TRIPLICATE DUAL SERIES OF DOUGALL–DIXON THEOREM

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Abstract. Applying the triplicate form of the extended Gould–Hsu inverse series relations to Dougall's summation theorem for the well–poised ${}_7F_6$-series, we establish, from the dual series, several interesting Ramanujan–like infinite series expressions for π^2 and $\pi^{\pm 1}$ with convergence rate "$-\frac{1}{27}$".

1. Introduction and Motivation

For an indeterminate x, the shifted factorial is defined by $(x)_0 \equiv 1$ and

$$(x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

This can also be expressed as a quotient $(x)_n = \Gamma(x+n)/\Gamma(x)$, where the Γ-function (see [41, §8] for example) is given by the beta integral

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\mathrm{d}u \quad \text{for} \quad \Re(x) > 0,$$

which admits Euler's reflection property

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{with} \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{1}$$

and the following asymptotic formula

$$\Gamma(x+n) \approx n^x(n-1)! \quad \text{as} \quad n \to \infty. \tag{2}$$

This last formula is simpler than Stirling's formula and will frequently be utilized in this paper to evaluate limits of Γ-function quotients.

About one century ago, Ramanujan [42] discovered seventeen remarkable infinite series for $1/\pi$. Three typical ones are reproduced as

$$\frac{4}{\pi} = \sum_{k=0}^\infty \frac{(\frac{1}{2})_k^3}{k!^3}\frac{1+6k}{4^k},$$

$$\frac{8}{\pi} = \sum_{k=0}^\infty \frac{(\frac{1}{2})_k(\frac{1}{4})_k(\frac{3}{4})_k}{k!^3}\frac{3+20k}{(-4)^k},$$

$$\frac{16}{\pi} = \sum_{k=0}^\infty \frac{(\frac{1}{2})_k^3}{k!^3}\frac{5+42k}{64^k}.$$

In the same paper, Ramanujan recorded also the formula (see also [3, 8, 45]) below

$$\frac{9801}{\pi\sqrt{8}} = \sum_{k=0}^\infty \frac{(4k)!}{k!^4}\frac{1103+26390k}{396^{4k}},$$

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which was used by Gosper in 1985 to compute 17×10^6 digits of π, a new world record at that time. However, a rigorous proof for this formula was provided 70 years later by Borwein brothers [6] in 1987. Afterwards, several faster convergent series for $1/\pi$ were found. Here we highlight the formula due to Chan–Liaw–Tan [11]

$$\frac{3\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!(3k)!}{k!^5}\frac{827+14151k}{27^k\times 1000^{2k+1}},$$

and Chudnovskys' famous formula [28] (see also [3,7,45])

$$\frac{1}{\pi}=12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!}{k!^3(3k)!}\frac{13591409+545140134k}{640320^{3k+3/2}}.$$

This latter one enabled Chudnovsky brothers to hold the world record for the calculation of π-digits from 1989 to 1994.

One of the recent advances in representing π by infinite series was made by Guillera [37–39]. By making use of the powerful WZ-method, he proved several beautiful formulae of Ramanujan–like for $1/\pi^2$, such as

$$\begin{aligned}
\frac{8}{\pi^2}&=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\frac{(\frac12)_k^5}{k!^5}\Big\{1+8k+20k^2\Big\},\\
\frac{32}{\pi^2}&=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\frac{(\frac12)_k^3(\frac14)_k(\frac34)_k}{k!^5}\Big\{3+34k+120k^2\Big\},\\
\frac{48}{\pi^2}&=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\frac{(\frac12)_k^3(\frac13)_k(\frac23)_k}{k!^5}\Big\{3+27k+74k^2\Big\},\\
\frac{128}{\pi^2}&=\sum_{k=0}^{\infty}\Big(\frac{-1}{1024}\Big)^k\frac{(\frac12)_k^5}{k!^5}\Big\{13+180k+820k^2\Big\}.
\end{aligned}$$

More comprehensive investigation has been made by Chu and Zhang [25–27] by manipulating the classical hypergeometric series.

There exist double series analogues for Ramanujan–like series. For example, Chan–Tanigawa–Yang–Zudilin [12] found, by deriving new analogues of Clausen's product identity, the following formula (see [10,29] for more series)

$$\frac{1}{\pi}=\frac{3\sqrt{6}}{1225}\sum_{k=0}^{\infty}\binom{2k}{k}\sum_{j=0}^{k}\binom{k}{j}^3\frac{561k+53}{39200^k}.$$

By employing quadratic transformations of hypergeometric series, Zudilin [44] derived the series below for $1/\pi^2$

$$\frac{10\sqrt{5}}{\pi^2}=\sum_{k=0}^{\infty}\frac{(4k)!}{k!^2(2k)!}\frac{18k^2-10k-3}{6400^k}\sum_{j=0}^{k}\binom{2j}{j}^3\binom{2k-2j}{k-j}16^{k-j}.$$

By manipulating Ramanujan's Eisenstein series, Baruah and Berndt [4] obtained more identities of similar type, for instance

$$\frac{12}{\pi^2}=\sum_{k=0}^{\infty}(-1)^k\frac{16k^2+16k+5}{64^k}\sum_{j=0}^{k}\binom{2j}{j}^3\binom{2k-2j}{k-j}^3.$$

In comparison with the existing methods, this paper will take a totally different way to approach similar series involving π. Our starting point will be the general summation theorem for the terminating well-poised ${}_7F_6$-series discovered by Dougall [30, 1907]. By making use of the triplicate form of the extended Gould–Hsu inverse series relations, we shall investigate the dual series of Dougall's well-poised sum, that will lead to a large class of summation formulae for π-related infinite series. Instead of different convergence rates, such as the series by Ramanujan [42], Guillera [37–39] and by others [3–5,7,8,10,28,29,45], our series converge exclusively at the rate "$\frac{1}{729}$". According to the bisection series method, most of these series are then reduced to simpler ones with the same convergence rate "$-\frac{1}{27}$". We remark that there are only few series of Ramanujan-like with this convergence rate in the literature (cf. [24–27]). Five elegant formulae are anticipated as follows:

$$\frac{9\sqrt{3}}{\pi\sqrt[3]{16}} = \sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k \frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{6})_k}{k!^3}\big\{2+21k\big\}, \tag{Example 21}$$

$$\frac{27\sqrt{3}}{\pi\sqrt[3]{32}} = \sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k \frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{5}{6})_k}{k!^3}\big\{5+42k\big\}, \tag{Example 22}$$

$$\frac{3\sqrt{3}}{\pi\sqrt[3]{4}} = \sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k \frac{(\frac{1}{3})_k(-\frac{1}{3})_k(-\frac{1}{6})_k}{k!^3}\big\{1-63k^2\big\}, \tag{Example 25}$$

$$\frac{\pi\Gamma^2(\frac{1}{3})}{6\Gamma^2(\frac{5}{6})} = \sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k \frac{k!(\frac{1}{2})_k(\frac{2}{3})_k}{(\frac{4}{3})_k^3}\big\{3+7k\big\}, \tag{Example 4}$$

$$\frac{48\pi\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})} = \sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k \frac{k!(\frac{1}{2})_k(\frac{1}{3})_k}{(\frac{5}{3})_k^3}\big\{9+28k+21k^2\big\}. \tag{Example 35}$$

The rest of the paper will be organized as follows. The next section will serve as the theoretical part, where the main theorems and proofs will be included. Then in Section 3, we shall present 35 infinite series expressions for π^2 and $\pi^{\pm 1}$ as applications.

2. Triplicate Inversion of Dougall's ${}_7F_6$-Series

A half century ago, Gould and Hsu [36] discovered a useful pair of inverse series relations, which can equivalently be reproduced below. Let $\{a_i, b_i\}$ be any two complex sequences such that the ϕ-polynomials defined by

$$\phi(x;0) \equiv 1 \quad \text{and} \quad \phi(x;n) = \prod_{k=0}^{n-1}(a_k + x b_k) \quad \text{for} \quad n \in \mathbb{N} \tag{3}$$

differ from zero for x, $n \in \mathbb{N}_0$. Then there hold the inverse series relations

$$f(n) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \prod_{i=0}^{n-1}(a_i + k b_i)\; g(k), \tag{4}$$

$$g(n) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{a_k + k b_k}{\prod_{i=0}^{k}(a_i + n b_i)} f(k). \tag{5}$$

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In terms of ϕ-polynomials, they can be expressed compactly as

$$f(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\phi(k;n)\ g(k), \tag{4'}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a_k+kb_k}{\phi(n;k+1)}f(k). \tag{5'}$$

The following inverse relations are a special case of inverse relations found independently by Chu [14,19] and Krattenthaler [40]:

$$f(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\prod_{j=0}^{n-1}(a_j+\lambda b_j+kb_j)(a_j-kb_j)\frac{\lambda+2k}{(\lambda+n)_{k+1}}g(k), \tag{6}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a_k+\lambda b_k+kb_k)(a_k-kb_k)}{\prod_{j=0}^{k}(a_j+\lambda b_j+nb_j)(a_j-nb_j)}(\lambda+k)_n f(k). \tag{7}$$

By making use of ϕ-polynomials, they can be written in a more symmetric form

$$f(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\phi(\lambda+k;n)\phi(-k;n)\frac{\lambda+2k}{(\lambda+n)_{k+1}}g(k), \tag{6'}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a_k+\lambda b_k+kb_k)(a_k-kb_k)}{\phi(\lambda+n;k+1)\phi(-n;k+1)}(\lambda+k)_n f(k); \tag{7'}$$

where the 'adjunct' factor can be expressed transparently as

$$(a_k+\lambda b_k+kb_k)(a_k-kb_k)=\frac{\phi(\lambda+k;k+1)\phi(-k;k+1)}{\phi(\lambda+k;k)\phi(-k;k)}.$$

These inverse series relations have been shown powerful in dealing with terminating hypergeometric series identities [14–17,19,33]. Their duplicate forms and the corresponding q-analogues due to Carlitz [9] with respective applications were extensively explored in [13,21,22] and [1,18,20,23,31,32,34,35].

By employing the above inverse pair, we shall work out several new π-related infinite series expressions. Recall the fundamental identity discovered by Dougall [30] (see also Bailey [2, §4.3]) for very–well–poised terminating ${}_7F_6$-series

$$\begin{aligned}\Omega_n(a;b,c,d)\ :=\ &\frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}\\ =\sum_{k=0}^{n}\frac{a+2k}{a}&\frac{(a)_k(b)_k(c)_k(d)_k(e)_k(-n)_k}{k!(1+a-b)_k(1+a-c)_k(1+a-d)_k(1+a-e)_k(1+a+n)_k},\end{aligned} \tag{8}$$

where the series is 2-balanced because $1+2a+n=b+c+d+e$.

For all $n\in\mathbb{N}_0$, it is well known that $\boxed{n=\lfloor\frac{n}{3}\rfloor+\lfloor\frac{1+n}{3}\rfloor+\lfloor\frac{2+n}{3}\rfloor}$, where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x. Then it is not difficult to check that Dougall's formula (8) is equivalent to the following one

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$$\begin{aligned}\Omega_n\big(a;b+\lfloor\tfrac{n}{3}\rfloor,c+\lfloor\tfrac{1+n}{3}\rfloor,d+\lfloor\tfrac{2+n}{3}\rfloor\big)&=\frac{(1+a)_n}{(b+c+d-a)_n}\\
\times\frac{(1+a-c-d)_{\lfloor\frac{n}{3}\rfloor}}{(b-a)_{\lfloor\frac{n}{3}\rfloor}}&\frac{(1+a-b-d)_{\lfloor\frac{1+n}{3}\rfloor}}{(c-a)_{\lfloor\frac{1+n}{3}\rfloor}}\frac{(1+a-b-c)_{\lfloor\frac{2+n}{3}\rfloor}}{(d-a)_{\lfloor\frac{2+n}{3}\rfloor}}\\
\times\frac{(b+c-a)_{\lfloor\frac{2n}{3}\rfloor}}{(1+a-d)_{\lfloor\frac{2n}{3}\rfloor}}&\frac{(b+d-a)_{\lfloor\frac{1+2n}{3}\rfloor}}{(1+a-c)_{\lfloor\frac{1+2n}{3}\rfloor}}\frac{(c+d-a)_{\lfloor\frac{2+2n}{3}\rfloor}}{(1+a-b)_{\lfloor\frac{2+2n}{3}\rfloor}},\end{aligned}$$

with its parameters subject to $\boxed{1+2a=b+c+d+e}$.

Reformulate the above equality as a binomial sum

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+2k}{(a+n)_{k+1}}\frac{(a)_k(b)_k(c)_k(d)_k(1+2a-b-c-d)_k(b+k)_{\lfloor\frac{n}{3}\rfloor}}{(1+a-b)_k(1+a-c)_k(1+a-d)_k(b+c+d-a)_k}\\
&\times(b-a-k)_{\lfloor\frac{n}{3}\rfloor}(c+k)_{\lfloor\frac{1+n}{3}\rfloor}(c-a-k)_{\lfloor\frac{1+n}{3}\rfloor}(d+k)_{\lfloor\frac{2+n}{3}\rfloor}(d-a-k)_{\lfloor\frac{2+n}{3}\rfloor}\\
&=(b)_{\lfloor\frac{n}{3}\rfloor}(1+a-c-d)_{\lfloor\frac{n}{3}\rfloor}(c)_{\lfloor\frac{1+n}{3}\rfloor}(1+a-b-d)_{\lfloor\frac{1+n}{3}\rfloor}(1+a-b-c)_{\lfloor\frac{2+n}{3}\rfloor}\\
&\times\frac{(d)_{\lfloor\frac{2+n}{3}\rfloor}(a)_n}{(b+c+d-a)_n}\frac{(b+c-a)_{\lfloor\frac{2n}{3}\rfloor}}{(1+a-d)_{\lfloor\frac{2n}{3}\rfloor}}\frac{(b+d-a)_{\lfloor\frac{1+2n}{3}\rfloor}}{(1+a-c)_{\lfloor\frac{1+2n}{3}\rfloor}}\frac{(c+d-a)_{\lfloor\frac{2+2n}{3}\rfloor}}{(1+a-b)_{\lfloor\frac{2+2n}{3}\rfloor}}.\end{aligned}$$

This equality matches exactly to (6) under the assignments $\lambda\to a$ and

$$\phi(x;n)\to(b-a+x)_{\lfloor\frac{n}{3}\rfloor}(c-a+x)_{\lfloor\frac{1+n}{3}\rfloor}(d-a+x)_{\lfloor\frac{2+n}{3}\rfloor}$$

as well as

$$\begin{aligned}f(n)&\to n!\,(a)_n\times\mathcal{F}(n),\\
g(k)&\to\frac{(a)_k(b)_k(c)_k(d)_k(1+2a-b-c-d)_k}{(1+a-b)_k(1+a-c)_k(1+a-d)_k(b+c+d-a)_k};\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}(n)=&\frac{(b+c-a)_{\lfloor\frac{2n}{3}\rfloor}}{(1+a-d)_{\lfloor\frac{2n}{3}\rfloor}}\frac{(b+d-a)_{\lfloor\frac{1+2n}{3}\rfloor}}{(1+a-c)_{\lfloor\frac{1+2n}{3}\rfloor}}\frac{(c+d-a)_{\lfloor\frac{2+2n}{3}\rfloor}}{(1+a-b)_{\lfloor\frac{2+2n}{3}\rfloor}}\\
&\times\frac{(b)_{\lfloor\frac{n}{3}\rfloor}(1+a-c-d)_{\lfloor\frac{n}{3}\rfloor}(c)_{\lfloor\frac{1+n}{3}\rfloor}(1+a-b-d)_{\lfloor\frac{1+n}{3}\rfloor}}{n!}\\
&\times\frac{(d)_{\lfloor\frac{2+n}{3}\rfloor}(1+a-b-c)_{\lfloor\frac{2+n}{3}\rfloor}}{(b+c+d-a)_n}.\end{aligned}\tag{9}$$

For the sake of brevity, we introduce the ψ-polynomials by

$$\begin{aligned}\psi(x;n)=\phi(a+x;n)\phi(-x;n)=&(b+x)_{\lfloor\frac{n}{3}\rfloor}(c+x)_{\lfloor\frac{1+n}{3}\rfloor}(d+x)_{\lfloor\frac{2+n}{3}\rfloor}\\
&\times(b-a-x)_{\lfloor\frac{n}{3}\rfloor}(c-a-x)_{\lfloor\frac{1+n}{3}\rfloor}(d-a-x)_{\lfloor\frac{2+n}{3}\rfloor}.\end{aligned}\tag{10}$$

Then the dual relation corresponding to (7) can explicitly be stated, after some simplifications, in the following lemma.

Lemma 1. *For the $\mathcal{F}$-quotient of shifted factorials and the ψ-polynomials defined respectively in* (9) *and* (10), *we have the summation formula*

$$\begin{aligned}&\frac{(b)_n(c)_n(d)_n(1+2a-b-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(b+c+d-a)_n}\\&=\sum_{k=0}^{n}\mathcal{F}(k)\frac{\psi(k;k+1)}{\psi(k;k)}\frac{(-n)_k(a+n)_k}{\psi(n;k+1)}.\end{aligned}$$

Observe that $\psi(n;k+1)$ is a polynomial of degree $2k+2$ in n with the leading coefficient equal to $(-1)^{k+1}$. Now multiply by "n^2" across the binomial relation in Lemma 1 and then let $n\to\infty$. We may evaluate the limits of the left member by (2) and of the corresponding right member through Weierstrass's M-test on uniformly convergent series (cf. Stromberg [43, §3.106]). After some routine simplifications, the resulting limiting relation can be expressed explicitly as follows.

Proposition 2. *Let $\Gamma(a,b,c,d)$ stand for the quotient of the Γ-function given by*

$$\Gamma(a,b,c,d)=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(b+c+d-a)}{\Gamma(b)\ \Gamma(c)\ \Gamma(d)\ \Gamma(1+2a-b-c-d)}. \tag{11}$$

Then for the $\mathcal{F}$-quotient of shifted factorials and the ψ-polynomials defined respectively in (9) *and* (10), *the following infinite series identity holds:*

$$\Gamma(a,b,c,d)=-\sum_{k=0}^{\infty}\frac{\psi(k;k+1)}{\psi(k;k)}\mathcal{F}(k).$$

Let $\varepsilon=0,1,2$ and $T(k)$ be the summand with the index k in the last series. Putting the initial ε terms aside and then classifying the remaining terms with respect to their indices modulo 3, we get the expressions

$$\begin{aligned}\sum_{k=0}^{\infty}T(k)&=\sum_{k=0}^{\varepsilon-1}T(k)+\sum_{k=0}^{\infty}\sum_{i=0}^{2}T(\varepsilon+i+3k)\\&=\sum_{k=0}^{\varepsilon-1}T(k)+\sum_{k=1}^{\infty}\sum_{j=1}^{3}T(\varepsilon-j+3k).\end{aligned}$$

Denote further by $\sigma(\varepsilon)$, $\Delta_k(\varepsilon)$ and $\nabla_k(\varepsilon)$ the sum of initial ε-terms and the "weight functions" (where the latter are clearly rational functions of k):

$$\sigma(\varepsilon)\ =\sum_{k=0}^{\varepsilon-1}\left\{\frac{\psi(k;k+1)}{-\psi(k;k)}\right\}\mathcal{F}(k), \tag{12}$$

$$\Delta_k(\varepsilon)=\sum_{i=0}^{2}\left\{\frac{\psi(\varepsilon+i+3k;1+\varepsilon+i+3k)}{-\psi(\varepsilon+i+3k;\varepsilon+i+3k)}\right\}\frac{\mathcal{F}(\varepsilon+i+3k)}{\mathcal{F}(3k)}, \tag{13}$$

$$\nabla_k(\varepsilon)=\sum_{j=1}^{3}\left\{\frac{\psi(\varepsilon-j+3k;1+\varepsilon-j+3k)}{-\psi(\varepsilon-j+3k;\varepsilon-j+3k)}\right\}\frac{\mathcal{F}(\varepsilon-j+3k)}{\mathcal{F}(3k)}. \tag{14}$$

Then the identity in Proposition 2 can be restated in the theorem below.

Theorem 3. *Assume that $\Gamma(a,b,c,d)$, $\sigma(\varepsilon)$, $\Delta_k(\varepsilon)$ and $\nabla_k(\varepsilon)$ are as in* (11), (12), (13) *and* (14) *respectively. Then for $\varepsilon=0,1,2$ and the $\mathcal{F}$-quotient of shifted*

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factorials defined in (9), *the following infinite series identities hold:*

$$\begin{aligned}\Gamma(a,b,c,d) &= \sigma(\varepsilon)+\sum_{k=0}^{\infty}\Delta_k(\varepsilon)\mathcal{F}(3k)\\ &= \sigma(\varepsilon)+\sum_{k=1}^{\infty}\nabla_k(\varepsilon)\mathcal{F}(3k).\end{aligned}$$

In the above theorem, the series is expressed in two different manners because it happens frequently that a series with its summation index initiating at $k=0$ has better looking than that at $k=1$, or vice versa. This will be seen from our examples in the next section. In the above series, $\mathcal{F}(3k)$ results in the dominant part

$$\begin{aligned}\mathcal{F}(3k) = &\frac{(b+c-a)_{2k}(b+d-a)_{2k}(c+d-a)_{2k}}{(1+a-b)_{2k}(1+a-c)_{2k}(1+a-d)_{2k}}\\ \times &\frac{(b)_k(c)_k(d)_k(1+a-b-c)_k(1+a-b-d)_k(1+a-c-d)_k}{(3k)!\ (b+c+d-a)_{3k}},\end{aligned}$$

which determines the convergence rate of the series to be "$\frac{1}{729}$". Instead, both $\Delta_k(\varepsilon)$ and $\nabla_k(\varepsilon)$ are perturbing parts consisting of only a few terms. Therefore for specific values of ε and $\{a,b,c,d\}$, in order to find the infinite series identity, it is enough to compute the corresponding $\sigma(\varepsilon)$ and $\Delta_k(\varepsilon)$ (or $\nabla_k(\varepsilon)$), and then to simplify the resulting expression.

3. Infinite Series of Ramanujan Type Involving π

By specifying the parameters $\{a,b,c,d\}$, we can derive numerous new infinite series identities of convergence rate "$\frac{1}{729}$" from Theorem 3 with $\varepsilon = 0,1,2$. In general, the presence of complicated weight polynomials in the summands make the corresponding series look less elegant. However, they may sometimes indicate that the results are somewhat deeper than the usual. In fact, there exist several such fast convergent series (cf. [4, 7, 8, 10, 28]) as illustrated in the introduction.

Now we present two examples to illustrate how to derive infinite series identities by making use of Theorem 3. Letting

$$\varepsilon=0 \quad\text{and}\quad \{a,b,c,d\}=\Big\{\frac12,\frac12,\frac12,\frac14\Big\},$$

we can compute with *Mathematica* commands

$$\begin{aligned}\mathcal{F}(3k) &= \frac{(\frac12)_{2k}(\frac14)^2_{2k}}{(2k)!^2(\frac54)_{2k}}\frac{(\frac12)^3_k(\frac14)_k(\frac34)^2_k}{(3k)!\ (\frac34)_{3k}}\frac{(\frac12)_k(\frac14)_k(\frac34)^3_k(\frac18)^2_k(\frac58)_k}{k!^3\ (\frac13)_k(\frac23)_k(\frac98)_k(\frac{7}{12})_k(\frac{11}{12})_k}\Big(\frac{1}{729}\Big)^k,\\ \Delta_k(0) &= \frac{(1+8k)(93184k^4+154432k^3+85840k^2+17484k+855)}{768(1+3k)(2+3k)(7+12k)},\\ \sigma(0) &= 0 \quad\text{and}\quad \Gamma(\tfrac12,\tfrac12,\tfrac12,\tfrac14)=\frac{1}{4\pi}.\end{aligned}$$

Substituting them into Theorem 3 and then multiplying by "4×2688" across the resulting equation, we derive the following infinite series identity.

Example 1 ($\boxed{\varepsilon=0:\frac12,\frac12,\frac12,\frac14}$).

$$\frac{2688}{\pi}=\sum_{k=0}^{\infty}\frac{(\frac12)_k(\frac14)_k(\frac34)_k^3(\frac18)_k(\frac58)_k}{k!^3\;(\frac43)_k(\frac53)_k(\frac{11}{12})_k(\frac{19}{12})_k}\frac{93184k^4+154432k^3+85840k^2+17484k+855}{729^k}.$$

Analogously, we have another infinite series identity of similar type.

Example 2 ($\boxed{\varepsilon=1:\frac32,\frac56,1,1}$).

$$\frac{20\pi}{3\sqrt3}=10+\sum_{k=1}^{\infty}\frac{k!(-\frac12)_k(\frac23)_k(-\frac13)_k^2(\frac16)_k(-\frac56)_k}{(\frac13)_k(\frac43)_k(\frac14)_k(\frac34)_k(\frac49)_k(\frac79)_k(\frac{10}{9})_k}\frac{19656k^4-7749k^3-3150k^2+613k+118}{729^k}.$$

When $\boxed{b+c+d-a}$ equals to a half integer, the corresponding series in Theorem 3 can be reformulated, by means of the bisection series method, as a simpler series with convergence rate "$\frac{-1}{27}$". In order to show how this approach works, we present demonstrations in details for two infinite series identities.

We start with the following strange evaluation of a hypergeometric ${}_3F_2$-series.

Example 3 ($\boxed{\varepsilon=0:\frac53,\frac53,\frac23,\frac56}$).

$$\sum_{k=0}^{\infty}\frac{(\frac23)_k(\frac73)_k(-\frac16)_k}{k!^2\;(k+1)!}\Big(\frac{-1}{27}\Big)^k=\frac{81\sqrt3}{28\cdot2^{2/3}\pi}.$$

Proof. By specifying the parameters in Theorem 3

$$\varepsilon=0\quad\text{and}\quad\big\{a,b,c,d\big\}=\Big\{\frac53,\frac53,\frac23,\frac56\Big\}$$

we have

$$\begin{aligned}\mathcal{F}(3k)&=\frac{(\frac23)_{2k}(\frac56)_{2k}(-\frac16)_{2k}}{(2k)!\;(2)_{2k}(\frac{11}{6})_{2k}}\frac{(\frac13)_k(\frac23)_k(\frac53)_k(\frac16)_k(\frac56)_k(\frac76)_k}{(3k)!\;(\frac32)_{3k}}\\&=\frac{(\frac13)_k(\frac53)_k(\frac16)_k(\frac56)_k(\frac{5}{12})_k^2(\frac{-1}{12})_k}{k!^3(\frac12)_k^2(\frac32)_k(\frac{17}{12})_k}\Big(\frac{1}{729}\Big)^k,\\ \Delta_k(0)&=\frac{7(1+6k)(5+12k)(5616k^3+11358k^2+7233k+1465)}{78732(1+k)(1+2k)^2},\\ \sigma(0)&=0\quad\text{and}\quad\Gamma(\tfrac53,\tfrac53,\tfrac23,\tfrac56)=\frac{15\sqrt3}{8\cdot2^{\frac23}\pi};\end{aligned}$$

which leads us to the following identity

$$\frac{59049\sqrt3}{14\cdot2^{2/3}\pi}=\sum_{k=0}^{\infty}\frac{(\frac13)_k(\frac53)_k(\frac56)_k(\frac76)_k(\frac{5}{12})_k(\frac{-1}{12})_k}{k!^2(2)_k(\frac32)_k^3}\frac{5616k^3+11358k^2+7233k+1465}{729^k}.$$

We claim that the above series is the bisection of the series below

$$\sum_{k=0}^{\infty}\Lambda_k\quad\text{for}\quad\Lambda_k:=\frac{(\frac23)_k(\frac73)_k(-\frac16)_k}{k!^2\;(2)_k}\Big(\frac{-1}{27}\Big)^k.$$

This can be justified by computing

$$\Lambda_{2k}+\Lambda_{2k+1}=\frac{(\frac13)_k(\frac53)_k(\frac56)_k(\frac76)_k(\frac{5}{12})_k(\frac{-1}{12})_k}{k!^2(2)_k(\frac32)_k^3}\frac{5616k^3+11358k^2+7233k+1465}{1458\cdot729^k}.$$

Therefore, we can evaluate the following simpler series

$$\sum_{k=0}^{\infty}\Lambda_k=\sum_{k=0}^{\infty}\Big\{\Lambda_{2k}+\Lambda_{2k+1}\Big\}=\frac{1}{1458}\times\frac{59049\sqrt{3}}{14\cdot 2^{2/3}\pi}=\frac{81\sqrt{3}}{28\cdot 2^{2/3}\pi}. \qquad \square$$

Next, we prove the following elegant formula for a Ramanujan–like series.

Example 4 ($\boxed{\varepsilon=0:\frac{4}{3},1,1,\frac{5}{6}}$).

$$\frac{\pi\Gamma^2(\frac{1}{3})}{6\Gamma^2(\frac{5}{6})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{k!(\frac{1}{2})_k(\frac{2}{3})_k}{(\frac{4}{3})_k^3}\{3+7k\}.$$

Proof. By specializing the parameters in Theorem 3

$$\varepsilon=0 \quad\text{and}\quad \{a,b,c,d\}=\Big\{\frac{4}{3},1,1,\frac{5}{6}\Big\}$$

we can compute

$$\begin{aligned}
\mathcal{F}(3k)&=\frac{(\frac{1}{2})_{2k}^2(\frac{2}{3})_{2k}}{(\frac{3}{2})_{2k}(\frac{4}{3})_{2k}^2}\,\frac{k!^2(\frac{1}{2})_k^2(\frac{1}{3})_k(\frac{5}{6})_k}{(3k)!\;(\frac{3}{2})_{3k}}\,\frac{k!(\frac{1}{2})_k(\frac{1}{3})_k(\frac{1}{4})_k^2(\frac{3}{4})_k(\frac{5}{6})_k}{(\frac{2}{3})_k^3(\frac{5}{4})_k(\frac{7}{6})_k^3}\Big(\frac{1}{729}\Big)^k,\\
\Delta_k(0)&=\frac{(1+4k)(4368k^4+9742k^3+7799k^2+2588k+283)}{72(2+3k)^3},\\
\sigma(0)&=0 \quad\text{and}\quad \Gamma(\tfrac{4}{3},1,1,\tfrac{5}{6})=\frac{\pi\Gamma^2(\frac{1}{3})}{36\Gamma^2(\frac{5}{6})};
\end{aligned}$$

which gives rise to the following identity

$$\frac{16\pi\Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{5}{6})}=\sum_{k=0}^{\infty}\frac{k!(\frac{1}{2})_k(\frac{1}{3})_k(\frac{1}{4})_k(\frac{3}{4})_k(\frac{5}{6})_k}{(\frac{5}{3})_k^3(\frac{7}{6})_k^3}\,\frac{4368k^4+9742k^3+7799k^2+2588k+283}{729^k}.$$

For the sequence defined by

$$\Lambda_k:=(3+7k)\frac{k!(\frac{1}{2})_k(\frac{2}{3})_k}{(\frac{4}{3})_k^3}\Big(\frac{-1}{27}\Big)^k,$$

it is routine to compute the sum of its two consecutive terms

$$\Lambda_{2k}+\Lambda_{2k+1}\frac{k!(\frac{1}{2})_k(\frac{1}{3})_k(\frac{1}{4})_k(\frac{3}{4})_k(\frac{5}{6})_k}{(\frac{5}{3})_k^3(\frac{7}{6})_k^3}\,\frac{4368k^4+9742k^3+7799k^2+2588k+283}{96\cdot 729^k}.$$

Hence the afore–displayed series is equivalent to the following simpler series

$$\sum_{k=0}^{\infty}\Lambda_k=\sum_{k=0}^{\infty}\Big\{\Lambda_{2k}+\Lambda_{2k+1}\Big\}=\frac{1}{96}\times\frac{16\pi\Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{5}{6})}=\frac{\pi\Gamma^2(\frac{1}{3})}{6\Gamma^2(\frac{5}{6})}.$$

This completes the proof of the formula given in Example 4. $\square$

By carrying out the same procedure, we shall evaluate further 31 Ramanujan–like series in closed forms. Compared of the other existing π-related series of convergence rate "$\frac{-1}{27}$" in the literature (cf. [24–27]), all the formulae recorded below are believed to be new, except for Examples 5, 21 and 22. We shall divide the series into four classes according to their values and display them as examples. In each example, the parameter setting $\boxed{\varepsilon:a,b,c,d}$ and eventual references will be highlighted in the header. Furthermore, all the formulae are experimentally checked by an appropriately devised *Mathematica* package in order to ensure the accuracy.

§3.1. **Series for π^2.**

Example 5 (Chu and Zhang [26,27]: $\boxed{\varepsilon = 0 : \frac{3}{2}, 1, 1, 1}$).

$$\frac{\pi^2}{2} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{k!^2}{(\frac{4}{3})_k (\frac{5}{3})_k} \frac{5+7k}{1+2k}.$$

Example 6 ($\boxed{\varepsilon = 1 : \frac{3}{2}, 1, 1, 1}$).

$$9\pi^2 = 89 + \sum_{k=1}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{1}{2})_k\, k!(3)_k}{(\frac{5}{2})_k (\frac{5}{3})_k (\frac{7}{3})_k} \big\{17 + 14k\big\}.$$

Example 7 ($\boxed{\varepsilon = 2 : \frac{5}{2}, 1, 1, 2}$).

$$\frac{1575\pi^2}{8} = 1960 - \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{1}{2})_k\, k!(5)_k}{(\frac{9}{2})_k (\frac{7}{3})_k (\frac{8}{3})_k} \big\{17 + 7k\big\}.$$

Example 8 ($\boxed{\varepsilon = 2 : \frac{3}{2}, 1, 1, 1}$).

$$675\pi^2 = 6600 + \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{3}{2})_k\, k!(3)_k}{(\frac{7}{2})_k (\frac{7}{3})_k (\frac{8}{3})_k} \big\{63 + 59k + 14k^2\big\}.$$

§3.2. **Series for π^2/Γ^3.**

Example 9 ($\boxed{\varepsilon = 0 : \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}}$).

$$\frac{2\pi^2}{\Gamma^3(\frac{2}{3})} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{2}{3})_k^2 (\frac{1}{6})_k}{k!\, (\frac{4}{3})_k (\frac{7}{6})_k} \big\{8 + 21k\big\}.$$

Example 10 ($\boxed{\varepsilon = 1 : \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}}$).

$$\frac{2\pi^2}{\Gamma^3(\frac{1}{3})} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{1}{3})_k^2 (-\frac{1}{6})_k}{k!\, (\frac{2}{3})_k (\frac{5}{6})_k} \big\{1 + 21k\big\}.$$

Example 11 ($\boxed{\varepsilon = 0 : \frac{3}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{3}}$).

$$\frac{45\pi^2}{\Gamma^3(\frac{1}{3})} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{1}{3})_k (\frac{7}{3})_k (-\frac{1}{6})_k}{k!\, (\frac{5}{3})_k (\frac{11}{6})_k} \big\{23 + 42k\big\}.$$

Example 12 ($\boxed{\varepsilon = 1 : \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}}$).

$$\frac{5\pi^2}{3\Gamma^3(\frac{2}{3})} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{5}{3})_k (-\frac{1}{3})_k (-\frac{5}{6})_k}{k!\, (\frac{1}{3})_k (\frac{7}{6})_k} \big\{5 - 42k\big\}.$$

Example 13 ($\boxed{\varepsilon = 0 : \frac{3}{2}, \frac{-1}{3}, \frac{2}{3}, \frac{5}{3}}$).

$$\frac{55\pi^2}{2\Gamma^3(\frac{1}{3})} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \frac{(\frac{10}{3})_k (-\frac{2}{3})_k (-\frac{7}{6})_k}{k!\, (\frac{2}{3})_k (\frac{17}{6})_k} \big\{16 + 21k\big\}.$$

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Example 14 ($\boxed{\varepsilon=2:\frac{3}{2},\frac{1}{3},\frac{1}{3},\frac{4}{3}}$).
$$\frac{91\pi^2}{16\Gamma^3(\frac{2}{3})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{11}{3})_k(-\frac{1}{3})_k(-\frac{5}{6})_k}{k!\,(\frac{4}{3})_k(\frac{19}{6})_k}\big\{23+21k\big\}.$$

Example 15 ($\boxed{\varepsilon=1:\frac{3}{2},\frac{4}{3},\frac{1}{3},\frac{1}{3}}$).
$$\frac{175\pi^2}{36\Gamma^3(\frac{2}{3})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{5}{3})_k^2(-\frac{5}{6})_k^2}{k!\,(\frac{1}{3})_k(\frac{7}{6})_k(\frac{13}{6})_k}\big\{25+42k\big\}.$$

Example 16 ($\boxed{\varepsilon=0:\frac{5}{2},\frac{2}{3},\frac{5}{3},\frac{5}{3}}$).
$$\frac{825\pi^2}{8\Gamma^3(\frac{1}{3})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{7}{3})_k^2(-\frac{1}{6})_k^2}{k!\,(\frac{5}{3})_k(\frac{11}{6})_k(\frac{17}{6})_k}\big\{53+42k\big\}.$$

Example 17 ($\boxed{\varepsilon=1:\frac{3}{2},\frac{5}{3},\frac{2}{3},\frac{2}{3}}$).
$$\frac{60\pi^2}{\Gamma^3(\frac{1}{3})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{4}{3})_k(-\frac{2}{3})_k(-\frac{1}{6})_k}{k!\,(\frac{2}{3})_k(\frac{11}{6})_k}\big\{32+111k+126k^2\big\}.$$

Example 18 ($\boxed{\varepsilon=0:\frac{3}{2},\frac{1}{3},\frac{4}{3},\frac{4}{3}}$).
$$\frac{21\pi^2}{\Gamma^3(\frac{2}{3})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{5}{3})_k(-\frac{1}{3})_k(\frac{1}{6})_k}{k!\,(\frac{4}{3})_k(\frac{13}{6})_k}\big\{83+195k+126k^2\big\}.$$

Example 19 ($\boxed{\varepsilon=1:\frac{3}{2},\frac{2}{3},\frac{-1}{3},\frac{5}{3}}$).
$$\frac{715\pi^2}{12\Gamma^3(\frac{1}{3})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{13}{3})_k(-\frac{5}{3})_k(-\frac{13}{6})_k}{k!\,(\frac{2}{3})_k(\frac{17}{6})_k}\big\{13+51k-126k^2\big\}.$$

Example 20 ($\boxed{\varepsilon=1:\frac{3}{2},\frac{2}{3},\frac{2}{3},\frac{2}{3}}$).
$$\frac{35\pi^2}{12\Gamma^3(\frac{1}{3})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(\frac{7}{3})_k(-\frac{7}{6})_k}{k!\,(\frac{2}{3})_k(\frac{11}{6})_k}\frac{7-75k-126k^2}{(1-6k)(5+6k)}.$$

§3.3. **Series for π^{-1}.**

Example 21 (Chu [25, 26]: $\boxed{\varepsilon=0:\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{6}}$).
$$\frac{9\sqrt{3}}{2^{\frac{4}{3}}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{6})_k}{k!^3}\big\{2+21k\big\}.$$

Example 22 (Chu [25, 26]: $\boxed{\varepsilon=0:\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{5}{6}}$).
$$\frac{27\sqrt{3}}{2^{\frac{5}{3}}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{5}{6})_k}{k!^3}\big\{5+42k\big\}.$$

Example 23 ($\boxed{\varepsilon=1:\frac{1}{3},\frac{1}{6},\frac{1}{3},\frac{1}{3}}$).
$$\frac{729\sqrt{3}}{20\sqrt[3]{2}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(\frac{8}{3})_k(\frac{1}{6})_k}{k!(2)_k^2}\big\{16+21k\big\}.$$

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Example 24 ($\boxed{\varepsilon=2:\frac{5}{3},\frac{5}{6},-\frac{1}{3},\frac{5}{3}}$).

$$\frac{2673\sqrt{3}}{14\sqrt[3]{4}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{2}{3})_k(\frac{13}{3})_k(\frac{5}{6})_k(-\frac{7}{6})_k}{k!^2(3)_k(\frac{17}{6})_k}\big\{65+42k\big\}.$$

Example 25 ($\boxed{\varepsilon=1:\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{-1}{6}}$).

$$\frac{3\sqrt{3}}{\pi\sqrt[3]{4}}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(-\frac{1}{3})_k(-\frac{1}{6})_k}{k!^3}\big\{1-63k^2\big\}.$$

Example 26 ($\boxed{\varepsilon=0:\frac{4}{3},\frac{1}{3},\frac{4}{3},\frac{7}{6}}$).

$$\frac{81\sqrt{3}}{2^{\frac{1}{3}}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(-\frac{1}{3})_k(\frac{7}{6})_k}{k!^2(2)_k}\big\{35+90k+63k^2\big\}.$$

Example 27 ($\boxed{\varepsilon=1:\frac{2}{3},\frac{5}{6},\frac{2}{3},\frac{2}{3}}$).

$$\frac{2187\sqrt{3}}{10\sqrt[3]{4}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{11}{6})_k}{k!(2)_k^2}\big\{77+144k+63k^2\big\}.$$

Example 28 ($\boxed{\varepsilon=1:\frac{1}{3},\frac{7}{6},\frac{1}{3},\frac{1}{3}}$).

$$\frac{2187\sqrt{3}}{14\sqrt[3]{2}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{4}{3})_k(-\frac{1}{3})_k(\frac{13}{6})_k}{k!(2)_k^2}\big\{65+195k+126k^2\big\}.$$

Example 29 ($\boxed{\varepsilon=1:\frac{4}{3},\frac{7}{6},\frac{1}{3},\frac{4}{3}}$).

$$\frac{6561\sqrt{3}}{40\sqrt[3]{2}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{4}{3})_k(\frac{11}{3})_k(\frac{1}{6})_k}{k!(2)_k(3)_k}\frac{143+285k+126k^2}{(1-3k)(2+3k)}.$$

Example 30 ($\boxed{\varepsilon=1:\frac{4}{3},\frac{1}{6},\frac{1}{3},\frac{4}{3}}$).

$$\frac{2187\sqrt{3}}{440\sqrt[3]{2}\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(\frac{1}{3})_k(\frac{14}{3})_k(-\frac{5}{6})_k}{k!(2)_k(3)_k}\frac{196+333k+126k^2}{(7+6k)(13+6k)}.$$

§3.4. **Series for π.**

Example 31 ($\boxed{\varepsilon=1:\frac{2}{3},\frac{1}{6},1,1}$).

$$\frac{9\pi\,\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})}=1+\sum_{k=1}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(3)_k(\frac{1}{2})_k(-\frac{2}{3})_k}{(\frac{2}{3})_k(\frac{5}{3})_k^2}\big\{13+21k\big\}.$$

Example 32 ($\boxed{\varepsilon=1:\frac{4}{3},\frac{5}{6},1,1}$).

$$\frac{4\pi\,\Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{5}{6})}=71+\sum_{k=1}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{(3)_k(\frac{1}{2})_k(\frac{2}{3})_k}{(\frac{4}{3})_k(\frac{7}{3})_k^2}\big\{23+21k\big\}.$$

Example 33 ($\boxed{\varepsilon=0:\frac{2}{3},1,1,\frac{1}{6}}$).

$$\frac{3\pi\,\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})}=\sum_{k=1}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{k!(\frac{1}{2})_k(-\frac{2}{3})_k}{(\frac{2}{3})_k^2(\frac{5}{3})_k}k\big\{13+21k\big\}.$$

Example 34 ($\boxed{\varepsilon=0:\frac{1}{3},1,1,\frac{-1}{6}}$).

$$\frac{\pi\Gamma^2(\frac{1}{3})}{12\Gamma^2(\frac{5}{6})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{k!(\frac{1}{2})_k(-\frac{4}{3})_k}{(\frac{1}{3})_k^2(\frac{4}{3})_k}\big\{21k^2+8k-3\big\}.$$

Example 35 ($\boxed{\varepsilon=0:\frac{5}{3},1,1,\frac{7}{6}}$).

$$\frac{48\pi\Gamma^2(\frac{2}{3})}{\Gamma^2(\frac{1}{6})}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\frac{k!(\frac{1}{2})_k(\frac{1}{3})_k}{(\frac{5}{3})_k^3}\big\{9+28k+21k^2\big\}.$$

Concluding Comments. There exist different ways to invert Dougall's ${}_7F_6$-sum through (6) and (7). However, all the dual series that we detected by *Mathematica* are ugly because of the presence of very complicated weight polynomials. Here is a couple of discouraging examples.

By examining another triplicate form of Dougall's ${}_7F_6$-sum

$$\Omega_n\big(a;b+\lfloor\tfrac{1+n}{3}\rfloor,c,d+\lfloor\tfrac{1+2n}{3}\rfloor\big)=\frac{(1+a-c-d)_{\lfloor\frac{1+n}{3}\rfloor}(b+c-a)_{\lfloor\frac{1+n}{3}\rfloor}}{(1+a-d)_{\lfloor\frac{1+n}{3}\rfloor}(b-a)_{\lfloor\frac{1+n}{3}\rfloor}}$$
$$\times\frac{(1+a)_n(b+d-a)_n}{(1+a-c)_n(b+c+d-a)_n}\frac{(1+a-b-c)_{\lfloor\frac{1+2n}{3}\rfloor}(c+d-a)_{\lfloor\frac{1+2n}{3}\rfloor}}{(1+a-b)_{\lfloor\frac{1+2n}{3}\rfloor}(d-a)_{\lfloor\frac{1+2n}{3}\rfloor}},$$

we can arrive, under the parameter setting $\varepsilon=1$ and $\{a,b,c,d\}=\Big\{\frac{5}{2},2,\frac{5}{4},1\Big\}$ and after a long and tedious computations, at the following series for π:

$$\frac{75\pi}{8}=30+\sum_{k=1}^{\infty}\Big(\frac{16}{729}\Big)^k\frac{k!(-\frac{1}{2})_k(\frac{1}{6})_k(-\frac{1}{6})_k(\frac{1}{8})_k(-\frac{1}{8})_k(-\frac{3}{8})_k(-\frac{5}{8})_k}{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{3}{4})_k(\frac{5}{4})_k(\frac{7}{12})_k(\frac{11}{12})_k(\frac{13}{12})_k(\frac{17}{12})_k}$$
$$\times\Big\{60-101k+1075k^2-4840k^3-49360k^4+136896k^5\Big\}.$$

Analogously, by specifying parameters $\varepsilon=0$ and $\{a,b,c,d\}=\Big\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{3}\Big\}$, we get another series for π^{-1}:

$$\frac{1485\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{16}{729}\Big)^k\frac{(\frac{1}{2})_k(\frac{2}{3})_k(\frac{1}{4})_k(\frac{3}{4})_k(\frac{1}{6})_k(\frac{1}{9})_k(\frac{4}{9})_k(\frac{7}{9})_k}{k!^3(\frac{4}{3})_k^2(\frac{17}{18})_k(\frac{23}{18})_k(\frac{29}{18})_k}$$
$$\times\ \Big\{812+20373k+169774k^2+634857k^3+1091016k^4+693036k^5\Big\}.$$

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