dim AS_1T(0) < \infty$.

21 Mar 2022 23:00:36 PDT
220321-Keskes Version 1 - Submitted to Rocky Mountain J. Math.

Now, the equality $\dim N(I - AS_1T) = \infty$ contradicts our hypothesis. Therefore the proof is completed. □

Theorem 3.1 represents an improvement of [2, Lemma 1.1 and Theorem 4.1] to linear relations.

Tarafdar [25] proved, for operators in $B(X)$, that inessential operators are improjective. Later, Aiena and González [3, Proposition 2.4] showed that such property is true for bounded operators acting between different Banach spaces. Next, we use this result to obtain an analogous to linear relations.

Lemma 3.4 *Let $T \in CR_F(X,Y)$. Then T is improjective if and only if Q_TT is improjective.*

Proof: Assume that Q_TT is not improjective. Let M be a closed infinite dimensional subspace of X such that $Q_TT_{|M}$ is injective and $R(Q_TT_{|M})$ is topologically complemented in $Y/T(0)$. Hence by Lemmas 2.6 (i) and 2.9 (ii) $T_{|M}$ is injective and TM is topologically complemented in Y. So that T is not improjective.

Suppose that T is not improjective and select a closed infinite dimensional subspace M of X for which $T_{|M}$ is injective and there is a closed subspace N of Y such that $TM \oplus N = Y$. Then $N(Q_TT_{|M}) = \{0\}$ (Lemma 2.6 (i)), Q_TTM and Q_TN are closed subspaces of $Y/T(0)$ (Lemma 2.5 (i)) and $Q_TTM \oplus Q_TN = Q_TY$ (Lemma 2.2 (ii)). So that Q_TT is not improjective. □

Proposition 3.1 *Let $T \in CR_F(X,Y)$. If T is inessential, then T is improjective.*

Proof: Assume that T is not improjective. According to Lemma 3.4 the bounded operator Q_TT is not improjective and thus we infer from the operator version of this Proposition 3.1 [3, Proposition 2.4] that there exists $S \in B(Y/T(0), X)$ such that $I - SQ_TT$ is not Fredholm and hence T is not inessential. □

The rest of this section is devoted to generalize the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ of Theorem 1.2 to linear relations. The analysis is essentially based on the previous results and the following lemma which gives sufficient conditions for the product of two linear relations to be improjective.

Lemma 3.5 *Let $T \in CR_F(X,Y)$ and $S \in CR_F(Y,Z)$. If one of the linear relations T, S is improjective then ST is improjective.*

Proof: Note that $ST \in CR_F(X,Z)$ by Corollary 2.1. First we shall verify that

$$Q_{ST}ST = (Q_{ST}SQ_T^{-1})(Q_TT) \text{ where } Q_{ST}SQ_T^{-1} \text{ and } Q_TT \text{ are bounded operators.} \quad (3.6)$$

The equality $Q_{ST}ST = (Q_{ST}SQ_T^{-1})(Q_TT)$ is established in Lemma 2.4 (vi) and it is evident that Q_TT is a bounded operator by Lemma 2.4 (v). Hence it remains only to verify that $Q_{ST}SQ_T^{-1}$ verify the required conditions. We have that $Q_{ST}SQ_T^{-1}(0) = Q_{ST}ST(0) = \{0\}$ and $D(Q_{ST}SQ_T^{-1}) = Q_TD(Q_{ST}S) = Y/T(0)$ and the continuity follows from Lemma

2.4 (i). Therefore (3.6) holds true.
Case 1: T is improjective. Then by Lemma 3.4, $Q_T T$ is a bounded improjective operator. Thus combining (3.6) with [3, Proposition 3.1] we obtain that $Q_{ST}ST$ is a bounded improjective operator. Hence ST is improjective (again by Lemma 3.4).
Case 2: S is improjective. The use of Lemma 3.4, the assertion (3.6) and [3, Proposition 3.1] allow us to say that, in order to show that ST is improjective, it is sufficient to prove that $Q_{ST}SQ_T^{-1}$ is improjective. To see this, we first note that applying the reasoning of the case 1, we have that the bounded operator $Q_{ST}S$ is improjective. Suppose that $Q_{ST}SQ_T^{-1} \in B(Y/T(0), X/ST(0))$ is not improjective. From [3, Theorem 2.3] we can find a closed infinite codimensional subspace V of $X/ST(0)$ for which $Q_V Q_{ST} S Q_T^{-1}$ is surjective and its null space is topologically complemented in $Y/T(0)$. Since $N(Q_V Q_{ST} S Q_T^{-1}) = (N(Q_V Q_{ST} S) + T(0))/T(0)$ one has that $N(Q_V Q_{ST}) + T(0)$ is topologically complemented in Y by virtue of Lemma 2.9 (ii). This fact together with the equality $\dim(N(Q_V Q_{ST} S) + T(0))/N(Q_V Q_S S) = \dim T(0)/(T(0) \cap N(Q_V Q_{ST} S)) < \infty$ ensures the complementability of $N(Q_V Q_{ST} S)$ and since clearly the surjectivity of $Q_V Q_{ST} S Q_T^{-1}$ implies that $Q_V Q_{ST} S$ is surjective we conclude from [3, Theorem 2.3] that $Q_{ST}S$ is not improjective, a contradiction. □
Now, we are in the position to establish the second main result of section 3.

Theorem 3.2 *Let $T \in CR_F(X, Y)$. The following statements are equivalent:*
(i) $T \in IN(X, Y)$.
(ii) $T \in IMP(X, Y)$ and $N(I - ST)$ is topologically complemented in X for every $S \in CR_F(Y, X)$.
(iii) $T \in IMP(X, Y)$ and $N(I - TS)$ is topologically complemented in Y for every $S \in CR_F(Y, X)$.

Proof: Assume that T is inessential. By Proposition 3.1 T is improjective. Moreover, by Theorem 3.1, for every $S \in CR_F(Y, X)$ we have $N(I - ST)$ and $N(I - TS)$ are finite dimensional. Hence they are topologically complemented in X and Y respectively. Therefore (i) implies the other assertions.
(ii) ⇒ (i) Assume (ii) holds true and define $M := N(I - ST)$. We claim that

$$\begin{cases} STQ^{-1}_{ST(0)\cap M} \,|_{Q_{ST(0)\cap M} M} \text{ is injective} \\ \text{and its range is topologically complemented in } X. \end{cases} \tag{3.7}$$

By Lemma 2.5 (i), $Q_{ST(0)\cap M} M$ is closed. Moreover, we have that
$N(STQ^{-1}_{ST(0)\cap M} \,|_{Q_{ST(0)\cap M} M}) = Q_{ST(0)\cap M} N(ST) \cap Q_{ST(0)\cap M} M = Q_{ST(0)\cap M}(N(ST) \cap M)$ (Lemma 2.2 (i))$=Q_{ST(0)\cap M}(ST(0) \cap M)$ (Lemma 3.1 (iii))$=\{0\}$.
$R(STQ^{-1}_{ST(0)\cap M} \,|_{Q_{ST(0)\cap M} M}) = STQ^{-1}_{ST(0)\cap M} Q_{ST(0)\cap M} M = STM$ (Lemma 2.1 (iii))$=M + ST(0)$ (Lemma 3.1 (ii)) and since M is topologically complemented in X by hypothesis and $\dim ST(0) < \infty$ we deduce from Lemma 2.9 (i) that $M + ST(0)$ is topologically complemented in X which completes the proof of (3.7).
On the other hand, a repeated application of Lemma 3.5 proves that ST and $STQ^{-1}_{ST(0)\cap M}$

are improjective linear relations with $STQ^{-1}_{ST(0)\cap M}(0) = ST(0)$ because $STQ^{-1}_{ST(0)\cap M}(0) = ST(ST(0)\cap M) = ST(ST(0)\cap N(ST))$ (Lemma 3.1 (iii))$=STST(0)\cap ST(ST)^{-1}(0) = ST(0)$ (Lemma 2.1 (iv)).
After that, according to (3.7) we have that $\dim Q_{ST(0)\cap M}M = \dim M/(ST(0)\cap M) < \infty$. Hence $\dim N(I-ST) < \infty$ for every $S \in CR_F(Y,X)$. Now, by Theorem 3.1 $(ii) \Rightarrow (i)$ we conclude that T is inessential.
$(iii) \Rightarrow (i)$ Proceeding as in the proof of the implication $(ii) \Rightarrow (i)$ we obtain that $TSQ^{-1}_{TS(0)\cap N(I-TS)}$ is an improjective linear relation such that its restriction to the closed subspace $Q_{TS(0)\cap N(I-TS)}N(I-TS)$ is injective with range $TSN(I-TS)$ topologically complemented in Y. Therefore for every $S \in CR_F(Y,X)$ $\dim N(I-TS) < \infty$ and thus T is inessential by Theorem 3.1 $(iii) \Rightarrow (i)$. □
This theorem generalizes the analogous result for bounded operators in Banach spaces of Aiena and González [3, Theorem 2.6 $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$].

4 Some characterizations of inessential linear relations in terms of the deficiencies and others among the improjective linear relations in terms of the complementability of some range spaces

In this section, our interest concentrates to establish several characterizations of the inessential linear relations in terms of the complementability of some range spaces, paralleling our investigation of the inessential linear relations in terms of their action on the complemented null spaces of the previous section. We begin our study with a purely algebraic lemma which will be crucial to obtain the main results of section 4.

Lemma 4.1 *Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,X)$. Then*
(i) $N(T) \subset N(ST) \subset R(I-ST)$ *and* $N(S) \subset N(TS) \subset R(I-TS)$.
(ii) $TR(I-ST) \subset R(I-TS)$ *and* $SR(I-TS) \subset R(I-ST)$.
(iii) Define $\widetilde{T} : X/R(I-ST) \to Y/R(I-TS)$ *and* $\widetilde{S} : Y/R(I-TS) \to X/R(I-ST)$ *by* $\widetilde{T}[x] := \{[y] : \ y \in Tx\}$ *and* $\widetilde{S}[y] := \{[x] : \ x \in Sy\}$ *respectively. Then* $\widetilde{T}$ *and* $\widetilde{S}$ *are injective operators.*
(iv) $\dim X/R(I-ST) < \infty$ *if and only if* $\dim Y/R(I-TS) < \infty$.

Proof: (i) Clearly $N(T) \subset N(ST)$. Let $x \in N(ST)$. Then it follows from Lemma 2.1 (i) that $x \in (I-ST)x \subset R(I-ST)$. So that $N(T) \subset N(ST) \subset R(I-ST)$ and similarly we obtain that $N(S) \subset N(TS) \subset R(I-TS)$.
(ii) Let $y \in Tx$, $x \in R(I-ST)$. From the definition of the product of linear relations we have that there exist $a \in X$ and $b \in Y$ such that $(a,b) \in G(T)$ and $(b, a-x) \in G(S)$. In this situation, the use of Lemma 2.1 (i) allows us to say that $TSb = b - y + TS(0)$. Hence $y \in (I-TS)b$ which completes the proof of the inclusion $TR(I-ST) \subset R(I-TS)$. In

a similar way we obtain that $SR(I-ST) \subset R(I-TS)$.
(iii) Note that the part (ii) permits to define $\widetilde{T}$ and $\widetilde{S}$. First, we verify that $\widetilde{T}$ and $\widetilde{S}$ are operators. Since $T(0) \subset TS(0) = (I-TS)(0) \subset R(I-TS)$ and similarly $S(0) \subset R(I-ST)$ we obtain that $\widetilde{T}[0] = [0]$ and $\widetilde{S}[0] = [0]$, that is, $\widetilde{T}$ and $\widetilde{S}$ are operators.
Now, let $\widetilde{T}[x] = \{[y] : \ y \in Tx\} = [0]$. Then $y \in Tx \cap R(I-TS)$, so that $Sy \subset STx \cap R(I-ST)$ by (ii). Let $a \in Sy$ and select $b \in X$ such that $a \in STx \cap (I-ST)b$. Applying Lemma 2.1 (i) we deduce that $(I-ST)x = x - (I-ST)b$ which implies that $(I-ST)(x+b) = x + (I-ST)(0)$, that is , $x \in (I-ST)(x+b) \subset R(I-ST)$. Hence $[x] = [0]$ and so the injectivity of $\widetilde{T}$ is proved. Reasoning in the same way we obtain that $\widetilde{S}$ is injective.
(iv) It is an immediate consequence of (iii). □

Corollary 4.1 *Let $T \in CR_F(X,Y)$. Then $T \in IN(X,Y)$ if and only if $I - TS \in \phi(Y)$ for all $S \in CR_F(Y,X)$.*

Proof: Assume that T is inessential, that is, $I - ST \in \phi(X)$ for all $S \in CR_F(Y,X)$. Then $\dim N(I-TS) < \infty$ by Lemma 3.1 (i) and $\dim Y/R(I-TS) < \infty$ by Lemma 4.1 (iv). So that it only remains to show that $R(I-TS)$ is closed. To see this, we note that $Q_{TS}(I-TS)$ is a bounded operator (Lemmas 2.4 (iii) and 2.10 (i) and (ii)), $\dim(Y/TS(0))/Q_{TS}R(I-TS) = \dim Y/R(I-TS)$ ([10, Exercise I.6.5]). Thus it follows from [15, Corollary IV.1.13] that $R(Q_{TS}(I-TS))$ is closed. Hence $R(I-TS)$ is closed by Lemma 2.6 (ii). The same arguments prove the other implication. □
In Theorem 3.1 we proved that the class of inessential linear relations can be characterized only by the nullity. Our next objective is to show that dually $IN(X,Y)$ can be characterized only means of the deficiencies β and $\bar{\beta}$ where for $T \in \mathcal{LR}(X,Y)$ $\bar{\beta} := \dim Y/\overline{R(T)}$. To this end, we shall need the above results and some lemmas concerning the adjoint of a linear relation.
We shall adopt the following notation: If M and L are subspaces of X and X' (the dual space of X) respectively, then $M^{\perp} := \{x' \in X' : \ x'(m) = 0 \text{ for all } \ m \in M\}$ and $L^{\top} := \{x \in X : \ x'(x) = 0 \text{ for all } \ x' \in L\}$.

Definition 4.1 *[10, Definitions III.1.1] The adjoint T' of $T \in \mathcal{LR}(X,Y)$ is defined by $G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'$.*

Lemma 4.2 *[10, Propositions III.1.2, III.1.4, III.1.5, III.4.6 and Corollary III.2.3] Let $T \in \mathcal{LR}(X,Y)$. Then*
(i) T' is closed.
(ii) $N(T') = R(T)^{\perp}$ and $T'(0) = D(T)^{\perp}$.
(iii) $(I-T)' = I - T'$ if $T \in \mathcal{LR}(X)$.
(iv) T is continuous if and only if $D(T') = T(0)^{\perp}$.
(v) If $E \subset D(T')$, then $(T'(E))^{\top} \cap D(T) = T^{-1}(E^{\top})$.

Definition 4.2 *[16, Definition 2.7] Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(X,Z)$. We say that T is S-bounded if $D(S) \subset D(T)$ and there exist $\alpha, \beta > 0$ such that $\|Tx\| \leq \alpha\|Sx\| + \beta\|x\|,\ x \in D(S)$.*

The following lemma gives sufficient conditions for $(ST)' = T'S'$ to holds.

Lemma 4.3 *Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,X)$ be closed and everywhere defined. Then*
(i) S' is $(ST)'$-bounded and T' is $(TS)'$-bounded.
(ii) $(ST)' = T'S'$ and $(TS)' = S'T'$.

Proof: (i) Note that S, T, ST and TS are bounded by Lemmas 2.4 (v) and 2.10 (i). So that, it follows from Lemma 4.2 (i), (iv) that $S', T', (ST)'$ and $(TS)'$ are closed with $D((ST)') = ST(0)^{\perp} \subset S(0)^{\perp} = D(S')$ and $D((TS)') \subset D(T')$. After that, we infer from [16, Lemma 2.11] that (i) holds true.
(ii) Combine the part (i) with [16, Theorem 3.1]. □
The following lemma is the multivalued version of the corresponding result for bounded operators proved in [1, Lemma 7.6].

Lemma 4.4 *Let $T \in CR_F(X,Y)$ and $S \in CR_F(Y,X)$. Then $\dim X/\overline{R(I-ST)} < \infty$ if and only if $\dim Y/\overline{R(I-TS)} < \infty$.*

Proof: We claim that

$$T' \text{ and } S' \text{ are operators.} \tag{4.1}$$

It is suffices to apply Lemma 4.2 (ii).
We claim that

$$\dim N(I - S'T') = \dim(Y/\overline{R(I-TS)})' \text{ and } \dim N(I - T'S') = \dim(X/\overline{R(I-ST)})'. \tag{4.2}$$

Both equalities are an immediate consequence of Lemmas 4.2 (ii), (iii) and 4.3 (ii). Now, using (4.1), (4.2) and Lemma 3.1 (i) we deduce the desired result. □
The following Lemmas are sometimes useful from which information concerning ϕ_- linear relations will follow.

Lemma 4.5 *[10, Proposition V.1.9 and Theorem V.5.5] Let $T \in \mathcal{LR}(X,Y)$ be closed. If $T \notin \phi_-(X,Y)$, then there exists a compact operator $K \in B(X,Y)$ such that $\overline{R(T+K)}$ has infinite codimension.*

Lemma 4.6 *Let $T \in CR_F(X)$ such that $\lambda - T \in \phi_-(X)$ for every $\lambda \in \mathbb{C}\backslash\{0\}$. Then $\lambda - T \in \phi_+(X)$ for every $\lambda \in \mathbb{C}\backslash\{0\}$.*

Proof: The proof proceeds exactly as in Lemma 3.3. □
Next we show that the class $IN(X,Y)$ presents a perfect symmetry with respect to the deficiencies β and $\bar{\beta}$.

Theorem 4.1 *Let $T \in CR_F(X,Y)$. The following statements are equivalent:*
(i) $T \in IN(X,Y)$.
(ii) $\dim X/\overline{R(I-ST)} < \infty$ *for every* $S \in CR_F(Y,X)$.
(iii) $\dim Y/\overline{R(I-TS)} < \infty$ *for every* $S \in CR_F(Y,X)$.

Proof: $(i) \Rightarrow (ii)$ Since $\bar{\beta}(I-ST) \leq \beta(I-ST)$, if $T \in IN(X,Y)$ then $\bar{\beta}(I-ST) < \infty$.
The equivalence $(ii) \Leftrightarrow (iii)$ is clear from Lemma 4.4.
$(iii) \Rightarrow (i)$ Suppose that $\bar{\beta}(I-TS) < \infty$ for all $S \in CR_F(Y,X)$. Let us consider two cases:
Case 1: $I-TS \in \phi_-(Y)$ for every $S \in CR_F(Y,X)$. In that case $I-TS \in \phi_+(Y)$ by Lemma 4.6. Thus it follows from Corollary 4.1 that T is inessential.
Case 2: $I-TS_1 \notin \phi_-(Y)$ for some $S_1 \in CR_F(Y,X)$. We have by Lemma 4.5 that

$$\text{there exists a bounded compact operator } K \text{ in } Y \text{ such that } \bar{\beta}(I-TS_1-K)=\infty. \quad (4.3)$$

We claim that there exists $B \in B(Y)$ and a bounded projection operator P_K of Y onto $R(I-K)$ such that

$$\dim N(P_K) < \infty \text{ and } (I-K)B = P_K. \quad (4.4)$$

Since $I-K \in B(Y) \cap \phi(Y)$ the assertion (4.4) follows from the Atkinson characterization of bounded Fredholm operators.
Put $N := \overline{R(I-TS_1-K)}$.
We claim that $Q_N TS_1$ and $Q_N(I-TS_1)$ are bounded operators satisfying

$$Q_N K = Q_N(I-TS_1) \text{ and } Q_N P_K = Q_N TS_1 B. \quad (4.5)$$

Indeed, $Q_N TS_1(0) = Q_N(I-TS_1)(0) = Q_N(I-TS_1-K)(0) \subset Q_N N = \{0\}$. These facts together with Lemma 2.10 ensure that $Q_N TS_1$ and $Q_N(I-TS_1)$ are bounded operators such that $Q_N K = Q_N(I-TS_1)$. Multiplying the right hand side of this last equality by B we obtain $Q_N KB = Q_N(I-TS_1)B = Q_N(B-TS_1B)$ ([10, Proposition I.3.1] since B is single valued) which implies $Q_N TS_1 B = Q_N(I-K)B = Q_N P_K$ by (4.4). Hence (4.5) holds true.
We claim that

$$Q_N(I-TS_1B)Y = \overline{Q_N(I-TS_1B)Y} \text{ finite dimensional.} \quad (4.6)$$

By (4.4) the bounded operator $Q_N(I-P_K)$ has finite dimensional range and thus the required properties are obtained from the equality $Q_N P_K = Q_N TS_1 B$ established in (4.5).
Now, $\dim Y/\overline{R(I-TS_1B)} \geq \dim(Y/N)/Q_N\overline{R(I-TS_1B)}$ ([10, Proposition I.6.1]) $= \dim(Y/N)/Q_N(I-TS_1B)Y = \infty$ ((4.6)). Hence $\bar{\beta}(I-TS_1B) = \infty$, contradicting our hypothesis. Therefore $I-TS \in \phi_-(Y)$ for all $S \in CR_F(Y,X)$. So that T is inessential by Corollary 4.1. □
The result represents an improvement of [2, Theorem 1.4] to linear relations.

Corollary 4.2 *Let $T \in CR_F(X,Y)$. The following statements are equivalent*
(i) $T \in IN(X,Y)$.
(ii) $\dim X/R(I-ST) < \infty$ *for every* $S \in CR_F(Y,X)$.
(iii) $\dim Y/R(I-TS) < \infty$ *for every* $S \in CR_F(Y,X)$.

Proof: $(i) \Leftrightarrow (ii)$ If $T \in IN(X,Y)$ then $\beta(I-ST) < \infty$ for all $S \in CR_F(Y,X)$. Conversely, if $\beta(I-ST) < \infty$ for all $S \in CR_F(Y,X)$, from $\bar{\beta}(I-ST) \leq \beta(I-ST)$ and Theorem 4.1 we have that T is inessential.
The equivalence $(ii) \Leftrightarrow (iii)$ is obvious by Lemma 4.1 (iv). □
The following lemma will be the key to characterize the inessential linear relations among the improjective linear relations.

Lemma 4.7 *Let $T \in CR_F(X,Y)$ and $S \in CR_F(Y,X)$. If $\overline{R(I-TS)}$ is topologically complemented in Y with closed complemented V then both subspaces $T^{-1}\overline{R(I-TS)}$ and SV are topologically complemented in X.*

Proof: Put $N := \overline{R(I-TS)}$ and let us denote by P_V the bounded projection from Y onto V along N.
We claim that

$$P_V TS = P_V. \tag{4.7}$$

It is evident noting that $P_V TS(0) = P_V(I-TS)(0) \subset P_V N = \{0\}$.
We claim that

$$N(S) \cap V = N(TS) \cap V = \{0\} \text{ and } T^{-1}N \cap SV = S(0). \tag{4.8}$$

The equalities $N(S) \cap V = N(TS) \cap V = \{0\}$ follows from Lemma 4.1 (i) and the fact $\overline{R(I-TS)} \cap V = \{0\}$. Let $x \in T^{-1}N \cap SV$. Then $x \in T^{-1}n \cap Sv$ for some $n \in N$ and $v \in V$. A repeated application of Lemma 2.1 (i) gives $Tx = n + T(0),\ Sv = x + S(0)$. So $(I-TS)v = v - n + (I-TS)(0)$ if and only if $v - n \in (I-TS)v$. Thus $v = v - n + n \in N \cap V = \{0\}$. Hence $x \in S(0)$. Therefore $T^{-1}N \cap SV \subset S(0)$.
Conversely, let $x \in S(0)$. Then $x \in SV$ and $Tx \subset TS(0) = (I-TS)(0) \subset N$. According the Lemma 2.1 (iii) $x \in T^{-1}Tx \subset T^{-1}N$ which completes the proof of (4.8).
Let $R := SP_VT$. We claim that R is a multivalued projection with

$$D(R) = X,\ N(R) = T^{-1}N,\ R(R) = SV \text{ and } R(0) = S(0). \tag{4.9}$$

R is a multivalued projection. Indeed, it is obvious that $D(R) = X$ and since $P_V = P_V TS$ by (4.7) we have that $R^2 = SP_VTSP_VT = SP_V^2T = R$.
$N(R) = T^{-1}N$. We have $N(R) = (P_VT)^{-1}S^{-1}(0) \subset (P_VT)^{-1}\overline{R(I-TS)}$ (Lemma 4.1 (i)) $= T^{-1}P_V^{-1}P_V^{-1}(0)$ (as $N(P_V) = \overline{R(I-TS)}$) $= T^{-1}P_V^{-1}(0)$ (Lemma 2.8 (i)) $= T^{-1}\overline{R(I-TS)} \subset (P_VT)^{-1}S^{-1}(0) = N(R)$.
$R(R) = SV$. Indeed, $R(R) = SR(P_VT) \subset SV = SR(P_V) = SR(P_VTS)$ (by (4.7)) $= R(RS) \subset R(R)$.

Finally, that $R(0) = S(0)$ is clear noting that $R(0) = SP_VT(0) \subset SP_V(I - TS)(0) \subset SP_VN = S(0) \subset SP_VT(0) = R(0)$. Therefore (4.9) holds true.
We claim that

$$T^{-1}\overline{R(I - TS)} \text{ and } SV \text{ are topologically complemented in } X. \tag{4.10}$$

We first verify that both subspaces are closed. Since T^{-1} closed with $R(T^{-1}) = D(T) = X$ and $N(T^{-1}) + \overline{R(I - TS)} = T(0) + \overline{R(I - TS)}$ is closed we infer from Lemma 2.7 (i) that $T^{-1}\overline{R(I - TS)}$ is closed subspace of X. Furthermore, since $S_{|V}(0) \subset T^{-1}\overline{R(I - TS)}$ by (4.8) and $X = SV + T^{-1}\overline{R(I - TS)}$ by (4.9) we obtain applying Lemma 2.7 (ii) that SV is closed. After that, a combination of (4.8), (4.9) and Lemma 2.8 leads to (4.10). Hence the lemma follows. □
For bounded operators in Banach spaces, the corresponding result was proved by Aiena and González [3, Lemma 2.5].
Now, we are ready to state our second main result of this section.

Theorem 4.2 *Let $T \in CR_F(X, Y)$. The following statements are equivalent*
(i) $T \in IN(X, Y)$.
(ii) $T \in IMP(X, Y)$ *and* $\overline{R(I - ST)}$ *is topologically complemented in* X *for every* $S \in CR_F(Y, X)$.
(iii) $T \in IMP(X, Y)$ *and* $\overline{R(I - TS)}$ *is topologically complemented in* Y *for every* $S \in CR_F(Y, X)$.

Proof: First we show that (i) implies the other statements. Assume that $T \in IN(X, Y)$. By Proposition 3.1 $T \in IMP(X, Y)$. Moreover by Theorem 4.1 for every $S \in CR_F(Y, X)$ the subspaces $\overline{R(I - ST)}$ and $\overline{R(I - TS)}$ are finite codimensional. Hence they are topologically complemented.
$(ii) \Rightarrow (i)$ Asume that $T \in IMP(X, Y)$ and $M := \overline{R(I - ST)}$ is topologically complemented in X. We claim that

$$S^{-1}M \text{ is closed, } X = R(ST) + M \text{ and } Y = R(T) + S^{-1}M. \tag{4.11}$$

The closedness of $S^{-1}M$ follows immediately from Lemma 2.7 (i) applied to S^{-1}. On the other hand, since $Q_MST(0) = Q_M(I - ST)(0) \subset Q_MM = \{0\}$ we have that $Q_M = Q_MST$ is a surjective operator. Thus $X = R(ST) + M$. This last equality together with Lemma 2.1 (ii) and (iii) gives $Y = D(S) = R(S^{-1}) = S^{-1}(R(ST) + M) = S^{-1}SR(T) + S^{-1}M = R(T) + S^{-1}M$. Hence (4.11) holds true.
We claim that

$$(ST)^{-1}M = M. \tag{4.12}$$

We have the chain of the equalities
$(ST)^{-1}M = (ST)^{-1}(M^{\perp})^{\top} = ((ST)'M^{\perp})^{\top} = ((ST)'N(I - (ST)'))^{\top} = (N(I - (ST)'))^{\top} = M$.
(Here we have combined Lemma 4.2 $(ii), (iii), (v)$ with the fact that ST is everywhere

defined).
We claim that

$$\begin{cases} Q_{Q_T S^{-1}M} Q_T T \text{ is a surjective operator} \\ \text{whose null space is topologically complemented in } X. \end{cases} \tag{4.13}$$

Firstly, we note that since $S^{-1}M$ is closed by (4.11) we have that $Q_T S^{-1}M$ is closed by virtue of Lemma 2.5 (i). Moreover, the equality $Y = R(T) + S^{-1}M$ established in (4.11) implies that $Q_{Q_T S^{-1}M} Q_T TX = Q_{Q_T S^{-1}M} Q_T (TX + S^{-1}M) = Q_{Q_T S^{-1}M} Q_T Y$. Thus $Q_{Q_T S^{-1}M} Q_T T$ is a surjective operator.
On the other hand, $N(Q_{Q_T S^{-1}M} Q_T T) = T^{-1} Q_T^{-1} Q_T S^{-1}M = T^{-1}(S^{-1}M + T(0)) = (ST)^{-1}M + N(T)$ (Lemma 2.1 (ii) and (iii)) $= (ST)^{-1}M$ (as $N(T) \subset M$ by Lemma 4.1 (i)) $= M$ (by (4.12)) which is topologically complemented by hypothesis. Therefore (4.13) holds true.
We claim that

$$S^{-1}M \text{ has finite codimension in } Y. \tag{4.14}$$

First, we verify that $\dim Y/(S^{-1}M + T(0)) < \infty$. To see this, we note that since T is improjective by hypothesis so is $Q_T T$ by virtue of Lemma 3.4. This fact combined with (4.13) and the characterization of bounded improjective operators in terms of quotient maps [2, Theorem 2.3] ensures that $Q_T S^{-1}M$ has finite codimension in $Y/T(0)$. Hence $\dim Y/(S^{-1}M + T(0)) < \infty$ by Lemma 2.5 (ii). Thus we have by Lemma 2.5 (ii) that $\dim(Y/S^{-1}M)/((S^{-1}M+T(0))/S^{-1}M) < \infty$ which implies that $\dim Y/S^{-1}M < \infty$ since trivially $\dim(S^{-1}M + T(0))/S^{-1}M = \dim T(0)/(T(0) \cap S^{-1}M) < \infty$. Hence (4.14) holds true.
In this situation, the assertion (4.14) together with [10, Proposition I.6.1] applied to T^{-1} and the equality $(ST)^{-1}M = M$ proved in (4.12) leads to $\dim X/M < \infty$. Therefore $\dim X/\overline{R(I - ST)} < \infty$ for every $S \in CR_F(Y, X)$. By Theorem 4.1, we may conclude that T is inessential.
(iii) $\Rightarrow$ (i) Assume that T is improjective and $N := \overline{R(I - TS)}$ is topologically complemented in Y. Then it is obvious that $T(0) \subset N$ and $Q_N T(0) \subset Q_N (I - TS)(0) \subset Q_N N = \{0\}$. So $Q_N = Q_N TS$.
We claim that

$$\begin{cases} Q_N T = Q_{Q_T N} Q_T T \text{ is a bounded surjective operator} \\ \text{whose null space is topologically complemented in } X. \end{cases} \tag{4.15}$$

Indeed, as an immediate consequence of Lemma 2.5 we obtain that $Q_T N$ is closed and $Q_N T = Q_{Q_T N} Q_T T$ and the surjectivity follows trivially from the equality $Q_N = Q_N TS$. Moreover, $N(Q_{Q_T N} Q_T T) = T^{-1}N$ which is topologically complemented in X by Lemma 4.7. Hence (4.15) holds true.
After that, using that $Q_T T$ is improjective by Lemma 3.4 together with (4.15) and [3, Theorem 2.3] we deduce that $Q_T N$is finite codimensional and thus applying again Lemma 2.5 (ii) we conclude that N is finite codimensional. Therefore (iii) implies that $\overline{R(I - TS)}$

is finite codimensional for every $S \in CR_F(Y, X)$. Hence by Theorem 4.1 we conclude that $T \in IN(X, Y)$. □

Theorem 4.2 is an extension of Theorem 1.2 $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ to the case of linear relations.

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