# COHOMOLOGY AND DEFORMATIONS OF GENERALIZED REYNOLDS OPERATORS ON LEIBNIZ ALGEBRAS

SHUANGJIAN GUO AND APURBA DAS

Abstract. In this paper, we introduce generalized Reynolds operators on Leibniz algebras as a generalization of twisted Poisson structures. We define the cohomology of a generalized Reynolds operator $K$ as the Loday-Pirashvili cohomology of a certain Leibniz algebra induced by $K$ with coefficients in a suitable representation. Then we consider formal deformations of generalized Reynolds operators from cohomological points of view. Finally, we introduce and study NS-Leibniz algebras as the underlying structure of generalized Reynolds operators.

## Introduction

Rota-Baxter operators on associative algebras were introduced in 1960 by Baxter [3] in his study of the fluctuation theory in probability. Recently, it has been found that Rota-Baxter operators are closely related with dendriform algebras, pre-Lie algebras and Yang-Baxter equations [1]. Also, it has applications in Connes-Kreimer's algebraic approach to the renormalization in quantum field theory [7]. The notion of (relative) Rota-Baxter operators on Lie algebras first appeared in the work of Kuperschmidt [15] while studying classical $r$-matrices. The notion of Leibniz algebras was introduced by Bloh as a noncommutative analogue of Lie algebras [5]. Relative Rota-Baxter operators on Leibniz algebras were studied in [24]. They are useful in twisting theory and the bialgebra theory for Leibniz algebras. More precisely, let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra and $(V, \rho^L, \rho^R)$ be a representation of it. A linear map $K : V \to \mathfrak{g}$ is a relative Rota-Baxter operator on $\mathfrak{g}$ with respect to the representation $V$ if it satisfies

$$[Ku, Kv] = K(\rho^L(Ku)v + \rho^R(Kv)u), \quad \text{for } u, v \in V. \tag{0. 1}$$

Such operators can be seen as the Leibniz algebraic analogue of Poisson structures. Generally, Rota-Baxter operators can be defined on algebraic operads which give rise to the splitting of operads [4,21]. For further details on Rota-Baxter operators, see [13].

Deformations of algebraic structures began with the seminal work of Gerstenhaber [12] for associative algebras. It has been extended to Lie algebras by Nijenhuis and Richardson [19,20]. In general, deformation theory was developed for algebras over binary quadratic operads by Balavoine [2]. Recently, deformations of relative Rota-Baxter operators (also called $\mathcal{O}$-operators) are developed in [9,25,26].

In [23] Ševera and Weinstein introduced a notion of twisted Poisson structure as a Dirac structure in a certain twisted Courant algebroid. Deformation quantizations of twisted Poisson structures are considered in [22]. Twisted Poisson structures are also studied by Klimčík and Strobl [14]. The corresponding algebraic notion, called twisted Rota-Baxter operators was introduced by Uchino [27] in the context of associative algebras and find relations with NS-algebras of Leroux [17] (see also [11]). Recently, one of the present authors introduced twisted Rota-Baxter operators on Lie algebras and considerd NS-Lie algebras as the underlying structure [10]. Reynolds operators on Lie algebras are natural examples of twisted Rota-Baxter operators. Thus, twisted Rota-Baxter operators can be thought of as generalized Reynolds operators. It is worth mentioning that Zhang, Gao and Guo [28] gave the construction of free objects of Reynolds algebras by bracketed words and rooted trees.

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Our aim in this paper is to consider generalized Reynolds operators on Leibniz algebras. On the one hand, such operators are Leibniz analogue of twisted Poisson structures. On the other hand, such operators are useful in the splitting of Leibniz algebras. Generalized Reynolds operators are also related to Nijenhuis operators on Leibniz algebras, and as a result are related to deformations of Leibniz algebras. We show that a generalized Reynolds operator $K$ induces a new Leibniz algebra structure and there is a suitable representation of it. The corresponding Loday-Pirashvili cohomology is called the cohomology of the generalized Reynolds operator $K$. This cohomology can be seen as the Leibniz analogue of the cohomology of twisted Poisson structures [16]. Motivated from deformations of twisted Poisson structures [22], here we study deformations of a generalized Reynolds operator $K$. We show that the infinitesimal in a formal deformation of $K$ is a 1-cocycle in the cohomology of $K$. Moreover, we define a notion of equivalence between two formal deformations of $K$. The infinitesimals corresponding to equivalent deformations are shown to be cohomologous. We introduce Nijenhuis elements associated with a generalized Reynolds operator $K$ and find a sufficient condition for the rigidity of $K$ in terms of Nijenhuis elements.

In the end, we introduce NS-Leibniz algebras as the underlying structure of generalized Reynolds operators on Leibniz algebras. We show that NS-Leibniz algebras split Leibniz algebras. We provide some examples of NS-Leibniz algebras. In particular, we construct NS-Leibniz algebras from Nijenhuis operators on Leibniz algebras. Further study on NS-Leibniz algebras is postponed to a forthcoming article.

The paper is organized as follows. In Section 2, we introduce generalized Reynolds operators on Leibniz algebras and study some properties. In Section 3, we define cohomology of a generalized Reynolds operator. This cohomology has been used in Section 4 to study deformations of a generalized Reynolds operator. Finally, in Section 5, we introduce NS-Leibniz algebras and find their relation with generalized Reynolds operators.

Throughout this paper, all vector spaces, (multi)linear maps are over the field $\mathbb{C}$ of complex numbers and all the vector spaces are finite-dimensional.

## 1. Leibniz algebras and Loday-Pirashvili cohomology

In this section, we recall Leibniz algebras and their cohomology. Our main references are [8,18].

1.1. **Definition.** A Leibniz algebra is a vector space $\mathfrak{g}$ together with a bilinear operation (called bracket) $[~,~] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]], \quad \text{for } x, y, z \in \mathfrak{g}. \tag{1.1}$$

A Leibniz algebra as above may be denoted by the pair $(\mathfrak{g}, [~,~])$ or simply by $\mathfrak{g}$ when no confusion arises. A Leibniz algebra whose bilinear bracket is skewsymmetric is nothing but a Lie algebra. Thus, Leibniz algebras are the non-skewsymmetric analogue of Lie algebras.

1.2. **Definition.** A representation of a Leibniz algebra $(\mathfrak{g}, [~,~])$ consists of a triple $(V, \rho^L, \rho^R)$ in which $V$ is a vector space and $\rho^L, \rho^R : \mathfrak{g} \to gl(V)$ are linear maps satisfying for $x, y \in \mathfrak{g}$,

$$\begin{cases} \rho^L([x,y]) = \rho^L(x) \circ \rho^L(y) - \rho^L(y) \circ \rho^L(x), \\ \rho^R([x,y]) = \rho^L(x) \circ \rho^R(y) - \rho^R(y) \circ \rho^L(x), \\ \rho^R([x,y]) = \rho^L(x) \circ \rho^R(y) + \rho^R(y) \circ \rho^R(x). \end{cases}$$

It follows that any Leibniz algebra $\mathfrak{g}$ is a representation of itself with

$$\rho^L(x) = L_x = [x,~~] \text{ and } \rho^R(x) = R_x = [~,x], \text{ for } x \in \mathfrak{g}.$$

Here $L_x$ and $R_x$ denotes the left and right multiplications by $x$, respectively. This is called the regular representation.

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Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra and $(V, \rho^L, \rho^R)$ be a representation of it. The cohomology of the Leibniz algebra $\mathfrak{g}$ with coefficients in $V$ is the cohomology of the cochain complex $\{C^*(\mathfrak{g}, V), \partial^*\}$, where $C^n(\mathfrak{g}, V) = \mathrm{Hom}(\mathfrak{g}^{\otimes n}, V)$ for $n \geq 0$, and the coboundary operator $\partial^n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V)$ given by

$$(1.2)\qquad \begin{aligned}&(\partial^n f)(x_1, \ldots, x_{n+1})\\ &= \sum_{i=1}^{n} (-1)^{i+1} \rho^L(x_i) f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) + (-1)^{n+1} \rho^R(x_{n+1}) f(x_1, \ldots, x_n)\\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^i f(x_1, \ldots, \hat{x}_i, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_{n+1}),\end{aligned}$$

for $x_1, \ldots, x_{n+1} \in \mathfrak{g}$. The corresponding cohomology groups are denoted by $H^*(\mathfrak{g}, V)$. This cohomology first appeared in [8] and was rediscovered by Loday and Pirashvili [18]. This cohomology is also called the Loday-Pirashvili cohomology.

1.3. **Remark.** The Leibniz algebras considered in Definition 1.1 are called left Leibniz algebras as the identity (1.1) is equivalent to the derivation property of the left multiplication maps $[x,~]$, for $x \in \mathfrak{g}$. In various papers of Loday (including the paper with Pirashvili [18]) he considered right Leibniz algebras. There is no major difference between left Leibniz algebras and right Leibniz algebras. More precisely, $(\mathfrak{g}, [~,~])$ is a left Leibniz algebra if and only if $(\mathfrak{g}, [~,~]^{op})$ is a right Leibniz algebra, where $[x, y]^{op} = [y, x]$, for $x, y \in \mathfrak{g}$. In this paper, we deal with left Leibniz algebras and the coboundary (1.2) represents the corresponding coboundary operator. This explicit formula is different than the one given in [18] as they have considered right Leibniz algebras. However, they are equivalent via the obvious correspondence between a representation of the left Leibniz algebra $\mathfrak{g}$ to a representation of the right Leibniz algebra $\mathfrak{g}^{op}$.

## 2. Generalized Reynolds operators

In this section, we introduce generalized Reynolds operators on Leibniz algebras, provide some examples and new constructions.

Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra and $(V, \rho^L, \rho^R)$ be a representation of it. Suppose $H \in C^2(\mathfrak{g}, V)$ is a 2-cocycle in the Loday-Pirashvili cochain complex, i.e., $H : \mathfrak{g} \otimes \mathfrak{g} \to V$ is a bilinear map satisfying

$$\rho^L(x) H(y, z) - \rho^L(y) H(x, z) - \rho^R(z) H(x, y) - H([x, y], z) - H(y, [x, z]) + H(x, [y, z]) = 0,$$

for $x, y, z \in \mathfrak{g}$.

2.1. **Definition.** A linear map $K : V \to \mathfrak{g}$ is said to a generalized Reynolds operator on $\mathfrak{g}$ with respect to the representation $V$ if the map $K$ satisfies

$$[Ku, Kv] = K(\rho^L(Ku)v + \rho^R(Kv)u + H(Ku, Kv)),$$

for all $u, v \in V$ and some fixed 2-cocycle $H$.

Let $(L, [~,~]_L)$ be a Lie algebra and $(V, \rho)$ be a representation of it. For any Chevalley-Eilenberg 2-cocycle $H \in C^2_{\mathrm{CE}}(L, V)$, a generalized Reynolds operator on $L$ with respect to the representation $V$ is a linear map $K : V \to L$ satisfying

$$[Ku, Kv]_L = K(\rho(Ku)v - \rho(Kv)u + H(Ku, Kv)), \quad \text{for } u, v \in V.$$

See [10] for more details. Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra and $(V, \rho^L, \rho^R)$ be a representation satisfying $\rho^L(x)v = -\rho^R(x)v$, for $x \in \mathfrak{g}$ and $x \in V$ (such representations are called symmetric representations in [18]). Consider the Lie algebra $\mathfrak{g}_{\mathrm{Lie}} = \mathfrak{g} / I$, where $I$ is the ideal of $\mathfrak{g}$ generated by elements of the form $[x, x]$, for $x \in \mathfrak{g}$. Then $(V, \rho)$ is a representation of the Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$, where

$$\rho : \mathfrak{g}_{\mathrm{Lie}} \otimes V \to V, \rho(|x|, v) = \rho^L(x)v, \text{ for all } x \in \mathfrak{g}, v \in V.$$

Moreover, if $H \in C^2_{\mathrm{CE}}(\mathfrak{g}_{\mathrm{Lie}}, V)$ is a 2-cocycle in the 2-cocycle in the Chevalley-Eilenberg cochain complex, then $H$ is obviously a 2-cocycle in the Loday-Pirashvili cochain complex.

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2.2. **Proposition.** *With the above notations, if $K : V \to \mathfrak{g}$ is a generalized Reynolds operator on the Leibniz algebra $\mathfrak{g}$, then $|K| : V \to \mathfrak{g}_{\mathrm{Lie}}$, $|K|(v) = |Kv|$ is a generalized Reynolds operator on the Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$.*

2.3. **Example.** Any relative Rota-Baxter operator (0. 1) is a generalized Reynolds operator with $H = 0$.

2.4. **Example.** Let $(V, \rho^L, \rho^R)$ be a representation of a Leibniz algebra $(\mathfrak{g}, [~,~])$. Suppose $h \in C^1(\mathfrak{g}, V)$ is an invertible 1-cochain in the Loday-Pirashvili cochain complex of $\mathfrak{g}$ with coefficients in $V$. Take $H = -\partial^1 h$. Then

$$H(Ku, Kv) = (-\partial^1 h)(Ku, Kv) = -\rho^L(K(u))v - \rho^R(K(v))u + h([Ku, Kv]), \quad \text{for } u, v \in V.$$

This shows that $K = h^{-1} : V \to \mathfrak{g}$ is a generalized Reynolds operator.

2.5. **Example.** Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra and $N : \mathfrak{g} \to \mathfrak{g}$ be a Nijenhuis operator on it, i.e., $N$ satisfies

$$[Nx, Ny] = N([Nx, y] + [x, Ny] - N[x, y]), \quad \text{for } x, y \in \mathfrak{g}.$$

In this case, the vector space $\mathfrak{g}$ carries a new Leibniz algebra structure with (deformed) bracket

$$[x, y]_N := [Nx, y] + [x, Ny] - N[x, y], \text{ for } x, y \in \mathfrak{g}. \tag{2. 1}$$

This deformed Leibniz algebra $\mathfrak{g}_N = (\mathfrak{g}, [~,~]_N)$ has a representation on $\mathfrak{g}$ by $\rho^L(x)y := [Nx, y]$ and $\rho^R(x)y := [y, Nx]$, for $x \in \mathfrak{g}_N, y \in \mathfrak{g}$. With this representation, the map $H : (\mathfrak{g}_N)^{\otimes 2} \to \mathfrak{g}$, $H(x, y) = -N[x, y]$ is a 2-cocycle in the Loday-Pirashvili cohomology of $\mathfrak{g}_N$ with coefficients in $\mathfrak{g}$. Moreover, the identity map $\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}_N$ is a generalized Reynolds operator.

2.6. **Example.** Let $(\mathfrak{g}, [~,~])$ be a three-dimensional Leibniz algebra whose bracket is given with respect to a basis $\{e_1, e_2, e_3\}$ by

$$[e_1, e_1] = e_3.$$

Let $H : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ be the bilinear map defined by

$$H(e_1, e_1) = e_1 \text{ and } H(e_3, e_1) = -e_3.$$

It is easy to check that $H$ is a 2-cocycle. Then $K = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a generalized Reynolds operator on $(\mathfrak{g}, [~,~])$ with respect to the regular representation if and only if

$$[Ke_i, Ke_j] = K([Ke_i, e_j] + [e_i, Ke_j] + H(Ke_i, Ke_j)), \quad \text{for } i, j = 1, 2, 3.$$

We have $[Ke_1, Ke_1] = [a_{11}e_1 + a_{21}e_2 + a_{31}e_3, a_{11}e_1 + a_{21}e_2 + a_{31}e_3] = a_{11}^2 e_3$. On the other hand,

$$\begin{aligned}
&K([Ke_1, e_1] + [e_1, Ke_1] + H(Ke_1, Ke_1)) \\
&= K([a_{11}e_1 + a_{21}e_2 + a_{31}e_3, e_1] + [e_1, a_{11}e_1 + a_{21}e_2 + a_{31}e_3] + a_{11}^2 e_1 - a_{31}a_{11}e_3) \\
&= 2a_{11}Ke_3 + a_{11}^2 Ke_1 - a_{31}a_{11}Ke_3 \\
&= (2a_{11}a_{13} - a_{31}a_{11}a_{13} + a_{11}^3)e_1 + (2a_{11}a_{23} + a_{11}^2 a_{21} - a_{31}a_{11}a_{23})e_2 \\
&+ (2a_{11}a_{33} + a_{11}^2 a_{31} - a_{31}a_{11}a_{33})e_3.
\end{aligned}$$

Thus, by $[Ke_1, Ke_1] = K([Ke_1, e_1] + [e_1, Ke_1] + H(Ke_1, Ke_1))$, we have

$$\begin{aligned}
&a_{11}^2 = 2a_{11}a_{33} + a_{11}^2 a_{31} - a_{31}a_{11}a_{33}, \ 2a_{11}a_{13} - a_{31}a_{11}a_{13} + a_{11}^3 = 0, \\
&2a_{11}a_{23} + a_{11}^2 a_{21} - a_{31}a_{11}a_{23} = 0.
\end{aligned}$$

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By considering other choices of $e_i$ and $e_j$, we obtain

$$\begin{aligned}
&a_{11}a_{12} = a_{12}a_{33} + a_{11}a_{12}a_{31} - a_{31}a_{12}a_{33},\ a_{11}^2a_{12} + a_{12}a_{13} - a_{31}a_{12}a_{23} = 0,\\
&a_{12}a_{23} + a_{11}a_{12}a_{21} - a_{31}a_{12}a_{23} = 0;\\
&a_{11}a_{13} = a_{13}a_{33} + a_{11}a_{13}a_{31} - a_{31}a_{13}a_{33},\ a_{11}^2a_{13} + a_{13}a_{13} - a_{31}a_{13}a_{23} = 0,\\
&a_{13}a_{23} + a_{11}a_{13}a_{21} - a_{31}a_{13}a_{23} = 0;\\
&a_{12}a_{11} = a_{12}a_{33} + a_{12}a_{11}a_{31} - a_{32}a_{11}a_{33},\ a_{12}a_{13} + a_{11}^2a_{12} - a_{32}a_{11}a_{13} = 0,\\
&a_{12}a_{23} + a_{12}a_{11}a_{21} - a_{32}a_{11}a_{23} = 0;\\
&a_{13}a_{11} = a_{13}a_{33} + a_{13}a_{11}a_{31} - a_{33}a_{11}a_{33},\ a_{13}a_{13} + a_{11}^2a_{13} - a_{33}a_{11}a_{13} = 0,\\
&a_{13}a_{23} + a_{13}a_{11}a_{21} - a_{33}a_{11}a_{23} = 0;
\end{aligned}$$

$$\begin{aligned}
&a_{12}^2 = a_{12}^2a_{31} - a_{32}a_{12}a_{33},\ a_{12}^2a_{21} - a_{32}a_{12}a_{23} = 0,\\
&a_{12}^2a_{31} - a_{32}a_{12}a_{33} = 0;\\
&a_{13}^2 = a_{13}^2a_{31} - a_{33}^2a_{13}a_{33},\ a_{13}^2a_{21} - a_{33}a_{13}a_{23} = 0,\\
&a_{13}^2a_{31} - a_{33}a_{13}a_{33} = 0;\\
&a_{12}a_{13} = a_{12}a_{13}a_{31} - a_{32}a_{13}a_{33},\ a_{12}a_{13}a_{11} - a_{32}a_{13}a_{13} = 0,\\
&a_{12}a_{13}a_{21} - a_{32}a_{13}a_{23} = 0.
\end{aligned}$$

Summarizing the above observations, we see that if $a_{11} = a_{12} = a_{13} = 0$, then the linear map $K$ is a generalized Reynolds operator on $(\mathfrak{g}, [\ ,\ ])$ with respect to the regular representation.

Let $K : V \to \mathfrak{g}$ be a generalized Reynolds operator. Suppose $(V', \rho'^L, \rho'^R)$ is a representation of another Leibniz algebra $(\mathfrak{g}', [\ ,\ ]')$ and $H' \in C^2(\mathfrak{g}', V')$ is a 2-cocycle. Let $K' : V' \to \mathfrak{g}'$ be an another generalized Reynolds operator.

2.7. **Definition.** A morphism of generalized Reynolds operators from $K$ to $K'$ consists of a pair $(\phi, \psi)$ of a Leibniz algebra morphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ and a linear map $\psi : V \to V'$ satisfying

$$\begin{aligned}
&\phi \circ K = K' \circ \psi,\\
&\psi(\rho^L(x)u) = \rho'^L(\phi(x))\psi(u), \quad \psi(\rho^R(x)u) = \rho'^R(\phi(x))\psi(u),\\
&\psi \circ H = H' \circ (\phi \otimes \phi), \text{ for } x \in \mathfrak{g}, u \in V.
\end{aligned}$$

Given a 2-cocycle $H$ in the Loday-Pirashvili cochain complex of $\mathfrak{g}$ with coefficients in $V$, one can construct the semidirect product algebra. More precisely, the direct sum $\mathfrak{g} \oplus V$ carries a Leibniz algebra structure with the bracket given by

$$[(x, u), (y, v)]_H := ([x, y], \rho^L(x)v + \rho^R(y)u + H(x, y)), \quad \text{for } x, y \in \mathfrak{g}, u, v \in V.$$

We denote this semidirect product Leibniz algebra by $\mathfrak{g} \ltimes_H V$. Using this semidirect product, one can characterize generalized Reynolds operators by their graph.

2.8. **Proposition.** *A linear map $K : V \to \mathfrak{g}$ is a generalized Reynolds operator if and only if its graph $Gr(K) = \{(Ku, u) |\ u \in V\}$ is a subalgebra of the semidirect product $\mathfrak{g} \ltimes_H V$.*

The proof of the above proposition is straightforward, hence we omit the details. Since $Gr(K)$ is isomorphic to $V$ as a vector space, as a consequence, we get the following.

2.9. **Proposition.** *Let $K : V \to \mathfrak{g}$ be a generalized Reynolds operator. Then the vector space $V$ carries a Leibniz algebra structure with the bracket*

$$[u, v]_K := \rho^L(Ku)v + \rho^R(Kv)u + H(Ku, Kv), \quad \textit{for } u, v \in V.$$

Some new constructions

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In this subsection, we construct new generalized Reynolds operators out of an old one by suitable modifications. We start with the following.

2.10. **Proposition.** *Let $(V, \rho^L, \rho^R)$ be a representation of a Leibniz algebra $(\mathfrak{g}, [\ ,\ ])$. For any 2-cocycle $H \in C^2(\mathfrak{g}, V)$ and 1-cochain $h \in C^1(\mathfrak{g}, V)$, the semidirect product Leibniz algebras $\mathfrak{g} \ltimes_H V$ and $\mathfrak{g} \ltimes_{H+\partial^1 h} V$ are isomorphic.*

**Proof.** We define an isomorphism $\Psi_h : \mathfrak{g} \ltimes_H V \to \mathfrak{g} \ltimes_{H+\partial^1 h} V$ of the underlying vector spaces by $\Psi_h(x, u) := (x, u - h(x))$, for $(x, u) \in \mathfrak{g} \ltimes_H V$. Then we have

$$\begin{aligned}
&\Psi_h([(x,u),(y,v)]_H) \\
&= \Psi_h([x,y], \rho^L(x)v + \rho^R(y)u + H(x,y)) \\
&= ([x,y], \rho^L(x)v + \rho^R(y)u + H(x,y) - h[x,y]) \\
&= ([x,y], \rho^L(x)v + \rho^R(y)u + H(x,y) - \rho^L(x)h(y) - \rho^R(y)h(x) + (\partial^1 h)(x,y)) \\
&= [(x, u - h(x)), (y, v - h(y))]_{H+\partial^1 h} \\
&= [\Psi_h(x,u), \Psi_h(y,v)]_{H+\partial^1 h}.
\end{aligned}$$

This shows that $\Psi_h$ is in fact an isomorphism of Leibniz algebras. □

2.11. **Proposition.** *Let $K : V \to \mathfrak{g}$ be a generalized Reynolds operator. For any 1-cochain $h \in C^1(\mathfrak{g}, V)$, if the linear map $(\mathrm{Id}_V - h \circ K) : V \to V$ is invertible, then the map $K \circ (\mathrm{Id}_V - h \circ K)^{-1} : V \to \mathfrak{g}$ is a generalized Reynolds operator.*

**Proof.** Consider the subalgebra $Gr(K) \subset \mathfrak{g} \ltimes_H V$ of the semidirect product. By Proposition 2.10, we get that

$$\Psi_h(Gr(K)) = \{(Ku, u - hK(u)) |\ u \in V\} \subset \mathfrak{g} \ltimes_{H+\partial^1 h} V$$

is a subalgebra. Since the map $(\mathrm{Id}_V - h \circ K) : V \to V$ is invertible, we have $\Psi_h(Gr(K))$ is the graph of the linear map $K \circ (\mathrm{Id}_V - h \circ K)^{-1}$. Hence by Proposition 2.8, the map $K \circ (\mathrm{Id}_V - h \circ K)^{-1}$ is a generalized Reynolds operator. □

Let $K : V \to \mathfrak{g}$ be a generalized Reynolds operator. Suppose $B \in C^1(\mathfrak{g}, V)$ is a 1-cocycle in the Loday-Pirashvili cochain complex of $\mathfrak{g}$ with coefficients in $V$. Then $B$ is said to be $K$-admissible if the linear map $(\mathrm{Id}_V + B \circ K) : V \to V$ is invertible. With this notation, we have the following.

2.12. **Proposition.** *Let $B \in C^1(\mathfrak{g}, V)$ be a $K$-admissible 1-cocycle. Then the map $K \circ (\mathrm{Id}_V + B \circ K)^{-1} : V \to g$ is a generalized Reynolds operator.*

**Proof.** Consider the deformed subspace

$$\tau_B(Gr(K)) = \{(Ku, u + B \circ K(u)) |\ u \in V\} \subset \mathfrak{g} \ltimes_H V.$$

Since $B$ is a 1-cocycle, $\tau_B(Gr(K)) \subset \mathfrak{g} \ltimes_H V$ turns out to be a subalgebra. Further, the map $(\mathrm{Id}_V + B \circ K)$ is invertible implies that $\tau_B(Gr(K))$ is the graph of the map $K \circ (\mathrm{Id}_V + B \circ K)^{-1}$. Hence the result follows from Proposition 2.8. □

The generalized Reynolds operator in the above proposition is called the gauge transformation of $K$ associated with $B$. We denote this generalized Reynolds operator simply by $K_B$.

2.13. **Proposition.** *Let $K$ be a generalized Reynolds operator and $B$ be a $K$-admissible 1-cocycle. Then the Leibniz algebra structures on $V$ induced from the generalized Reynolds operators $K$ and $K_B$ are isomorphic.*

**Proof.** Consider the linear isomorphism $(\mathrm{Id}_V + B \circ K) : V \to V$. Moreover, for any $u, v \in V$, we have

$$\begin{aligned}
&[(\mathrm{Id}_V + B\circ K)(u), (\mathrm{Id}_V + B\circ K)(v)]_{K_B}\\
&= \rho^L(K(u))(\mathrm{Id}_V + B\circ K)(v) + \rho^R(K(v))(\mathrm{Id}_V + B\circ K)(u) + H(Ku, Kv)\\
&= \rho^L(K(u))v + \rho^R(K(v))u + \rho^L(K(u))(B\circ K(v)) + \rho^R(K(v))(B\circ K(u)) + H(Ku, Kv)\\
&= \rho^L(K(u))v + \rho^R(K(v))u + B[Ku, Kv] + H(Ku, Kv)\\
&= [u, v]_K + B\circ K([u, v]_K)\\
&= (\mathrm{Id}_V + B\circ K)([u, v]_K).
\end{aligned}$$

This shows that $(\mathrm{Id}_V + B\circ K) : (V, [\ ,\ ]_K) \to (V, [\ ,\ ]_{K_B})$ is a Leibniz algebra isomorphism. □

## 3. Cohomology of Generalized Reynolds operators

In this section, we define the cohomology of a generalized Reynolds operator $K$ as the Loday-Pirashvili cohomology of the Leibniz algebra $(V, [\ ,\ ]_K)$ constructed in Proposition 2.9 with coefficients in a suitable representation on $\mathfrak{g}$. In the next section, we will use this cohomology to study deformations of $K$. We start with the following result.

3.1. **Proposition.** *Let $K : V \to \mathfrak{g}$ be a generalized Reynolds operator. Define maps $\overline{\rho}^L, \overline{\rho}^R : V \to gl(\mathfrak{g})$ by*

$$\overline{\rho}^L(u)x = [Ku, x] - K(\rho^R(x)u) - KH(Ku, x) \quad \text{and} \quad \overline{\rho}^R(u)x = [x, Ku] - K(\rho^L(x)u) - KH(x, Ku),$$

*for $u \in V$ and $x \in \mathfrak{g}$. Then $(\mathfrak{g}, \overline{\rho}^L, \overline{\rho}^R)$ is a representation of the Leibniz algebra $(V, [\ ,\ ]_K)$.*

**Proof.** For $u, v \in V$ and $x \in \mathfrak{g}$ , we have

$$\begin{aligned}
&\overline{\rho}^L(u)\overline{\rho}^L(v)x - \overline{\rho}^L(v)\overline{\rho}^L(u)x\\
&= \overline{\rho}^L(u)([Kv, x] - K(\rho^R(x)v) - KH(Kv, x)) - \overline{\rho}^L(v)([Ku, x] - K(\rho^R(x)u) - KH(Ku, x))\\
&= [Ku, [Kv, x]] - [Ku, K(\rho^R(x)v)] - [Ku, KH(Kv, x)] - K(\rho^R([Kv, x])u) + K(\rho^R(K\rho^R(x)v)u)\\
&\quad + K(\rho^R(KH(Kv, x))u) - KH(Ku, [Kv, x]) + KH(Ku, K(\rho^R(x)v)) + KH(Ku, KH(Kv, x))\\
&\quad - [Kv, [Ku, x]] + [Kv, K(\rho^R(x)u)] + [Kv, KH(Ku, x)] + K(\rho^R([Ku, x])v) - K(\rho^R(K\rho^R(x)u)v)\\
&\quad - K(\rho^R(KH(Ku, x))v) + KH(Kv, [Ku, x]) - KH(Kv, K(\rho^R(x)u)) - KH(Kv, KH(Ku, x))\\
&= [[Ku, Kv], x] - K(\rho^R(x)\rho^L(Ku)v) - K(\rho^R(x)\rho^R(Kv)u) - K(\rho^R(x)H(Ku, Kv)) - KH(K[u, v]_K, x)\\
&= [K[u, v]_K, x] - K(\rho^R(x)[u, v]_K) - KH(K[u, v]_K, x)\\
&= \overline{\rho}^L([u, v]_K)x.
\end{aligned}$$

The third equality is obtained by some cancellations and using the fact that $H$ is a 2-cocycle. Thus, we deduce that

$$\overline{\rho}^L([u, v]_K) = \overline{\rho}^L(u)\overline{\rho}^L(v) - \overline{\rho}^L(v)\overline{\rho}^L(u).$$

We also have

$$\begin{aligned}
&\overline{\rho}^L(u)\overline{\rho}^R(v)x - \overline{\rho}^R(v)\overline{\rho}^L(u)x\\
&= \overline{\rho}^L(u)([x, Kv] - K(\rho^L(x)v) - KH(x, Kv)) - \overline{\rho}^R(v)([Ku, x] - K(\rho^R(x)u) - KH(Ku, x))\\
&= [Ku, [x, Kv]] - [Ku, K(\rho^L(x)v)] - [Ku, KH(x, Kv)] - K(\rho^R([x, Kv])u) + K(\rho^R(K\rho^L(x)v)u)\\
&\quad + K(\rho^R(KH(x, Kv))u) - KH(Ku, [x, Kv]) + KH(Ku, K(\rho^L(x)v)) + KH(Ku, KH(x, Kv))\\
&\quad - [[Ku, x], Kv] + [K(\rho^R(x)u), Kv] + [KH(Ku, x), Kv] + K(\rho^L([Ku, x])v) - K(\rho^L(K\rho^R(x)u)v)\\
&\quad - K(\rho^L(KH(Ku, x))v) + KH([Ku, x], Kv) - KH(K(\rho^R(x)u), Kv) - KH(KH(Ku, x), Kv)\\
&= [x, [Ku, Kv]] - K(\rho^L(x)\rho^L(Ku)v) - K(\rho^L(x)\rho^R(Kv)u) - K\rho^L(x)H(Ku, Kv)) - KH(x, K[u, v]_K)\\
&= [x, K[u, v]_K] - K(\rho^L(x)[u, v]_K) - KH(x, K[u, v]_K)\\
&= \overline{\rho}^R([u, v]_K)x,
\end{aligned}$$

which shows that

$$\overline{\rho}^R([u,v]_K) = \overline{\rho}^L(u)\overline{\rho}^R(v) - \overline{\rho}^R(v)\overline{\rho}^L(u).$$

Similarly, we can show that

$$\overline{\rho}^R([u,v]_K) = \overline{\rho}^R(v) \circ \overline{\rho}^R(u) + \overline{\rho}^L(u) \circ \overline{\rho}^R(v).$$

Therefore, $(\mathfrak{g}, \overline{\rho}^L, \overline{\rho}^R)$ is a representation of the Leibniz algebra $(V, [~,~]_K)$. □

It follows from the above proposition that we may consider the Loday-Pirashvili cohomology of the Leibniz algebra $(V, [~,~]_K)$ with coefficients in the representation $(\mathfrak{g}, \overline{\rho}^L, \overline{\rho}^R)$. More precisely, we define

$$C^n_K(V, \mathfrak{g}) := \mathrm{Hom}(V^{\otimes n}, \mathfrak{g}), \quad \text{for } n \geq 0$$

and the differential $\partial^n_K : C^n_K(V, \mathfrak{g}) \to C^{n+1}_K(V, \mathfrak{g})$ by

$$\begin{aligned}
&(\partial^n_K f)(u_1, \ldots, u_{n+1}) \\
=& \sum_{i=1}^{n} (-1)^{i+1} [Ku_i, f(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1})] - \sum_{i=1}^{n} (-1)^{i+1} K(\rho^R(f(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1}))u_i) \\
&- \sum_{i=1}^{n} (-1)^{i+1} KH(Ku_i, f(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1})) + (-1)^{n+1} [f(u_1, \ldots, u_n), Ku_{n+1}] \\
&+ (-1)^n K(\rho^L(f(u_1, \ldots, u_n))u_{n+1}) + (-1)^n KH(f(u_1, \ldots, u_n), Ku_{n+1}) \\
&+ \sum_{1 \leq i < j \leq n+1} (-1)^i f(u_1, \ldots, \hat{u}_i, \ldots, u_{j-1}, \rho^L(Ku_i)u_j + \rho^R(Ku_j)u_i + H(Ku_i, Ku_j), u_{j+1}, \ldots, u_{n+1}),
\end{aligned}$$

for $f \in C^n_K(V, \mathfrak{g})$ and $u_1, \ldots, u_{n+1} \in V$. Then $\{C^*_K(V, \mathfrak{g}), \partial^*_K\}$ is a cochain complex. We denote by

$$Z^n_K(V, \mathfrak{g}) = \{f \in C^n_K(V, \mathfrak{g}) \mid \partial^n_K f = 0\} \quad \text{and} \quad B^n_K(V, \mathfrak{g}) = \{\partial^{n-1}_K g \mid g \in C^{n-1}_K(V, \mathfrak{g})\}$$

the space of $n$-cocycles and $n$-coboundaries, respectively. The corresponding quotients

$$H^n_K(V, \mathfrak{g}) := \frac{Z^n_K(V, \mathfrak{g})}{B^n_K(V, \mathfrak{g})}, \text{ for } n \geq 0$$

are called the cohomology of the generalized Reynolds operator $K$.

## 4. Deformations of generalized Reynolds operators

In this section, we will apply the classical deformation theory of Gerstenhaber to generalized Reynolds operators. We will introduce certain elements (called Nijenhuis elements) associated with a generalized Reynolds operator that arise from trivial linear deformations. We also consider rigidity of a generalized Reynolds operator and find a sufficient condition for rigidity in terms of Nijenhuis elements.

### Linear deformations

Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra, $(V, \rho^L, \rho^R)$ be a representation of it, and $H \in C^2(\mathfrak{g}, V)$ be a 2-cocycle in the Loday-Pirashvili cochain complex. Suppose $K : V \to \mathfrak{g}$ is a generalized Reynolds operator.

4.1. **Definition.** A linear map $K_1 : V \to \mathfrak{g}$ is said to generate a linear deformation of the generalized Reynolds operator $K$ if for all $t \in \mathbb{C}$, the sum $K_t = K + tK_1$ is still a generalized Reynolds operator. In this case, $K_t = K + tK_1$ is said to be a linear deformation of $K$.

Suppose $K_1$ generates a linear deformation of $K$. Then we have

$$[K_t u, K_t v] = K_t\big(\rho^L(K_t u)v + \rho^R(K_t v)u + H(K_t u, K_t v)\big), \text{ for } u, v \in V.$$

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This is equivalent to the following conditions

$$[Ku, K_1v] + [K_1u, Kv] = K_1(\rho^L(Ku)v + \rho^R(Kv)u + H(Ku, Kv)) + K(\rho^L(K_1u)v + \rho^R(K_1v)u + H(K_1u, Kv) + H(Ku, K_1v)), \tag{4. 1}$$

$$[K_1u, K_1v] = K_1(\rho^L(K_1u)v + \rho^R(K_1v)u + H(Ku, K_1v) + H(K_1u, Kv)) + KH(K_1u, K_1v), \tag{4. 2}$$

$$K_1(H(K_1(u), K_1(v))) = 0. \tag{4. 3}$$

Note that Eq. (4.1) means that $K_1$ is a 1-cocycle in the cohomology of $K$. Hence $K_1$ induces an element in $H^1_K(V, \mathfrak{g})$.

4.2. **Definition.** Two linear deformations $K_t = K + tK_1$ and $K'_t = K + tK'_1$ of $K$ are said to be equivalent if there exists an element $x \in \mathfrak{g}$ such that

$$(\phi_t = \mathrm{Id}_\mathfrak{g} + tL_x,\ \psi_t = \mathrm{Id}_V + t(\rho^L(x) + H(x, K-))$$

is a morphism of generalized Reynolds operators from $K_t$ to $K'_t$.

The condition that $\phi_t = \mathrm{Id}_\mathfrak{g} + tL_x$ is a Leibniz algebra morphism of $(\mathfrak{g}, [\ ,\ ])$ is equivalent to

$$[[x, y], [x, z]] = 0, \quad \text{for } y, z \in \mathfrak{g}. \tag{4. 4}$$

Further, the conditions $\psi_t(\rho^L(y)u) = \rho^L(\phi_t(y))\psi_t(u)$ and $\psi_t(\rho^R(y)u) = \rho^R(\phi_t(y))\psi_t(u)$, for $y \in \mathfrak{g}, u \in V$ are respectively equivalent to

$$\begin{cases} H(x, K(\rho^L(y)u)) = \rho^L(y)H(x, Ku), \\ \rho^L([x, y])(\rho^L(x)u + H(x, Ku)) = 0, \end{cases} \tag{4. 5}$$

$$\begin{cases} H(x, K(\rho^R(y)u)) = \rho^R(y)H(x, Ku), \\ \rho^R([x, y])(\rho^L(x)u + H(x, Ku)) = 0. \end{cases} \tag{4. 6}$$

Similarly, the conditions $\psi_t \circ H = H \circ (\phi_t \otimes \phi_t)$ and $\phi_t \circ K_t = K'_t \circ \psi_t$ are respectively equivalent to

$$\begin{cases} \rho^L(x)H(y, z) + H(x, KH(y, z)) = H(x, [y, z]) + H(y, [x, z]), \\ H([x, y], [x, z]) = 0, \end{cases} \tag{4. 7}$$

$$\begin{cases} K_1(u) + [x, Ku] = K(\rho^L(x)u + H(x, Ku)) + K'_1(u), \\ [x, K_1u] = K'_1(\rho^L(x)u + H(x, Ku)). \end{cases} \tag{4. 8}$$

It follows from the first identity in (4. 8) that $K_1(u) - K'_1(u) = \partial^0_K(x)(u)$. Hence we obtain the following.

4.3. **Theorem.** *If two linear deformations $K_t = K + tK_1$ and $K'_t = K + tK'_1$ of a generalized Reynolds operator $K$ are equivalent, then $K_1$ and $K'_1$ are in the same cohomology class of $H^1_K(V, \mathfrak{g})$.*

4.4. **Definition.** A linear deformation $K_t = K + tK_1$ of a generalized Reynolds operator $K$ is said to be trivial if $K_t$ is equivalent to the undeformed deformation $K'_t = K$.

We will now define Nijenhuis elements associated with a generalized Reynolds operator $K$ in a way that a trivial deformation of $K$ induces a Nijenhuis element.

4.5. **Definition.** Let $K$ be a generalized Reynolds operator. An element $x \in \mathfrak{g}$ is called a Nijenhuis element associated with $K$ if $x$ satisfies

$$[x, \overline{\rho}^R(u)(x)] = 0, \quad \text{for } u \in V$$

and Equations (4. 4), (4. 5), (4. 6), (4. 7) hold.

The set of all Nijenhuis elements associated with $K$ is denoted by $\mathrm{Nij}(K)$. As mentioned earlier, a trivial (linear) deformation induces a Nijenhuis element. In the next subsection, we give a sufficient condition for the rigidity of a generalized Reynolds operator in terms of Nijenhuis elements.

Formal deformations

Let $\mathbb{C}[[t]]$ be the ring of power series in one variable $t$. For any $\mathbb{C}$-linear space $V$, we let $V[[t]]$ denotes the vector space of formal power series in $t$ with coefficients in $V$. Moreover, if $(\mathfrak{g}, [~,~])$ is a Leibniz algebra over $\mathbb{C}$, then one can extend the Leibniz bracket on $\mathfrak{g}[[t]]$ by $\mathbb{C}[[t]]$-bilinearity. Furthermore, if $(V, \rho^L, \rho^R)$ is a representation of the Leibniz algebra $(\mathfrak{g}, [~,~])$, then there is a representation $(V[[t]], \rho^L, \rho^R)$ of the Leibniz algebra $\mathfrak{g}[[t]]$. Here, $\rho^L$ and $\rho^R$ are also extended by $\mathbb{C}[[t]]$-bilinearity. Similarly, a 2-cocycle $H$ can be extended to a 2-cocycle (which we denote by the same notation $H$) on the Leibniz algebra $\mathfrak{g}[[t]]$ with coefficients in $V[[t]]$.

Let $K : V \to \mathfrak{g}$ be a generalized Reynolds operator on the Leibniz algebra $(\mathfrak{g}, [~,~])$ with respect to the representation $(V, \rho^L, \rho^R)$. We consider a power series of the form

$$K_t = \sum_{i=0}^{+\infty} K_i t^i, \quad \text{for } K_i \in \mathrm{Hom}(V, \mathfrak{g}) \quad \text{with } K_0 = K.$$

Extend $K_t$ to a linear map from $V[[t]]$ to $\mathfrak{g}[[t]]$ by $\mathbb{C}[[t]]$-linearity which we still denote by $K_t$.

4.6. **Definition.** A formal deformation of $K$ is given by a formal power series $K_t = \sum_{i=0}^{+\infty} K_i t^i$ with $K_0 = K$, such that $K_t$ satisfies the following:

$$[K_t u, K_t v] = K_t\big(\rho^L(K_t u)v + \rho^R(K_t v)u + H(K_t(u), K_t(v))\big), \quad \text{for } u, v \in V. \tag{4. 9}$$

It follows that $K_t$ is a generalized Reynolds operator on the Leibniz algebra $\mathfrak{g}[[t]]$ with respect to the representation $V[[t]]$.

4.7. **Remark.** If $K_t = \sum_{i=0}^{+\infty} K_i t^i$ is a formal deformation of the generalized Reynolds operator $K$ on the Leibniz algebra $(\mathfrak{g}, [~,~])$ with respect to the representation $(V, \rho^L, \rho^R)$, then $[~,~]_{K_t}$ defined by

$$[u, v]_{K_t} := \sum_{i=0}^{+\infty} \Big(\rho^L(K_i u)v + \rho^R(K_i v)u + \sum_{j+k=i} H(K_j u, K_k v)\Big) t^i, \quad \text{for } u, v \in V,$$

is a formal deformation of the associated Leibniz algebra $(V, [~,~]_K)$.

By expanding the identity (4. 9) and comparing coefficients of various powers of $t$, we obtain for $n \geq 0$,

$$\sum_{i+j=n} [K_i u, K_j v] = \sum_{i+j=n} K_i(\rho^L(K_j u)v + \rho^R(K_j v)u) + \sum_{i+j+k=n} K_i H(K_j(u), K_k(v)),$$

for $u, v \in V$. It holds for $n = 0$ as $K$ is a generalized Reynolds operator. For $n = 1$, we obtain

$$\begin{aligned}[Ku, K_1 v] + [K_1 u, Kv] = {} & K_1(\rho^L(Ku)v + \rho^R(Kv)u + H(Ku, Kv)) \\ & + K(\rho^L(K_1 u)v + \rho^R(K_1 v)u + H(K_1(u), Kv) + H(K(u), K_1 v)).\end{aligned}$$

This condition is equivalent to $(\partial^1_K(K_1))(u, v) = 0$, for $u, v \in V$.

Next, we define an equivalence between two formal deformations of a generalized Reynolds operator.

4.8. **Definition.** Two formal deformations $K_t = \sum_{i=0}^{+\infty} K_i t^i$ and $K'_t = \sum_{i=0}^{+\infty} K'_i t^i$ of a generalized Reynolds operator $K$ are said to be equivalent if there exists an element $x \in \mathfrak{g}$, linear maps $\phi_i \in gl(\mathfrak{g})$ and $\psi_i \in gl(V)$ for $i \geq 2$ such that the pair

$$\big(\phi_t = \mathrm{Id}_{\mathfrak{g}} + tL_x + \sum_{i=2}^{+\infty} \phi_i t^i, ~ \psi_t = \mathrm{Id}_V + t(\rho^L(x) + H(x, K-)) + \sum_{i=2}^{+\infty} \psi_i t^i\big)$$

is a morphism of $H$-Generalized Reynolds operators from $K_t$ to $K'_t$.

By equating coefficients of $t$ from both sides of the identity $\phi_t \circ K_t = K'_t \circ \psi_t$, we obtain

$$K_1(u) - K'_1(u) = K(\rho^L(x)u + H(x, Ku)) - [x, Ku] = \partial^1_K(x)(u), \text{ for } u \in V.$$

As a summary, we get the following.

4.9. **Theorem.** *The linear term of a formal deformation of a generalized Reynolds operator $K$ is a 1-cocycle in the cohomology of $K$, and the corresponding cohomology class depends only on the equivalence class of the deformation of $K$.*

4.10. **Definition.** A generalized Reynolds operator $K$ is said to be rigid if any formal deformation of $K$ is equivalent to the undeformed deformation $K'_t = K$.

In the next theorem, we give a sufficient condition for the rigidity of a generalized Reynolds operator in terms of Nijenhuis elements.

4.11. **Theorem.** *Let $K$ be a generalized Reynolds operator. If $Z^1_K(V,\mathfrak{g}) = \partial^0_K(\mathrm{Nij}(K))$ then $K$ is rigid.*

**Proof.** Let $K_t = \sum_{i=0}^{+\infty} K_i t^i$ be any formal deformation of $K$. Then it follows from Theorem 4.9 that the linear term $K_1$ is a 1-cocycle in the cohomology of $K$, i.e., $K_1 \in Z^1_K(V,\mathfrak{g})$. Thus, by the hypothesis, there is a Nijenhuis element $x \in \mathrm{Nij}(K)$ such that $K_1 = -\partial^0_K(x)$. We take

$$\phi_t = \mathrm{Id}_{\mathfrak{g}} + tL_x \quad \text{and} \quad \psi_t = \mathrm{Id}_V + t(\rho^L(x) + H(x, K-)),$$

and define $K'_t = \phi_t \circ K_t \circ \psi_t^{-1}$. Then $K'_t$ is a formal deformation equivalent to $K_t$. For $u \in V$, we observe that

$$\begin{aligned} K'_t(u) &= (\mathrm{Id}_{\mathfrak{g}} + tL_x)(K_t(u - t\rho^L(x)u - tH(x,Ku) + \text{ power of } t^{\geq 2})) \\ &= K(u) + t(K_1u - K\rho^L(x)u - KH(x,Ku) + [x,Ku]) + \text{ power of } t^{\geq 2}. \\ &= K(u) + t^2 K'_2(u) + \cdots \qquad (\text{as } K_1 = -\partial^0_K(x)). \end{aligned}$$

Hence the coefficient of $t$ in the expression of $K'_t$ is trivial. Applying the same process repeatedly, we get that $K_t$ is equivalent to $K$. Therefore, $K$ is rigid. □

## 5. NS-Leibniz algebras

In this section, we introduce NS-Leibniz algebras as the underlying structure of generalized Reynolds operators. We study some properties of NS-Leibniz algebras and give some examples. Further study on NS-Leibniz algebras is postponed to a forthcoming paper.

5.1. **Definition.** An NS-Leibniz algebra is a quadruple $(A, \triangleright, \triangleleft, \diamond)$ consisting of a vector space $A$ together with three bilinear operations $\triangleright, \triangleleft, \diamond : A \otimes A \to A$ satisfying for all $x, y, z \in A$,

$$(A1) \qquad x \triangleright (y * z) = (x \triangleright y) \triangleright z + y \triangleleft (x \triangleright z),$$

$$(A2) \qquad x \triangleleft (y \triangleright z) = (x \triangleleft y) \triangleright z + y \triangleright (x * z),$$

$$(A3) \qquad x \triangleleft (y \triangleleft z) = (x * y) \triangleleft z + y \triangleleft (x \triangleleft z),$$

$$(A4) \qquad x \triangleleft (y \diamond z) + x \diamond (y * z) = (x \diamond y) \triangleright z + (x * y) \diamond z + y \triangleleft (x \diamond z) + y \diamond (x * z),$$

where $x * y = x \triangleright y + x \triangleleft y + x \diamond y$.

NS-Leibniz algebras are more general than Leibniz-dendriform algebras introduced in [24]. More precisely, an NS-Leibniz algebra $(A, \triangleright, \triangleleft, \diamond)$ in which the bilinear operation $\diamond$ is trivial is a Leibniz-dendriform algebra.

In the following, we show that NS-Leibniz algebras split Leibniz algebras.

5.2. **Proposition.** *Let $(A, \triangleright, \triangleleft, \diamond)$ be an NS-Leibniz algebra. Then the vector space $A$ with the bilinear operation*

$$[~,~]_* : A \otimes A \to A, \;\; [x,y]_* := x * y$$

*is a Leibniz algebra.*

**Proof.** By summing up the left hand sides of the identities (A1)-(A4), we simply get $[x, [y, z]_*]_*$. On the other hand, by summing up the right hand sides of the identities (A1)-(A4), we get $[[x, y]_*, z]_* + [y, [x, z]_*]_*$. Hence the result follows. □

5.3. **Proposition.** *Let $A$ be an associative algebra and $P : A \to A$ be a linear map satisfying $P(x)P(y) = P(P(x)y) = P(xP(y))$, for any $x, y \in A$. Define bilinear operations $\triangleright, \triangleleft, \diamond : A \otimes A \to A$ by*

$$x \triangleright y = -yP(x), x \triangleleft y = P(x)y, \text{ and } x \diamond y = 0, \text{ for } x, y \in A.$$

*Then $(A, \triangleright, \triangleleft, \diamond)$ is an NS-Leibniz algebra.*

**Proof.** For any $x, y, z \in A$, we have

$$\begin{aligned}(x \triangleright y) \triangleright z + y \triangleleft (x \triangleright z) &= zP(yP(x)) - P(y)zP(x)\\ &= zP(y)P(x) - P(y)zP(x)\\ &= x \triangleright (y * z).\end{aligned}$$

Aiso, we have

$$\begin{aligned}x \triangleleft (y \triangleright z) - (x \triangleleft y) \triangleright z &= -P(x)zP(y) + zP(P(x)y)\\ &= -P(x)zP(y) + zP(x)P(y)\\ &= y \triangleright (x * z).\end{aligned}$$

Thus (A1) and (A2) hold. Similarly, we can check that (A3) holds and (A4) holds obviously. This completes the proof. □

NS-Leibniz algebras also arise from weighted Rota-Baxter operators on Leibniz algebras. Let $(\mathfrak{g}, [\ ,\ ])$ be a Leibniz algebra. A linear map $T : \mathfrak{g} \to \mathfrak{g}$ is said to be a Rota-Baxter operator of weight $\lambda$ on the Leibniz algebra if $T$ satisfies

$$[Tx, Ty] = T([Tx, y] + [x, Ty] + \lambda[x, y]), \text{ for } x, y \in \mathfrak{g}.$$

Note that the identity map $\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$ is a Rota-Baxter operator of weight -1. If $T$ is a Rota-Baxter operator of weight $\lambda$, then $-\lambda \mathrm{Id} - T$ is so. In the following result, we show that Rota-Baxter operators of weight $\lambda$ induce NS-Leibniz algebras.

5.4. **Proposition.** *Let $T : \mathfrak{g} \to \mathfrak{g}$ be a Rota-Baxter operator of weight $\lambda$ on the Leibniz algebra $(\mathfrak{g}, [\ ,\ ])$. Then there is an NS-Leibniz algebra structure on the vector space $\mathfrak{g}$ with bilinear operations*

$$x \triangleright y = [x, Ty], x \triangleleft y = [Tx, y], \text{ and } x \diamond y = \lambda[x, y], \text{ for } x, y \in \mathfrak{g}.$$

**Proof.** For any $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}(x \triangleright y) \triangleright z + y \triangleleft (x \triangleright z) &= [[x, Ty], Tz] + [Ty, [x, Tz]]\\ &= [x, [Ty, Tz]]\\ &= [x, T(y * z)]\\ &= x \triangleright (y * z).\end{aligned}$$

Also,

$$\begin{aligned}x \triangleleft (y \triangleright z) - (x \triangleleft y) \triangleright z &= [Tx, [y, Tz]] - [[x, Ty], Tz]\\ &= [y, [Tx, Tz]]\\ &= [y, T(x * z)]\\ &= y \triangleright (x * z).\end{aligned}$$

Similarly, we have

$$\begin{aligned} x \triangleleft (y \triangleleft z) - y \triangleleft (x \triangleleft z) &= [Tx, [Ty, z]] - [Ty, [Tx, z]] \\ &= [[Tx, Ty], z] \\ &= [T(x * y), z] \\ &= (x * y) \triangleleft z. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & (x \diamond y) \triangleright z + (x * y) \diamond z + y \triangleleft (x \diamond z) + y \diamond (x * z) \\ = \; & \lambda[[x, y], Tz] + \lambda[[Tx, y], z] + \lambda[[x, Ty], z] + \lambda^2[[x, y], z] + \lambda[Ty, [x, z]] \\ & + \lambda[y, [Tx, z]] + \lambda[y, [x, Tz]] + \lambda^2[y, [x, z]] \\ = \; & \lambda[Tx, [y, z]] + \lambda[x, [y, Tz]] + \lambda[x, [Ty, z]] + \lambda^2[x, [y, z]] \\ = \; & x \triangleleft (y \diamond z) + x \diamond (y * z). \end{aligned}$$

This completes the proof. □

The Leibniz algebra $(A, [~,~]_*)$ of the above proposition is called the subadjacent Leibniz algebra of $(A, \triangleright, \triangleleft, \diamond)$ and $(A, \triangleright, \triangleleft, \diamond)$ is called a compatible NS-Leibniz algebra structure on $(A, [~,~]_*)$.

5.5. **Proposition.** *Let $V$ be a vector space. On the vector space $gl(V) \oplus V$, define three bilinear operations $\triangleright, \triangleleft, \diamond : (gl(V) \oplus V) \otimes (gl(V) \oplus V) \rightarrow gl(V) \oplus V$ by*

$$(A + u) \triangleright (B + v) = -BA, \quad (A + u) \triangleleft (B + v) = AB + Av, \quad (A + u) \diamond (B + v) = -Bu.$$

*Then $(gl(V) \oplus V, \triangleright, \triangleleft, \diamond)$ is an NS-Leibniz algebra.*

**Proof.** For any $x = A + u,~ y = B + v$ and $z = C + w \in gl(V) \oplus V$, we have

$$\begin{aligned} (x \triangleright y) \triangleright z + y \triangleleft (x \triangleright z) = CBA - BCA &= x \triangleright (BC - CB + Bw - Cv) \\ &= x \triangleright (y * z). \end{aligned}$$

Similarly, we have

$$\begin{aligned} (x \triangleleft y) \triangleright z + y \triangleright (x * z) &= (AB + Av) \triangleright z + y \triangleright (AC - CA + Aw - Cu) \\ &= -CAB - ACB + CAB \\ &= -ACB = x \triangleleft (y \triangleright z) \end{aligned}$$

and

$$\begin{aligned} (x * y) \triangleleft z + y \triangleleft (x \triangleleft z) &= (AB - BA + Av - Bu) \triangleleft z + y \triangleleft (AC + Aw) \\ &= ABC - BAC + ABw - BAw + BAC + BAw \\ &= ABC + ABw = x \triangleleft (y \triangleleft z). \end{aligned}$$

Finally,

$$\begin{aligned} & (x \diamond y) \triangleright z + (x * y) \diamond z + y \triangleleft (x \diamond z) + y \diamond (x * z) \\ & = -Bu \triangleright z + (AB - BA + Av - Bu) \diamond z + y \triangleleft -Cu + y \diamond (AC - CA + Aw - Cu) \\ & = -CAv + CBu - BCu - ACv + CAv \\ & = CBu - BCu - ACv = x \triangleleft (y \diamond z) + x \diamond (y * z). \end{aligned}$$

This shows that $(gl(V) \oplus V, \triangleright, \triangleleft, \diamond)$ is an NS-Leibniz algebra. □

5.6. **Proposition.** *Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra and $N : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Nijenhuis operator on it. Then the bilinear operations*

$$x \triangleright y = [x, Ny], \quad x \triangleleft y = [Nx, y] \quad \textit{and} \quad x \diamond y = -N[x, y], \quad \textit{for } x, y \in \mathfrak{g}$$

*defines an NS-Leibniz algebra structure on* $\mathfrak{g}$.

**Proof.** For any $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned} x \triangleright (y * z) = [x, N(y * z)] &= [x, [Ny, Nz]] \\ &= [[x, Ny], Nz] + [Ny, [x, Nz]] \\ &= (x \triangleright y) \triangleright z + y \triangleleft (x \triangleright z). \end{aligned}$$

Hence the identity (A1) of Definition 5.1 holds. Similarly,

$$\begin{aligned} x \triangleleft (y \triangleright z) = [Nx, [y, Nz]] &= [[Nx, y], Nz] + [y, [Nx, Nz]] \\ &= (x \triangleleft y) \triangleright z + y \triangleright (x * z), \end{aligned}$$

and

$$\begin{aligned} x \triangleleft (y \triangleleft z) = [Nx, [Ny, z]] &= [[Nx, Ny], z] + [Ny, [Nx, z]] \\ &= (x * y) \triangleleft z + y \triangleleft (x \triangleleft z). \end{aligned}$$

Therefore, the identities (A2) and (A3) also hold. To prove the identity (A4), we first recall from [6] that the given Leibniz bracket $[~,~]$ and the deformed Leibniz bracket $[~,~]_N$ given in (2. 1) are compatible in the sense that their sum also defines a Leibniz bracket on $\mathfrak{g}$. This is equivalent to the fact that

$$(5.~1) \qquad [x, [y, z]]_N + [x, [y, z]_N] = [[x, y], z]_N + [[x, y]_N, z] + [y, [x, z]]_N + [y, [x, z]_N],$$

for $x, y, z \in \mathfrak{g}$. The identity (A4) of Definition 5.1 simply follows from (5. 1). Hence $(\mathfrak{g}, \triangleright, \triangleleft, \diamond)$ is an NS-Leibniz algebra. □

Let $(A, \triangleright, \triangleleft, \diamond)$ be an NS-Leibniz algebra. Define two linear maps $L_\triangleleft : A \to gl(A)$, $R_\triangleright : A \to gl(A)$ and a bilinear map $H : A \otimes A \to A$ by

$$L_\triangleleft(x) y = x \triangleleft y, \quad R_\triangleright(x) y = y \triangleright x \quad \text{and} \quad H(x, y) = x \diamond y, \quad \text{for } x, y \in A.$$

With these notations, we have the following.

5.7. **Proposition.** *Let* $(A, \triangleright, \triangleleft, \diamond)$ *be an NS-Leibniz algebra. Then* $(A, L_\triangleleft, R_\triangleright)$ *is a representation of the subadjacent Leibniz algebra* $(A, [~,~]_*)$*, and* $H$ *defined above is a* 2*-cocycle. Moreover, the identity map* $\mathrm{Id} : A \to A$ *is a generalized Reynolds operator on the Leibniz algebra* $(A, [~,~]_*)$ *with respect to the representation* $(A, L_\triangleleft, R_\triangleright)$.

**Proof.** For any $x, y, z \in A$, we have

$$\begin{aligned} L_\triangleleft([x, y]_*) z = [x, y]_* \triangleleft z &\overset{(A3)}{=} x \triangleleft (y \triangleleft z) - y \triangleleft (x \triangleleft z) \\ &= \big(L_\triangleleft(x) \circ L_\triangleleft(y) - L_\triangleleft(y) \circ L_\triangleleft(x)\big) z. \end{aligned}$$

Similarly,

$$\begin{aligned} R_\triangleright([x, y]_*) z = z \triangleright [x, y]_* &\overset{(A2)}{=} x \triangleleft (z \triangleright y) - (x \triangleleft z) \triangleright y \\ &= L_\triangleleft(x) \circ R_\triangleright(y)) z - R_\triangleright(y) \circ L_\triangleleft(x) z, \end{aligned}$$

and

$$\begin{aligned} R_\triangleright([x, y]_*) z = z \triangleright [x, y]_* &\overset{(A1)}{=} (z \triangleright x) \triangleright y + x \triangleleft (z \triangleright y) \\ &= \big(R_\triangleright(y) \circ R_\triangleright(x) + L_\triangleleft(x) R_\triangleright(y)\big) z. \end{aligned}$$

Therefore, $(A, L_\triangleleft, R_\triangleright)$ is a representation of the subadjacent Leibniz algebra $(A, [~,~]_*)$. Moreover, the condition (A4) is equivalent that $H$ is a 2-cocycle in the Loday-Pirashvili cochain complex of the Leibniz algebra $(A, [~,~]_*)$ with coefficients in the representation $(A, L_\triangleleft, R_\triangleright)$. Finally, we have

$$\mathrm{Id}(L_\triangleleft(\mathrm{Id}~x) y + R_\triangleright(\mathrm{Id}~y) x + H(\mathrm{Id}~x, \mathrm{Id}~y)) = x \triangleleft y + x \triangleright y + x \circ y = [\mathrm{Id}~x, \mathrm{Id}~y]_*,$$

which shows that $\mathrm{Id} : A \to A$ is a generalized Reynolds operator on the Leibniz algebra $(A, [~,~]_*)$ with respect to the representation $(A, L_{\triangleleft}, R_{\triangleright})$. □

5.8. **Proposition.** *Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra, $(V, \rho^L, \rho^R)$ be a representation and $H \in C^2(\mathfrak{g}, V)$ be a 2-cocycle. Let $K : V \to \mathfrak{g}$ be a generalized Reynolds operator. Then there is an NS-Leibniz algebra structure on $V$ with bilinear operations given by*

$$u \triangleright v := \rho^R(Kv)u, \quad u \triangleleft v := \rho^L(Ku)v \quad \text{and} \quad u \diamond v := H(Ku, Kv), \quad \text{for } u, v \in V.$$

**Proof.** For any $u, v, w \in V$, we have

$$\begin{aligned} u \triangleright (v * w) = \rho^R(K(v * w))u &= \rho^R([Kv, Kw])u \\ &= \rho^L(Kv)\rho^R(Kw)u + \rho^R(Kw)\rho^R(Kv)u \\ &= v \triangleleft (u \triangleright w) + (u \triangleright v) \triangleright w. \end{aligned}$$

Similarly,

$$\begin{aligned} u \triangleleft (v \triangleright w) = \rho^L(Ku)\rho^R(Kw)v &= \rho^R([Ku, Kz])v + \rho^R(Kw)\rho^L(Ku)v \\ &= v \triangleright (u * w) + (u \triangleleft v) \triangleright w, \end{aligned}$$

and

$$\begin{aligned} u \triangleleft (v \triangleleft w) = \rho^L(Ku)\rho^L(Kv)(w) &= \rho^L([Ku, Kv])w + \rho^L(Kv)\rho^L(Ku)w \\ &= (u * v) \triangleleft w + v \triangleleft (u \triangleleft w). \end{aligned}$$

Hence (A1), (A2) and (A3) of Definition 5.1 hold. Since $H$ is a 2-cocycle, we have $(\partial^2 H)(Ku, Kv, Kz) = 0$, i.e.,

$$\begin{aligned} &\rho^L(Ku)H(Kv, Kw) - \rho^L(Kv)H(Ku, Kw) - \rho^R(Kw)H(Ku, Kv) \\ &\qquad -H([Ku, Kv], Kw) - H(Kv, [Ku, Kw]) + H(Ku, [Kv, Kw]) = 0. \end{aligned}$$

This is equivalent to the condition (A4) of Definition 5.1. Hence the proof. □

5.9. **Remark.** The subadjacent Leibniz algebra of the NS-Leibniz algebra constructed in Proposition 5.8 is given by

$$[u, v]_* = \rho^L(Ku)v + \rho^R(Kv)u + H(Ku, Kv), \quad \text{for } u, v \in V.$$

This Leibniz algebra structure on $V$ coincides with the one given in Proposition 2.9.

In the following, we give a necessary and sufficient condition for the existence of a compatible NS-Leibniz algebra structure on a Leibniz algebra.

5.10. **Proposition.** *Let $(\mathfrak{g}, [~,~])$ be a Leibniz algebra. Then there is a compatible NS-Leibniz algebra structure on $\mathfrak{g}$ if and only if there exists an invertible generalized Reynolds operator $K : V \to \mathfrak{g}$ on $\mathfrak{g}$ with respect to a representation $(V, \rho^L, \rho^R)$ and a 2-cocycle $H$. Furthermore, the compatible NS-Leibniz algebra structure on $\mathfrak{g}$ is given by*

$$x \triangleright y := K(\rho^R(y)K^{-1}x), \quad x \triangleleft y := K(\rho^L(x)K^{-1}y) \quad \text{and} \quad x \diamond y = KH(x, y), \quad \text{for } x, y \in \mathfrak{g}.$$

**Proof.** Let $K : V \to \mathfrak{g}$ be an invertible generalized Reynolds operator on $\mathfrak{g}$ with respect to a representation $(V, \rho^L, \rho^R)$ and a 2-cocycle $H$. By Proposition 5.8, there is an NS-Leibniz algebra structure on $V$ given by

$$u \,\bar{\triangleright}\, v := \rho^R(Kv)u, \ \bar{\triangleleft}\, v := \rho^L(Ku)v \ \text{ and } \ u \,\bar{\diamond}\, v := H(Ku, Kv), \quad \text{for } u, v \in V.$$

Since $K$ is an invertible map, the bilinear operations

$$\begin{aligned} x \triangleright y &:= K(K^{-1}x \,\bar{\triangleright}\, K^{-1}y) = K(\rho^R(y)K^{-1}x), \\ x \triangleleft y &:= K(K^{-1}x \,\bar{\triangleleft}\, K^{-1}y) = K(\rho^L(x)K^{-1}y), \\ x \diamond y &:= K(K^{-1}x \,\bar{\diamond}\, K^{-1}y) = KH(x, y), \quad \text{for } x, y \in \mathfrak{g} \end{aligned}$$

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defines an NS-Leibniz algebra on $\mathfrak{g}$. Moreover, we have

$$\begin{aligned} & x \triangleright y + x \triangleleft y + x \diamond y \\ & = K(\rho^R(y)K^{-1}x) + K(\rho^L(x)K^{-1}y) + KH(x,y) \\ & = K(\rho^R(K \circ K^{-1}y)K^{-1}x) + K(\rho^L(K \circ K^{-1}x)K^{-1}y) + KH(K \circ K^{-1}x, K \circ K^{-1}y) \\ & = [K \circ K^{-1}x, K \circ K^{-1}y] = [x,y]. \end{aligned}$$

Conversely, let $(\mathfrak{g}, \triangleright, \triangleleft, \diamond)$ be a compatible NS-Leibniz algebra structure on $\mathfrak{g}$. By Proposition 5.7, $(\mathfrak{g}, L_\triangleleft, R_\triangleright)$ is a representation of the Leibniz algebra $(\mathfrak{g}, [~,~])$, and the identity map $\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$ is a generalized Reynolds operator on the Leibniz algebra $(\mathfrak{g}, [~,~])$ with respect to the representation $(\mathfrak{g}, L_\triangleleft, R_\triangleright)$. Hence the proof. □

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School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, P. R. of China

*E-mail address*: shuangjianguo@126.com

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, West Bengal India

*E-mail address*: apurbadas348@gmail.com