Regarding r-orthogonal factorizations in bipartite graphs

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Abstract

Let m, t, r and k_i $(1 \leq i \leq m)$ be positive integers with $k_i \geq (2r-1)t+1$. Let G be a graph, H be an mr-subgraph of G, and $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ be a (g,f)-factorization of G. If for any partition $\{A_1, A_2, \cdots, A_m\}$ of $E(H)$ with $|A_i| = r$, G has a (g,f)-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ with $A_i \subseteq E(F_i)$, $1 \leq i \leq m$, then we say that G has (g,f)-factorizations randomly r-orthogonal to H. Let $H_1, H_2, \cdots, H_t$ be t vertex-disjoint mr-subgraphs of a bipartite graph G with $\Delta(G) \leq k_1+k_2+\cdots+k_m-m+1$. In this paper, it is demonstrated that a bipartite graph G with $\Delta(G) \leq k_1+k_2+\cdots+k_m-m+1$ possesses a $[0,k_i]_1^m$-factorization randomly r-orthogonal to every H_i, $1 \leq i \leq t$.

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1 Introduction

Lots of real-world networks can be simulated by networks or graphs. Henceforth we replace *network* by *graph*. An important example of such a network is a communication network with nodes corresponding to cities and links standing for communication channels. Other examples include the World Wide Web with nodes acting for web pages and links simulating hyperlinks between web pages, or an online social network with nodes modelling persons and links standing for personal contacts of each user. Many real-life problems on network design and optimization, e. g. the file transfer problems on computer networks, building blocks and so on, are related to the factors, factorizations and orthogonal factorizations of graphs [2]. Horton [8] first claimed that a Room square of order $2n$ is equivalent to an orthogonal 1-factorization of K_{2n}. Euler [4] first discovered that a pair of orthogonal Latin squares of order n is related to two orthogonal 1-factorizations of $K_{n,n}$.

All graphs discussed in this article will be finite, undirected and simple graphs. Let G be a graph. We use $V(G)$ to denote the vertex set of G and use $E(G)$ to denote the edge set of G. For any $x \in V(G)$, the degree of x in G is defined as the number of edges which are adjacent to x, and denoted by $d_G(x)$. We denote by $\Delta(G)$ the maximum degree in a graph G. For $X \subseteq V(G)$, $G[X]$

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denotes the subgraph of G induced by X, and $G-X=G[V(G)\setminus X]$. Let X and Y be two disjoint vertex subsets of G. We denote by $E_G(X,Y)$ the set of edges with one end in X and the other in Y, and write $e_G(X,Y)=|E_G(X,Y)|$. Let E' be a subset of $E(G)$. We denote by $G-E'$ the subgraph derived from G by removing the edges in E', and by $G[E']$ the subgraph of G induced by E'. For convenience, we let $\varphi(X)=\sum\limits_{x\in X}\varphi(x)$ for any function φ. Especially, $\varphi(\emptyset)=0$ and $d_{G-X}(Y)=\sum\limits_{x\in Y}d_{G-X}(x)$. Let $\mathbb{N}\cup\{0\}$ denote the set of nonnegative integers. For two functions $g,f:V(G)\to\mathbb{N}\cup\{0\}$ with $0\leq g(x)\leq f(x)$ for all $x\in V(G)$, a spanning subgraph F of G is called a (g,f)-factor if $g(x)\leq d_F(x)\leq f(x)$ for all $x\in V(G)$. In particular, G is called a (g,f)-graph if G itself is a (g,f)-factor. A (g,f)-factorization of G is a decomposition of the edge set of G into edge-disjoint (g,f)-factors $F_1,F_2,\cdots,F_m$. We call a subgraph H of G an mr-subgraph of G if $|E(H)|=mr$. Assume that H is an mr-subgraph of G and $\mathcal{F}=\{F_1,F_2,\cdots,F_m\}$ is a (g,f)-factorization of G. Then $\mathcal{F}$ is r-orthogonal to H if $|E(H)\cap E(F_i)|=r$ for $1\leq i\leq m$. If for any partition $\{A_1,A_2,\cdots,A_m\}$ of $E(H)$ with $|A_i|=r$, G has a (g,f)-factorization $\mathcal{F}=\{F_1,F_2,\cdots,F_m\}$ with $A_i\subseteq E(F_i)$, $1\leq i\leq m$, then we say that G has (g,f)-factorizations randomly r-orthogonal to H. Let a and b be two positive integers. Similarly, we may define $[a,b]$-factor, $[a,b]$-factorization, r-orthogonal $[a,b]$-factorization and randomly r-orthogonal $[a,b]$-factorization. Let $k_1,k_2,\cdots,k_m$ be m positive integers. A $[0,k_i]_1^m$-factorization $\mathcal{F}$ of G is a decomposition of the edge set of G into edge-disjoint factors $F_1,F_2,\cdots,F_m$, where each F_i is a $[0,k_i]$-factor for $1\leq i\leq m$. A $[0,k_i]_1^m$-factorization $\mathcal{F}=\{F_1,F_2,\cdots,F_m\}$ of G is r-orthogonal to an mr-subgraph H of G if $|E(H)\cap E(F_i)|=r$ for $1\leq i\leq m$. If for any partition $\{A_1,A_2,\cdots,A_m\}$ of $E(H)$ with $|A_i|=r$, G has a $[0,k_i]_1^m$-factorization $\mathcal{F}=\{F_1,F_2,\cdots,F_m\}$ with $A_i\subseteq E(F_i)$, $1\leq i\leq m$, then we call that G has $[0,k_i]_1^m$-factorizations randomly r-orthogonal to an mr-subgraph H of G. In particular, randomly 1-orthogonal is equivalent to 1-orthogonal, and 1-orthogonal is also said to be orthogonal. A graph, denoted by $G=(A,B,E(G))$, is a bipartite graph with bipartition $\{A,B\}$ and edge $E(G)$.

Kano, Katona and Király [10], Zhou [28], Zhou, Bian and Pan [32], Zhou [30], Zhou, Sun and Liu [36], Zhou, Wu and Bian [37], Zhou, Wu and Xu [38], Zhou and Bian [31], Wang and Zhang [21], Wu [23] investigated the existence of $[1,2]$-factors in graphs and obtained some results for graphs admitting $[1,2]$-factors. Matsubara, Matsuda, Matsuo, Noguchi and Ozeki [17], Zhou and Liu [34], Zhou [26, 27] put forward some sufficient conditions for graphs to possess $[a,b]$-factors. Egawa and Kano [3], Wang and Zhang [22], Zhou [29], Gao, Wang and Guirao [7] showed some results for graphs having (g,f)-factors. Kano [9] demonstrated some results with relation to the existence of $[a,b]$-factorizations in graphs. Yan, Pan, Wong and Tokuda [25] discussed the problem on (g,f)-factorizations in graphs and derived some results for graphs to admit (g,f)-factorizations.

Alspach, Heinrich and Liu [2] put forward the following open problem: Given a subgraph H of G, does there exist a factorization $\mathcal{F}$ of G with a given property orthogonal to H?

Recently, more and more results on the above problem have been derived: Liu [14], Yan [24], Li and Liu [13], Liu and Long [15], Lam, Liu, Li and Shiu [11] investigated orthogonal factorizations in $(mg+m-1,mf-m+1)$-graphs. Li, Chen and Yu [12], Wang [20] discussed orthogonal factorizations in $(mg+k,mf-k)$-graphs. Feng [5] verified the existence of orthogonal factorizations in $(0,mf-m+1)$-graphs. Feng and Liu [6] proved the existence of orthogonal $[0,k_i]_1^m$-factorizations in graphs. Zhou, Liu and Zhang [35], Liu and Zhu [16] studied orthogonal factorizations in bipartite graphs. Some

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other results on the existence of orthogonal factorizations in graphs can be discovered in [18, 19, 33].

In what follows, we shall deal with the more general problem: Given t vertex-disjoint nr-subgraphs $H_1, H_2, \cdots, H_t$ of G, does there exist a factorization $\mathcal{F}$ of G with a given property randomly r-orthogonal to every H_i for $1 \le i \le t$? The purpose of this paper is to study the above problem, and derive the following result.

Theorem 1.1. Let m, t, r and k_i $(1 \le i \le m)$ be positive integers with $k_i \ge (2r-1)t+1$, G be a bipartite graph with $\Delta(G) \le k_1 + k_2 + \cdots + k_m - m + 1$, and $H_1, H_2, \cdots, H_t$ be t vertex-disjoint mr-subgraphs of G. Then G possesses a $[0, k_i]_1^m$-factorization randomly r-orthogonal to every H_i for $1 \le i \le t$.

If $t = 1$ in Theorem 1.1, then we derive the following corollary.

Corollary 1.1. Let m, r and k_i $(1 \le i \le m)$ be positive integers with $k_i \ge 2r$, G be a bipartite graph with $\Delta(G) \le k_1 + k_2 + \cdots + k_m - m + 1$, and H be an mr-subgraphs of G. Then G possesses a $[0, k_i]_1^m$-factorization randomly r-orthogonal to H.

If $r = 1$ in Theorem 1.1, then we obtain the following corollary.

Corollary 1.2. Let m, t and k_i $(1 \le i \le m)$ be positive integers with $k_i \ge t+1$, G be a bipartite graph with $\Delta(G) \le k_1 + k_2 + \cdots + k_m - m + 1$, and $H_1, H_2, \cdots, H_t$ be t vertex-disjoint m-subgraphs of G. Then G possesses a $[0, k_i]_1^m$-factorization orthogonal to every H_i for $1 \le i \le t$.

In what follows, we provide an example of an orthogonal factorization: Let $m = 2$, $t = 1$ and $k_i = t + 1 = 2$ for $1 \le i \le m$. Let $G = (X, Y, E(G)) = K_{n,n}$, $n = 3$, be a complete bipartite graph where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Let H be a subgraph of G with $V(H) = \{x_1, x_2, y_1, y_2\}$ and $E(H) = \{x_1y_1, x_2y_2\}$. Set $E_1 = \{x_1y_1\}$ and $E_2 = \{x_2y_2\}$. G is a bipartite graph with $\Delta(G) = k_1+k_2+\cdots+k_m-m+1$, where $k_1 = k_2 = \cdots = k_m = t+1$ and $(m, t) = (2, 1)$. We easily see that G has a $[0, 2]$-factorization $\{F_1, F_2\}$ such that $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$, where $F_1 = \{x_1y_1, y_1x_2, x_2y_3, y_3x_3, x_3y_2\}$ and $F_2 = \{x_1y_2, x_1y_3, x_2y_2, x_3y_1\}$. That is to say, G possesses a $[0, k_i]_1^m$-factorization orthogonal to H. Similarly, for any 2-subgraph H' of G, we easily find a $[0, k_i]_1^m$-factorization of G orthogonal to H'.

2 Preliminary Lemmas

Folkman and Fulkerson gave a criterion for a bipartite graph with a (g, f)-factor (see Theorem 6.8 in [1]).

Lemma 2.1. Let $G = (A, B, E(G))$ be a bipartite graph, and $g, f : V(G) \to \mathbb{N} \cup \{0\}$ be two functions with $0 \le g(x) \le f(x)$ for each $x \in V(G)$. Then G admits a (g, f)-factor if and only if

$$\gamma_{1G}(X, Y; g, f) = f(X) + d_{G-X}(Y) - g(Y) \ge 0$$

and

$$\gamma_{2G}(X, Y; g, f) = f(Y) + d_{G-Y}(X) - g(X) \ge 0$$

for any $X \subseteq A$ and $Y \subseteq B$.

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We easily see that $d_{G-Y}(X) = e_G(X, B \setminus Y)$ and $d_{G-X}(Y) = e_G(Y, A \setminus X)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$, and let $X \subseteq A$ and $Y \subseteq B$. Put

$$E_1^{X,B\setminus Y} = |E_1 \cap E_G(X, B \setminus Y)|, \quad E_1^{Y,A\setminus X} = |E_1 \cap E_G(Y, A \setminus X)|$$

$$E_2^{X,B\setminus Y} = |E_2 \cap E_G(X, B \setminus Y)|, \quad E_2^{Y,A\setminus X} = |E_2 \cap E_G(Y, A \setminus X)|$$

Note that $E_1^{X,B\setminus Y} \le d_{G-Y}(X)$, $E_1^{Y,A\setminus X} \le d_{G-X}(Y)$, $E_2^{X,B\setminus Y} \le d_{G-Y}(X)$ and $E_2^{Y,A\setminus X} \le d_{G-X}(Y)$.

Using Lemma 2.1, Liu and Zhu [16] showed a characterization for a bipartite graph to admit a (g, f)-factor including E_1 and excluding E_2, which plays an important role in the proof of our theorem.

Lemma 2.2 (Liu and Zhu [16]). Let $G = (A, B, E(G))$ be a bipartite graph, let $g, f : V(G) \to \mathbb{N} \cup \{0\}$ be two functions with $0 \le g(x) \le f(x)$ for each $x \in V(G)$, and let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G possesses a (g, f)-factor F with $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if

$$\gamma_{1G}(X, Y; g, f) \ge E_1^{X,B\setminus Y} + E_2^{Y,A\setminus X}$$

and

$$\gamma_{2G}(X, Y; g, f) \ge E_1^{Y,A\setminus X} + E_2^{X,B\setminus Y}$$

for any $X \subseteq A$ and $Y \subseteq B$.

3 The Proof of Theorem 1.1

In what follows, we always assume that G is a bipartite graph with $\Delta(G) \le k_1 + k_2 + \cdots + k_m - m + 1$, where m and k_i $(1 \le i \le m)$ are positive integers with $k_i \ge (2r - 1)t + 1$. For every isolated vertex x of G and every $[0, k_i]$-factor F_i, we possess $d_{F_i}(x) = 0$. We denote by I the set of all isolated vertices of G. Obviously, G possesses a $[0, k_i]$-factor if $G - I$ has a $[0, k_i]$-factor. Hence, we may assume that G does not possess isolated vertices. Next, we define

$$p(x) = \max\{0, d_G(x) - (k_1 + k_2 + \cdots + k_{m-1} - m + 2)\}$$

and

$$q(x) = \min\{k_m, d_G(x)\}$$

for any $x \in V(G)$. In light of the definitions of $p(x)$ and $q(x)$, we admit $0 \le p(x) \le q(x)$ for each $x \in V(G)$.

Let $H_1, H_2, \cdots, H_t$ be t vertex-disjoint mr-subgraphs of G. Choose arbitrary $A_i \subseteq E(H_i)$ with $|A_i| = r$ for $1 \le i \le t$. Let $E_1 = \bigcup_{i=1}^{t} A_i$ and $E_2 = \left(\bigcup_{i=1}^{t} E(H_i)\right) \setminus E_1$. Then $|E_1| = rt$ and $|E_2| = (m - 1)rt$.

The proof of Theorem 1.1 depends heavily on the following lemma.

Lemma 3.1. Let m, t, r and k_i $(1 \le i \le m)$ be positive integers with $2 \le m$ and $k_i \ge (2r - 1)t + 1$, $G = (A, B, E(G))$ be a bipartite graph with $\Delta(G) \le k_1 + k_2 + \cdots + k_m - m + 1$. Then G possesses a (p, q)-factor F_m with $E_1 \subseteq E(F_m)$ and $E_2 \cap E(F_m) = \emptyset$, where E_1 and E_2 are defined as the above.

Proof. In light of Lemma 2.2, it suffices to justify that

$$\gamma_{1G}(X', Y'; p, q) \geq E_1^{X', B\setminus Y'} + E_2^{Y', A\setminus X'}$$

and

$$\gamma_{2G}(X', Y'; p, q) \geq E_1^{Y', A\setminus X'} + E_2^{X', B\setminus Y'}$$

for any $X' \subseteq A$ and $Y' \subseteq B$. We justify only the first inequality. The second one can be justi1ed similarly.

We now choose two subsets $X \subseteq A$ and $Y \subseteq B$ such that
(a) $\gamma_{1G}(X, Y; p, q) - E_1^{X, B\setminus Y} - E_2^{Y, A\setminus X}$ is minimum;
(b) $|X|$ is minimum subject to (a).
By the definition of $E_1^{X,B\setminus Y}$, $E_1^{Y,A\setminus X}$, $E_2^{X,B\setminus Y}$ and $E_2^{Y,A\setminus X}$, we derive

$$E_1^{X,B\setminus Y} \leq \min\{rt, r|X|\}, \qquad E_2^{Y,A\setminus X} \leq \min\{(m-1)rt, (m-1)r|Y|\},$$

$$E_1^{Y,A\setminus X} \leq \min\{rt, r|Y|\}, \qquad E_2^{X,B\setminus Y} \leq \min\{(m-1)rt, (m-1)r|X|\}\}.$$

Claim 1. If $X \neq \emptyset$, then $q(x) \leq d_G(x) - 1$ for each $x \in X$, and so $q(x) = k_m$ for each $x \in X$.
Proof. Let $X_1 = \{x \in X : q(x) \geq d_G(x)\}$. Next, we justify $X_1 = \emptyset$.

On the contrary, we let $X_1 \neq \emptyset$. Write $X_0 = X \setminus X_1$. Hence, we derive

$$\begin{aligned}\gamma_{1G}(X, Y; p, q) &= q(X) + d_{G-X}(Y) - p(Y)\\ &= q(X_0) + q(X_1) + d_{G-X_0}(Y) - e_G(X_1, Y) - p(Y)\\ &\geq q(X_0) + d_{G-X_0}(Y) - p(Y) + d_G(X_1) - e_G(X_1, Y)\\ &= \gamma_{1G}(X_0, Y; p, q) + d_{G-Y}(X_1). \end{aligned} \tag{3.1}$$

Note that

$$E_1^{X,B\setminus Y} + E_2^{Y,A\setminus X} \leq E_1^{X_0,B\setminus Y} + E_2^{Y,A\setminus X_0} + E_1^{X_1,B\setminus Y} \tag{3.2}$$

and

$$d_{G-Y}(X_1) \geq E_1^{X_1,B\setminus Y}. \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} & \gamma_{1G}(X, Y; p, q) - E_1^{X,B\setminus Y} - E_2^{Y,A\setminus X}\\ \geq\ & \gamma_{1G}(X_0, Y; p, q) + d_{G-Y}(X_1) - E_1^{X_0,B\setminus Y} - E_2^{Y,A\setminus X_0} - E_1^{X_1,B\setminus Y}\\ \geq\ & \gamma_{1G}(X_0, Y; p, q) - E_1^{X_0,B\setminus Y} - E_2^{Y,A\setminus X_0}, \end{aligned}$$

which contradicts the choice of X (See condition (b)). Thus, we admit $X_1 = \emptyset$, and so if $X \neq \emptyset$, then $q(x) \leq d_G(x) - 1$ for each $x \in X$. Combining this with the definition of $q(x)$, we admit $q(x) = k_m$ for each $x \in X$ if $X \neq \emptyset$. This completes the proof of Claim 1. □

Next, we let $d = k_1 + k_2 + \cdots + k_{m-1} - m + 2$, $Y_1 = \{x : d_G(x) - d \geq 1, x \in Y\}$ and $Y_0 = Y \setminus Y_1$. By the definition of $p(x)$, it is obvious that

$$p(x) = 0 \tag{3.4}$$

for any $x \in Y_0$, and

$$p(x) = d_G(x) - d \tag{3.5}$$

for any $x \in Y_1$. By the definition of $E_2^{Y,A\setminus X}$, we have

$$E_2^{Y_0,A\setminus X} + E_2^{Y_1,A\setminus X} = E_2^{Y,A\setminus X}. \tag{3.6}$$

From Claim 1, we easily see that $q(X) = k_m|X|$ for those $X \subseteq A$ that satisfy conditions (a) and (b). If $Y_1 = \emptyset$, then by (3.4), $E_1^{X,B\setminus Y} \le \min\{rt, r|X|\} \le r|X|$, $E_2^{Y,A\setminus X} \le d_{G-X}(Y)$ and $k_m \ge (2r-1)t+1$ we derive

$$\begin{aligned}
\gamma_{1G}(X,Y;p,q) &= q(X) + d_{G-X}(Y) - p(Y) \\
&= k_m|X| + d_{G-X}(Y) - p(Y_0) - p(Y_1) \\
&= k_m|X| + d_{G-X}(Y) \\
&\ge ((2r-1)t+1)|X| + d_{G-X}(Y) \\
&\ge r|X| + d_{G-X}(Y) \\
&\ge E_1^{X,B\setminus Y} + E_2^{Y,A\setminus X}.
\end{aligned}$$

If $X = \emptyset$, then $E_1^{X,B\setminus Y} = 0$. Using (3.4), (3.5), (3.6), $k_i \ge (2r-1)t+1$ $(1 \le i \le m)$, $2 \le m$ and $d_G(Y_0) = d_{G-X}(Y_0) \ge E_2^{Y_0,A\setminus X}$, we admit

$$\begin{aligned}
\gamma_{1G}(X,Y;p,q) &= q(X) + d_{G-X}(Y) - p(Y) \\
&= d_G(Y_0) + d_G(Y_1) - p(Y_0) - p(Y_1) \\
&= d_G(Y_0) + d_G(Y_1) - p(Y_1) \\
&= d_G(Y_0) + d_G(Y_1) - (d_G(Y_1) - d|Y_1|) \\
&= d_G(Y_0) + d|Y_1| \\
&= d_G(Y_0) + (k_1 + k_2 + \cdots + k_{m-1} - m + 2)|Y_1| \\
&\ge d_G(Y_0) + ((m-1)((2r-1)t+1) - m + 2)|Y_1| \\
&= d_G(Y_0) + ((m-1)(2r-1)t+1)|Y_1| \\
&\ge d_G(Y_0) + (m-1)r|Y_1| \\
&\ge E_2^{Y_0,A\setminus X} + E_2^{Y_1,A\setminus X} \\
&= E_2^{Y,A\setminus X} \\
&= E_1^{X,B\setminus Y} + E_2^{Y,A\setminus X}.
\end{aligned}$$

Next, we always assume that $X \ne \emptyset$ and $Y_1 \ne \emptyset$. The following proof will be divided into two cases.

Case 1. $|X| \ge |Y_1|$.

Since G is a graph with $\Delta(G) \le k_1 + k_2 + \cdots + k_m - m + 1$, we derive $d_G(Y_1) \le (k_1 + k_2 + \cdots + k_m - m + 1)|Y_1| = (d + k_m - 1)|Y_1|$. Combining this with (3.4), (3.5) and Claim 1, we admit

$$\gamma_{1G}(X,Y;p,q) = q(X) + d_{G-X}(Y) - p(Y)$$

$$
\begin{aligned}
&= q(X) + d_{G-X}(Y) - p(Y_0) - p(Y_1) \\
&= k_m|X| + d_{G-X}(Y) - p(Y_1) \\
&= k_m|X| + d_{G-X}(Y) + d|Y_1| - d_G(Y_1) \\
&= k_m(|X| - |Y_1|) + d_{G-X}(Y) + (d + k_m)|Y_1| - d_G(Y_1) \\
&\geq k_m(|X| - |Y_1|) + d_{G-X}(Y) + d_G(Y_1) + |Y_1| - d_G(Y_1) \\
&= k_m(|X| - |Y_1|) + |Y_1| + d_{G-X}(Y) \\
&= (k_m - 1)(|X| - |Y_1|) + |X| + d_{G-X}(Y). \qquad (3.7)
\end{aligned}
$$

Subcase 1.1. $|X| \geq rt$.

Note that $E_1^{X,B\setminus Y} \leq \min\{rt, r|X|\} \leq rt$ and $d_{G-X}(Y) \geq E_2^{Y,A\setminus X}$. By (3.7), $|X| \geq |Y_1|$ and $k_m \geq (2r-1)t + 1$, we obtain

$$
\begin{aligned}
\gamma_{1G}(X, Y; p, q) &\geq (k_m - 1)(|X| - |Y_1|) + |X| + d_{G-X}(Y) \\
&\geq |X| + d_{G-X}(Y) \\
&\geq rt + d_{G-X}(Y) \\
&\geq E_1^{X,B\setminus Y} + E_2^{Y,A\setminus X}.
\end{aligned}
$$

Subcase 1.2. $|X| \leq rt - 1$.

Note that $Y_1 \neq \emptyset$. Hence, $|Y_1| \geq 1$. Next, we shall consider two cases.

Subcase 1.2.1. $|Y_1| = 1$.

Let $Y_1 = \{y\}$. Note that $E_1^{X,B\setminus Y} \leq \min\{rt, r|X|\} \leq r|X|$, $E_2^{Y,A\setminus X} \leq \min\{(m-1)rt, (m-1)r|Y|\} \leq (m-1)r|Y|$ and $d_{G-X}(Y) \geq E_2^{Y,A\setminus X}$. According to (3.5), (3.6), (3.7), $X \neq \emptyset$, $2 \leq m$ and $k_i \geq (2r-1)t + 1$ $(1 \leq i \leq m)$, we get

$$
\begin{aligned}
\gamma_{1G}(X, Y; p, q) &\geq (k_m - 1)(|X| - |Y_1|) + |X| + d_{G-X}(Y) \\
&= (k_m - 1)(|X| - 1) + |X| + d_{G-X}(Y_1) + d_{G-X}(Y_0) \\
&= (k_m - 1)(|X| - 1) + |X| + d_{G-X}(y) + d_{G-X}(Y_0) \\
&\geq (k_m - 1)(|X| - 1) + d_G(y) + d_{G-X}(Y_0) \\
&\geq (k_m - 1)(|X| - 1) + d + 1 + d_{G-X}(Y_0) \\
&= (k_m - 1)(|X| - 1) + k_1 + k_2 + \cdots + k_{m-1} - m + 3 + d_{G-X}(Y_0) \\
&\geq (2r - 1)t(|X| - 1) + (m - 1)((2r - 1)t + 1) - m + 3 + d_{G-X}(Y_0) \\
&\geq r(|X| - 1) + (m - 1)((2r - 1) + 1) - m + 3 + d_{G-X}(Y_0) \\
&= r(|X| - 1) + (m - 1)r + (m - 1)(r - 1) + 2 + d_{G-X}(Y_0) \\
&\geq r(|X| - 1) + (m - 1)r + r + 1 + d_{G-X}(Y_0) \\
&> r|X| + (m - 1)r + d_{G-X}(Y_0) \\
&= r|X| + (m - 1)r|Y_1| + d_{G-X}(Y_0) \\
&\geq E_1^{X,B\setminus Y} + E_2^{Y_1,A\setminus X} + E_2^{Y_0,A\setminus X} \\
&= E_1^{X,B\setminus Y} + E_2^{Y,A\setminus X}.
\end{aligned}
$$

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Subcase 1.2.2. $|Y_1| \geq 2$.

If $r = 1$, then $E_1^{X,B\setminus Y} \leq \min\{t, |X|\} \leq |X|$. Note that $d_{G-X}(Y) \geq E_2^{Y,A\setminus X}$. In light of (3.7) and $|X| \geq |Y_1|$, we derive

$$\begin{aligned}
\gamma_{1G}(X,Y;p,q) &\geq (k_m-1)(|X|-|Y_1|)+|X|+d_{G-X}(Y) \\
&\geq |X|+d_{G-X}(Y) \\
&= E_1^{X,B\setminus Y}+E_2^{Y,A\setminus X}.
\end{aligned}$$

In the following, we consider $r \geq 2$. Note that $E_1^{X,B\setminus Y} \leq \min\{rt, r|X|\} \leq rt$ and $E_2^{Y,A\setminus X} \leq \min\{(m-1)rt, (m-1)r|Y|\} \leq (m-1)rt$. Since $|Y_1| \geq 2$, there exist $y_1, y_2 \in Y_1$. In terms of (3.5), (3.7), $|X| \geq |Y_1|$, $|X| \leq rt-1$, $2 \leq m$ and $k_i \geq (2r-1)t+1$ $(1 \leq i \leq m)$, we have

$$\begin{aligned}
\gamma_{1G}(X,Y;p,q) &\geq (k_m-1)(|X|-|Y_1|)+|X|+d_{G-X}(Y) \\
&\geq |X|+d_{G-X}(Y_1) \\
&\geq 2|X|+d_{G-X}(Y_1)-(rt-1) \\
&\geq 2|X|+d_{G-X}(y_1)+d_{G-X}(y_2)-(rt-1) \\
&\geq d_G(y_1)+d_G(y_2)-rt+1 \\
&\geq 2(d+1)-rt+1 \\
&> 2d-rt \\
&= 2(k_1+k_2+\cdots+k_{m-1}-m+2)-rt \\
&\geq 2((m-1)((2r-1)t+1)-m+2)-rt \\
&> (2m-2)(2r-1)t-rt \\
&\geq m(2r-1)t-rt \\
&= mrt+m(r-1)t-rt \\
&\geq mrt \\
&= rt+(m-1)rt \\
&\geq E_1^{X,B\setminus Y}+E_2^{Y,A\setminus X}.
\end{aligned}$$

Case 2. $|X| \leq |Y_1|-1$.

Since G is a graph with $\Delta(G) \leq k_1+k_2+\cdots+k_m-m+1$, we possess $d_G(X) \leq (k_1+k_2+\cdots+k_m-m+1)|X| = (d+k_m-1)|X|$. Note that $d_{G-Y}(X) \geq E_1^{X,B\setminus Y}$ and $E_2^{Y,A\setminus X} \leq \min\{(m-1)rt, (m-1)r|Y|\} \leq (m-1)rt$. By (3.4), (3.5), Claim 1, $2 \leq m$ and $k_i \geq (2r-1)t+1$ $(1 \leq i \leq m)$, we get

$$\begin{aligned}
\gamma_{1G}(X,Y;p,q) &= q(X)+d_{G-X}(Y)-p(Y) \\
&= q(X)+d_G(Y)-e_G(X,Y)-p(Y_0)-p(Y_1) \\
&= k_m|X|+d_G(Y)-e_G(X,Y)-p(Y_1) \\
&= k_m|X|+d_G(Y)-e_G(X,Y)+d|Y_1|-d_G(Y_1) \\
&\geq k_m|X|-e_G(X,Y)+d|Y_1| \\
&= (d+k_m)|X|-e_G(X,Y)+d(|Y_1|-|X|)
\end{aligned}$$

$$
\begin{aligned}
&\geq d_G(X)+|X|-e_G(X,Y)+d \\
&= d_{G-Y}(X)+|X|+k_1+k_2+\cdots+k_{m-1}-m+2 \\
&\geq d_{G-Y}(X)+|X|+(m-1)((2r-1)t+1)-m+2 \\
&= d_{G-Y}(X)+|X|+(m-1)(2r-1)t+1 \\
&\geq d_{G-Y}(X)+|X|+(m-1)rt+1 \\
&> d_{G-Y}(X)+(m-1)rt \\
&\geq E_1^{X,B\setminus Y}+E_2^{Y,A\setminus X}.
\end{aligned}
$$

In conclusion, $\gamma_{1G}(X,Y;p,q)\geq E_1^{X,B\setminus Y}+E_2^{Y,A\setminus X}$. In terms of the choice of X and Y, we possess $\gamma_{1G}(X',Y';p,q)\geq E_1^{X',B\setminus Y'}+E_2^{Y',A\setminus X'}$ for any $X'\subseteq A$ and $Y'\subseteq B$. It follows from Lemma 2.2 that G admits a (p,q)-factor F_m with $E_1\subseteq E(F_m)$ and $E_2\cap E(F_m)=\emptyset$. This finishes the proof of Lemma 3.1. □

Proof of Theorem 1.1. We verify Theorem 1.1 by induction on m and n. Obviously, Theorem 1.1 is true when $m=1$. Therefore, we may assume that $m\geq 2$ in the following. For the inductive step, let Theorem 1.1 be true for arbitrary bipartite graph G' with $\Delta(G')\leq k_1+k_2+\cdots+k_{m'}-m'+1$ and $1\leq m'<m$, and arbitrary t vertex-disjoint $m'r$-subgraphs $H_1',H_2',\cdots,H_t'$ of G'. Next, we discuss a bipartite graph G with $\Delta(G)\leq k_1+k_2+\cdots+k_m-m+1$ and arbitrary t vertex-disjoint mr-subgraphs $H_1,H_2,\cdots,H_t$ of G.

We select any $A_{i,m}\subseteq E(H_i)$ with $|A_{i,m}|=r$ for $1\leq i\leq t$. Write $E_1=\bigcup_{i=1}^{t}A_{i,m}$ and $E_2=\left(\bigcup_{i=1}^{t}E(H_i)\right)\setminus E_1$. In terms of Lemma 3.1, G admits a (p,q)-factor F_m with $E_1\subseteq E(F_m)$ and $E_2\cap E(F_m)=\emptyset$. Obviously, F_m is also a $[0,k_m]$-factor of G. Set $G'=G-E(F_m)$. By the definition of $p(x)$, we derive

$$
\begin{aligned}
0\leq d_{G'}(x) &= d_G(x)-d_{F_m}(x)\leq d_G(x)-p(x) \\
&\leq d_G(x)-(d_G(x)-(k_1+k_2+\cdots+k_{m-1}-m+2)) \\
&= k_1+k_2+\cdots+k_{m-1}-(m-1)+1
\end{aligned}
$$

for any $x\in V(G)$. And so G' is a bipartite graph with $\Delta(G')\leq k_1+k_2+\cdots+k_{m-1}-(m-1)+1$. Write $H_i'=H_i-A_{i,m}$ for $1\leq i\leq t$. It is obvious that $H_1',H_2',\cdots,H_t'$ are t vertex-disjoint $(m-1)r$-subgraphs of G'. By the induction hypothesis, G' possesses a $[0,k_i]_1^{m-1}$-factorization randomly r-orthogonal to every H_i', $1\leq i\leq t$. Hence, G admits a $[0,k_i]_1^m$-factorization randomly r-orthogonal to every H_i, $1\leq i\leq t$. We complete the proof of Theorem 1.1. □

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