Upper and Lower Solutions Methods for Impulsive Caputo-Hadamard Fractional Differential Inclusions

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Abstract

In this paper, we investigate the existence of solutions for a class of initial value problems for impulsive Caputo-Hadamard fractional differential inclusions of order $r \in (1,2]$. We apply the concept of lower and upper solutions combined with the fixed point theorem of Bohnnenblust-Karlin.

Key words and phrases: Initial value problem, fractional differential inclusion, impulsive, Caputo-Hadamard fractional derivative, fractional integral, fixed point theorem, upper and lower solutions.
AMS (MOS) Subject Classifications: 26A33, 34A08, 34A37

1 Introduction

This paper deals with the existence of solutions for the initial value problem (IVP for short), for the fractional order differential inclusion with impulses,

$$^{CH}D^r y(t) \in F(t, y(t)), \text{ for a.e., } t \in J = [a, T], t \neq t_k, \ k = 1, ..., m, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, ..., m, \tag{2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, ..., m, \tag{3}$$

$$y(a) = y_1, y'(a) = y_2, \tag{4}$$

where $^{CH}D^r$ is the Caputo-Hadamard fractional derivative $1 < r \leq 2$, $a > 0$, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I_k \, \overline{I_k} : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$, are continuous functions, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, $y(t_k^+) = \lim_{\varepsilon \to 0^+} y(t_k + \varepsilon)$

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and $y(t_k^-) = \lim_{\varepsilon \to 0^-} y(t_k - \varepsilon)$ represent the right and left limits of y at $t = t_k$, $k = 1, \ldots, m$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. There has been a significant development in theory of fractional calculus and fractional ordinary and partial differential equations in recent years; see e.g., the monographs of Hilfer [23], Kilbas *et al.* [25], Podlubny [31], Momani *et al.* [30], and the papers by Agarwal *et al.* [2] and Benchohra *et al.* [8]. Applied problems require the definitions of fractional derivatives allowing the utilization of physically interpretable initial data, which contain $y(0)$, $y'(0)$, and so on. Caputo's fractional derivative satisfies these demands. For more details concerning geometric and physical interpretation of fractional derivatives of Riemann-Liouville type and Caputo type, see [31, 32, 22].

However, the literature on Hadamard-type fractional differential equations has not undergone as much development; see [4, 34]. The fractional derivative that Hadamard [17] introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent. Detailed descriptions of the Hadamard fractional derivative and integral can be found in [11, 12, 13]. Recently, Hadamard fractional calculus is attracting attention which is important to the theory of fractional calculus [25]. The works in [4, 11, 12, 13, 24, 27, 34] are fundamental in the development of Hadamard fractional calculus. A Caputo-type modification of the Hadamard fractional derivative which is called the Caputo-Hadamard fractional derivative was given in [16], and its fundamental theorems were proved in [15, 1].

Impulsive differential equations (for $r \in \mathbb{N}$) have become important in recent years as mathematical models of phenomena in both physical and social sciences. There has been a significant development in impulsive theory, especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Ait Dads *et al* [3], Bainov and Simeonov [5], Belhannache *et al* [6] , Benchohra *et al* [8, 10], Lakshmikantham *et al*[28], and Samoilenko and Perestyuk [33], Hammou *et al*[18, 19, 20] and the references therein. In [9], Benchohra and Slimani initiated the study of fractional differential equations with impulses.

The method of upper and lower solutions plays an important role in the investigation of solutions for differential equations and inclusions. See the monographs by Benchohra et *al* [10], Heikkila and Lakshmikantham [21], Ladde *et al* [29] and the references therein.

By means of the concept of lower and upper solutions combined with the fixed point theorem of Bohnenblust-Karlin, we show the existence of solutions for the problem (1)-(4). This paper initiates the applications of the upper and lower solutions method for impulsive fractional differential inclusions involving the Caputo-Hadamard fractional derivative at fixed moments of impulse.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used in the remainder of this paper.

Let $[a,b]$ be a compact interval, and $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : a \leq t \leq b\},$$

and we denote by $L^1([a,b],\mathbb{R})$ the Banach space of functions $y : [a,b] \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_a^b |y(t)|dt.$$

Let $AC([a,b],\mathbb{R})$ be a space of functions $y : [a,b] \to \mathbb{R}$, which are absolutely continuous.
Let $(X, \|\cdot\|)$ be a Banach space.
Let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$
$P_{cl,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed and convex}\}$
and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \underset{x\in B}{\cup} G(x)$ is bounded in X for all $B \in P_b(X)$, i.e.

$$\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty.$$

G is called upper semi-continuous (u.s.c.) on X, if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denote by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.
Let $G : X \to \mathcal{P}(X)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in X.

Definition 2.1 *A function $F : [a, b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1- Carathéodory if*

(1) $t \to F(t, u)$ is measurable for each $u \in \mathbb{R}$;

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in [a, b]$;

(3) for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{\|v\| \ : v \in F(t, u)\} \leq \varphi_q \text{ for all } \|u\| \leq q \text{ and for a.e. } t \in J.$$

For each $y \in C([a, b], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1([a, b], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [a, b]\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$.
Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [26]).

In the sequel, we need the following fixed point theorem:

Theorem 2.2 *(Bohnenblust-Karlin)[14] Let X be a Banach space and $K \in P_{cl,c}(X)$, and suppose that the operator $G : K \to P_{cl,c}(X)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

Let us recall some definitions and properties of Hadamard fractional integration and differentiation.
Let $AC^n_\delta[a, b] = \{g : [a, b] \to \mathbb{R} \mid \delta^{n-1}g \in AC[a, b]\}$ where $\delta = t\frac{d}{dt}$, $0 < a < b < \infty$.

Definition 2.3 *[25] The Hadamard fractional integral of order $r > 0$ for a function $h \in L^1([1, +\infty), \mathbb{R})$ is defined as*

$${}^H I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds,$$

provided the integral exists for a.e. $t > 1$.

Example 2.4 *Let $q > 0$. Then*

$$^{H}I_1^q \ln t = \frac{1}{\Gamma(2+q)}(\ln t)^{1+q};\ \textit{for a.e. } t \in [1,+\infty).$$

Definition 2.5 *[25] The Hadamard fractional derivative of order $r > 0$ applied to the function $h \in AC_\delta^n([1,+\infty),\mathbb{R})$ is defined as*

$$(^{H}D_1^q h)(t) = \delta^n(^{H}I_1^{n-r}h)(t),$$

where $n-1 < r < n,\ n = [r]+1,$ and $[r]$ is the integer part of r.

Definition 2.6 *[16] For a given function $h \in AC_\delta^n([a,b],\mathbb{R})$, such that $0 < a < b$, the Caputo-Hadamard fractional derivative of order $r > 0$ is defined as follows:*

$$^{Hc}D^r y(t) =^{H} D^r \left[y(s) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!}\left(\log\frac{s}{a}\right)^k \right](t),$$

where $Re(\alpha) \geq 0$ and $n = [Re(\alpha)]+1$.

Lemma 2.7 *[16] Let $y \in AC_\delta^n([a,b],\mathbb{R})$ or $C_\delta^n([a,b],\mathbb{R})$ and Then*

$$^{H}I^r(^{Hc}D^r y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!}\left(\log\frac{t}{a}\right)^k.$$

Definition 2.8 *[25]. The Hadamard fractional integral of order r for a function $h : [a,b] \to \mathbb{R}$, where $a, b \geq 0$, is defined by*

$$I_a^r h(t) = \frac{1}{\Gamma(r)} \int_a^t \left(\log\frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds,\ r > 0,$$

provided the integral exists.

Definition 2.9 *[16]. For a function $g \in AC_\delta^n[a,b]$ the Caputo type Hadamard derivative of fractional order r is defined as follows*

(i) *If $r \notin \mathbb{N}$, then for $n-1 < [Re(r)] < n$, where $[Re(r)]$ denotes the integer part of $Re(r)$,*

$$(^{CH}D_a^r g)(t) = \frac{1}{\Gamma(n-r)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-r-1} \delta^n g(s)\frac{ds}{s}.$$

(ii) *If $r \in \mathbb{N}$, then $(^{CH}D_a^r g)(t) = \delta^n g(t)$,*

where $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.10 *[16] Let $y \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$,*
Then

$$I_a^r(^{CH}D_a^r y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!}\left(\log\frac{t}{a}\right)^k. \tag{5}$$

3 Main results

Consider the following space

$$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} \mid y \in C^2_\delta((t_k,t_{k+1}],\mathbb{R}),\ k = 0,\dots,m+1,$$

$$\text{there exist}\, y(t_k^+)\,\text{and}\, y(t_k^-),\ k = 1,\dots,m,\ \text{with}\, y(t_k^-) = y(t_k)\}.$$

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in J}|y(t)|.$$

Set $J' = J \setminus \{t_1,...,t_m\}$.

Definition 3.1 *A function $y \in PC(J,\mathbb{R}) \bigcap \bigcup_{k=0}^m AC^2_\delta((t_k,t_{k+1}],\mathbb{R})$ is said to be a solution of (1)-(4) if there exists a function $v \in L^1([a,T],\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. $t \in J$, for which ${}^{CH}D^r y(t) = v(t)$ on J', and y also satisfies the impulsive conditions (2)-(4).*

Definition 3.2 *A function $u \in PC(J,\mathbb{R}) \bigcap \bigcup_{k=0}^m AC^2_\delta((t_k,t_{k+1}],\mathbb{R})$ is said to be a lower solution of (1)-(4) if there exists a function $v_1 \in L^1([a,T],\mathbb{R})$ such that $v_1(t) \in F(t,u(t))$ a.e. $t \in J$, for which ${}^{CH}D^\alpha u(t) \le v_1(t)$ on J', and u also satisfies the conditions $\Delta u|_{t=t_k} \le I_k(u(t_k^-))$, $\Delta u|_{t=t_k} \le \overline{I_k}(u(t_k^-))$, $k = 1,\dots,m$, and $u(a) \le y_1$, $u'(a) \le y_2$.*

Similarly, a function $w \in PC(J,\mathbb{R}) \bigcap \bigcup_{k=0}^m AC^2_\delta((t_k,t_{k+1}],\mathbb{R})$ is said to be an upper solution of(1)-(4) if there exists a function $v_2 \in L^1([a,T],\mathbb{R})$ such that $v_2(t) \in F(t,w(t))$ a.e. $t \in J$, for which ${}^{CH}D^r w(t) \ge v_2(t)$ on J' and w also satisfies the conditions $\Delta w|_{t=t_k} \ge I_k(w(t_k^-)), \Delta w|_{t=t_k} \ge \overline{I_k}(w(t_k^-))$, $k = 1,\dots,m$, and $w(a) \ge y_1$, $w'(a) \ge y_2$.

To prove the existence the solution to (1)-(4), we need the following auxiliary lemma.

Lemma 3.3 *Let $1 < r \le 2$ and let $\rho \in AC(J',\mathbb{R})$. A function y is a solution of the* *fractional integral equation*

$$y(t)=\begin{cases} y_1+ay_2\log\left(\frac{t}{a}\right)+\frac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}\nu(s)\frac{ds}{s}, & \text{if } t\in[a,t_1],\\ y_1+ay_2\log\left(\frac{t}{a}\right)+\sum_{k=1}^{m}\frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-2}\nu(s)\frac{ds}{s} & \\ +\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\nu(s)\frac{ds}{s} & \\ +\sum_{k=1}^{m}I_k(y(t_k^-)) & \\ +\sum_{k=1}^{m}t_k\log\left(\frac{t}{t_k}\right)\overline{I}_k(y(t_k^-)), & \text{if } t\in(t_k,t_{k+1}], k=1,\dots,m, \end{cases} \tag{6}$$

if and only if y *is a solution of the fractional IVP*

$$^{CH}D_a^r y(t)=\nu(t),\ \text{for each},\ t\in J^{'}, \tag{7}$$

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)),\quad k=1,...,m, \tag{8}$$

$$\Delta y'|_{t=t_k}=\overline{I}_k(y(t_k^-)),\quad k=1,...,m, \tag{9}$$

$$y(a)=y_1, y'(a)=y_2. \tag{10}$$

Proof: Let y be a solution of (7)-(10). Applying the Hadamard fractional integral of order r to both sides of (7), using conditions (8)-(10) and Lemma 2.10 we get:

If $t\in[a,t_1]$

$$y(t)=c_1+c_2\log\left(\frac{t}{a}\right)+\frac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}\nu(s)\frac{ds}{s}.$$

Hence $c_1=y(a)=y_1$ *and* $c_2=y'(a)=ay_2$, and

$$y(t)=y_1+ay_2\log\left(\frac{t}{a}\right)+\frac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}\nu(s)\frac{ds}{s}.$$

If $t\in(t_1,t_2]$

$$y(t)=c_1+c_2\log\left(\frac{t}{a}\right)+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\log\frac{t}{s}\right)^{r-1}\nu(s)\frac{ds}{s}. \tag{11}$$

We have

$$\Delta y|_{t=t_1}=y(t_1^+)-y(t_1^-)$$

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and

$$\begin{aligned} I_1(y(t_1^-)) &= c_1 + c_2 \log\left(\frac{t_1}{a}\right) \\ &- \left(y_1 + a y_2 \log\left(\frac{t_1}{a}\right) + \frac{1}{\Gamma(r)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{r-1}\nu(s)\frac{ds}{s}\right). \end{aligned}$$

Hence

$$c_1 + c_2\log\left(\frac{t_1}{a}\right) = y_1 + a y_2\log\left(\frac{t_1}{a}\right) + \frac{1}{\Gamma(r)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{r-1}\nu(s)\frac{ds}{s} + I_1(y(t_1^-).$$

Also, we have

$$\Delta y'|_{t=t_1} = y'(t_1^+) - y'(t_1^-),$$

and

$$\begin{aligned} \overline{I}_1(y(t_1^-)) &= \frac{c_2}{t_1} - \left(\frac{a}{t_1}\right)y_2 \\ &+ \frac{1}{t_1\Gamma(r-1)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{r-2}\nu(s)\frac{ds}{s}. \end{aligned}$$

Hence

$$c_2 = a y_2 + \frac{1}{t_1\Gamma(r-1)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{r-2}\nu(s)\frac{ds}{s} + \overline{I}_1(y(t_1^-)) \tag{12}$$

and

$$\begin{aligned} c_1 &= y_1 - \frac{\log\left(\frac{t_1}{a}\right)}{t_1\Gamma(r-1)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{r-2}\nu(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{r-1}\nu(s)\frac{ds}{s} + I_1(y(t_1^-)) - t_1\log\left(\frac{t_1}{a}\right)\overline{I}_1(y(t_1^-). \end{aligned} \tag{13}$$

Then by (12)-(13) and (11), we have

$$\begin{aligned} y(t) &= y_1 + a y_2\log\left(\frac{t}{a}\right) + \frac{\log\left(\frac{t}{t_1}\right)}{t_1\Gamma(r-1)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{r-2}\nu(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\log\frac{t}{s}\right)^{r-1}\nu(s)\frac{ds}{s} + I_1(y(t_1^-)) + t_1\log\left(\frac{t}{t_1}\right)\overline{I}_1(y(t_1^-). \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 2.10 we obtain (6).

Conversely, assume that y satisfies the impulsive fractional integral equation (6). If $t \in [a, t_1]$, then $y(a) = y_1, y'(a) = y_2$, and using that ${}^{CH}D_a^r$ is the left inverse of I_a^r, we get

$${}^{CH}D_a^r y(t) = \nu(t), \text{ for all } t \in [a, t_1].$$

Next, let $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$. We have ${}^{CH}D_a^r C = 0$, for any constant C, and then

$$^{CH}D_a^r y(t) = \nu(t), \text{ for all } t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\begin{aligned} \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ \Delta y'|_{t=t_k} &= \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(a) = y_1&, \ y'(a) = y_2. \end{aligned}$$

□

Theorem 3.4 *Assume the following hypotheses hold:*

(H1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map.*

(H2) *There exist* $u, w \in PC(J, \mathbb{R}) \bigcap AC_\delta^2((t_k, t_{k+1}], \mathbb{R})$, $k = 0, \dots, m$, *lower and upper solutions, respectively, for the problem (1)-(4) such that* $u(t) \leq w(t)$ *for each* $t \in J$.

(H3)

$$u(t_k^+) \leq \min_{y \in [u(t_k^-), w(t_k^-)]} I_k(y) \leq \max_{y \in [u(t_k^-), w(t_k^-)]} I_k(y) \leq w(t_k^+), \ k = 1, \dots, m.$$

(H4) *There exist* $u', w' \in PC(J, \mathbb{R}) \bigcap AC_\delta^1((t_k, t_{k+1}], \mathbb{R})$, $k = 0, \dots, m$, *lower and upper solutions, respectively, for the problem (1)-(4) such that* $u'(t) \leq w'(t)$ *for each* $t \in J$.

(H5)

$$u'(t_k^+) \leq \min_{y \in [u'(t_k^-), w'(t_k^-)]} \overline{I_k}(y) \leq \max_{y \in [u'(t_k^-), w'(t_k^-)]} \overline{I_k}(y) \leq w'(t_k^+), \ k = 1, \dots, m.$$

(H6) *There exists* $l \in L^1(J, \mathbb{R}^+)$, *such that*

$$H_d(F(t, u), F(t, \bar{u})) \leq l(t)|u - \bar{u}| \qquad \textit{for every } u, \bar{u} \in \mathbb{R},$$

and

$$d(0, F(t, 0)) \leq l(t), \quad \textit{a.e. } t \in J.$$

Then the problem (1)-(4) has at least one solution y *such that*

$$u(t) \leq y(t) \leq w(t) \textit{ for all } t \in J.$$

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Remark 3.5 *(1) $\tau : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is the truncation operator defined by*

$$(\tau y)(t) = \begin{cases} u(t), & y(t) < u(t), \\ y(t), & u(t) \leq y(t) \leq w(t), \\ w(t), & y(t) > w(t). \end{cases} \tag{14}$$

(2) For each $y \in PC(J,\mathbb{R})$, the set $\widetilde{S}^1_{F,\tau y}$ is nonempty. In fact, (H_1) implies that there exists $v_3 \in S^1_{F,\tau y}$, so we set

$$v = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3},$$

where

$$A_3 = \{t \in J \ : u(t) \leq y(t) \leq w(t)\}$$

Then, by decomposability, $v \in \widetilde{S}^1_{F,\tau y}$.

(3) By the definition of τ it is clear that $F(\cdot, \tau y(\cdot))$ is an L^1- Carathéodory multi-valued map with compact convex values and there exists $\phi_1 \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t, \tau y(t))\|_{\mathcal{P}} \leq \phi_1(t) \text{ for each } \ y \in \mathbb{R}.$$

(4) By the definition of τ and from (H3), (H5) we have

$$u(t_k^+) \leq I_k(y(\tau(t_k^-)) \leq w(t_k^+), \ k = 1, \dots, m.$$

and

$$u'(t_k^+) \leq \overline{I}_k(y(\tau(t_k^-)) \leq w'(t_k^+), \ k = 1, \dots, m.$$

Proof: Transform the problem (1)-(4) into a fixed point problem. Consider the following modified problem, for $1 < r \leq 2$ and $a > 0$,

$$^{CH}D^r_a y(t) \in F(t, \tau(y(t))), \ \text{ for a.e. } \ t \in J = [a,T], t \neq t_k, \ k = 1, \dots, m, \tag{15}$$

$$\Delta y|_{t=t_k} = I_k(\tau(t_k^-, y(t_k^-))), \quad k = 1, \dots, m, \tag{16}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(\tau(t_k^-, y(t_k^-))), \quad k = 1, \dots, m, \tag{17}$$

$$y(a) = y_1, y'(a) = y_2, \tag{18}$$

A solution to (15)-(18) is a fixed point of the operator $N : PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ defined by

$$N(y) \ = \ \{h \in PC(J,\mathbb{R}), h(t) = y_1 + ay_2 \log\left(\frac{t}{a}\right)$$

$$
\begin{aligned}
&+ \sum_{k=1}^{m} \frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \nu(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \nu(s)\frac{ds}{s} \\
&+ \sum_{k=1}^{m} I_k(y(\tau(t_k^-)) + \sum_{k=1}^{m} t_k \log\left(\frac{t}{t_k}\right) \overline{I}_k(y(\tau(t_k^-)), \nu \in S_{F,y}\}.
\end{aligned}
$$

Where

$$\nu \in \widetilde{S}^1_{F,\tau y} = \{v \in S^1_{F,\tau y} : v(t) \geq v_1(t) \text{ on } A_1 \text{ and } v(t) \leq v_2(t) \text{ on } A_2\},$$

$$S^1_{F,\tau y} = \{v \in L^1(J,\mathbb{R}) : v \in F(t,(\tau y)(t))\},$$

$$A_1 = \{t \in J : y(t) < u(t) \leq w(t)\},\ A_2 = \{t \in J : u(t) \leq w(t) < y(t)\}.$$

Set

$$
\begin{aligned}
R &= |y_1| + |y_2|\log T + \frac{(\log T)^r \|\phi_1\|_{L_1}}{\Gamma(r)} \sum_{k=1}^{m} \left(\log \frac{t_k}{t_{k-1}}\right)^{r-1} + \frac{\|\phi_1\|_{L_1}}{\Gamma(r+1)} \left(\log \frac{T}{a}\right)^{r} \\
&+ \textstyle\sum_{k=1}^{m} \max\{|u(t_k^+)|, |w(t_k^+|\} + T\log T \sum_{k=1}^{m} \max\{|u'(t_k^+)|, |w'(t_k^+|\}.
\end{aligned}
$$

and

$$B = \{y \in PC(J,\mathbb{R}) : \|y\|_{PC} \leq R\}.$$

Clearly B is a closed convex subset of $PC(J,\mathbb{R})$ and that N maps B into B. We shall show that B satisfies the assumption of Theorem 2.2.The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in B$.*

Let h_1, h_2 belong to $N(y)$. Then there exist $\nu_1, \nu_2 \in \widetilde{S}^1_{F,y}$ such that for each $t \in J$ and for each $i = 1, 2$, we have

$$
\begin{aligned}
h_i(t) &= y_1 + y_2 \log\left(\frac{t}{a}\right) \\
&+ \sum_{a<t_k<t} \frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \nu_i(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \nu_i(s)\frac{ds}{s} \\
&+ \sum_{a<t_k<t} I_k(y(\tau(t_k^-)) + \sum_{a<t_k<t} t_k \log\left(\frac{t}{t_k}\right) \overline{I}_k(y(\tau(t_k^-)).
\end{aligned}
$$

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Let $0 \leq \lambda \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned}
(\lambda h_1 + (1-\lambda)h_2)(t) &= y_1 + y_2 \log\left(\frac{t}{a}\right) \\
&+ \sum_{a<t_k<t} \frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log\frac{t_k}{s}\right)^{r-2} (\lambda\nu_1 + (1-\lambda)\nu_2)(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log\frac{t}{s}\right)^{r-1} (\lambda\nu_1 + (1-\lambda)\nu_2)(s)\frac{ds}{s} \\
&+ \sum_{a<t_k<t} I_k(y(\tau(t_k^-))) + \sum_{a<t_k<t} t_k \log\left(\frac{t}{t_k}\right) \overline{I}_k t_k^-(y(\tau(t_k^-))).
\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\lambda h_1 + (1-\lambda)h_2 \in N(y).$$

Step 2: *N maps bounded sets into bounded sets in B.*

For each $h \in N(y)$, there exists $\nu \in \widetilde{S}^1_{F,y}$ such that

$$\begin{aligned}
h(t) &= y_1 + y_2 \log\left(\frac{t}{a}\right) \\
&+ \sum_{a<t_k<t} \frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log\frac{t_k}{s}\right)^{r-2} \nu(s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log\frac{t}{s}\right)^{r-1} \nu(s)\frac{ds}{s} \\
&+ \sum_{a<t_k<t} I_k(y(\tau(t_k^-))) + \sum_{a<t_k<t} t_k \log\left(\frac{t}{t_k}\right) \overline{I}_k(y(\tau(t_k^-))).
\end{aligned}$$

By (H1)-(H5) , we have, for each $t \in J$

$$
\begin{aligned}
|h(t)| \leq\; & |y_1| + |y_2| \left|\log\left(\frac{t}{a}\right)\right| + \frac{1}{\Gamma(r-1)} \sum_{a<t_k<t} \left| \frac{\log\left(\frac{t}{t_k}\right)}{t_k} \right| \\
& \int_{t_{k-1}}^{t_k} \left|\left(\log\frac{t_k}{s}\right)\right|^{r-2} |\nu(s)| \frac{ds}{s} \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left|\left(\log\frac{t}{s}\right)\right|^{r-1} |\nu(s)| \frac{ds}{s} \\
& + \sum_{a<t_k<t} |I_k(y(\tau(t_k^-))| \\
& + \sum_{a<t_k<t} \left| t_k \log\left(\frac{t}{t_k}\right)\right| |\overline{I}_k(y(\tau(t_k^-))| \\
\leq\; & |y_1| + |y_2| \log T + \frac{(\log T)^r \|\phi_1\|_{L_1}}{\Gamma(r)} \sum_{k=1}^{m} \left(\log\frac{t_k}{t_{k-1}}\right)^{r-1} + \frac{\|\phi_1\|_{L_1}}{\Gamma(r+1)} \left(\log\frac{T}{a}\right)^r \\
+\; & \textstyle\sum_{k=1}^m \max\{|u(t_k^+)|, |w(t_k^+|\} + T\log T \sum_{k=1}^m \max\{|u'(t_k^+)|, |w'(t_k^+|\}.
\end{aligned}
$$

Thus $\|h\|_\infty \leq R$.

Step 3: *N maps bounded sets into equicontinuous sets of B*

Let $\lambda_1, \lambda_2 \in J$, $\lambda_1 < \lambda_2$, Let $y \in B$ and $h \in N(y)$. Then

$$
\begin{aligned}
|h(\lambda_2)-h(\lambda_1)| &\leq \left|y_2\log\left(\frac{\lambda_2}{\lambda_1}\right)\right| \\
&+\frac{\|\phi_1\|_{L_1}}{\Gamma(r-1)}\sum_{0<t_k<(\lambda_2-\lambda_1)}\left|\frac{\log\left(\frac{\lambda_2}{t_k}\right)}{t_k}\right|\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}\frac{ds}{s} \\
&+\frac{\left|\frac{\log\left(\frac{\lambda_2}{\lambda_1}\right)}{s}\right|\|\phi_1\|_{L_1}}{\Gamma(r-1)}\sum_{0<t_k<\lambda_1}\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}\frac{ds}{s} \\
&+\frac{\|\phi_1\|_{L_1}}{\Gamma(r)}\int_{t_k}^{t}\left|\left(\log\frac{t}{s}\right)\right|^{r-1}\frac{ds}{s} \\
&+\frac{\|\phi_1\|_{L_1}}{\Gamma(r)}\int_{t_k}^{\lambda_1}\left[\left(\log\frac{\lambda_2}{s}\right)^{r-1}-\left(\log\frac{\lambda_1}{s}\right)^{r-1}\right]\frac{ds}{s} \\
&+\frac{\|\phi_1\|_{L_1}}{\Gamma(r)}\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{r-1}\frac{ds}{s} \\
&+\sum_{0<t_k<(\lambda_2-\lambda_1)}\|I_k(y(\tau(t_k^-))\| \\
&+\sum_{0<t_k<(\lambda_2-\lambda_1)}|t_k\log\left(\frac{\lambda_2}{t_k}\right)|\|\overline{I}_k(y(\tau(t_k^-))\| \\
&+\log\left(\frac{\lambda_2}{\lambda_1}\right)\sum_{0<t_k<\lambda_1}|t_k|\|\overline{I}_k(y(\tau(t_k^-))\|.
\end{aligned}
$$

As $\tau_1\to\tau_2$, the right hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzel-Ascoli theorem, we can conclude that $N: PC(J,\mathbb{R})\to\mathcal{P}(PC(J,\mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n\to y_*$, $h_n\in N(y_n)$ and $h_n\to h_*$. We need to show that $h_*\in N(y_*)$. $h_n\in N(y_n)$ means that there exists $\nu_n\in\widetilde{S}^1_{F,y_n}$, such that, for each $t\in J$

$$
\begin{aligned}
h_n(t) &= y_1+y_2\log\left(\frac{t}{a}\right) \\
&+ \sum_{a<t_k<t}\frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-2}\nu_n(s)\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \nu_n(s) \frac{ds}{s} \\
&+ \sum_{a<t_k<t} I_k(y(\tau(t_k^-)) + \sum_{a<t_k<t} t_k \log\left(\frac{t}{t_k}\right) \overline{I}_k(y(\tau(t_k^-)).
\end{aligned}
$$

We must show that there exists $\nu_* \in \widetilde{S}^1_{F,y_*}$ such that, for each $t \in J$,

$$
\begin{aligned}
h_*(t) &= y_1 + y_2 \log\left(\frac{t}{a}\right) \\
&+ \sum_{a<t_k<t} \frac{\log\left(\frac{t}{t_k}\right)}{t_k \Gamma(r-1)} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-2} \nu_*(s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \nu_*(s) \frac{ds}{s} \\
&+ \sum_{a<t_k<t} I_k(y(\tau(t_k^-)) + \sum_{a<t_k<t} t_k \log\left(\frac{t}{t_k}\right) \overline{I}_k(y(\tau(t_k^-)).
\end{aligned}
$$

Since $F(t,\cdot)$ is upper semi-continuous, then for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n \geq n_0$, we have

$$\nu_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $\nu_{n_m}(\cdot)$ such that

$$\nu_{n_m}(\cdot) \to \nu_*(\cdot) \ \text{ as } \ m \to \infty,$$

and

$$\nu_*(t) \in F(t, y_*(t)), \ \text{ a.e. } t \in J.$$

For every $w \in F(t, y_*(t))$, we have

$$|\nu_{n_m}(t) - \nu_*(t)| \leq |\nu_{n_m}(t) - w| + |w - \nu_*(t)|.$$

Then

$$|\nu_{n_m}(t) - \nu_*(t)| \leq d(\nu_{n_m}(t), F(t, y_*(t)).$$

We obtain an analogous relation by interchanging the roles of ν_{n_m} and ν_*, and it follows that

$$|\nu_{n_m}(t) - \nu_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned}
|h_{n_m}(t)-h_*(t)| &\leq \frac{1}{\Gamma(r-1)}\sum_{k=1}^{m}\frac{\log\left(\frac{t}{t_k}\right)}{t_k}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-2}|\nu_{n_m}(s)-\nu_*(s)|\frac{ds}{s}\\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}|\nu_{n_m}(s)-\nu_*(s)|\frac{ds}{s}\\
&+ \sum_{k=1}^{m}|I_k(y_{n_m}(\tau(t_k^-))-I_k(y_*(\tau(t_k^-))|\\
&+ \sum_{k=1}^{m}t_k\log\left(\frac{t}{t_k}\right)|\overline{I}_k(y_{n_m}(\tau(t_k^-))-\overline{I}_k(y_*(\tau(t_k^-))|.\\
&\leq \frac{1}{\Gamma(r-1)}\sum_{k=1}^{m}\frac{\log\left(\frac{t}{t_k}\right)}{t_k}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-2}l(s)\|y_{nm}-y_*\|_\infty\frac{ds}{s}\\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}l(s)\|y_{nm}-y_*\|_\infty\frac{ds}{s}\\
&+ \sum_{k=1}^{m}|I_k(y_{n_m}(\tau(t_k^-))-I_k(y_*(\tau(t_k^-))|\\
&+ \sum_{k=1}^{m}t_k\log\left(\frac{t}{t_k}\right)|\overline{I}_k(y_{n_m}(\tau(t_k^-))-\overline{I}_k(y_*(\tau(t_k^-))|.\\
&\leq \frac{m(\log T)^r}{\Gamma(r)}\int_a^T l(s)ds\|y_{nm}-y_*\|_\infty\\
&+ \frac{(\log T)^r}{r\Gamma(r)}\int_a^T l(s)ds\|y_{nm}-y_*\|_\infty\\
&+ \sum_{k=1}^{m}|I_k(y_{n_m}(\tau(t_k^-))-I_k(y_*(\tau(t_k^-))|\\
&+ \sum_{k=1}^{m}t_k\log\left(\frac{t}{t_k}\right)|\overline{I}_k(y_{n_m}(\tau(t_k^-))-\overline{I}_k(y_*(\tau(t_k^-))|.
\end{aligned}$$

So, as $m\to\infty$,

$$\begin{aligned}
\|h_{n_m}(t)-h_*(t)\|_\infty &\leq \frac{m(\log T)^r}{\Gamma(r)}\int_a^T l(s)ds\|y_{nm}-y_*\|_\infty\\
&+ \frac{(\log T)^r}{r\Gamma(r)}\int_a^T l(s)ds\|y_{nm}-y_*\|_\infty\\
&+ \sum_{k=1}^{m}|I_k(\tau t_k^-,y_{n_m}(t_k^-))-I_k(\tau t_k^-,y_*(t_k^-))|\\
&+ \sum_{k=1}^{m}t_k\log\left(\frac{t}{t_k}\right)|\overline{I}_k(\tau t_k^-,y_{n_m}(t_k^-))-\overline{I}_k(\tau t_k^-,y_*(t_k^-))|\\
&\longrightarrow 0.
\end{aligned}$$

Step 5: *The solution y of (15)-(18) satisfies*

$$u(t) \leq y(t) \leq w(t) \text{ for all } t \in J.$$

Let y be the above solution to (15)-(18). We prove that

$$y(t) \leq w(t) \text{ for all } t \in J.$$

Assume that $y - w$ attains a positive maximum on $[t_k^+, t_{k+1}^-]$ at $\bar{t}_k \in [t_k^+, t_{k+1}^-]$ for some $k = 0, \ldots, m$; that is

$$(y - w)(\bar{t}_k) = \max\{y(t) - w(t) \ : t \in [t_k^+, t_{k+1}^-]\} > 0, \text{ for some } \ k = 0, \ldots, m.$$

We distinguish the following cases.

Case 1. If $\bar{t}_k \in (t_k^+, t_{k+1}^-)$ there exists $t_k^* \in (t_k^+, t_{k+1}^-)$ such that

$$y(t_k^*) - w(t_k^*) \leq 0, \tag{19}$$

and

$$y(t) - w(t) > 0, \text{ for all } t \in (t_k^*, \bar{t}_k]. \tag{20}$$

By the definition of τ, we have

$${}^{CH}D_a^\alpha y(t) \in F(t, w(t)) \text{ for all } t \in [t_k^*, \bar{t}_k].$$

An integration on $[t_k^*, \bar{t}_k]$ yields

$$y(t) - y(t_k^*) = \frac{1}{\Gamma(\alpha)} \int_{t_k^*}^t \left(\log \frac{t}{s}\right)^{\alpha-1} \nu(s) \frac{ds}{s}, \tag{21}$$

where $\nu(t) \in F(t, w(t))$. From (21) and using the fact that w is an upper solution to (1)-(4) we get

$$y(t) - y(t_k^*) \leq w(t) - w(t_k^*). \tag{22}$$

From (19),(20) and (22) we obtain the contradiction

$$0 < y(t) - w(t) \leq y((t_k^*) - w((t_k^*) \leq 0, \text{ for all } t \in [t_k^*, \bar{t}_k].$$

Case 2. If $\bar{t}_k = t_k^+, \ k = 1, \ldots, m$, then

$$w(t_k^+) < I_k(y(\tau(t_k^-)) < w(t_k^+),$$

and

$$w'(t_k^+) < \overline{I}_k(y(\tau(t_k^-)) < w'(t_k^+),$$

which is a contradiction. Thus

$$y(t) \leq w(t) \text{ for all } \ t \in [1, T].$$

Analogously, we can prove that

$$y(t) \geq u(t) \text{ for all } \ t \in [1, T].$$

This shows that the problem (15)-(18) has a solution in the interval $[u, w]$ which is a solution of (1)-(4). □

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4 An example

We conclude this paper with an example to illustrate our main result. we consider the fractional differential inclusion with impulse

$$ {}^{CH}D^r y(t) \in F(t, y(t)), \ \text{for a.e., } \ t \in J = [1, e], \, t \neq \frac{3}{2}, \ 1 < r \leq 2, \tag{23} $$

$$ \Delta y|_{t=\frac{3}{2}} = \frac{1}{3 + y(\frac{3}{2}^{-})}, \tag{24} $$

$$ \Delta y'|_{t=\frac{3}{2}} = \frac{1}{5 + y(\frac{3}{2}^{-})} \tag{25} $$

$$ y(1) = 0, y'(0) = 0, \tag{26} $$

where ${}^{CH}D^r$ is the Caputo-Hadamard fractional derivative , $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map satisfying

$$ F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\} $$

where $f_1, f_2 : [1, e] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1, e]$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : \mho_{\nvDash}(\approx, \curvearrowright) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [1, e]$, $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : \mho_{\nvDash}(\approx, \curvearrowright) < \mu\}$ is open for each $\mu \in \mathbb{R}$). And

$$ I_k(y) = \frac{1}{3 + y}, $$

$$ \overline{I}_k(y) = \frac{1}{5 + y}, $$

Assume that there exists $p \in C([1, e], \mathbb{R}^+)$ such that,

$$ \begin{aligned} \|F(t, u)\|_{\mathcal{P}} &= \sup\{|v|, v(t) \in F(t, y(t))\} \\ &= \max(|f_1(t, y(t))|, |f_2(t, y(t))|) \\ &\leq p(t), \text{ for each } t \in [1, e], \ y \in \mathbb{R}. \end{aligned} $$

It is clear that F is compact and convex-valued, and it is upper semi-continuous, and furthermore, we assume that there exists $h_1(.)$, $h_2(.) \in L^1(J, \mathbb{R})$ such that

$$ h_1(t) \leq \max(|f_1(t, y)|, |f_2(t, y)|) \leq h_2(t), \quad \text{for all } t \in J, \text{ and } y \in \mathbb{R}, $$

and for each $t \in J$

$$ \int_1^t h_1(s) \frac{ds}{s} \leq y_1 \text{ and } \int_1^t h_1(s) \frac{ds}{s} \leq y_e, $$

$$ \int_1^t h_2(s) \frac{ds}{s} \geq y_1 \text{ and } \int_1^t h_2(s) \frac{ds}{s} \geq y_e, $$

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Consider the functions $w(t)=\displaystyle\int_1^t h_1(s)\frac{ds}{s}$, $u(t)=\displaystyle\int_1^t h_2(s)\frac{ds}{s}$.
Clearly, w and u are lower and upper solutions of the problem (23)-(26), respectively; that is,

$$ {}_{C}^{H}D^{r}w(t)\leq f_1(t,w(t)),\qquad \text{for all } t\in J \text{ and all } y\in\mathbb{R}, $$

and

$$ {}_{C}^{H}D^{r}u(t)\geq f_2(t,u(t)),\qquad \text{for all } t\in J \text{ and all } y\in\mathbb{R}, $$

Since all the conditions of the Theorem (3.4) are satisfied, problem (23)-(26) has at least one solution y on J with $w<y<u$.

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