GENERALIZATIONS OF AN IDENTITY OF N-P. SKORUPPA

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Abstract. N-P. Skoruppa gave a completely elementary proof for identities involving the divisor sums showing the hidden pattern behind all of those identities. In a similar spirit, we extend his ideas to obtain other identities.

1. Introduction

In [Sko] the following identity was proved.

Theorem 1. ***(Skoruppa's Identity)*** *Let $h(x,y)$ be a function of integer arguments such that $h(x,y) = h(y, y-x)$. Then for any positive integer ℓ*

$$\sum_{ax+by=\ell} h(a,b) - h(a,-b) = \sum_{t|\ell} \left(\frac{\ell}{t} h(t,0) - \sum_{j=0}^{t-1} h(t,j) \right),$$

where the sum on the left is understood to run over all quadruples of positive integers which satisfy the given condition and on the right t runs over all positive divisors of ℓ.

This beautiful identity can be used to give quick proofs from scratch of Eisenstein series identities avoiding the use of modular forms. Proofs of such identities had been given by J. Liouville, S. Ramanujan, B. van der Pol and R. Rankin, see bibliography in [Sko], chapter 4 and bibliography in [Be]. Also see [Ra] and [Ra1]. A generalization of the above identity was obtained in [By]. Also, Huard, Ou, Spearman and William proved a far reaching generalization of Liouville's identity. In fact Skoruppa's identity is a special case of their result. See chapter 13 of the masterpiece [W] and the references therein.

The proof of the theorem is based on the following lemma for which we need some definitions. Let ℓ be a positive integer. For any pair a, b of positive integers we define $\Lambda_\ell(a,b)$ to be the number of pairs of positive integers (x,y) such that $ax + by = \ell$. We extend the definition of $\Lambda_\ell(a,b)$ for any pair of integers as follows: if $ab \neq 0$ then

$$\Lambda_\ell(a,b) = sign(ab)\Lambda_\ell(|a|,|b|),$$

and $\Lambda_\ell(a,b) = 0$ if $ab = 0$.

Lemma 1. *(Lemma,* [Sko]*) For any triple of integers a, b, c such that $a+b+c = 0$ one has*

$$\Lambda_\ell(a,b) + \Lambda_\ell(b,c) + \Lambda_\ell(a,c) = \begin{cases} 1, & \text{if } abc \neq 0 \text{ and } t|\ell \\ 1 - \ell/t, & \text{if } abc = 0 \text{ and } t \neq 0, t|\ell \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

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where $t=\max\{|a|,|b|,|c|\}$.

The aim of this note is to give some generalizations of this lemma and the above theorem. Our first result is the following theorem.

Theorem 2. *Let* $h(x,y)$ *be a function of integer arguments such that* $h(x,y)=h(y,y-x)$ *and let* r *be a real parameter. Then for any positive integer* ℓ

$$\begin{aligned}&\sum_{ax+by=\ell} h(a,b)x^r+\sum_{ax+by=\ell;\ b>a} h(a,b)(y^r-x^r)\\&+\sum_{ax+by=\ell} h(a,b)(x+y)^r-\sum_{ax+by=\ell}\left(h(a,-b)+h(b,-a)\right)y^r\\&=\sum_{t|\ell}\left(\left\{\left(\sum_{i=1}^{\ell/t-1} i^r\right)+\left(\frac{\ell}{t}\right)^r\left(\frac{\ell}{t}-1\right)\right\}h(t,0)-2\left(\frac{\ell}{t}\right)^r\sum_{j=1}^{t-1}h(t,j)\right),\end{aligned}$$

where the sums on the left are understood to run over all quadruples of positive integers which satisfy the given conditions and on the right t *runs over all positive divisors of* ℓ.

Here and in what follows we denote, as usual, $\sigma_j(n):=\sum_{a|n}a^j$, the sum of the j powers of the divisors of n.

Observe that for a generic function $f(a,b)$ one has

$$\sum_{ax+by=\ell} f(a,b)y^r x^{r'}=\sum_{k=1}^{\ell-1}\sum_{a|(\ell-k)}\sum_{b|k} f(a,b)\left(\frac{k}{b}\right)^r\left(\frac{\ell-k}{a}\right)^{r'},$$

(this is easy to prove if one sets $by=k$, $ax=\ell-k$) and therefore

$$\sum_{ax+by=\ell} a^{r_1}b^{r_2}y^r x^{r'}=\sum_{k=1}^{\ell-1}k^r(\ell-k)^{r'}\sigma_{r_2-r}(k)\sigma_{r_1-r'}(\ell-k).$$

If $2\le n$ is an even positive integer and B_n is the nth Bernoulli number then

$$G_n=G_n(q):=-\frac{B_n}{2n}+\sum_{\ell=1}^{\infty}\sigma_{n-1}(\ell)q^\ell,$$

is the nth Eisenstein series. Any Eisenstein series or derivatives of the form $(qd/dq)^kG_n$ can be expressed as a polynomial in G_2,G_4 and G_6 ([Sko], [Be]).

For $n=0,1,3$ we set

$$G_n=G_n(q):=\sum_{\ell=1}^{\infty}\sigma_{n-1}(\ell)q^\ell,$$

and (here $e(x):=e^{2\pi ix}$)

$$G_n^{\pm}=G_n^{\pm}(q,x):=\sum_{k=1}^{\infty}\left(\sum_{a|k}a^{n-1}e(\pm ax)\right)q^k.$$

Write for short $G_n^{\pm}(q,x) = G_n^{\pm}$ and set

$$F(q,x) := G_3^+ q\frac{d}{dq}G_0^- - G_2^+ q\frac{d}{dq}G_1^- + G_1^+ q\frac{d}{dq}G_2^- \\ - \left(q\frac{d}{dq}G_2^+\right)G_1^- + \left(q\frac{d}{dq}G_1^+\right)G_2^- - \left(q\frac{d}{dq}G_0^+\right)G_3^-.$$

The following corollary is a particular case of Theorem 2.

Corollary 1. *Let $e(x) := e^{2\pi i x}$ and $0 < r < 1$. The following formula holds:*

$$-5\left(q\frac{d}{dq}G_1\right)G_2 + G_2G_3 + \left(q\frac{d}{dq}G_2\right)G_1 + \lim_{r\to 1-}\int_0^1 \frac{F(q,x)}{1-re(x)}dx \\ = q\frac{d}{dq}\left(\frac{3}{2}q\frac{d}{dq}G_1 - \frac{1}{8}G_1 + \frac{1}{2}G_2 - \frac{5}{3}G_3\right) - \frac{G_3}{24}.$$

We also prove the following theorems, the first is an easy consequence of Skoruppa's identity.

Theorem 3. *Let $h(x,y,z)$ be a function of integer arguments such that $h(x,y,z) = h(y,y-x,z)$. Then for any positive integer ℓ*

$$\sum_{ax+by+cz=\ell} h(a,b,c) - h(a,-b,c) = \sum_{cz=1}^{\ell-2}\sum_{t|(\ell-cz)}\left(\frac{(\ell-cz)}{t}h(t,0,c) - \sum_{j=0}^{t-1}h(t,j,c)\right),$$

where the sum on the left is understood to run over all sextuples of positive integers which satisfy the given condition. The first sum on the right-hand side is understood over all positive integers c and z whose product runs from 1 to $\ell-2$.

Set

$$\alpha(x,y,z) := \begin{cases} 1, & \text{if } x \neq y, y \neq z, x \neq z, \\ 0, & \text{if only two variables are equal}, \\ -2, & \text{if } x = y = z. \end{cases} \tag{2}$$

With this definition one has the following result.

Theorem 4. *Assume that ℓ is a positive integer and that $h(x,y,z)$ is an integer-valued function which satisfies $h(x,y,z) = h(y,-2x+y-z,x-y+z)$. Then*

$$\sum_{\substack{ax+by+cz=\ell \\ a<b<c}} h(a+c,a,-b) + \sum_{\substack{ax+by+cz=\ell \\ a<c<b}} h(a,-c-b,c-a) \\ + \sum_{\substack{ax+by+cz=\ell \\ b<a<c}} h(a+c-b,a-b,-a) + \sum_{\substack{ax+by+cz=\ell \\ b<a<c}} h(b,2c-a,c-2b) \\ + \sum_{\substack{ax+by+cz=\ell \\ a<b<c}} h(b-a,-b-c,a) + \sum_{\substack{ax+by+cz=\ell \\ a<b<c}} h(b-a,2c-b,c-2b+2a) \\ - \sum_{ax+by+cz=\ell} \alpha(x,y,z)h(a,-2a-b-2c,c) \\ = 2\sum_{t|\ell}\sum_{i,j=1}^{i+j<t} h(i,-i-j-t,j), \tag{3}$$

where the sums on the left are understood to run over all sextuples of positive integers which satisfy the given conditions and on the right t runs over all positive divisors of ℓ.

Concerning the condition on h on the last theorem one should observe that if $p(x,y,z)$ is **any** integer-valued function then

$$\begin{aligned} h(x,y,z) := {} & p(y,-2x+y-z,x-y+z) + p(-2x+y-z,-3x-2z,3x-y+2z) \\ & + p(-3x-2z,-2x-y-2z,4x+3z) + p(-2x-y-2z,-y-z,3x+y+3z) \\ & + p(-y-z,x,x+y+2z) + p(x,y,z), \end{aligned}$$

verifies the condition of the theorem. For example, taking $p(x,y,z)$ as $-x/6$, $x^2/2$, $xy/2$ one obtains

$$\begin{aligned} h(x,y,z) &= x+z, \\ h(x,y,z) &= 9x^2+2y^2+12xz+2yz+5z^2, \\ h(x,y,z) &= 6x^2+y^2+9xz+yz+4z^2, \end{aligned}$$

respectively.

To state our final result we need a definition: if i is an integer and given a polynomial

$$p(u) = d_n u^n + d_{n-1}u^{n-1} + \cdots + d_0,$$

we define

$$p_{\ell,i} := d_n\sigma_{n+i}(\ell) + d_{n-1}\sigma_{n+i-1}(\ell) + \cdots + d_0\sigma_i(\ell).$$

Theorem 5. *If ℓ is a positive integer and $p(u)$ is a polynomial then*

$$\begin{aligned} -\sum_{ax+by+cz=\ell} \alpha(x,y,z)p(a+b+c) + 3 \sum_{\substack{ax+by+cz=\ell \\ c>\max(a,b);\ a\neq b}} p(c) \\ = p_{\ell,2} - 3\, p_{\ell,1} + 2\, p_{\ell,0}. \end{aligned}$$

2. Proof of Theorem 2

With r a real parameter and given a,b,ℓ positive integers we set $\Lambda^r_{\ell,1}(a,b) := \sum_{ax+by=\ell} y^r$ and $\Lambda^r_{\ell,2}(a,b) := \sum_{ax+by=\ell}(x+y)^r$ where the sums are understood as above, that is, over all pairs of positive integers (x,y) such that $ax+by=\ell$. Observe that $\Lambda^0_{\ell,1}(a,b) = \Lambda_\ell(a,b)$, $\Lambda^r_{\ell,2}(a,b) = \Lambda^r_{\ell,2}(b,a)$ and in general $\Lambda^r_{\ell,1}(a,b) \neq \Lambda^r_{\ell,1}(b,a)$. If $a,b \geq 0$ and $ab=0$ then we extend the definition as $\Lambda^r_{\ell,1}(a,b) = \Lambda^r_{\ell,2}(a,b) = 0$.

Lemma 2. *For integers $a,b \geq 0$ the following holds:*

$$\Lambda^r_{\ell,1}(a,b) - \Lambda^r_{\ell,1}(a,a+b) - \Lambda^r_{\ell,2}(b,a+b) = \begin{cases} \left(\frac{\ell}{a+b}\right)^r, & \text{if } ab\neq 0 \text{ and } (a+b)|\ell, \\ -\sum_{i=1}^{\ell/a-1} i^r, & \text{if } b=0 \text{ and } a\neq 0, a|\ell, \\ -\left(\frac{\ell}{b}\right)^r\left(\frac{\ell}{b}-1\right), & \text{if } a=0 \text{ and } b\neq 0, b|\ell, \\ 0, & \text{otherwise.} \end{cases}$$

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Proof. If $a = b = 0$ the lemma is trivially true. If a, b are positive integers then one has $\sum_{az+bw=\ell;z>w} w^r = \sum_{ax+(a+b)y=\ell} y^r = \Lambda^r_{\ell,1}(a, a+b)$ via $(x, y) = (z-w, w)$. Also $\sum_{az+bw=\ell;w>z} w^r = \sum_{bx+(a+b)y=\ell}(x+y)^r = \Lambda^r_{\ell,2}(b, a+b)$ via $(x, y) = (w-z, z)$. Therefore

$$\Lambda^r_{\ell,1}(a, b) - \Lambda^r_{\ell,1}(a, a+b) - \Lambda^r_{\ell,2}(b, a+b) = \sum_{az+bw=\ell;z=w} w^r,$$

which is $\left(\frac{\ell}{a+b}\right)^r$ or 0 as $(a+b)|\ell$ or not. This proves the first case of the lemma.

If $a = 0, b > 0$ then the left-hand side of the lemma is

$$-\Lambda^r_{\ell,2}(b, b) = -\sum_{bx+by=\ell}(x+y)^r,$$

while if $b = 0, a > 0$ the left-hand side of the lemma is

$$-\Lambda^r_{\ell,1}(a, a) = -\sum_{ax+ay=\ell} y^r,$$

which proves the lemma. □

Proof. (of the theorem) Interchange a and b in the last lemma and add to the same lemma to yield that (*): if $a, b \geq 0$

$$\Lambda^r_{\ell,1}(a, b) + \Lambda^r_{\ell,1}(b, a) - \Lambda^r_{\ell,1}(a, a+b) - \Lambda^r_{\ell,1}(b, a+b) - \Lambda^r_{\ell,2}(b, a+b) - \Lambda^r_{\ell,2}(a, a+b)$$

$$= \begin{cases} 2\left(\frac{\ell}{a+b}\right)^r, & \text{if } ab \neq 0 \text{ and } (a+b)|\ell, \\ -\sum_{i=1}^{\ell/(a+b)-1} i^r - \left(\frac{\ell}{a+b}\right)^r\left(\frac{\ell}{a+b} - 1\right), & \text{if } ab = 0, a+b \neq 0 \text{ and } (a+b)|\ell, \\ 0, \text{ otherwise.} \end{cases}$$

Let $\delta_{a,b}(x, y)$ be an integer argument function whose value is 1 or 0 as $(a, b) = (x, y)$ or not (i.e. a Dirac delta at (a, b)). We write for short $\delta_{a,b} = \delta_{a,b}(x, y)$. For integers a, b we define the operator $U\delta_{a,b}(x, y) = \delta_{b,b-a}(x, y)$ and given a Dirac delta $\delta_{a,b}$ we define an integer-valued function by $L\delta_{a,b} := L\delta_{a,b}(x, y) = \sum_{i=0}^{5} U^i\delta_{a,b}(x, y)$. Observe that $U^6\delta_{a,b}(x, y) = \delta_{a,b}(x, y)$ and $L\delta_{a,b}(y, y-x) = L\delta_{a,b}(x, y)$. In other words $L\delta_{a,b}(x, y)$ is a function which equal to 1 if (x, y) belongs to the *orbit* of (a, b) and is equal to zero if not. The *orbit* of (a, b) is the set of points obtanied by applying $(a, b) \to (b, b-a)$ repeatedly (the operator U above) and has six points unless $a = b = 0$. The orbits of two different points either are equal or are disjoint.

Now take $a, b \geq 0$ and

$$h(x, y) = L\delta_{a,-b} = \delta_{a,-b} + \delta_{-b,-b-a} + \delta_{-b-a,-a} + \delta_{-a,b} + \delta_{b,b+a} + \delta_{b+a,a},$$

so that $h(x, y) = h(y, y-x)$. Set

$$S_1 := \sum_{a_0x+b_0y=\ell;a_0>b_0} h(a_0, b_0)x^r + \sum_{a_0x+b_0y=\ell;a_0<b_0} h(a_0, b_0)(x+y)^r - \sum_{a_0x+b_0y=\ell} h(a_0, -b_0)y^r,$$

$$S_2 := \sum_{a_0x+b_0y=\ell;b_0>a_0} h(a_0, b_0)y^r + \sum_{a_0x+b_0y=\ell;a_0>b_0} h(a_0, b_0)(x+y)^r - \sum_{a_0x+b_0y=\ell} h(b_0, -a_0)y^r,$$

so that with the above choice of h

$$S_1 = \Lambda^r_{\ell,1}(a, a+b) + \Lambda^r_{\ell,2}(b, a+b) - \Lambda^r_{\ell,1}(a, b),$$

$$S_2 = \Lambda^r_{\ell,1}(b, a+b) + \Lambda^r_{\ell,2}(a, a+b) - \Lambda^r_{\ell,1}(b, a),$$

if $a, b > 0$. Also $S_1 = S_2 = 0$ if $ab = 0$.

Finally,

$$S_3 := \sum_{a_0x+b_0y=\ell; a_0=b_0} h(a_0, b_0)x^r + \sum_{a_0x+b_0y=\ell; a_0=b_0} h(a_0, b_0)(x+y)^r$$

is equal to $\Lambda^r_{\ell,1}(b,b)+\Lambda^r_{\ell,2}(b,b)$ if $a = 0, b > 0$ (or $\Lambda^r_{\ell,1}(a,a)+\Lambda^r_{\ell,2}(a,a)$ if $b = 0, a > 0$) or is equal to 0 otherwise.

Note that $S_1+S_2+S_3$ is the left-hand side of the formula stated in the theorem. Formula (*) yields that the theorem is true for this particular choice of h. By linearity the function $h_0 := \sum_{(a,b);a,b\geq 0} const(a, b)h$, a linear sum of such h, satisfies the theorem too.

Finally, notice that for any h_1 with $h_1(x, y) = h_1(y, y-x)$ one can find constants $const(a, b)$ so that $h_0(x, y) = h_1(x, y)$ whenever (x, y) is in the range of the theorem. This ends the proof.

□

3. Proof of the corollary

Set $h(x, y) = x^2 - xy + y^2$, $r = 1$ in Theorem 2. One obtains: *If ℓ is a positive integer then*

$$\begin{aligned}
&-5\sum_{k=1}^{\ell-1}(\ell-k)\,\sigma_0(\ell-k)\sigma_1(k) + \sum_{k=1}^{\ell-1} k\,\sigma_{-1}(k)\sigma_2(\ell-k)\\
&+\sum_{k=1}^{\ell-1} k\,\sigma_1(k)\sigma_0(\ell-k) + \sum_{k=1}^{\ell-1}\sum_{a|(\ell-k);\, b|k;\, b>a}(a^2-ab+b^2)\left(\frac{k}{b}-\frac{\ell-k}{a}\right)\\
&=\frac{3}{2}\ell^2\sigma_0(\ell)-\frac{1}{3}\ell\sigma_0(\ell)+\frac{1}{2}\ell\sigma_1(\ell)-\frac{5}{3}\ell\sigma_2(\ell),
\end{aligned} \tag{4}$$

where $\sigma_j(n) := \sum_{i|n} i^j$. We remark that the double sum in this formula corresponds to the second sum of the left hand side of the theorem.

Trivially $n\sigma_{-1}(n) = \sigma_1(n)$. Observe that the coefficient of q^ℓ in

$$\left(G_2+\frac{1}{24}\right)G_3,$$
$$\left(q\frac{d}{dq}G_2\right)G_1,$$
$$\left(q\frac{d}{dq}G_1\right)\left(G_2+\frac{1}{24}\right),$$

respectively is

$$\begin{aligned}
&\sum_{k=1}^{\ell-1}\sigma_1(k)\sigma_2(\ell-k) = \sum_{k=1}^{\ell-1} k\,\sigma_{-1}(k)\sigma_2(\ell-k),\\
&\sum_{k=1}^{\ell-1} k\,\sigma_1(k)\sigma_0(\ell-k),\\
&\sum_{k=1}^{\ell-1}(\ell-k)\sigma_0(\ell-k)\,\sigma_1(k),
\end{aligned} \tag{5}$$

respectively. Also ($e(x) := e^{2\pi i x}$)

$$G_{n+1}^{+}(q,x)G_{m+1}^{-}(q,x) = \sum_{\ell=2}^{\infty}\left(\sum_{k=1}^{\ell-1}\sum_{a|\ell-k;b|k} a^n b^m e((a-b)x)\right) q^{\ell}.$$

Let $0<r<1$. By orthogonality one has $\int_0^1 \frac{e(-nx)}{1-re(x)}dx = r^n$ if $n=0,1,2,\ldots$ ($=0$ if $n=-1,-2,\ldots$). Thus the coefficient of q^ℓ in

$$\lim_{r\to 1-}\int_0^1 \frac{G_{n+1}^{+}(q,x)G_{m+1}^{-}(q,x)}{1-re(x)}dx,\quad \lim_{r\to 1-}\int_0^1 \frac{\left(q\frac{d}{dq}G_{n+1}^{+}(q,x)\right)G_{m+1}^{-}(q,x)}{1-re(x)}dx,$$

$$\lim_{r\to 1-}\int_0^1 \frac{G_{n+1}^{+}(q,x)\left(q\frac{d}{dq}G_{m+1}^{-}(q,x)\right)}{1-re(x)}dx,$$

respectively is

$$\sum_{k=1}^{\ell-1}\sum_{a|(\ell-k);\,b|k;\,b>a} a^n b^m,\quad \sum_{k=1}^{\ell-1}\sum_{a|(\ell-k);\,b|k;\,b>a} (\ell-k)a^n b^m,\quad \sum_{k=1}^{\ell-1}\sum_{a|(\ell-k);\,b|k;\,b>a} a^n k b^m,$$

respectively.

Write for short $G_n^{\pm}(q,x) = G_n^{\pm}$ and set

$$\begin{aligned} F(q,x) := {} & G_3^{+}q\frac{d}{dq}G_0^{-} - G_2^{+}q\frac{d}{dq}G_1^{-} + G_1^{+}q\frac{d}{dq}G_2^{-} \\ & - \left(q\frac{d}{dq}G_2^{+}\right)G_1^{-} + \left(q\frac{d}{dq}G_1^{+}\right)G_2^{-} - \left(q\frac{d}{dq}G_0^{+}\right)G_3^{-}. \end{aligned}$$

Using the above, the coefficient of q^ℓ in

$$\lim_{r\to 1-}\int_0^1 \frac{F(q,x)}{1-re(x)}dx,$$

is

$$\sum_{k=1}^{\ell-1}\sum_{a|(\ell-k);\,b|k;\,b>a} (a^2-ab+b^2)\left(\frac{k}{b}-\frac{\ell-k}{a}\right). \tag{6}$$

Using (4), (5), (6) yields

$$-5\left(q\frac{d}{dq}G_1\right)\left(G_2+\frac{1}{24}\right) + \left(G_2+\frac{1}{24}\right)G_3 + \left(q\frac{d}{dq}G_2\right)G_1 + \lim_{r\to 1-}\int_0^1 \frac{F(q,x)}{1-re(x)}dx$$

$$= q\frac{d}{dq}\left(\frac{3}{2}q\frac{d}{dq}G_1 - \frac{1}{3}G_1 + \frac{1}{2}G_2 - \frac{5}{3}G_3\right),$$

which proves the corollary after some simplification.

4. Proof of Theorem 3

Observe that

$$\sum_{ax+by+cz=\ell} h(a,b,c)-h(a,-b,c)=\sum_{c,z;\ cz=1}^{\ell-2}\ \sum_{ax+by=\ell-cz} h(a,b,c)-h(a,-b,c),$$

where the first sum on the right is understood to be in positive integers c and z and cz running from 1 to $\ell-2$. We apply Theorem 1 to the inner sum (with the change $\ell\to\ell-cz$).

5. Generalization of Lemma 1

Let ℓ be a positive integer. For any triple of positive integers (a,b,c) we define $\Lambda_\ell(a,b,c)$ to be the number of pairs of positive integers (x,y,z) such that $ax+by+cz=\ell$. Notice that $\Lambda_\ell(a,b,c)$ is invariant under permutations of $\{a,b,c\}$.

Lemma 3. *Let ℓ,a,b,c be positive integers. Define $\Delta_\ell^1(a,b,c):=$*

$$\begin{aligned}\Lambda_\ell(a,b,c)&-\Lambda_\ell(a+b+c,b+c,c)-\Lambda_\ell(a+b+c,b+c,b)-\Lambda_\ell(a+b+c,a+c,c)\\&-\Lambda_\ell(a+b+c,a+c,a)-\Lambda_\ell(a+b+c,a+b,a)-\Lambda_\ell(a+b+c,a+b,b),\end{aligned}$$

and $\Delta_\ell^2(a,b,c):=$

$$\begin{aligned}\Lambda_\ell(a+b,a+b+c)&+\Lambda_\ell(c,a+b+c)+\Lambda_\ell(b+c,a+b+c)\\&+\Lambda_\ell(a,a+b+c)+\Lambda_\ell(a+c,a+b+c)+\Lambda_\ell(b,a+b+c).\end{aligned}$$

Then $\Delta_\ell^1(a,b,c)-\Delta_\ell^2(a,b,c)$ is equal to 1 or 0 as $a+b+c|\ell$ or not.

Proof. 1) The solutions of $au+bv+cw=\ell$ which satisfy any of the conditions

$$\begin{aligned}u&<v<w,\\u&<w<v,\\v&<u<w,\\v&<w<u,\\w&<u<v,\\w&<v<u,\end{aligned}$$

are in $1-1$ correspondence with the solutions of

$$\begin{aligned}(a+b+c)x+(b+c)y+cz&=\ell,\\(a+b+c)x+(b+c)y+bz&=\ell,\\(a+b+c)x+(a+c)y+cz&=\ell,\\(a+b+c)x+(a+c)y+az&=\ell,\\(a+b+c)x+(a+b)y+bz&=\ell,\\(a+b+c)x+(a+b)y+az&=\ell,\end{aligned}$$

respectively via

$$\begin{aligned}(x,y,z)&=(u,v-u,w-v),\\(x,y,z)&=(u,w-u,v-w),\\(x,y,z)&=(v,u-v,w-u),\\(x,y,z)&=(v,w-v,u-w),\\(x,y,z)&=(w,u-w,v-u),\\(x,y,z)&=(w,v-w,u-v),\end{aligned}$$

respectively.

2) The solutions of $au+bv+cw=\ell$ which satisfy any of the conditions

$$\begin{aligned}u=v>w,\\u=v<w,\end{aligned}$$

(that is, we are looking for solutions of $(a+b)u+cw=\ell$ with $u\neq w$) are in $1-1$ correspondence with the solutions of

$$\begin{aligned}(a+b)x+(a+b+c)y&=\ell,\\cx+(a+b+c)y&=\ell,\end{aligned}$$

respectively via

$$\begin{aligned}(x,y)&=(u-w,w),\\(x,y)&=(w-u,u).\end{aligned}$$

Similarly the solutions of $au+bv+cw=\ell$ which satisfy any of the conditions

$$\begin{aligned}u=w>v,\\u=w<v,\end{aligned}$$

are in $1-1$ correspondence with the solutions of

$$\begin{aligned}(a+c)x+(a+b+c)y&=\ell,\\bx+(a+b+c)y&=\ell,\end{aligned}$$

respectively via

$$\begin{aligned}(x,y)&=(u-v,v),\\(x,y)&=(v-u,u).\end{aligned}$$

Finally, the solutions of $au+bv+cw=\ell$ which satisfy any of the conditions

$$\begin{aligned}v=w>u,\\v=w<u,\end{aligned}$$

are in $1-1$ correspondence with the solutions of

$$\begin{aligned}(b+c)x+(a+b+c)y&=\ell,\\ax+(a+b+c)y&=\ell,\end{aligned}$$

respectively via

$$\begin{aligned}(x,y)&=(v-u,u),\\(x,y)&=(u-v,v).\end{aligned}$$

3) If $u = v = w$ then $au + bv + cw = \ell$ has one or no solution acordingly as $a + b + c|\ell$ or not.

The lemma follows because any positive integer solution of $au + bv + cw = \ell$ satisfies exactly one of the above conditions given in the three steps. □

Corollary 2. *Let ℓ, a, b, c be a positive integers and define $\Delta(a, b, c) := \Lambda_\ell(a, b + c) + \Lambda_\ell(b, a + c) + \Lambda_\ell(c, a + b)$. Then $\Delta(a, b, c) - \Delta^1_\ell(a, b, c)$ is 2 or 0 as $a + b + c|\ell$ or not.*

Proof. Using Lemma 1 one can see that $\Delta(a, b, c) - \Delta^2_\ell(a, b, c)$ is 3 or 0 as $a+b+c|\ell$ or not. (Observe that the sum $\Delta(a, b, c)-\Delta^2_\ell(a, b, c)$ is equal to $\Delta_\ell(a, b+c)-\Delta_\ell(b+c, a + b + c) - \Delta_\ell(a, a + b + c)$ plus two similar sums for which Lemma 1 may be applied.)

Also from Lemma 3, $\Delta^1_\ell(a, b, c) - \Delta^2_\ell(a, b, c)$ is equal to 1 or 0 as $a + b + c|\ell$ or not.

These two last results yield that $\Delta(a, b, c) - \Delta^1_\ell(a, b, c)$ is 2 or 0 as $a + b + c|\ell$ or not. □

6. Proof of Theorem 4

The proof follows three steps. In the first step we prove that the theorem is true for a particular kind of function and in the second step we show that this is the only kind of function that matters. The theorem then follows by linearity.

I) Let $\delta_{a,b,c}(x, y, z)$ be an integer argument function whose value is 1 or 0 as $(a, b, c) = (x, y, z)$ or not (i.e. a Dirac delta at (a, b, c)). We write for short $\delta_{a,b,c} = \delta_{a,b,c}(x, y, z)$. From now on, we will deal only with integer-valued triplets (a, b, c).

We define the operator $U\delta_{a,b,c}(x, y, z) = \delta_{b,-2a+b-c,a-b+c}(x, y, z)$ and we define an integer-valued function by

$$L\delta_{a,b,c} := L\delta_{a,b,c}(x, y, z) = \sum_{i=0}^{5} U^i\delta_{a,b,c}(x, y, z).$$

Observe that $U^6\delta_{a,b,c}(x, y, z) = \delta_{a,b,c}(x, y, z)$ and

$$L\delta_{a,b,c}(y, -2x + y - z, x - y + z) = L\delta_{a,b,c}(x, y, z).$$

The *orbit* of (a, b, c) is the set of points obtained by applying $(a, b, c) \to (b, -2a + b - c, a - b + c)$ sucesively (the operator U above) and it has six points unless one starts at a point of the form $(a, a, -2a)$ in which case the orbit only has one point. (This is easy to check.)

In other words, if (a, b, c) is not of the form $(a', a', -2a')$ for some a', then $L\delta_{a,b,c}(x, y, z)$ is a function which equal to 1 if (x, y, z) belongs to the *orbit* of (a, b, c) and is equal to zero if not. The following observation will be used several times: if $L\delta_{a',b',c'}(a'', b'', c'') \neq 0$ then $L\delta_{a',b',c'} = L\delta_{a'',b'',c''}$

The proof of the theorem is then as follows. One computes

$$\begin{aligned} L\delta_{a,-2a-b-2c,c} =&\delta_{a,-2a-b-2c,c} + \delta_{b,2a+b+c,a-b+c} + \delta_{-4a-b-3c,-3a-2c,5a+b+4c} \\ &+ \delta_{-3a-2c,b,4a+3c} + \delta_{-2a-b-2c,-4a-b-3c,3a+b+3c} + \delta_{2a+b+c,a,-a-b}, \end{aligned} \tag{7}$$

and therefore

$$L\delta_{a,-2a-b-2c,c} = L\delta_{b,2a+b+c,a-b+c} = L\delta_{2a+b+c,a,-a-b}. \tag{8}$$

Then

$$S_1 := \sum_{\substack{a_0x+b_0y+c_0z=\ell\\ a_0<b_0<c_0}} h(a_0+c_0,a_0,-b_0) + \sum_{\substack{a_0x+b_0y+c_0z=\ell\\ a_0<c_0<b_0}} h(a_0,-c_0-b_0,c_0-a_0)$$

(9)

$$+ \sum_{\substack{a_0x+b_0y+c_0z=\ell\\ b_0<a_0<c_0}} h(a_0+c_0-b_0,a_0-b_0,-a_0) + \sum_{\substack{a_0x+b_0y+c_0z=\ell\\ b_0<a_0<c_0}} h(b_0,2c_0-a_0,c_0-2b_0)$$

$$+ \sum_{\substack{a_0x+b_0y+c_0z=\ell\\ a_0<b_0<c_0}} h(b_0-a_0,-b_0-c_0,a_0) + \sum_{\substack{a_0x+b_0y+c_0z=\ell\\ a_0<b_0<c_0}} h(b_0-a_0,2c_0-b_0,c_0-2b_0+2a_0)$$

is, using (7), equal to

$$\Lambda_\ell(a+b+c,a+b,a)+\Lambda_\ell(a+b+c,a+c,a)+\Lambda_\ell(a+b+c,a+b,b)$$
$$+\Lambda_\ell(a+b+c,b+c,b)+\Lambda_\ell(a+b+c,a+c,c)+\Lambda_\ell(a+b+c,b+c,c),$$

in case that $h = L\delta_{a,-2a-b-2c,c}$ and a,b,c are positive integers (observe that $h \neq L\delta_{a',a',-2a'}$ for any integer a' using (7)).

Observe that

$$\sum_{a_0x+b_0y+c_0z=\ell} L\delta_{a,-2a-b-2c,c}(a_0,-2a_0-b_0-2c_0,c_0) = \Lambda_\ell(a,b,c).$$

Also if $\delta_0(x)$ is the delta Dirac at zero defined on the integers then

$$\sum_{a_0x+b_0y+c_0z=\ell} \delta_0(x-y)L\delta_{a,-2a-b-2c,c}(a_0,-2a_0-b_0-2c_0,c_0) = \Lambda_\ell(a+b,c),$$

because the left sum represents the positive solutions x,y,z with $x=y$ of $ax+by+cz=\ell$. Therefore

$$\sum_{a_0x+b_0y+c_0z=\ell} \{\delta_0(x-y)+\delta_0(y-z)+\delta_0(z-x)\}\, L\delta_{a,-2a-b-2c,c}(a_0,-2a_0-b_0-2c_0,c_0)$$
$$= \Lambda_\ell(a,b+c)+\Lambda_\ell(b,a+c)+\Lambda_\ell(c,a+b) = \Delta(a,b,c).$$

As $\alpha(x,y,z) = 1-\{\delta_0(x-y)+\delta_0(y-z)+\delta_0(z-x)\}$ is the function given by (2) we have reached the following result:

$$S_2 := \sum_{a_0x+b_0y+c_0z=\ell} \alpha(x,y,z)h(a_0,-2a_0-b_0-2c_0,c_0)$$

is $\Lambda_\ell(a,b,c)-\Delta(a,b,c)$ is case that $h = L\delta_{a,-2a-b-2c,c}$ and a,b,c are positive integers .

Thus we have proved with this choice of h that the sum S_1-S_2, which is the left-hand side of the formula (3), is equal to $\Delta(a,b,c)-\Delta^1_\ell(a,b,c)$, which is, using the corollary, 2 or 0 if $a+b+c|\ell$ or not.

The right-hand side of (3) is

$$S_3 := 2\sum_{t|\ell}\sum_{i,j=1}^{i+j<t} h(i,-i-j-t,j),$$

which is 2 or 0 if $a+b+c|\ell$ or not if $h = L\delta_{a,-2a-b-2c,c}$, with a,b,c positive integers using (7). Thus we have proved that the theorem for such choice of h.

II) Next we we show that the following statement is true: *if $S_1 \neq 0$ for $h = L\delta_{a',b',c'}$ then for some positive integers a,b,c one has $h = L\delta_{a,-2a-b-2c,c}$* .

The proof is as follows. We look at the first summand of (9). If

$$\sum_{\substack{a_0x+b_0y+c_0z=\ell\\ a_0<b_0<c_0}} L\delta_{a',b',c'}(a_0+c_0,a_0,-b_0)\neq 0,$$

then $L\delta_{a',b',c'} = L\delta_{a_0+c_0,a_0,-b_0}$ for some positive integers a_0, b_0, c_0. Using that $a_0 < b_0 < c_0$ then one can write $a_0 = a, b_0 = a+b, c_0 = a+b+c$ with a, b, c positive, so that $L\delta_{a_0+c_0,a_0,-b_0} = L\delta_{2a+b+c,a,-a-b} = L\delta_{a,-2a-b-2c,c}$ using (8).

The second summand of (9) can be treated in a similar way: if

$$\sum_{\substack{a_0x+b_0y+c_0z=\ell\\ a_0<c_0<b_0}} L\delta_{a',b',c'}(a_0,-c_0-b_0,c_0-a_0)\neq 0,$$

then using $a_0 < c_0 < b_0$ one can write $a_0 = a, c_0 = a+c, b_0 = a+b+c$ with a, b, c positive and therefore $L\delta_{a',b',c'} = L\delta_{a_0,-c_0-b_0,c_0-a_0} = L\delta_{a,-2a-b-2c,c}$.

If the third summand in nonzero for some $L\delta_{a',b',c'}$ then as $b_0 < a_0 < c_0$ write $a_0 = a+b, b_0 = b, c_0 = a+b+c$ with a, b, c positive and therefore $L\delta_{a',b',c'} = L\delta_{a_0+c_0-b_0,a_0-b_0,-a_0} = L\delta_{2a+b+c,a,-a-b} = L\delta_{a,-2a-b-2c,c}$ using (8). If the fourth summand is nonzero then using $b_0 < a_0 < c_0$ one can write $b_0 = b, a_0 = c+b, c_0 = a+b+c$ and then $L\delta_{a',b',c'} = L\delta_{b_0,2c_0-a_0,c_0-2b_0} = L\delta_{b,2a+b+c,a-b+c} = L\delta_{a,-2a-b-2c,c}$ using (8).

The fifth and sixth summands can be treated in a similar way and they are left to the reader: this proves the statement for if $S_1 \neq 0$ for some $h = L\delta_{a',b',c'}$ then at least one out of the six summands is nonzero and the proof follows.

It is easy to prove that if $S_2 \neq 0$ for $h = L\delta_{a',b',c'}$ then $h = L\delta_{a,-2a-b-2c,c}$ for some positive integers a, b, c.

Finally, if $S_3 \neq 0$ for some $h = L\delta_{a',b',c'}$ then $L\delta_{a',b',c'} = L\delta_{i,-2i-2j-k,j}$ because as $i+j < t, 1 \leq i, j$ one may write $t = i+j+k$ with $1 \leq k$.

In other words we have proved: *assume that $h = L\delta_{a',b',c'} \neq L\delta_{a,-2a-b-2c,c}$ for any choice of positive integers a, b, c. With this choice of h, the right and left-hand side of* (3) *is zero.*

III) To prove the theorem assume that h satisfies $h(x,y,z) = h(y,-2x+y-z,x-y+z)$, this condition ensures that h is constant on the orbits of points (x,y,z).

Write

$$h := \sum_{(a,b,c)} const(a,b,c)L\delta_{a,-2a-b-2c,c} + \sum_{(a',b',c')} const(a',b',c')L\delta_{a',b',c'},$$

where in the first sum a, b, c are positive and in the last sum, each summand $L\delta_{a',b',c'} \neq L\delta_{a,-2a-b-2c,c}$ for any choice of positive integers a, b, c (by (II) its contribution is then zero). The theorem follows by linearity using (I).

7. Proof of Theorem 5

The proof follows by linearity from the following result (case $p(u) = u^n$ of the theorem): *if ℓ is a positive integer and $n = 0, 1, 2, \ldots$*

$$-\sum_{ax+by+cz=\ell} \alpha(x,y,z)(a+b+c)^n + 3\sum_{\substack{ax+by+cz=\ell\\ c>\max(a,b);\ a\neq b}} c^n$$

$$= 2\sum_{3\leq t;\ t|\ell} t^n\left(\frac{1}{2}t^2-\frac{3}{2}t+1\right) = \sigma_{n+2}(\ell) - 3\sigma_{n+1}(\ell) + 2\sigma_n(\ell).$$

This last formula follows taking $h(x,y,z) = (x-y+z)^n$ in the following lemma.

Lemma 4. *Assume that ℓ is a positive integer and that $h(x,y,z)$ is an integer-valued function which satisfies $h(x,y,z) = h(y, y-x, z-y)$ and for any positive integers a,b,c satisfies $h(a,-b,c) = h(c,-a,b) = h(b,-c,a)$. Then*

$$-\sum_{ax+by+cz=\ell} \alpha(x,y,z) h(b, a+b, 2a+b+c) + 3 \sum_{\substack{ax+by+cz=\ell \\ c>\max(a,b);\ a\neq b}} h(b,a,c+a-b)$$

$$= 2\sum_{t|\ell} \sum_{i,j=1}^{i+j<t} h(i, i+j, t+j). \tag{10}$$

Proof. The proof follows the same lines as before.

I) For integers a,b,c we define the operator $U\delta_{a,b,c}(x,y,z) = \delta_{b,b-a,c-b}(x,y,z)$ and we define an integer-valued function by $L\delta_{a,b,c} := L\delta_{a,b,c}(x,y,z) = \sum_{i=0}^{5} U^i \delta_{a,b,c}(x,y,z)$. Observe that $U^6\delta_{a,b,c}(x,y,z) = \delta_{a,b,c}(x,y,z)$ and $L\delta_{a,b,c}(y, y-x, z-y) = L\delta_{a,b,c}(x,y,z)$. The *orbit* of a point has six points unless one starts at a point of the form $(0,0,c)$ in which case it has only one point (this is easy to check). Observe that

$$\begin{aligned} L\delta_{a,-b,c} =& \delta_{b,a+b,2a+b+c} + \delta_{a+b,a,a+c} + \delta_{a,-b,c} \\ &+ \delta_{-a-b,-a,a+2b+c} + \delta_{-a,b,2a+2b+c} + \delta_{-b,-a-b,b+c}. \end{aligned} \tag{11}$$

which yields

$$L\delta_{a,-b,c} = L\delta_{b,a+b,2a+b+c} = L\delta_{a+b,a,a+c}. \tag{12}$$

Define $h_0 = h_0(x,y,z)$ by

$$\begin{aligned} h_0 :&= L\delta_{a,-b,c} + L\delta_{b,-c,a} + L\delta_{c,-a,b} \\ &= \delta_{b,a+b,2a+b+c} + \delta_{a+b,a,a+c} + \delta_{a,-b,c} + \delta_{c,b+c,a+2b+c} + \delta_{b+c,b,a+b} + \delta_{b,-c,a} \\ &\quad + \delta_{a,a+c,a+b+2c} + \delta_{a+c,c,b+c} + \delta_{c,-a,b} + \dots, \end{aligned}$$

where the dots indicate that the remaining terms are of the form $\delta_{a',b',c'}$ with a' negative whenever a,b,c are positive (their contribution is zero in the following sums below). Also notice that $h_0(y, y-x, z-y) = h_0(x,y,z)$.

From now on let a,b,c be positive integers in the definition of h_0. A check yields

$$\sum_{a_0x+b_0y+c_0z=\ell} h_0(b_0, a_0+b_0, 2a_0+b_0+c_0) - 3 \sum_{\substack{a_0x+b_0y+c_0z=\ell \\ c_0>\max(a_0,b_0);\ a_0\neq b_0}} h_0(b_0, a_0, c_0+a_0-b_0)$$

$$= 3\Delta_\ell^1(a,b,c) \tag{13}$$

where $\Delta_\ell^1(a,b,c)$ is defined as in Lemma 3.

Also

$$\sum_{a_0x+b_0y+c_0z=\ell} \{\delta_0(x-y) + \delta_0(y-z) + \delta_0(z-x)\}\, h_0(b_0, a_0+b_0, 2a_0+b_0+c_0)$$

$$= 3\left(\Lambda_\ell(a, b+c) + \Lambda_\ell(b, a+c) + \Lambda_\ell(c, a+b)\right) = 3\Delta(a,b,c). \tag{14}$$

Using the definition of $\alpha(x,y,z)$ given by (2) we have reached, using (13) and (14), the following result: the left-hand side of formula (10) is equal to $3(\Delta(a,b,c) - \Delta_\ell^1(a,b,c))$ if $h = h_0$. The last corollary yields that this is equal to 6 or 0 as $a+b+c|\ell$ or not.

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To prove the equality notice that the right-hand side of formula (10) is 6 or 0 as $a+b+c|\ell$ or not, if $h = h_0$. This proves the theorem for the particular case $h = h_0$.

II) Next we will show that the following statement is true: *if the right or left-hand side of formula* (10) *is non zero for some* $h = L\delta_{a',b',c'}$ *with integers* a', b', c', *then one must have* $h = L\delta_{a,-b,c}$ *with* a, b, c *positive integers.*

To prove this, one starts observing that if $\sum_{a_0x+b_0y+c_0z=\ell} L\delta_{a',b',c'}(b_0, a_0 + b_0, 2a_0 + b_0 + c_0) \neq 0$ then the orbit of (a', b', c') must be the orbit of $(b_0, a_0 + b_0, 2a_0 + b_0 + c_0)$. In other words, $L\delta_{a',b',c'} = L\delta_{b_0,a_0+b_0,2a_0+b_0+c_0} = L\delta_{a_0,-b_0,c_0}$ with a_0, b_0, c_0 positive using (12).

The second formula of the left-hand side of (10) can be treated in a similar way. If

$$\sum_{\substack{a_0x+b_0y+c_0z=\ell \\ c_0>\max(a_0,b_0);\ a_0\neq b_0}} L\delta_{a',b',c'}(b_0, a_0, c_0 + a_0 - b_0) \neq 0,$$

then two cases are possible:

i) $a_0 < b_0 < c_0$, and the orbit of (a', b', c') is the orbit of $(b_0, a_0, c_0 + a_0 - b_0) = (b+a, a, c+a)$ with $a = a_0, b = b_0 - a_0, c = c_0 - b_0$ positive integers. In other words $L\delta_{a',b',c'} = L\delta_{b+a,a,c+a} = L\delta_{a,-b,c}$ using (12).

ii) $b_0 < a_0 < c_0$, and the orbit of (a', b', c') is the orbit of $(b_0, a_0, c_0 + a_0 - b_0) = (b, b + a, 2a + b + c)$ with $a = a_0 - b_0, b = b_0, c = c_0 - a_0$ positive integers. In other words $L\delta_{a',b',c'} = L\delta_{b,b+a,2a+b+c} = L\delta_{a,-b,c}$ using (12).

Finally, if $\sum_{t|\ell} \sum_{i,j=1}^{i+j<t} L\delta_{a',b',c'}(i, i + j, t + j) \neq 0$ we can write $t = i + j + k$ with i, j, k positive integers and then $L\delta_{a',b',c'} = L\delta_{i,i+j,i+2j+k} = L\delta_{j,-i,k}$ using (12). This ends the proof of the statement.

III) To prove the theorem assume that h satisfies $h(x, y, z) = h(y, y - x, z - y)$ and $h(a, -b, c) = h(c, -a, b) = h(b, -c, a)$ for any positive integers a, b, c. The first condition ensures that h is constant on the orbit of (x, y, z) while the second says that this constant is the same for the orbits of $(a, -b, c), (c, -a, b), (b, -a, c)$.

Therefore we can find constants such that

$$h := \sum_{(a,b,c)} const(a, b, c)h_0 + \sum_{(a,b,c)} const(a, b, c)L\delta_{a,-b,c},$$

where in the first sum a, b, c are all positive integers with h_0 defined as in (I) and in the last sum at least one of a, b, c is zero or negative (its contribution being zero by (II)) proving the theorem by linearity. □

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