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ON THE LINEAR SPACE OF THE TWO-SIDED GENERALIZED FIBONACCI SEQUENCES

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ABSTRACT. In this paper, we study the linear space of all two-sided generalized Fibonacci sequences $\{F_n\}_{n\in\mathbb{Z}}$ that satisfy the recurrence equation of order k: $F_n = F_{n-1} + F_{n-2} + \cdots + F_{n-k}$. We give two types of explicit formula, one is based on generalized binomial coefficients and the other based on generalized multinomial coefficients.

1. Introduction

The Fibonacci sequence, $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, have been generalized in many ways. One of the generalizations [12, 5, 17] is to change the recurrence equation to $F_n = \alpha F_{n-1} + \beta F_{n-2}$, thus keeping the characteristic equation remained in order 2. Another common generalization is to extend the recurrence equation to a higher order. For a fixed integer $k \geq 2$, a sequence is called a Fibonacci sequence of order k if it satisfies the following recurrence equation

$$F_n = F_{n-1} + F_{n-2} + \cdots + F_{n-k}. \tag{1}$$

For some particular values of k, the sequence has a special name. It is called a tribonacci sequence, a tetranacci sequence and a pentanacci sequence for $k = 3, 4, 5$, respectively.

A Fibonacci sequence of order k is uniquely determined by a list of values of k consecutive terms. For instance, if the values of $F_0, F_1, \ldots, F_{k-1}$ are given then using the recurrence equation (1), we can work out the values of all other terms F_n for $n \geq k$, as well as for negative indices $n < 0$. Here is an example of a Fibonacci sequence of order 5:

$$\ldots, F_{-4} = -2, F_{-3} = 7, F_{-2} = -3, F_{-1} = -4,$$
$$F_0 = \mathbf{3}, F_1 = \mathbf{1}, F_2 = \mathbf{4}, F_3 = \mathbf{1}, F_4 = \mathbf{5}, F_5 = 14, F_6 = 25, \ldots \quad .$$

Since we have $F_0 = 0$ and $F_1 = 1$ in the original Fibonacci sequence, there are two common ways to set the initial conditions: (i) $F_0 = F_1 = \cdots = F_{k-2} = 0$, $F_{k-1} = 1$ as in [18, 9, 19, 13, 4, 6]; or (ii) $F_0 = 0, F_1 = \cdots = F_{k-2} = F_{k-1} = 1$ as in [14, 21, 3]. Another initial condition $F_0 = F_1 = \cdots = F_{k-1} = 1$ appears in Ferguson [8] arisen in the study of polyphase merge-sorting. Various formulas have been found for Fibonacci sequences with these three initial conditions which can be grouped into three types: Binet formula [7, 13], binomial coefficients [8, 1] and multinomial coefficents [18, 13]. We note that these formulas of F_n are only restricted to the integer indices $n \geq 0$. The Binet type of formula is algebraic in nature and remains valid when we extend to negative indices $n < 0$. However, formulas

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involved binomial coefficients and multinomial coefficents are limited to non-negative indices and it is not trivial to extend to negative indices.

While most authors only consider sequences F_n with $n \geq 0$, in this paper, we will study two-sided sequences. Those are sequences $\{F_n\}$ where the index $n \in \mathbb{Z}$, that is, we allow n to be a *negative integer*. Instead of looking for explicit formula for a Fibonacci sequence with a particular initial condition, our aim is to find explicit formulas for a general Fibonacci sequence that has an *arbitrary initial condition* $(F_0, F_1, \ldots, F_{k-1})$. To do that, we consider the set of all Fibonacci sequences of order k. This forms a k-dimensional linear space. We will study the standard basis of this linear space which is denoted by $B^{(0)}, B^{(1)}, \ldots, B^{(k-1)}$. For $0 \leq j \leq k-1$, each $B^{(j)}$ is a Fibonacci sequence whose initial values are all zero except $B_j^{(j)} = 1$. We will find explicit formula for the basis sequences $B^{(0)}, B^{(1)}, \ldots, B^{(k-1)}$, and thus, any Fibonacci sequence F can be determined by a linear combination $F = F_0 B^{(0)} + F_1 B^{(1)} + \cdots + F_{k-1} B^{(k-1)}$.

Our aim is to find explicit formulas for two-sided Fibonacci sequences that are expressed in terms of binomial coefficients and multinomial coefficients, respectively. Since the classical binomial coefficients and multinomial coefficients are only associated with non-negative integers, to use these for our two-sided sequences we need to extend the binomial notation and multinomial notation to include negative integers. To this end, we extend the binomial notation $\binom{n}{i}$ to negative values of n and i, writing this as $\left\langle \binom{n}{i} \right\rangle$. Subjected to the two conditions $\left\langle \binom{n}{n} \right\rangle = 1$ and $\left\langle \binom{n-1}{i} \right\rangle + \left\langle \binom{n-1}{i-1} \right\rangle = \left\langle \binom{n}{i} \right\rangle$, the latter is called *the Pascal Recursion equation*, the value of the generalized binomial notation is uniquely determined. In Theorem 7, we will show that

$$B_n^{(j)} = -\sum_{i \in \mathbb{Z}} (-1)^i \left\langle \binom{n-ik}{i-1} \right\rangle 2^{n+1-i(k+1)} + \sum_{i \in \mathbb{Z}} (-1)^i \left\langle \binom{n-j-1-ik}{i-1} \right\rangle 2^{n-j-i(k+1)} \text{ for all } n \in \mathbb{Z}.$$

We extend the multinomial notation $\binom{n}{i_1, i_2, \ldots, i_t}$ to negative values of n and $i_1, \ldots, i_t$, writing this as $\left\langle \binom{n}{i_1, i_2, \ldots, i_t} \right\rangle$. The generalization is done as follows.

Using the generalized binomial notation we extend the traditional multinomial notation

$$\binom{n}{i_1, i_2, \ldots, i_k} = \binom{n}{i_2 + \cdots + i_t} \binom{i_2 + \cdots + i_t}{i_3 + \cdots + i_t} \cdots \binom{i_{t-2} + i_{t-1} + i_t}{i_{t-1} + i_t} \binom{i_{t-1} + i_t}{i_t},$$

to

$$\left\langle \binom{n}{i_1, i_2, \ldots, i_t} \right\rangle = \left\langle \binom{n}{i_2 + \cdots + i_t} \right\rangle \left\langle \binom{i_2 + \cdots + i_t}{i_3 + \cdots + i_t} \right\rangle \cdots \left\langle \binom{i_{t-2} + i_{t-1} + i_t}{i_{t-1} + i_t} \right\rangle \left\langle \binom{i_{t-1} + i_t}{i_t} \right\rangle.$$

Using this generalized multinomial notation, in Theorem 12, we will show that

$$B_n^{(j)} = \sum_{n-k-j \leq a_1 + 2a_2 + \cdots + ka_k \leq n-k} \left\langle \binom{a_1 + a_2 + \cdots + a_k}{a_1, a_2, \ldots, a_k} \right\rangle, \text{ for all } n \in \mathbb{Z}.$$

The rest of the paper is organised as follows. In section 2, we study the linear space of Fibonacci sequences of order k in general, especially looking at the linear automorphisms of this space. Formulas based on the generalized binomial notation are derived in section 3. Formulas based on the generalized multinomial notation are derived in section 4. Finally, in section 5, we remark on how the generalized Fibonacci sequences are related to a tiling problem.

2. The Fibonacci linear space of order k

Definition 1. *Let $k \geq 2$ be a fixed integer. A sequence $\{F_n\}_{n\in\mathbb{Z}}$ is called a Fibonacci sequence of order k if it satisfies the following recurrence equation*

$$F_n = F_{n-1} + F_{n-2} + \cdots + F_{n-k}, \quad \text{for all } n \in \mathbb{Z}. \tag{2}$$

We can see that, given k values $(F_0, F_1, \ldots, F_{k-1})$, then using the Fibonacci recurrence equation (2), all other values F_n for $n \in \mathbb{Z}$ are determined uniquely. We will refer to $(F_0, F_1, \ldots, F_{k-1})$ as the initial values of the sequence. The set of all Fibonacci sequences of order k forms a k-dimensional vector space (either over the field $\mathbb{R}$ or $\mathbb{C}$). We will use $\mathsf{Fibonacci}^{(k)}$ to denote this vector space of all Fibonacci sequences of order k. We now define the standard basis for the Fibonacci vector space $\mathsf{Fibonacci}^{(k)}$.

Definition 2. *Let $k \geq 2$ be a fixed integer. For each integer $0 \leq j \leq k-1$, the sequence $B^{(j)} \in \mathsf{Fibonacci}^{(k)}$ is defined by the initial values*

$$B_n^{(j)} = \begin{cases} 0, & \text{if } 0 \leq n \leq k-1 \text{ and } n \neq j \\ 1, & \text{if } n = j. \end{cases}$$

The special sequences $B^{(0)}, B^{(1)}, \ldots, B^{(k-1)}$ defined above form a standard basis for the space $\mathsf{Fibonacci}^{(k)}$. Any member of this Fibonacci vector space is a linear combination of the standard basis and we have the following theorem.

Theorem 1. *Let $k \geq 2$ be a fixed integer. Let $\{F_n\}_{n\in\mathbb{Z}}$ be a Fibonacci sequence of order k. Then*

$$F_n = \sum_{j=0}^{k-1} B_n^{(j)} F_j \quad \text{for all } n \in \mathbb{Z}.$$

By Theorem 1, we can see that in order to determine an explicit formula for any Fibonacci sequence $\{F_n\}_{n\in\mathbb{Z}}$, it suffices to derive formula for the k basis sequences $B^{(0)}, B^{(1)}, \ldots, B^{(k-1)}$.

2.1. *Linear operators on the Fibonacci space.*

Here we list some standard linear operators on two-sided sequences.

- Identity operator $\mathtt{I}$.
- Left shift operator $\mathtt{L}$: $\mathtt{L}(X) = Y$ iff $Y_n = X_{n+1}$ for all $n \in \mathbb{Z}$.
- Right shift operator $\mathtt{R}$: $\mathtt{R}(X) = Y$ iff $Y_n = X_{n-1}$ for all $n \in \mathbb{Z}$. The left shift and the right shift are inverse of each other: $\mathtt{LR} = \mathtt{RL} = \mathtt{I}$.
- Forward difference operator Δ: $\Delta(X) = Y$ iff $Y_n = X_{n+1} - X_n$ for all $n \in \mathbb{Z}$. Here $\Delta = \mathtt{L} - \mathtt{I}$.
- Backward difference operator ∇: $\nabla(X) = Y$ iff $Y_n = X_n - X_{n-1}$ for all $n \in \mathbb{Z}$. Here $\nabla = \mathtt{I} - \mathtt{R} = \mathtt{I} - \mathtt{L}^{-1}$, $\mathtt{L}\nabla = \Delta$ and $\mathtt{R}\Delta = \nabla$.

We have the following theorem concerning the above operators.

Theorem 2. *All operators $\mathtt{I}$, $\mathtt{L}$, $\mathtt{R}$, Δ and ∇ when restricted to the space $\mathsf{Fibonacci}^{(k)}$ are linear automorphisms $\mathsf{Fibonacci}^{(k)} \to \mathsf{Fibonacci}^{(k)}$ and satisfy the following relations:*

(i) $\mathtt{L}^k = \mathtt{I}+\mathtt{L}+\mathtt{L}^2+\cdots+\mathtt{L}^{k-1}$.
(ii) $\mathtt{R} = \mathtt{L}^{-1} = -\mathtt{I}-\mathtt{L}-\mathtt{L}^2-\cdots-\mathtt{L}^{k-2}+\mathtt{L}^{k-1}$
(iii) $\mathtt{R}^k = \mathtt{I}-\mathtt{R}-\mathtt{R}^2-\cdots-\mathtt{R}^{k-1}$.
(iv) $\mathtt{L} = \mathtt{R}^{-1} = \mathtt{I}+\mathtt{R}+\mathtt{R}^2+\cdots+\mathtt{R}^{k-1}$.
(v) $\mathtt{L}^{k+1} = 2\mathtt{L}^k-\mathtt{I}$.
(vi) $\mathtt{R}^{k+1} = 2\mathtt{R}-\mathtt{I}$.
(vii) $\Delta(\mathtt{I}+(k-1)\mathtt{R}+(k-2)\mathtt{R}^2+(k-3)\mathtt{R}^3+\cdots+2\mathtt{R}^{k-2}+\mathtt{R}^{k-1}) = (k-1)\mathtt{I}$.
(viii) $\nabla(k\mathtt{I}+(k-1)\mathtt{R}+(k-2)\mathtt{R}^2+\cdots+2\mathtt{R}^{k-2}+\mathtt{R}^{k-1}) = (k-1)\mathtt{I}$.
(ix) $\sum_{i=0}^{k}\binom{k+1}{i+1}\frac{k-1-2i}{k+1}\Delta^i = 0$.
(x) $(k-1)\mathtt{I}+\sum_{i=1}^{k}\binom{k+1}{i+1}(-1)^i\nabla^i = 0$.

Proof. It is easy to see that all these operators $\mathtt{I}$, $\mathtt{L}$, $\mathtt{R}$, Δ and ∇ are linear. Each maps a Fibonacci sequence to another Fibonacci sequence. The bijectivity of $\mathtt{I}$, $\mathtt{L}$, $\mathtt{R}$ is obvious, whereas, the bijectivity of Δ and ∇ follows from (vii) and (viii), respectively.

(i) For any $X \in \mathsf{Fibonacci}^{(k)}$, let $(\mathtt{I}+\mathtt{L}+\mathtt{L}^2+\cdots+\mathtt{L}^{k-1})(X) = Y$ then $Y_n = X_n+X_{n+1}+X_{n+2}+\cdots+X_{n+k-1} = X_{n+k}$, therefore, $Y = \mathtt{L}^k(X)$. This proves that, restricted to the linear space $\mathsf{Fibonacci}^{(k)}$, $\mathtt{I}+\mathtt{L}+\mathtt{L}^2+\cdots+\mathtt{L}^{k-1} = \mathtt{L}^k$.

(ii) For any $X \in \mathsf{Fibonacci}^{(k)}$, let $(-\mathtt{I}-\mathtt{L}-\mathtt{L}^2-\cdots-\mathtt{L}^{k-2}+\mathtt{L}^{k-1})(X) = Y$ then $Y_n = -X_n-X_{n+1}-X_{n+2}-\cdots-X_{n+k-2}+X_{n+k-1} = X_{n-1}$. Hence, $Y = \mathtt{R}(X)$, and therefore, $-\mathtt{I}-\mathtt{L}-\mathtt{L}^2-\cdots-\mathtt{L}^{k-2}+\mathtt{L}^{k-1} = \mathtt{R} = \mathtt{L}^{-1}$.

(iii) For any $X \in \mathsf{Fibonacci}^{(k)}$, let $(\mathtt{I}-\mathtt{R}-\mathtt{R}^2-\cdots-\mathtt{R}^{k-1})(X) = Y$ then $Y_n = X_n-X_{n-1}-X_{n-2}-\cdots-X_{n-k+1} = X_{n-k}$. Hence, $Y = \mathtt{R}^k(X)$, and therefore, $\mathtt{I}-\mathtt{R}-\mathtt{R}^2-\cdots-\mathtt{R}^{k-1} = \mathtt{R}^k$.

(iv) For any $X \in \mathsf{Fibonacci}^{(k)}$, let $(\mathtt{I}+\mathtt{R}+\mathtt{R}^2+\cdots+\mathtt{R}^{k-1})(X) = Y$ then $Y_n = X_n+X_{n-1}+X_{n-2}+\cdots+X_{n-k+1} = X_{n+1}$. Hence, $Y = \mathtt{L}(X)$, and therefore, $\mathtt{I}+\mathtt{R}+\mathtt{R}^2+\cdots+\mathtt{R}^{k-1} = \mathtt{L} = \mathtt{R}^{-1}$.

(v) By (i), $\mathtt{L}^{k+1} = \mathtt{L}\mathtt{L}^k = \mathtt{L}(\mathtt{I}+\mathtt{L}+\mathtt{L}^2+\cdots+\mathtt{L}^{k-1}) = \mathtt{L}+\mathtt{L}^2+\cdots+\mathtt{L}^{k-1}+\mathtt{L}^k = (\mathtt{I}+\mathtt{L}+\mathtt{L}^2+\cdots+\mathtt{L}^{k-1})+\mathtt{L}^k-\mathtt{I} = \mathtt{L}^k+\mathtt{L}^k-\mathtt{I} = 2\mathtt{L}^k-\mathtt{I}$.

(vi) By (iii), $\mathtt{R}^{k+1} = \mathtt{R}\mathtt{R}^k = \mathtt{R}(\mathtt{I}-\mathtt{R}-\mathtt{R}^2-\cdots-\mathtt{R}^{k-1}) = \mathtt{R}-\mathtt{R}^2-\mathtt{R}^3-\cdots-\mathtt{R}^{k-1}-\mathtt{R}^k = \mathtt{R}-\mathtt{R}^2-\mathtt{R}^3-\cdots-\mathtt{R}^{k-1}-(\mathtt{I}-\mathtt{R}-\mathtt{R}^2-\cdots-\mathtt{R}^{k-1}) = 2\mathtt{R}-\mathtt{I}$.

(vii) We have

$$\begin{aligned}
&\Delta(\mathtt{I}+(k-1)\mathtt{R}+(k-2)\mathtt{R}^2+(k-3)\mathtt{R}^3+\cdots+2\mathtt{R}^{k-2}+\mathtt{R}^{k-1})\\
&=(\mathtt{L}-\mathtt{I})(\mathtt{I}+(k-1)\mathtt{R}+(k-2)\mathtt{R}^2+(k-3)\mathtt{R}^3+\cdots+2\mathtt{R}^{k-2}+\mathtt{R}^{k-1})\\
&=\mathtt{L}+(k-2)\mathtt{I}-\mathtt{R}-\mathtt{R}^2-\cdots-\mathtt{R}^{k-2}-\mathtt{R}^{k-1}\\
&=(k-1)\mathtt{I}\quad \text{by (iv)}.
\end{aligned}$$

(viii) We have

$$\begin{aligned}
&\nabla(k\mathtt{I}+(k-1)\mathtt{R}+(k-2)\mathtt{R}^2+\cdots+2\mathtt{R}^{k-2}+\mathtt{R}^{k-1})\\
&=(\mathtt{I}-\mathtt{R})(k\mathtt{I}+(k-1)\mathtt{R}+(k-2)\mathtt{R}^2+\cdots+2\mathtt{R}^{k-2}+\mathtt{R}^{k-1})\\
&=k\mathtt{I}-\mathtt{R}-\mathtt{R}^2-\cdots-\mathtt{R}^{k-1}-\mathtt{R}^k\\
&=(k-1)\mathtt{I}\quad \text{by (iii)}.
\end{aligned}$$

(ix) Substituting $\mathtt{L} = \mathtt{I} + \Delta$ into (i), we have

$$(\mathtt{I}+\Delta)^k = \mathtt{I} + (\mathtt{I}+\Delta) + (\mathtt{I}+\Delta)^2 + \cdots + (\mathtt{I}+\Delta)^{k-1}$$

$$\sum_{i=0}^{k}\binom{k}{i}\Delta^i = \sum_{j=0}^{k-1}\sum_{i=0}^{j}\binom{j}{i}\Delta^i = \sum_{i=0}^{k-1}\sum_{j=i}^{k-1}\binom{j}{i}\Delta^i = \sum_{i=0}^{k-1}\binom{k}{i+1}\Delta^i.$$

Therefore,

$$\Delta^k = \sum_{i=0}^{k-1}\left(\binom{k}{i+1} - \binom{k}{i}\right)\Delta^i = \sum_{i=0}^{k-1}\binom{k+1}{i+1}\frac{k-1-2i}{k+1}\Delta^i.$$

(x) Substituting $\mathtt{R} = \mathtt{I} - \nabla$ into (iii), we have

$$(\mathtt{I}-\nabla)^k = \mathtt{I} - (\mathtt{I}-\nabla) - (\mathtt{I}-\nabla)^2 - \cdots - (\mathtt{I}-\nabla)^{k-1}.$$

So

$$\sum_{i=1}^{k}\binom{k}{i}(-\nabla)^i = -\sum_{j=1}^{k-1}\sum_{i=0}^{j}\binom{j}{i}(-\nabla)^i = -(k-1)\mathtt{I} - \sum_{i=1}^{k-1}\sum_{j=i}^{k-1}\binom{j}{i}(-\nabla)^i$$
$$= -(k-1)\mathtt{I} - \sum_{i=1}^{k-1}\binom{k}{i+1}(-\nabla)^i.$$

Therefore,

$$(-\nabla)^k = -(k-1)\mathtt{I} - \sum_{i=1}^{k-1}\left(\binom{k}{i+1} + \binom{k}{i}\right)(-\nabla)^i$$
$$= -(k-1)\mathtt{I} - \sum_{i=1}^{k-1}\binom{k+1}{i+1}(-\nabla)^i$$

and

$$\sum_{i=1}^{k}\binom{k+1}{i+1}(-\nabla)^i = -(k-1)\mathtt{I}. \quad \blacksquare$$

Theorem 3. *Denote $S = B^{(0)} + B^{(1)} + \cdots + B^{(k-1)} \in \mathsf{Fibonacci}^{(k)}$. We have*

(i) $B^{(j)} - B^{(j-1)} = \mathtt{R}^j(B^{(0)})$ for all $1 \le j \le k-1$.

(ii) $B^{(j)} = \sum_{i=0}^{j}\mathtt{R}^i(B^{(0)})$ for all $0 \le j \le k-1$.

(iii) $B^{(0)} = \mathtt{R}(B^{(k-1)})$ and $B^{(k-1)} = \mathtt{L}(B^{(0)})$.

(iv) $B^{(j)} = \sum_{i=0}^{j}\mathtt{R}^{i+1}(B^{(k-1)})$ for all $0 \le j \le k-1$.

(v) $S = (k\mathtt{I} + (k-1)\mathtt{R} + (k-2)\mathtt{R}^2 + \cdots + \mathtt{R}^{k-1})(B^{(0)})$.

(vi) $\nabla(S) = (k-1)B^{(0)}$.

(vii) $(\mathtt{I} - \mathtt{R}^{j+1})(S) = (k-1)B^{(j)}$ for all $0 \le j \le k-1$.

Proof. (i) Both $B^{(j)} - B^{(j-1)}$ and $\mathtt{R}^j(B^{(0)})$ are members of $\mathsf{Fibonacci}^{(k)}$ and their initial values are equal, therefore, $B^{(j)} - B^{(j-1)} = \mathtt{R}^j(B^{(0)})$.

(ii) It follows from (i).

(iii) By (ii), $B^{(k-1)} = \sum_{i=0}^{k-1} \mathtt{R}^i(B^{(0)})$ and since $\mathtt{L} = \mathtt{R}^{-1} = \mathtt{I} + \mathtt{R} + \mathtt{R}^2 + \cdots + \mathtt{R}^{k-1}$ (Theorem 2(iv)), we have $B^{(k-1)} = \mathtt{L}(B^{(0)})$ and so $B^{(0)} = \mathtt{R}(B^{(k-1)})$.

(iv) It follows from (ii) and (iii).

(v) It follows from (ii).

(vi) It follows from (v) and Theorem 2(viii).

(vii) We have

$$\begin{aligned}
(k-1)B^{(j)} &= (k-1)\sum_{i=0}^{j} \mathtt{R}^i(B^{(0)}) \quad \text{by (ii)} \\
&= \sum_{i=0}^{j} \mathtt{R}^i(\nabla(S)) \quad \text{by (vi)} \\
&= \sum_{i=0}^{j} (\mathtt{R}^i(1-\mathtt{R}))(S) = (1-\mathtt{R}^{j+1})(S).
\end{aligned}$$

Another direct way to prove (vii) is by observing that both $(k-1)B^{(j)}$ and $(1-\mathtt{R}^{j+1})(S)$ are members of $\mathsf{Fibonacci}^{(k)}$ and their initial values are equal. ■

3. Explicit formulas based on binomials

In this section, we will derive explicit formula for the two-sided Fibonacci basis sequences $B^{(0)}$, $B^{(1)}, \ldots, B^{(k-1)}$ expressed in terms of binomial coefficients. Since the traditional binomial notation is associated with non-negative integers, to use these for our two-sided sequences we need to extend the binomial notation to include negative integers. To this end, we extend the binomial notation $\binom{n}{i}$ to negative values of n and i.

The binomial notation $\binom{n}{i}$ can be generalized to $\left\langle \binom{n}{i} \right\rangle$ for all integers n and i by enforcing two conditions:

- $\left\langle \binom{n}{n} \right\rangle = 1$ for all $n \in \mathbb{Z}$; and
- Pascal Recursion relation

$$\left\langle \binom{n-1}{i} \right\rangle + \left\langle \binom{n-1}{i-1} \right\rangle = \left\langle \binom{n}{i} \right\rangle. \tag{3}$$

With these two conditions, $\left\langle \binom{n}{i} \right\rangle$ is uniquely determined as

$$\left\langle \binom{n}{i} \right\rangle = \begin{cases} \frac{n^{\underline{n-i}}}{(n-i)!} = \frac{n(n-1)(n-2)\ldots(i+1)}{(n-i)!}, & \text{if } n \geq i \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

$$= \begin{cases} \binom{n}{i}, & \text{if } n \geq i \geq 0 \\ (-1)^{i+n}\binom{-i-1}{-n-1}, & \text{if } -1 \geq n \geq i\,. \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Refer to [15, 16] for detailed discussion on various generalizations of binomial notation. The following table shows some values of $\left\langle \binom{n}{i} \right\rangle$:

$\left\langle\binom{n}{i}\right\rangle$	$i=-6$	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$n=6$	0	0	0	0	0	0	1	6	15	20	15	6	1
5	0	0	0	0	0	0	1	5	10	10	5	1	0
4	0	0	0	0	0	0	1	4	6	4	1	0	0
3	0	0	0	0	0	0	1	3	3	1	0	0	0
2	0	0	0	0	0	0	1	2	1	0	0	0	0
1	0	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0
-1	-1	1	-1	1	-1	1	0	0	0	0	0	0	0
-2	5	-4	3	-2	1	0	0	0	0	0	0	0	0
-3	-10	6	-3	1	0	0	0	0	0	0	0	0	0
-4	10	-4	1	0	0	0	0	0	0	0	0	0	0
-5	-5	1	0	0	0	0	0	0	0	0	0	0	0
-6	1	0	0	0	0	0	0	0	0	0	0	0	0

In the following theorem, we define an auxiliary sequence $\{A_n\}_{n\in\mathbb{Z}}$ which will be useful in the sequel. Note that this sequence is not a member of the linear space $\mathsf{Fibonacci}^{(k)}$. The proof of the theorem is a consequence of the Pascal Recursion relation (3).

Theorem 4. *Let $k \geq 2$ and the sequence $\{A_n\}$ defined as*

$$A_n = \sum_{i\in\mathbb{Z}} (-1)^i \left\langle\binom{n-ik}{i-1}\right\rangle 2^{n+1-i(k+1)} \qquad \text{for all } n \in \mathbb{Z}. \tag{6}$$

Then $A_0 = A_1 = A_2 = \cdots = A_{k-1} = 0$, $A_n = A_{n-1} + A_{n-2} + \cdots + A_{n-k} - 1$ and $A_n = 2A_{n-1} - A_{n-k-1}$.

Proof. Note that the above summation in the formula of A_n only has a finite number of non-zero terms. This is because $\left\langle\binom{n-ik}{i-1}\right\rangle = 0$ except for $1 \leq i \leq \frac{n+1}{k+1}$ when $n \geq 0$ and $\frac{n+1}{k} \leq i \leq \frac{n+1}{k+1}$ for $n \leq -1$. It follows that $A_0 = A_1 = A_2 = \cdots = A_{k-1} = 0$ and $A_k = -1$.

We have

$$\begin{aligned}
2A_{n-1} - A_{n-k-1} =& 2\sum (-1)^i \left\langle\binom{n-1-ik}{i-1}\right\rangle 2^{n-i(k+1)} \\
&- \sum (-1)^i \left\langle\binom{n-k-1-ik}{i-1}\right\rangle 2^{n-k-i(k+1)} \\
=& \sum (-1)^i \left\langle\binom{n-1-ik}{i-1}\right\rangle 2^{n+1-i(k+1)} \\
&+ \sum (-1)^{i+1} \left\langle\binom{n-1-(i+1)k}{i-1}\right\rangle 2^{n+1-(i+1)(k+1)}.
\end{aligned}$$

In the last summation, let $i := i+1$, we have

$$\begin{aligned}2A_{n-1}-A_{n-k-1} =&\sum(-1)^i\left\langle\binom{n-1-ik}{i-1}\right\rangle 2^{n+1-i(k+1)}\\ &+\sum(-1)^i\left\langle\binom{n-1-ik}{i-2}\right\rangle 2^{n+1-i(k+1)}\end{aligned}$$

and by the Pascal Recursion (3),

$$\begin{aligned}2A_{n-1}-A_{n-k-1} =&\sum(-1)^i\left\langle\binom{n-ik}{i-1}\right\rangle 2^{n+1-i(k+1)}\\ =&A_n.\end{aligned}$$

Therefore, $(\mathtt{R}^{k+1}-2\mathtt{R}+\mathtt{I})(A)=0$.

As $\mathtt{R}^{k+1}-2\mathtt{R}+\mathtt{I}=(\mathtt{R}-\mathtt{I})(\mathtt{R}^k+\mathtt{R}^{k-1}+\cdots+\mathtt{R}-\mathtt{I})$, it follows that $(\mathtt{R}^k+\mathtt{R}^{k-1}+\cdots+\mathtt{R}-\mathtt{I})(A)$ is a constant sequence, so $A_{n-1}+A_{n-2}+\cdots+A_{n-k}-A_n=A_0+A_1+\cdots+A_{k-1}-A_k=1$. ■

Recall that in Theorem 3 we define the sequence $S=B^{(0)}+B^{(1)}+\cdots+B^{(k-1)}\in\mathsf{Fibonacci}^{(k)}$. The following theorem gives an explicit formula for the sequence S.

Theorem 5. *Let $k\ge 2$. The k-order Fibonacci sequence S (determined by the first k terms $(1,1,\dots,1)$) satisfies the following formula*

$$S_n=1-(k-1)\sum_{i\in\mathbb{Z}}(-1)^i\left\langle\binom{n-ik}{i-1}\right\rangle 2^{n+1-i(k+1)}\quad \textit{for all } n\in\mathbb{Z}. \tag{7}$$

Proof. Let S'_n denote the sequence on the RHS of (7) then $S'_n=1-(k-1)A_n$ where $\{A_n\}$ is the auxiliary sequence defined in Theorem 4. It follows from Theorem 4 that $S'_0=S'_1=\cdots=S'_{k-1}=1$, $S'_k=k$ and $S'_n=2S'_{n-1}-S'_{n-k-1}$. By Theorem 2(vi), the sequence S also satisfies the same recursion equation $S_n=2S_{n-1}-S_{n-k-1}$. Since $S_i=S'_i$ for all $0\le i\le k$, it follows that $S_i=S'_i$ for all $i\in\mathbb{Z}$. ■

Theorem 6. *Let $k\ge 2$. The k-order Fibonacci sequence S (determined by the first k terms $(1,1,\dots,1)$) satisfies the following formula*

$$S_n=1-(k-1)\sum_{1\le i\le\frac{n+1}{k+1}}(-1)^i\binom{n-ik}{i-1}2^{n+1-i(k+1)}\quad \textit{for all } n\ge 0, \tag{8}$$

$$S_n=1-(k-1)\sum_{\frac{n+1}{k}\le i\le\frac{n+1}{k+1}}(-1)^i\left\langle\binom{n-ik}{i-1}\right\rangle 2^{n+1-i(k+1)}\quad \textit{for all } n\le -1. \tag{9}$$

Proof. Since $\left\langle\binom{n-ik}{i-1}\right\rangle=0$ except for $1\le i\le\frac{n+1}{k+1}$ when $n\ge 0$ and $\frac{n+1}{k}\le i\le\frac{n+1}{k+1}$ for $n\le -1$, the theorem follows from Theorem 5. ■

Theorem 7. *Let $k\ge 2$, $0\le j\le k-1$. The k-order Fibonacci sequence $B^{(j)}$ satisfies the following formula*

$$B_n^{(j)}=-\sum_{i\in\mathbb{Z}}(-1)^i\left\langle\binom{n-ik}{i-1}\right\rangle 2^{n+1-i(k+1)}+\sum_{i\in\mathbb{Z}}(-1)^i\left\langle\binom{n-j-1-ik}{i-1}\right\rangle 2^{n-j-i(k+1)}\ \textit{for all } n\in\mathbb{Z}.$$

Proof. By Theorem 3(vii), $B^{(j)} = \frac{1}{k-1}(\mathtt{I} - \mathtt{R}^{j+1})(S)$, thus, using the formula (7) for S_n in Theorem 5, we obtain the desired formula for $B_n^{(j)}$. ∎

The formula (8) for S_n in Theorem 6 is equivalent to a formula in Ferguson [8] (formula (3) for $V_{n,a(n+1)+b}$). Theorem 7 for the case $j = k-1$ and positive indices is proved in Benjamin et al. [1].

4. Explicit formula based on multinomials

In this section, we will derive explicit formula for the two-sided Fibonacci basis sequences $B^{(0)}$, $B^{(1)}, \ldots, B^{(k-1)}$ expressed in terms of multinomial coefficients. Since the traditional multinomial notation is associated with non-negative integers, to use these for our two-sided sequences we need to extend the multinomial notation to include negative integers. To this end, we extend the multinomial notation $\binom{n}{i_1, i_2, \ldots, i_t}$ to negative values of n and $i_1, i_2, \ldots, i_t$.

A multinomial is defined as

$$(i_1, i_2, \ldots, i_t) = \binom{i_1 + i_2 + \cdots + i_t}{i_1, i_2, \ldots, i_t} = \frac{(i_1 + i_2 + \cdots + i_t)!}{i_1! i_2! \ldots i_t!}.$$

We observe that

$$(i_1, i_2, \ldots, i_t) = \binom{i_1 + \cdots + i_t}{i_2 + \cdots + i_t}\binom{i_2 + \cdots + i_t}{i_3 + \cdots + i_t} \cdots \binom{i_{t-2} + i_{t-1} + i_t}{i_{t-1} + i_t}\binom{i_{t-1} + i_t}{i_t}.$$

We will use this formula to extend multinomial notation for negative integers.

Definition 3. *Let $t \geq 2$ be an integer. For any integers $i_1, i_2, \ldots, i_t$, the generalized multinomial $\langle (i_1, i_2, \ldots, i_t) \rangle$ is defined as*

$$\begin{aligned}\langle (i_1, i_2, \ldots, i_t) \rangle &= \left\langle \binom{i_1 + i_2 + \cdots + i_t}{i_1, i_2, \ldots, i_t} \right\rangle \\ &= \left\langle \binom{i_1 + \cdots + i_t}{i_2 + \cdots + i_t} \right\rangle \left\langle \binom{i_2 + \cdots + i_t}{i_3 + \cdots + i_t} \right\rangle \cdots \left\langle \binom{i_{t-2} + i_{t-1} + i_t}{i_{t-1} + i_t} \right\rangle \left\langle \binom{i_{t-1} + i_t}{i_t} \right\rangle.\end{aligned}$$

Using the following formula for the generalized binomial coefficient

$$\left\langle \binom{n}{i} \right\rangle = \begin{cases} \frac{n^{\underline{n-i}}}{(n-i)!} = \frac{n(n-1)(n-2)\ldots(i+1)}{(n-i)!}, & \text{if } n \geq i \\ 0, & \text{otherwise} \end{cases},$$

we obtain the following formula for the generalized multinomial

$$\begin{aligned}\langle (i_1, i_2, \ldots, i_t) \rangle &= \left\langle \binom{i_1 + i_2 + \cdots + i_t}{i_1, i_2, \ldots, i_t} \right\rangle \\ &= \begin{cases} \dfrac{(i_1 + \cdots + i_t)^{\underline{i_1}} (i_2 + \cdots + i_t)^{\underline{i_2}} \ldots (i_{t-1} + i_t)^{\underline{i_{t-1}}}}{i_1! i_2! \ldots i_{t-1}!}, & \text{if } i_1, i_2, \ldots, i_{t-1} \geq 0 \\ 0, & \text{otherwise} \end{cases}.\end{aligned}$$

When $t = 2$, the Pascal Recursion relation becomes

$$\langle (i_1, i_2) \rangle = \langle (i_1 - 1, i_2) \rangle + \langle (i_1, i_2 - 1) \rangle.$$

For a general $t \geq 2$, we have the following generalized Pascal Recursion relation for multinomials:

$$\langle (i_1, i_2, \ldots, i_t) \rangle = \langle (i_1 - 1, i_2, \ldots, i_t) \rangle + \langle (i_1, i_2 - 1, \ldots, i_t) \rangle + \cdots + \langle (i_1, i_2, \ldots, i_t - 1) \rangle . \tag{10}$$

Since $\left\langle \binom{n}{i} \right\rangle$ is non-zero only for $n \geq i \geq 0$ or $-1 \geq n \geq i$, the generalized multinomial $\langle (i_1, i_2, \ldots, i_t) \rangle$ is non-zero only for $i_1 + \cdots + i_t \geq i_2 + \cdots + i_t \geq \cdots \geq i_{t-1} + i_t \geq i_t \geq 0$ or $-1 \geq i_1 + \cdots + i_t \geq i_2 + \cdots + i_t \geq \cdots \geq i_{t-1} + i_t \geq i_t$. Using the formula (5) for $\left\langle \binom{n}{i} \right\rangle$, we can derive the formula for the generalized multinomial in these two separate cases.

Case 1. If $i_1 + \cdots + i_t \geq i_2 + \cdots + i_t \geq \cdots \geq i_{t-1} + i_t \geq i_t \geq 0$, i.e. $i_1, i_2, \ldots, i_t \geq 0$, then

$$\langle (i_1, i_2, \ldots, i_t) \rangle = \left\langle \binom{i_1 + i_2 + \cdots + i_t}{i_1, i_2, \ldots, i_t} \right\rangle = \binom{i_1 + i_2 + \cdots + i_t}{i_1, i_2, \ldots, i_t} = (i_1, i_2, \ldots, i_t).$$

Case 2. If $-1 \geq i_1 + \cdots + i_t \geq i_2 + \cdots + i_t \geq \cdots \geq i_{t-1} + i_t \geq i_t$ then

$$\begin{aligned} \langle (i_1, i_2, \ldots, i_t) \rangle &= \left\langle \binom{i_1 + i_2 + \cdots + i_t}{i_1, i_2, \ldots, i_t} \right\rangle \\ &= (-1)^{i_1 + \cdots + i_{t-1}} \binom{-i_t - 1}{i_1, i_2, \ldots, i_{t-1}, -i_1 - \cdots - i_t - 1} \\ &= (-1)^{i_1 + \cdots + i_{t-1}} (i_1, i_2, \ldots, i_{t-1}, -i_1 - \cdots - i_t - 1). \end{aligned}$$

Thus, we obtain the following theorem that connects the generalized multinomial to the classical multinomial.

Theorem 8. *For any integer $t \geq 2$ and $i_1, i_2, \ldots, i_t \in \mathbb{Z}$, we have*

$$\langle (i_1, i_2, \ldots, i_t) \rangle = \begin{cases} (i_1, i_2, \ldots, i_t), & \text{if } i_1, i_2, \ldots, i_t \geq 0 \\ (-1)^{i_1 + \cdots + i_{t-1}} (i_1, i_2, \ldots, i_{t-1}, -i_1 - \cdots - i_t - 1) & \text{if } i_1, i_2, \ldots, i_{t-1} \geq 0 \text{ and } i_1 + \cdots + i_t \leq -1 \\ 0, & \text{otherwise.} \end{cases}$$

In the following theorem, we define an auxiliary sequence $\{X_n\}_{n \in \mathbb{Z}}$. Note that X is a member of the linear space $\mathsf{Fibonacci}^{(k)}$.

Theorem 9. *Let $k \geq 2$, $c \in \mathbb{Z}$ any constant, and*

$$\begin{aligned} X_n &= \sum_{a_1 + 2a_2 + \cdots + k a_k = n + c} \langle (a_1, a_2, \ldots, a_k) \rangle \\ &= \sum_{s_1 + s_2 + \cdots + s_k = n + c} \left\langle \binom{s_1}{s_2} \right\rangle \left\langle \binom{s_2}{s_3} \right\rangle \cdots \left\langle \binom{s_{k-1}}{s_k} \right\rangle . \end{aligned}$$

Then $\{X_n\}_{n \in \mathbb{Z}}$ is a Fibonacci sequence of order k.

Proof. The two formulas on the RHS are equivalent by using the variables $s_1 = a_1 + \cdots + a_k$, $s_2 = a_2 + \cdots + a_k$, ..., $s_{k-1} = a_{k-1} + a_k$ and $s_k = a_k$.

Note that the summation only has a finite number of non-zero terms. This is because $\langle (a_1, a_2, \ldots, a_k) \rangle$ is non-zero only if $s_1 \geq s_2 \geq \cdots \geq s_k \geq 0$ or $-1 \geq s_1 \geq s_2 \geq \cdots \geq s_k$, and there are only a finite number of choices for $s_1, s_2, \ldots, s_k$ that have the same sign whose sum $s_1 + s_2 + \cdots + s_k = n + c$ is fixed.

By Pascal Recursion relation (10),

$$\begin{aligned} X_n = & \sum_{a_1+2a_2+\cdots+ka_k=n+c} \langle (a_1-1, a_2, \ldots, a_k)\rangle \\ & + \sum_{a_1+2a_2+\cdots+ka_k=n+c} \langle (a_1, a_2-1, \ldots, a_k)\rangle \\ & + \cdots + \sum_{a_1+2a_2+\cdots+ka_k=n+c} \langle (a_1, a_2, \ldots, a_k-1)\rangle. \end{aligned}$$

Let $a_1' = a_1 - 1$, $a_2' = a_2 - 1$, ..., $a_k' = a_k - 1$. We have

$$\begin{aligned} X_n = & \sum_{a_1'+2a_2+\cdots+ka_k=n+c-1} \left\langle (a_1', a_2, \ldots, a_k)\right\rangle \\ & + \sum_{a_1+2a_2'+\cdots+ka_k=n+c-2} \left\langle (a_1, a_2', \ldots, a_k)\right\rangle \\ & + \cdots + \sum_{a_1+2a_2+\cdots+ka_k'=n+c-k} \left\langle (a_1, a_2, \ldots, a_k')\right\rangle \\ & = X_{n-1} + X_{n-2} + \cdots + X_{n-k}, \end{aligned}$$

therefore, $\{X_n\}$ is a Fibonacci sequence of order k. ∎

Theorem 10. *Let $k \geq 2$. Then*

$$B_n^{(0)} = \sum_{a_1+2a_2+\cdots+ka_k=n-k} \langle (a_1, a_2, \ldots, a_k)\rangle, \quad \text{for all } n \in \mathbb{Z}. \tag{11}$$

Proof. Let B' denote the RHS, then by Theorem 9, B' is a Fibonacci sequence. We only need to show its initial values match with those of $B^{(0)}$.

Again, as in the proof of Theorem 9, we use the variables $s_1 = a_1 + \cdots + a_k$, $s_2 = a_2 + \cdots + a_k$, ..., $s_{k-1} = a_{k-1} + a_k$ and $s_k = a_k$, then $s_1 + s_2 + \cdots + s_k = n - k$. When $n = 0$, $s_1 + s_2 + \cdots + s_k = -k < 0$, so $\langle (a_1, a_2, \ldots, a_k)\rangle$ is non-zero only if $-1 \geq s_1 \geq s_2 \geq \cdots \geq s_k$. The only possibility is $s_1 = s_2 = \cdots = s_k = -1$ and this gives $a_1 = a_2 = \cdots = a_{k-1} = 0$, $a_k = -1$ and $B_0' = \langle (0, \ldots, 0, -1)\rangle = 1$.

When $1 \leq n \leq k-1$, $-(k-1) \leq s_1 + s_2 + \cdots + s_k = n - k < 0$. There are no such $-1 \geq s_1 \geq s_2 \geq \cdots \geq s_k$ that satisfy this condition, so the summation is empty and $B_n' = 0$ for $1 \leq n \leq k-1$. ∎

Theorem 11. *Let $k \geq 2$. Then*

$$B_n^{(k-1)} = \sum_{a_1+2a_2+\cdots+ka_k=n-k+1} \langle (a_1, a_2, \ldots, a_k)\rangle, \quad \text{for all } n \in \mathbb{Z}.$$

Proof. By Theorem 3(iii), $B^{(k-1)} = \mathtt{L}(B^{(0)})$, so using the formula for $B_n^{(0)}$ in Theorem 10 we obtain the desired formula for $B_n^{(k-1)}$. ∎

The formula in Theorem 11 is proved in Miles [18] for natural number $n \geq k-1$. Our Theorem 11 extends it to $n < k-1$ and negative integer n.

The Tribonacci sequence $\{T_n\}_{n\geq 0}$ studied in Rabinowitz [20] is a Fibonacci sequence of order $k = 3$ with initial values $T_0 = 0$, $T_1 = 1$, $T_2 = 1$. Solving for T_{-1}, we have $T_{-1} = 0$, so $T = \mathtt{L}(B^{(2)})$. The formula in Theorem 11 is proved in Rabinowitz [20] for $k=3$ and $n\geq 2$. Our Theorem 11 extends it to all order $k\geq 2$ and all index $n\in\mathbb{Z}$.

The next theorem give an explicit formula for all basis Fibonacci sequences of order k.

Theorem 12. *Let $k\geq 2$. For any $0\leq j\leq k-1$,*

$$B_n^{(j)}=\sum_{n-k-j\leq a_1+2a_2+\cdots+ka_k\leq n-k}\langle(a_1,a_2,\ldots,a_k)\rangle,\ \textit{for all } n\in\mathbb{Z}.$$

Proof. By Theorem 3(ii), $B^{(j)}=\sum_{i=0}^{j}\mathtt{R}^i(B^{(0)})$, so using the formula for $B_n^{(0)}$ in Theorem 10 we obtain the desired formula for $B_n^{(j)}$. ■

Theorem 11 and Theorem 12 give rise to two different formulas for the sequence $B^{(k-1)}$. It would be interesting to see a combinatorial proof of the equality of these two formulas.

5. A remark on a tiling problem

It is well known that the classical Fibonacci sequence, $F_0=0$, $F_1=1$, $F_n=F_{n-1}+F_{n-2}$, has a close relation with the tiling problem. The value F_n counts the number of tilings of an $1\times n$-board with square-tiles 1×1 and domino-tiles 1×2. This is because for $n\geq 2$, by considering the first tile, if the first tile is a square then there are F_{n-1} ways to cover the remaining strip of length $n-1$, and if the first tile is a domino then there are F_{n-2} ways to cover the remaining strip of length $n-2$. That is how the recursion equation $F_n=F_{n-1}+F_{n-2}$ arises.

If we allow tiles of length up to k, then the result is a sequence $\{C_n\}_{n\geq 0}$. We have $C_0=0$, $C_1=1$, $C_2=C_0+C_1$, $C_3=C_0+C_1+C_2,\ldots$, $C_{k-1}=C_0+C_1+\cdots+C_{k-2}$, and for $n\geq k$, $C_n=C_{n-1}+C_{n-2}+\cdots+C_{n-k}$. Of course, if we extend the index to negative integers and set $C_{-1}=C_{-2}=\cdots=C_{-(k-2)}=0$ then we have the Fibonacci recursion equation $C_n=C_{n-1}+C_{n-2}+\cdots+C_{n-k}$ holds for all $n\geq 2$. This sequence C is just a left shift of the basis sequence $B^{(k-1)}$. Indeed, $C=\mathtt{R}^{k-2}(B^{(k-1)})$. Many authors such as Gabai, Philippou, Muwafi, Benjamin, Heberle, Quinn and Su [19, 9, 1, 2] have studied this tiling problem and here we decide to use the letter C to denote this sequence since it is related to a *combinatorial* problem.

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