Stability analysis for a time-fractional SIR epidemic disease model with varying population sizes

S. Hariharan[a*], J. Manimaran[a†] and L. Shangerganesh[a‡]

[a] Department of Applied Sciences, National Institute of Technology Goa
Goa - 403 401, India.

Abstract

This paper introduces fractional-order derivatives in the SIR epidemic disease model with varying population sizes. First, we study the existence, uniqueness, and boundedness of solutions of the considered model. Then the basic reproduction number (BRN) R_0 is derived. Local and global asymptotic stability established for disease-free equilibrium point (DFEP). Under certain conditions, we obtain the model's endemic equilibrium point (EEP) and demonstrate the local asymptotic stability. Further, we study the sensitivity analysis for BRN. Finally, different numerical results are provided to study the effects of fractional derivatives and validate the theoretical results.

Keyword: SIR epidemic model; Fractional mathematical model; Stability analysis; Basic reproduction number; Endemic equilibrium point; Sensitivity analysis.

AMS Subject Classifications: 34A08, 34D20, 92B05, 92D30.

1 Introduction

Epidemic models are an essential subject in mathematical ecology and have also received significant attention among researchers due to the recent pandemic disease COVID-19. In the literature, many mathematical models demonstrate the transmission of disease and its speed of spread; see, for example, [1, 4, 6, 7, 9, 10, 11, 26, 28, 29] and also reference therein. In particular, some diseases are spread from human to human and spread the infection quickly, and new variations are also formed in one location and spread to another. Further, it easily transfers from one group to another group. This behavior may be understood through mathematical models using the SIR epidemic model with varying population sizes. There are very few papers for the SIR model with various population sizes accessible; see [2, 13, 17, 18, 23].

There has been a lot of interest in researching the SIR model with integer and fractional order derivatives recently. SIR model for epidemic spread among a community of people taken into account with random perturbations in [1]. Also carried out were stability studies and numerical

*Email: hariharan@nitgoa.ac.in
†Email: manimaranj@nitgoa.ac.in
‡Email: shangerganesh@nitgoa.ac.in

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

simulations in [1]. The existence of non-negative solutions and the asymptotic stability of the equilibrium point are demonstrated for the SIR model with stochastic perturbation in [9]. The local and global asymptotic stabilities of the disease-free equilibrium are investigated using a SIR model with two delays and a general non-linear incidence rate in [10]. Stability analysis for the equilibrium point was addressed while taking into account the SIR epidemic model for the Hepatitis B virus in [11]. Additionally, the same model's optimum control and sensitivity analyses were investigated in [11]. The dynamics of the SIR epidemic model with a discrete-time lag, effects of delay on reproduction number, and local stability analysis for equilibrium point were all investigated in [28]. Additionally, the identical model was taken into account, and global stability for the equilibrium point was investigated using the Lyapunov functional approach in [29]. The interaction dynamics between susceptible and infected individuals in the community under consideration are described by a non-linear SIR model, and stability analysis for equilibrium points is also covered in [31].

There aren't many studies that look at the SIR model with various groups in the literature. A two-group stochastic SIRS epidemic model is proposed with standard incidence rates and deduced sufficient conditions for the existence of a positive solution in [2]. The conditions for disease extinction and persistence in the mean were studied for the integer-order derivative SIR epidemic model with different populations in [13]. The global stability of the SIRS epidemic was investigated using a multi-group SIRS epidemic model with a variety of population sizes in [17]. A multi-group SIRS epidemic model with variable overall population size, infection between various groups, and enough conditions to achieve the highest recovery rate was researched in [18]. It was thought about using the SIR-epidemic model for populations with heterogeneous compositions, and stability analysis for equilibrium points and epidemiological inference for disease transmission were both covered in [23].

On the other hand, fractional-order differential equations with applications in science and engineering have been the subject of extensive study. The Hastings-Powell food chain model with fractional order is taken into consideration, and the necessary and adequate conditions for the discretized system's stability are discussed in [14]. The fractional SIR model for the measles virus was put forth in [15], and stability analysis for the equilibrium point was also investigated. The Adams-Bash forth-Moulton scheme was used to demonstrate the chaotic attractors for the SIR epidemic model with fractional derivatives of childhood diseases model was suggested in [16].

The presence and uniqueness of positive and bounded solutions were examined using the fractional-order SVEIR model in [19]. Additionally, it was determined that the equilibrium values for the same model had global stability in [19]. The implementation of the non-local fractional-order epidemic model to an infection with the human respiratory syncytial virus was examined, along with the best controls in [22]. For the SIR epidemic model with delay, local and global stability of the trivial and EEP were investigated in the context of the fractional derivative in [25]. The fractional-order SIR epidemic model had a approximate solution found in [27]. In this work, we suggested a fractional-order derivative SIR epidemic disease model

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

with varying populations, in contrast to the papers cited above. We also look at the BRN's derivation, stability analysis of equilibrium points, and presence of solutions.

This paper considers a mathematical model of SIR epidemic disease with varying population sizes proposed in [13]. Further, here we extend the same model with time-fractional derivatives. Therefore, the proposed fractional model in this paper consists of three different population variables, namely susceptible $S_i(t)$, infected $I_i(t)$ and recovered individuals $R_i(t)$ in the group where $i = 1, 2$. We considered the following SIR mathematical model with fractional-order derivatives $0 < \alpha < 1$ and varying populations in two groups,

$$\begin{aligned}
{}^cD_t^\alpha S_1 &= \sigma_1 - (\xi_{11} I_1 + \xi_{12} I_2 + \gamma_1) S_1, \\
{}^cD_t^\alpha I_1 &= \xi_{11} S_1 I_1 + \xi_{12} S_1 I_2 - (\gamma_1 + \nu_1 + \rho_1) I_1, \\
{}^cD_t^\alpha R_1 &= \rho_1 I_1 - \gamma_1 R_1, \\
{}^cD_t^\alpha S_2 &= \sigma_2 - (\xi_{21} I_1 + \xi_{22} I_2 + \gamma_2) S_2, \\
{}^cD_t^\alpha I_2 &= \xi_{21} S_2 I_1 + \xi_{22} S_2 I_2 - (\gamma_2 + \nu_2 + \rho_2) I_2, \\
{}^cD_t^\alpha R_2 &= \rho_2 I_2 - \gamma_2 R_2.
\end{aligned} \tag{1.1}$$

Here $\sigma_i,\ i = 1, 2$ represent the recruitment rate of the population into the group and natural death rate is given by $\gamma_i,\ i = 1, 2$. Further, $\rho_i,\ i = 1, 2$ represent the natural recovery rate and death rate due to disease is given by $\nu_i,\ i = 1, 2$. Moreover, the transmission rate of incidence from S_i to I_i and S_i to I_j respectively given as ξ_{ii} and ξ_{ij} where $i, j = 1, 2\ i \neq j$. Without loss of generality, we assume that ν_i, ρ_i, ξ_{ii}, ξ_{ij} are non-negative constant and σ_i and γ_i are positive constant.

The paper is arranged as follows: In Section 2, we provide some preliminaries, it is helpful throughout the article. In Section 3, we prove the existence and uniqueness of the solutions of the model (1.1). Further, the non-negativity and boundedness of solutions are also proved. The calculation of BRN and stability analysis of a DFEP is presented in Section 4. Conditions for the existence of EEP and their stability are also discussed in Section 5. Finally, in Section 6, we perform sensitivity analysis for the BRN, and some numerical results are provided to show the effects of fractional derivative for the model (1.1).

2 Mathematical preliminaries

In this section, we recall some basic definitions, and lemmas are very useful to prove the paper's main results.

Definition 2.1. *[21] Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a function. Then*

$${}^cD_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g(s) ds$$

is said to be the Caputo fractional-order derivative of $g(t)$, where $\alpha \in (n-1, n)$ and $\Gamma(\alpha)$ is the Euler Gamma function.

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

Definition 2.2. *[15]*
Normalized sensitivity index for R_0 with respect to $\aleph$ is given by

$$\mathbb{S}_{\aleph} = \frac{\aleph}{R_0}\frac{\partial R_0}{\partial \aleph},$$

where R_0 is the BRN and $\aleph$ is a given parameter in the model equation.

Lemma 2.1. *(Generalized mean-value theorem) [20]*
Let $g(x) \in C[a,b]$ and ${}^cD_a^{\alpha} g(x) \in C(a,b]$ for $0 < \alpha \leq 1$ then

$$g(x) = g(a) + \frac{1}{\Gamma(\alpha)} {}^cD_a^{\alpha} g(\epsilon)(x-a)^{\alpha}.$$

Lemma 2.2. *[30]*
Let $v(t) \in \mathbb{R}^+$ be a continuous and differentiable function. Then, for any time $t \geq t_0$,

$${}^cD^{\alpha}\left[v(t) - v_0 - v_0 ln\frac{v(t)}{v_0}\right] \leq \left(1 - \frac{v_0}{v(t)}\right) {}^cD^{\alpha} v(t),$$

$v_0 \in \mathbb{R}^+$ is known data.

Lemma 2.3. *[8]*
If V is a bounded closed set, then every solution of ${}^cD^{\alpha}x(t) = f(x)$ take the initial value from V and remains in V for all time. If there exists a function $U(x) : V \to \mathbb{R}$, which has a continuous first partial derivatives with

$${}^cD^{\alpha}U|_{{}^cD^{\alpha}x(t)=f(x)} \leq 0.$$

Let $Q = \{x|{}^cD^{\alpha}U|_{{}^cD^{\alpha}x(t)=f(x)} = 0\}$ and L be the largest invariant set of Q. Then every solution of $x(t)$ inizitated in $V \to L$ as $t \to \infty$. In particular, when $L = 0$, then $x \to 0$ as $t \to \infty$.

Lemma 2.4. *[12]*
Let $v(t)$ be a continuous function on $[a, \infty)$ and satisfy

$${}^cD^{\alpha}v(t) \leq -\mu v(t) + \lambda,$$

$$v(a) = v_a.$$

Here $(\mu, \lambda) \in \mathbb{R}^2, \mu \neq 0$ and $a \geq 0$ is the initial time. Then, the solution has the form

$$v(t) \leq (v_a - \frac{\lambda}{\mu})E_{\alpha}[-\mu(t-a)^{\alpha}] + \frac{\lambda}{\mu}.$$

Here $E_{\alpha}[\cdot]$ denotes the Mittag-Leffler functions [24].

3 Solvability of fractional SIR epidemic disease model with varying population sizes

In this section, we prove the well possedness of solutions for the proposed SIR epidemic disease fractional-order system with varying populations. Here, we use the contraction mapping principle to prove the desired result. Then, we study the non-negativity of solutions of the proposed model. Finally, we establish the boundedness of the solutions of the model (1.1).

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

In order to prove that there exists a solution $(S_1(t), I_1(t), R_1(t), S_2(t), I_2(t), R_2(t))$ for (1.1), we rewrite the given system (1.1) as follows:

$$^cD_t^\alpha U(t) = F(U(t)),\ 0 < \alpha < 1,\ t \in (0, T],\ U(0) = U_0,$$

where the nonlinear function $F : \Omega \to \mathbb{R}$ is defined as below:

$$U(t) = \begin{bmatrix} S_1 \\ I_1 \\ R_1 \\ S_2 \\ I_2 \\ R_2 \end{bmatrix}, \qquad U_0 = \begin{bmatrix} S_{1_0} \\ I_{1_0} \\ R_{1_0} \\ S_{2_0} \\ I_{2_0} \\ R_{2_0} \end{bmatrix}, \qquad F(U(t)) = \begin{bmatrix} \sigma_1 - (\xi_{11} I_1 + \xi_{12} I_2 + \gamma_1) S_1 \\ \xi_{11} S_1 I_1 + \xi_{12} S_1 I_2 - (\gamma_1 + \nu_1 + \rho_1) I_1 \\ \rho_1 I_1 - \gamma_1 R_1 \\ \sigma_2 - (\xi_{21} I_1 + \xi_{22} I_2 + \gamma_2) S_2 \\ \xi_{21} S_2 I_1 + \xi_{22} S_2 I_2 - (\gamma_2 + \nu_2 + \rho_2) I_2 \\ \rho_2 I_2 - \gamma_2 R_2 \end{bmatrix}.$$

Here Ω is defined as follows:

$$\Omega = \{(S_1, I_1, R_1, S_2, I_2, R_2) \in \mathbb{R}_+^6 : max(|S_1|, |I_1|, |R_1|, |S_2|, |I_2|, |R_2|) \leq A\}. \tag{3.1}$$

Further $X = C([0, T], \mathbb{R})$ is the Banach space of continuous functions from $[0, T]$ into $\mathbb{R}$ and $(X, \|.\|_\infty)$ endowed with the supremum norm $\|U(t)\|_\infty = \sup_{0 \leq t \leq T} |U(t)|$.

Theorem 3.1. *Suppose that*

(i) there exists a constant $0 < M < 1$ *such that* $|F(U(t)) - F(V(t))| \leq M\|U - V\|$ *where*

$$M = \frac{T^\alpha}{\Gamma(\alpha + 1)}\ max\ \{(\xi_{11} + \xi_{12})2A + \gamma_1, 2A(\xi_{11} + \xi_{12}) + (\gamma_1 + \nu_1 + \rho_1), \rho_1 + \gamma_1,$$
$$(\xi_{21} + \xi_{22})2A + \gamma_1, 2A(\xi_{21} + \xi_{22}) + (\gamma_2 + \nu_2 + \rho_2), \rho_2 + \gamma_2\},$$

(ii) there exists $U_0 \in X$ *then the operator* $\Theta : X \to X$ *is defined by*

$$\Theta(U(t)) = U_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} F(U(\tau)) d\tau,$$

satisfies

$$\|\Theta(U(t)) - \Theta(V(t)))\| \leq M\|U - V\|,$$

where M *is defined as before.*

Then there exists a unique solution for the system (1.1) in the region $\Omega \times (0, T]$ *with the initial conditions* $F(0) = F_0$ *and* $t \in (0, T]$.

Proof. Consider the solution of the system (1.1), which is given from the Lemma (2.1), as follows:

$$\begin{aligned} \Theta(U(t)) &= U_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} F(U(\tau)) d\tau, \\ \Theta(U(t)) - \Theta(V(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} (F(U(\tau)) - F(V(\tau))) d\tau, \\ |\Theta(U(t)) - \Theta(V(t))| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} |(F(U(\tau)) - F(V(\tau)))| d\tau. \end{aligned}$$

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

The norm of the matrix $P = |p_{i,j}(t)|$, is denoted by

$$\|P\|_\infty = \sum_{i,j} \sup_{t\in(0,T])} |p_{i,j}(t)|.$$

Now we get

$$\begin{aligned}
\|\Theta(U(t)) - \Theta(V(t))\| &\leq \left(\frac{1}{\Gamma(\alpha)}\right)\left(\frac{t^\alpha}{\alpha}\right) \max\{(\xi_{11}+\xi_{12})2A+\gamma_1, 2A(\xi_{11}+\xi_{12})+G_1, \rho_1+\gamma_1, \\
&\qquad (\xi_{21}+\xi_{22})2A+\gamma_1, 2A(\xi_{21}+\xi_{22})+G_2, \rho_2+\gamma_2\}\|U-V\| \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \max\{(\xi_{11}+\xi_{12})2A+\gamma_1, 2A(\xi_{11}+\xi_{12})+G_1, \rho_1+\gamma_1, \\
&\qquad (\xi_{21}+\xi_{22})2A+\gamma_1, 2A(\xi_{21}+\xi_{22})+G_2, \rho_2+\gamma_2\}\|U-V\| \\
&\leq M\|U-V\|.
\end{aligned}$$

Here $G_1 = \gamma_1 + \nu_1 + \rho_1$ and $G_2 = \gamma_2 + \nu_2 + \rho_2$. If $M < 1$, then $U = F(U)$ is contraction mapping, and this becomes the sufficient condition for the existence and uniqueness of the solution for the model (1.1). □

Theorem 3.2. *Suppose system* (1.1) *has a unique solution for all time* $t \geq 0$ *with non-negative initial conditions then all state variables* $S_i(t), I_i(t), R_i(t), (i = 1, 2)$ *are also non-negative. Further the total population* $Q(t) = \sum_{i=1}^{2} S_i(t) + I_i(t) + R_i(t)$ *remain bounded.*

Proof. It is easy to understand that from Theorem 3.1 there exists a unique solution for the system (1.1). Next, we have to prove that solutions of (1.1) are non-negative. From the system, we have

$$\begin{aligned}
&{}^cD_t^\alpha S_1|_{S_1=0} = \sigma_1 \geq 0, \\
&{}^cD_t^\alpha I_1|_{I_1=0} = \xi_{12} S_1 I_2 \geq 0, \\
&{}^cD_t^\alpha R_1|_{R_1=0} = \rho_1 I_1 \geq 0, \\
&{}^cD_t^\alpha S_2|_{S_2=0} = \sigma_2 \geq 0, \\
&{}^cD_t^\alpha I_2|_{I_2=0} = \xi_{21} S_2 I_1 \geq 0, \\
&{}^cD_t^\alpha R_2|_{R_2=0} = \rho_2 I_2 \geq 0.
\end{aligned} \tag{3.2}$$

By (3.2) and by Lemma 2.1, we say that $(S_1(t), I_1(t), R_1(t), S_2(t), I_2(t), R_2(t)) \geq 0$ for all $t \geq 0$.

Next, we want to prove that non-negative solutions of (1.1) also bounded. Adding all the equations of fractional SIR epidemic model (1.1) and using the definition $Q(t)$, we get

$${}^cD_t^\alpha Q(t) = \sigma_1 - \gamma_1 S_1 - (\gamma_1 + \nu_1) I_1 - \gamma_1 R_1 + \sigma_2 - \gamma_2 S_2 - (\gamma_2 + \nu_2) I_2 - \gamma_2 R_2.$$

For $\mu > 0$, we get

$$\begin{aligned}
{}^cD_t^\alpha Q(t) + \mu Q(t) &= \sigma_1 - (\gamma_1 - \mu) S_1 - (\gamma_1 + \nu_1 - \mu) I_1 - (\gamma_1 - \mu) R_1 + \sigma_2 - (\gamma_2 - \mu) S_2 \\
&\quad -(\gamma_2 + \nu_2 - \mu) I_2 - (\gamma_2 - \mu) R_2.
\end{aligned}$$

Suppose we assume that $\mu \leq \min\{\gamma_1, \gamma_2\}$ then

$${}^cD_t^\alpha Q(t) + \mu Q(t) \leq \sigma_1 + \sigma_2 = g.$$

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

Then by Lemma 2.4, we have

$$Q_t \leq \left(Q_0 - \frac{g}{\mu}\right) E_\alpha[-\mu t^\alpha] + \frac{g}{\mu},$$

if $t \to \infty$ then $Q(t) \to \frac{g}{\mu}$. This shows that $0 < Q(t) \leq \frac{g}{\mu}$. Hence, all the solutions of the system beginning with $\mathbb{R}_6^+$ are restricted to the region

$$\Omega = \left\{(S_1, I_1, R_1, S_2, I_2, R_2) \in \mathbb{R}_6^+ | Q(t) \leq \frac{g}{\mu} + \delta, \text{ for any } \delta > 0\right\}.$$

It is clear that Ω obtained above satisfies as in (3.1). □

4 Stability analysis of a disease free equilibrium point

In this section, first, we find the DFEP for the considered model. Then, using the method proposed in [3] BRN of the multiple group model derived. Finally, we conclude the section with stability analysis of the DFEP of the model (1.1).

4.1 Equlibrium point

Suppose $E^* := (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*) \in \mathbb{R}_+^6$ is the equilibrium point of the model (1.1). Then

$$\begin{aligned} {}^cD_t^\alpha S_1(E^*) = 0, \quad {}^cD_t^\alpha I_1(E^*) = 0, \quad {}^cD_t^\alpha R_1(E^*) = 0, \\ {}^cD_t^\alpha S_2(E^*) = 0, \quad {}^cD_t^\alpha I_2(E^*) = 0, \quad {}^cD_t^\alpha R_2(E^*) = 0. \end{aligned} \tag{4.1}$$

From the above, we obtain one equilibrium point $E_0 = \left(\frac{\sigma_1}{\gamma_1}, 0, 0, \frac{\sigma_2}{\gamma_2}, 0, 0\right)$ and it is called a DFEP. Apart from the DFEP, there are some other possible equilibrium points that exist, and those will be discussed later.

4.2 Basic reproductive number

Obtaining BRN for SIR model with varying population is important calculation to analyze the model behaviour. However, it is not straightforward for the considered model (1.1) as in the basic SIR model. We require a more systematic approach as in [3]. Therefore, we follow a method proposed in [3] and then calculate the BRN of (1.1). To compute the BRN R_0, first, we distinguish the new infection from all other changing individuals. Let $\mathbb{F}_i$ denotes the rate of arrival of new infections in the compartment i. $\mathbb{V}_i^-$ refer the transformation rate of individuals from the compartment i to other and $\mathbb{V}_i^+$ represents the transformation rate of individuals from other to compartment i.

$${}^cD_t^\alpha = f_i(x) = \mathbb{F}_i - \mathbb{V}_i,$$

where $\mathbb{V}_i = \mathbb{V}_i^- - \mathbb{V}_i^+$. To define a next generation matrix, we have to calculate F_i and V_i. To find F_i and V_i, compute the first partial derivatives with respect to the infected compartments and then we form a next generation matrix as FV^{-1}.

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

We divide the model (1.1) into two sub-models. We consider sub-model i as a fractional derivative of S_i, I_i, R_i for $i = 1, 2$ respectively. For the sub-model i

$$\mathbb{F}_i = (\sigma_i, \xi_{i1} S_i I_1 + \xi_{i2} S_i I_2, 0),$$

$$\mathbb{V}_i = (\gamma_i S_1 + \xi_{i1} S_i I_1 + \xi_{i2} S_i I_2, (\gamma_i + \nu_i + \rho_i) I_i, \gamma_i R_i - \rho_i I_i).$$

Here I_i (i = 1, 2) is the only infected compartment in the sub-model i. So the next generation matrix for the disease free equilibrium point $\left(\frac{\sigma_i}{\gamma_i}, 0, 0\right)$ is

$$F_i = \frac{\partial \mathbb{F}_i}{\partial I_i} \text{ and } V_i = \frac{\partial \mathbb{V}_i}{\partial I_i}.$$

Then we get, $F_i = [S_i \xi_{ii}] = \left[\frac{\xi_{ii}\sigma_i}{\gamma_i}\right]$ and $V_i = [\gamma_i + \nu_i + \rho_i] = [G_i]$.

$$R_{0i} = F_i V_i^{-1} = \frac{\xi_{ii}\sigma_i}{\gamma_i G_i} = \frac{K_{ii}}{G_i},$$

where $K_{ij} = \frac{\xi_{ij}\sigma_i}{\gamma_i}$, $G_i = \gamma_i + \nu_i + \rho_i$. Thus the BRN is

$$R_0 = R_{01} + R_{02} = \frac{K_{11}}{G_1} + \frac{K_{22}}{G_2}. \tag{4.2}$$

Now, we discuss about the stability of DFEP of the model (1.1).

Theorem 4.1. *Suppose $K_{ij} = \frac{\xi_{ij}\sigma_i}{\gamma_i}$, $G_i = \gamma_i + \nu_i + \rho_i$ for $i, j = 1, 2$ where ξ_{ij}, σ_i γ_i, ν_i ρ_i are defined as in the model (1.1). If the following conditions $R_0 < 1$ and*

$$K_{11}K_{22} + G_1G_2 > G_1K_{22} + G_2K_{11} + K_{12}K_{21}, \tag{4.3}$$

hold true then the DFEP is locally asymptotic stable.

Proof. Consider a function $f : \mathbb{R}^6_+ \to \mathbb{R}^6_+$, where

$$f(U) = (f_1(U), f_2(U), f_3(U), f_4(U), f_5(U), f_6(U)),\ U = (S_1, I_1, R_1, S_2, I_2, R_2) \in \mathbb{R}^+_6.$$

Suppose RHS of (1.1) is taken as f_i, $(i = 1, \cdots, 6)$, then (1.1) is rewritten as $D^\alpha(U) = f_i(U)$.

Thus the Jacobian matrix is given by

$$J_f = \frac{\partial(f_1, f_2, f_3, f_4, f_5, f_6)}{\partial(S_1, I_1, R_1, S_2, I_2, R_2)} = \begin{bmatrix} \frac{\partial f_1}{\partial S_1} & \cdots & \frac{\partial f_1}{\partial R_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_6}{\partial S_1} & \cdots & \frac{\partial f_6}{\partial R_2} \end{bmatrix}.$$

Now, we give the Jacobian matrix for the model (1.1) at the disease free equilibrium $E_0 = \left(\frac{\sigma_1}{\gamma_1}, 0, 0, \frac{\sigma_2}{\gamma_2}, 0, 0\right)$.

$$J_f(E_0) = \begin{bmatrix} -\gamma_1 & -K_{11} & 0 & 0 & -K_{12} & 0 \\ 0 & K_{11} - G_1 & 0 & 0 & K_{12} & 0 \\ 0 & \rho_1 & -\gamma_1 & 0 & 0 & 0 \\ 0 & -K_{21} & 0 & -\gamma_2 & -K_{22} & 0 \\ 0 & k_{21} & 0 & 0 & K_{22} - G_2 & 0 \\ 0 & 0 & 0 & 0 & \rho_2 & -\gamma_2 \end{bmatrix}.$$

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

Characteristic equation of the above matrix is given as

$$(-\lambda-\gamma_1)(-\lambda-\gamma_1)(-\lambda-\gamma_2)(-\lambda-\gamma_2)(\lambda^2+P_1\lambda+P_2)=0,$$

where

$$\begin{aligned} P_1 &= (G_1+G_2-(K_{11}+K_{22}),\\ P_2 &= K_{11}K_{22}-G_1K_{22}-G_2K_{11}+G_1G_2-K_{12}K_{21}. \end{aligned}$$

By simple calculation, we get the following eigenvalues,

$$\lambda_1=-\gamma_1;\ \lambda_2=-\gamma_1;\ \lambda_3=-\gamma_2;\ \lambda_4=-\gamma_2.$$

We find the remaining eigenvalues by solving the equation

$$\lambda^2+P_1\lambda+P_2=0. \tag{4.4}$$

We know that from Routh-Hurtwiz condition equation (4.4) has a negative root of real parts if P_1 and P_2 are greater than 0. If $R_0<1$ then it is obvious that $P_1>0$ and similarly if condition (4.3) hold true then $P_2>0$. Thus, the DFEP is locally asymptotic stable. □

Next, we prove that DFEP is globally asymptotically stable.

Theorem 4.2. *Suppose* $K_{ij}=\dfrac{\xi_{ij}\sigma_i}{\gamma_i}$ *for* $i,j=1,2$ *where* ξ_{ij}, σ_i γ_i *are defined as in the model* (1.1). *If the following conditions*

$$\begin{aligned} k_{11}+k_{21}&<\gamma_1+\nu_1,\\ k_{12}+k_{22}&<\gamma_2+\nu_2, \end{aligned} \tag{4.5}$$

hold true then the DFEP is globally asymptotic stable.

Proof: Consider the Lyapunov function as

$$\begin{aligned} &L(S_1,I_1,R_1,S_2,I_2,R_2)\\ &\quad=\left(S_1-\frac{\sigma_1}{\gamma_1}-\frac{\sigma_1}{\gamma_1}\ln\left(\frac{\gamma_1S_1}{\sigma_1}\right)\right)+I_1+R_1+\left(S_2-\frac{\sigma_2}{\gamma_2}-\frac{\sigma_2}{\gamma_2}\ln\left(\frac{\gamma_2S_2}{\sigma_2}\right)\right)+I_2+R_2. \end{aligned}$$

From the above definition, it is easy to understand that

$$L(S_1,I_1,R_1,S_2,I_2,R_2)\begin{cases} =0 \text{ only at } \left(\dfrac{\sigma_1}{\gamma_1},0,0,\dfrac{\sigma_2}{\gamma_2},0,0\right),\\ >0\ \Omega\neq\left(\dfrac{\sigma_1}{\gamma_1},0,0,\dfrac{\sigma_2}{\gamma_2},0,0\right). \end{cases}$$

First compute the fractional order α^{th} derivative for L, and use Lemma 2.1, we get

$$\begin{aligned} {}^cD_t^\alpha L(t) &\leq \left(1-\frac{\sigma_1}{\gamma_1S_1}\right){}^cD_t^\alpha S_1+{}^cD_t^\alpha I_1+{}^cD_t^\alpha R_1+\left(1-\frac{\sigma_2}{\gamma_2S_2}\right){}^cD_t^\alpha S_2+{}^cD_t^\alpha I_2+{}^cD_t^\alpha R_2\\ &\leq -\frac{(\sigma_1-S_1\gamma_1)^2}{S_1\gamma_1}-\frac{(\sigma_2-S_2\gamma_2)^2}{S_2\gamma_2}-\gamma_1R_1-\gamma_2R_2-\left(\gamma_1+\nu_1-\frac{\xi_{21}\sigma_2}{\gamma_2}-\frac{\xi_{11}\sigma_{case1}}{\gamma_1}\right)I_1\\ &\quad-\left(\gamma_2+\nu_2-\frac{\xi_{12}\sigma_1}{\gamma_1}-\frac{\xi_{22}\sigma_2}{\gamma_2}\right)I_2\\ &\leq -\left(\gamma_1+\nu_1-\frac{\xi_{21}\sigma_2}{\gamma_2}-\frac{\xi_{11}\sigma_1}{\gamma_1}\right)I_1-\left(\gamma_2+\nu_2-\frac{\xi_{12}\sigma_1}{\gamma_1}-\frac{\xi_{22}\sigma_2}{\gamma_2}\right)I_2\\ &\leq -(\gamma_1+\nu_1-K_{21}-K_{11})\,I_1-(\gamma_2+\nu_2-K_{12}-K_{22})\,I_2. \end{aligned}$$

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

Then we get, ${}^cD_t^\alpha L(t) \leq 0$ for all $(S_1, I_1, R_1, S_2, I_2, R_2) \in \mathbb{R}_+^6$ if (4.5) hold true. Further, ${}^cD_t^\alpha L(t) = 0$ only at DFEP.

Then using Lemma 2.3, it follows that every solution belongs to $\mathbb{R}_+^6$ tends to E_0. These shows that the equilibrium point E_0 is globally asymptotically stable.

5 Stability analysis of endemic equilibrium point

In this section, first we prove that there exists at-least one EEP for the considered fractional SIR model with varying population sizes (1.1). Further, we prove that the EEP is locally asymptotic stable for BRN $R_0 > 1$.

Theorem 5.1. *Consider $R_0 > 1$, $H_{ij} = \dfrac{\xi_{ij}\gamma_j}{\rho_j}$ and $G_i = \gamma_i + \nu_i + \rho_i$. Further, assume that at least one of the following conditions are satisfied*

$$\begin{array}{rcl} \sigma_2\rho_2\sigma_1\rho_1(H_{21}H_{12} - H_{11}H_{22}) + G_1\gamma_2^2\rho_2H_{22}\sigma_2 & < & G_1\gamma_1^2G_2\gamma_2^2 - G_2\gamma_2^2\rho_1H_{11}\sigma_1 \\ \textit{and} \qquad H_{11}H_{22} & > & H_{21}H_{12}, \end{array} \tag{5.1}$$

$$\begin{array}{rcl} \sigma_2\rho_2\sigma_1\rho_1(H_{21}H_{12} - H_{11}H_{22}) + G_1\gamma_2^2\rho_2H_{22}\sigma_2 & > & G_1\gamma_1^2G_2\gamma_2^2 - G_2\gamma_2^2\rho_1H_{11}\sigma_1 \\ \textit{and} \qquad H_{11}H_{22} & < & H_{21}H_{12}. \end{array} \tag{5.2}$$

Then there exists at least one EEP other than DFEP $\left(\dfrac{\sigma_1}{\gamma_1}, 0, 0, \dfrac{\sigma_2}{\gamma_2}, 0, 0\right)$.

Proof. Equating the fractional derivatives of (1.1) to zero, for $i = 1, 2$, we get

$$S_i = \left(\frac{\sigma_i}{\xi_{i1}I_1 + \xi_{i2}I_2 + \gamma_i}\right), \tag{5.3}$$

$$S_i(\xi_{i1}I_1 + \xi_{i2}I_2) = G_iI_i, \tag{5.4}$$

$$I_i = \left(\frac{\gamma_i}{\rho_i}\right)R_i. \tag{5.5}$$

Substituting (5.3) and (5.5) in (5.4), we get

$$A_1R_1^2 - A_2R_1 - A_3 = 0, \tag{5.6}$$

$$B_1R_2^2 - B_2R_2 - B_3 = 0, \tag{5.7}$$

respectively for $i = 1$ & $i = 2$. Here, A_k, B_k for $k = 1, 2, 3$ are defined as follows:

$$\begin{array}{ll} A_1 = G_1\gamma_1^2\rho_2\xi_{11}; & B_1 = G_2\gamma_2^2\rho_1\xi_{22}; \\ A_2 = \xi_{11}\sigma_1\gamma_1\rho_2\rho_1 - G_1\gamma_1\gamma_2\xi_{12}\rho_1R_2 - G_1\gamma_1^2\rho_1\rho_2; & B_2 = \xi_{22}\sigma_2\gamma_2\rho_2\rho_1 - G_2\gamma_1\gamma_2\xi_{21}\rho_2R_1 - G_2\gamma_2^2\rho_2\rho_1; \\ A_3 = \xi_{12}\sigma_1\gamma_2\rho_1^2R_2; & B_3 = \xi_{21}\sigma_2\gamma_1\rho_2^2R_1. \end{array}$$

It is easy to see that by Descartes' rules of signs , if $R_1 > 0$ in (5.7) then R_2 has atleast one positive solution. Next our claim is that R_1 has atleast one positive solution. Substituting A_k, B_k for $k = 1, 2, 3$ in (5.6) and (5.7), we get

$$R_2 = \frac{(G_1\gamma_1H_{11}R_1 + G_1\gamma_1^2 - \rho_1H_{11}\sigma_1)R_1}{H_{12}\sigma_1\rho_1 - G_1\gamma_1R_1H_{12}}, \tag{5.8}$$

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

$$R_1 = \frac{(G_2\gamma_2 H_{22} R_2 + G_2\gamma_2^2 - \rho_2 H_{22}\sigma_2)R_2}{H_{21}\sigma_2\rho_2 - G_2\gamma_2 R_2 H_{21}}. \tag{5.9}$$

Now, substitute (5.8) in (5.9) and solving the resulting algebraic equation, we get

$$R_1 = 0, \tag{5.10}$$

$$a_1 R_1^3 + a_2 R_1^2 + a_3 R_1 + a_4 = 0, \tag{5.11}$$

where we assume a_1, a_2, a_3 and a_4 are as follows:

$$\begin{aligned}
a_1 &= (G_1\gamma_1 H_{11})^2 G_2\gamma_2 H_{22} - (G_1\gamma_1)^2 H_{12} H_{11} G_2\gamma_2 H_{21},\\
a_2 &= (G_1\gamma_1 H_{12})^2 H_{21}\sigma_2\rho_2 - H_{21} H_{12}\sigma_1\rho_1 G_1\gamma_1 H_{11} G_2\gamma_2 + G_1^2\gamma_1^3 G_2\gamma_2 H_{21} H_{12} -\\
&\quad \rho_1\sigma_1 H_{11} G_1\gamma_1 H_{12} G_2\gamma_2 H_{21} - G_1\gamma_1 H_{11}(G_2\gamma_2 H_{22} G_1\gamma_1^2 - G_2\gamma_2 H_{22}\rho_1\sigma_1 H_{11}\\
&\quad -G_1\gamma_1 H_{12} G_2\gamma_2^2 + \rho_2\sigma_2 H_{22} G_1\gamma_1 H_{12}) - (G_1\gamma_1^2 - \rho_1 H_{11}\sigma_1) G_1\gamma_1 H_{11} G_2\gamma_2 H_{22},\\
a_3 &= H_{12}\sigma_1\rho_1(\rho_1\sigma_1 H_{11} G_2\gamma_2 H_{21} - G_1\gamma_1 H_{12} H_{21}\sigma_2\rho_2 - G_1\gamma_1^2 G_2\gamma_2 H_{21})\\
&\quad -G_1\gamma_1 H_{11} H_{12}\sigma_1\rho_1(G_2\gamma_2^2 - \rho_2 H_{22}\sigma_2) - (G_1\gamma_1^2 - \rho_1 H_{11}\sigma_1)(G_2\gamma_2 H_{22} G_1\gamma_1^2\\
&\quad -G_2\gamma_2 H_{22}\rho_1\sigma_1 H_{11} - G_1\gamma_1 H_{12} G_2\gamma_2^2 + \rho_2\sigma_2 H_{22} G_1\gamma_1 H_{12}) - G_1\gamma_1 H_{12}^2\sigma_1\rho_1 H_{21}\sigma_2\rho_2,\\
a_4 &= (H_{12}\sigma_1\rho_1)^2 H_{21}\sigma_2\rho_2 - (G_1\gamma_1^2 - \rho_1 H_{11}\sigma_1)(H_{12}\sigma_1\rho_1(G_2\gamma_2^2 - \rho_2 H_{22}\sigma_2)).
\end{aligned}$$

The one root $R_1 = 0$ gives the DFEP. Then, from (5.11), we look for other possible roots .

Suppose if $a_1 > 0$ and $a_4 < 0$ or $a_1 < 0$ and $a_4 > 0$ then at least one of the conditions (5.2) are satisfied. Therefore, the Descartes rule of signs implies that there exists at least on positive real root for (5.11). □

Theorem 5.2. *Suppose $R_0 > 1$ and*

$$\begin{aligned}
\phi_i &> 0, \forall i = 1, \cdots, 6,\\
\phi_1\phi_2 &> \phi_3,\\
\phi_1\phi_2\phi_3 + \phi_1\phi_5 &> \phi_1^2\phi_4 + \phi_3^2,\\
\phi_1^2(\phi_2(\phi_3\phi_4 + \phi_1\phi_6)) + \phi_5\phi_3^2 &> \phi_5(\phi_1\phi_2 - \phi_3)^2 + \phi_1(\phi_3(\phi_1\phi_6 + \phi_2\phi_5 +\\
&\quad \phi_4\phi_3) + \phi_5^2 + \phi_1^2\phi_4^2),\\
(\phi_5(\phi_1\phi_2 - \phi_3) - \phi_1^2\phi_6)^2\phi_3 &> \phi_1\phi_6(\phi_3^2 + \phi_1^2\phi_4) + \phi_1\phi_2(\phi_5(\phi_1\phi_2 - \phi_3) -\\
+\phi_1^2\phi_6(\phi_2\phi_3 + \phi_5) &\quad \phi_1^2\phi_6)^2 + \phi_1\phi_6(\phi_5\phi_1^2 + (\phi_1\phi_2 - \phi_3)\phi_3 - \phi_1^2\phi_4)^2.
\end{aligned} \tag{5.12}$$

are satisfied then the EEP $E_1 = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*)$ of the model (1.1) is locally asymptotic stable. Here, assume that

$a_{11} = -\xi_{11} I_1^* - \xi_{12} I_2^* - \gamma_1$, $a_{12} = -\xi_{11} S_1^*$, $a_{15} = -\xi_{12} S_1^*$,

$a_{21} = \xi_{11} I_1^* + \xi_{12} I_2^*$, $a_{22} = \xi_{11} S_1^* - G_1$, $a_{25} = \xi_{12} S_1^*$,

$a_{32} = \rho_1$, $a_{33} = -\gamma_1$,

$a_{42} = -\xi_{21} S_2^*$ $a_{44} = -\xi_{21} I_1^* - \xi_{22} I_2^* - \gamma_2$, $a_{45} = -\xi_{22} S_2^*$,

$a_{52} = \xi_{21} S_2^*$ $a_{54} = \xi_{21} I_1^* + \xi_{22} I_2^*$ $a_{55} = \xi_{22} S_2^* - G_2$,

$a_{65} = \rho_2$, $a_{66} = -\gamma_2$.

Further, we also assume that

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

$$
\begin{aligned}
\phi_1 &= -(a_{11}+a_{22}+a_{33}+a_{44}+a_{55}+a_{66}),\\
\phi_2 &= a_{11}(a_{22}+a_{33}+a_{44}+a_{55}+a_{66})+a_{22}(a_{33}+a_{44}+a_{55}+a_{66})+a_{33}(a_{44}+a_{55}+a_{66})\\
&\quad +a_{44}(a_{55}+a_{66})+a_{55}a_{66}-a_{12}a_{21}-a_{45}a_{54},\\
\phi_3 &= (a_{21}a_{12}(a_{33}+a_{44}+a_{55}+a_{66})+a_{45}a_{54}(a_{33}+a_{66}+a_{22}+a_{11})+a_{52}a_{25}(a_{11}+a_{33}\\
&\quad +a_{44}+a_{66}))-(a_{11}(a_{22}(a_{33}+a_{44}+a_{55}+a_{66})+a_{33}(a_{44}+a_{55}+a_{66})+a_{44}(a_{55}+a_{66})\\
&\quad +a_{55}a_{66})+a_{22}(a_{33}(a_{44}+a_{55}+a_{66})+a_{44}(a_{55}+a_{66})+a_{55}a_{66})+a_{33}(a_{44}(a_{55}+a_{66})\\
&\quad +a_{55}a_{66})+a_{44}a_{55}a_{66}+a_{21}a_{52}a_{15}+a_{25}a_{42}a_{54}),\\
\phi_4 &= a_{11}(a_{22}(a_{33}(a_{44}+a_{55}+a_{66})+a_{44}(a_{55}+a_{66})+a_{55}a_{66})+a_{33}(a_{44}(a_{55}+a_{66})+a_{55}\\
&\quad a_{66})+a_{44}a_{55}a_{66})+a_{22}(a_{33}(a_{44}(a_{55}+a_{66})+a_{55}a_{66})+a_{44}a_{55}a_{66})+a_{33}a_{44}a_{55}a_{55}\\
&\quad -(a_{12}a_{21}(a_{33}(a_{44}+a_{55}+a_{66})+a_{44}(a_{55}+a_{66})+a_{55}a_{66})+a_{11}a_{25}a_{52}(a_{33}+a_{44}+a_{66})\\
&\quad +a_{11}a_{25}a_{42}a_{54}+a_{21}a_{15}a_{52}(a_{33}+a_{44}+a_{66})-a_{54}a_{45}(a_{11}(a_{33}+a_{22}+a_{66})-a_{21}a_{12})\\
&\quad -a_{22}a_{33}a_{45}a_{54}-a_{25}a_{33}a_{52}a_{44}-a_{25}a_{33}a_{52}a_{66}-a_{22}a_{45}a_{54}a_{66}-a_{25}a_{44}a_{52}a_{66}\\
&\quad -a_{33}a_{45}a_{54}a_{66}-a_{21}a_{15}a_{42}a_{54}+a_{25}a_{54}a_{42}(a_{33}+a_{66}),\\
\phi_5 &= a_{66}(a_{21}a_{12}-a_{11}a_{22})(a_{44}a_{55}-a_{45}a_{54})+(a_{44}a_{52}-a_{54}a_{42})(a_{11}a_{25}-a_{21}a_{15})\\
&\quad a_{33}a_{66}(a_{21}a_{12}a_{44}-a_{11}a_{22}a_{44}-a_{11}a_{22}a_{55}+a_{11}a_{25}a_{52}-a_{21}a_{15}a_{52}+a_{21}a_{12}a_{55}\\
&\quad -a_{11}a_{44}a_{55}+a_{11}a_{45}a_{54}-a_{22}a_{44}a_{55}+a_{22}a_{45}a_{54}-a_{25}a_{42}a_{54}+a_{25}a_{52}a_{44})\\
&\quad -(a_{33}((a_{11}a_{22}-a_{21}a_{12})(a_{44}a_{55}-a_{54}a_{45}))+(a_{11}a_{25}-a_{21}a_{15})(a_{42}a_{54}-a_{52}a_{44})),\\
\phi_6 &= a_{33}a_{66}(a_{11}a_{22}a_{44}a_{55}-a_{11}a_{22}a_{45}a_{54}-a_{11}a_{25}a_{42}a_{54}+a_{44}a_{25}a_{11}a_{52}\\
&\quad -a_{21}a_{12}a_{44}a_{55}+a_{21}a_{12}a_{45}a_{54}-a_{15}a_{42}a_{54}a_{21}+a_{21}a_{15}a_{52}a_{44}).
\end{aligned}
$$

Proof. Jacobian matrix for the system at E_1 is

$$
J(E_1)=\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & a_{15} & 0\\
a_{21} & a_{22} & 0 & 0 & a_{25} & 0\\
0 & a_{32} & a_{33} & 0 & 0 & 0\\
0 & a_{42} & 0 & a_{44} & a_{45} & 0\\
0 & a_{52} & 0 & a_{54} & a_{55} & 0\\
0 & 0 & 0 & 0 & a_{65} & a_{66}
\end{bmatrix}.
$$

The characteristics equation of the above matix is

$$\lambda^6+\phi_1\lambda^5+\phi_2\lambda^4+\phi_3\lambda^3+\phi_4\lambda^2+\phi_5\lambda+\phi_6=0.$$

where the values of a_{ij} and ϕ_i are defined above. If it stastifies the Routh-Hurwitz criterion, then the EEP of the system is locally asymptotic stable. □

6 Numerical simulations

In this section, first we perform the sensitivity analysis for the BRN R_0. Then, we analyse the considered fractional SIR epidemic model with varying population using numerical simulations. In the numerical computations, we consider various order of fractional derivatives and compared the results with integer order derivative. The numerical simulations in the fractional order

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

system (1.1) are carried out by using Garrappa's MATLAB code "flmm2.m" [5]. Further, we also analyse effects of the models parameters using a sequence of numerical simulations.

6.1 Sensitivity analysis

In this section, we perform sensitivity analysis of the model (1.1). Using the Definition 2.2, we do analysis for the parameters in the BRN R_0. Here

$$R_0 = \frac{\xi_{11}\sigma_1}{\gamma_1(\gamma_1+\nu_1+\rho_1)} + \frac{\xi_{22}\sigma_2}{\gamma_2(\gamma_2+\nu_2+\rho_2)}.$$

Suppose the normalized sensitivity index for a parameter is positive then the R_0 value increases if there is increase in the given parameters. Similarly the R_0 value decreases if the given parameter value decreases. On other hand, the normalized sensitivity index for a parameter is negative then the R_0 value increases (or decreases) if the parameter value decreases (or increases).

$$\begin{aligned}
\mathbb{S}_{\xi_{11}} &= \frac{\xi_{11}}{R_0}\left(\frac{\partial R_0}{\partial \xi_{11}}\right) &&= \frac{1}{1+\frac{\xi_{22}\sigma_2\gamma_1 G_1}{\xi_{11}\sigma_1\gamma_2 G_2}} > 0,\\
\mathbb{S}_{\sigma_1} &= \frac{\sigma_1}{R_0}\left(\frac{\partial R_0}{\partial \sigma_1}\right) &&= \frac{1}{1+\frac{\xi_{22}\sigma_2\gamma_1 G_1}{\xi_{11}\sigma_1\gamma_2 G_2}} > 0,\\
\mathbb{S}_{\xi_{22}} &= \frac{\xi_{22}}{R_0}\left(\frac{\partial R_0}{\partial \xi_{22}}\right) &&= \frac{1}{1+\frac{\xi_{11}\sigma_1\gamma_2 G_2}{\xi_{22}\sigma_2\gamma_1 G_1}} > 0,\\
\mathbb{S}_{\sigma_2} &= \frac{\sigma_2}{R_0}\left(\frac{\partial R_0}{\partial \sigma_2}\right) &&= \frac{1}{1+\frac{\xi_{11}\sigma_1\gamma_2 G_2}{\xi_{22}\sigma_2\gamma_1 G_1}} > 0,\\
\mathbb{S}_{\gamma_1} &= \frac{\gamma_1}{R_0}\left(\frac{\partial R_0}{\partial \gamma_1}\right) &&= -\left(\frac{\gamma_1+G_1}{G_1\left(1+\frac{\xi_{22}\sigma_2\gamma_1 G_1}{\xi_{11}\sigma_1\gamma_2 G_2}\right)}\right) < 0,\\
\mathbb{S}_{\nu_1} &= \frac{\nu_1}{R_0}\left(\frac{\partial R_0}{\partial \nu_1}\right) &&= -\left(\frac{\nu_1}{G_1\left(1+\frac{\xi_{22}\sigma_2\gamma_1 G_1}{\xi_{11}\sigma_1\gamma_2 G_2}\right)}\right) < 0,\\
\mathbb{S}_{\rho_1} &= \frac{\rho_1}{R_0}\left(\frac{\partial R_0}{\partial \rho_1}\right) &&= -\left(\frac{\rho_1}{G_1\left(1+\frac{\xi_{22}\sigma_2\gamma_1 G_1}{\xi_{11}\sigma_1\gamma_2 G_2}\right)}\right) < 0,\\
\mathbb{S}_{\gamma_2} &= \frac{\gamma_2}{R_0}\left(\frac{\partial R_0}{\partial \gamma_2}\right) &&= -\left(\frac{\gamma_2+G_2}{G_2\left(1+\frac{\xi_{11}\sigma_1\gamma_2 G_2}{\xi_{22}\sigma_2\gamma_1 G_1}\right)}\right) < 0,\\
\mathbb{S}_{\nu_2} &= \frac{\nu_2}{R_0}\left(\frac{\partial R_0}{\partial \nu_2}\right) &&= -\left(\frac{\nu_2}{G_2\left(1+\frac{\xi_{11}\sigma_1\gamma_2 G_2}{\xi_{22}\sigma_2\gamma_1 G_1}\right)}\right) < 0,\\
\mathbb{S}_{\rho_2} &= \frac{\rho_2}{R_0}\left(\frac{\partial R_0}{\partial \rho_2}\right) &&= -\left(\frac{\rho_2}{G_2\left(1+\frac{\xi_{11}\sigma_1\gamma_2 G_2}{\xi_{22}\sigma_2\gamma_1 G_1}\right)}\right) < 0.
\end{aligned}$$

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

In order to perform the sensitivity analysis for R_0, we assume the following values for the model parameters,

$$\text{Case A:} \quad \begin{array}{lllllll} \sigma_1 = 0.7, & \xi_{11} = 0.5, & \xi_{12} = 0.3, & \gamma_1 = 0.2, & \nu_1 = 0.5, & \rho_1 = 0.3, \\ \sigma_2 = 0.5, & \xi_{21} = 0.1, & \xi_{22} = 0.3, & \gamma_2 = 0.1, & \nu_2 = 0.2\ , & \rho_2 = 0.3, \end{array}$$

Moreover, using the values defined above, we obtain the normalized sensitivity index for every parameter in R_0 as follows:

$$\begin{array}{ll} \mathbb{S}_{\xi_{11}} & 0.411\cdots > 0, \\ \mathbb{S}_{\sigma_2} & 0.588\cdots > 0, \\ \mathbb{S}_{\gamma_1} & -0.494\cdots < 0, \\ \mathbb{S}_{\rho_1} & -0.123\cdots < 0, \\ \mathbb{S}_{\nu_2} & -0.196\cdots < 0, \end{array} \qquad \begin{array}{ll} \mathbb{S}_{\xi_{22}} & 0.588\cdots > 0, \\ \mathbb{S}_{\sigma_1} & 0.411\cdots > 0, \\ \mathbb{S}_{\nu_1} & -0.205\cdots < 0, \\ \mathbb{S}_{\gamma_2} & -0.686\cdots < 0, \\ \mathbb{S}_{\rho_2} & -0.294\cdots < 0. \end{array}$$

In the above mentioned results $\mathbb{S}_{\xi_{11}}, \mathbb{S}_{\xi_{22}}, \mathbb{S}_{\sigma_1}, \mathbb{S}_{\sigma_2}$ are positive. Therefore, R_0 value increase if ξ_{11},ξ_{22},σ_1,σ_2 are increasing. Similarly $\mathbb{S}_{\gamma_1}, \mathbb{S}_{\nu_1}, \mathbb{S}_{\rho_1}, \mathbb{S}_{\gamma_2}, \mathbb{S}_{\nu_2}, \mathbb{S}_{\rho_2}$ are negative. Therefore, R_0 value increasing if γ_1,ν_1,ρ_1,γ_2,ν_2,ρ_2 are decreasing. Otherwise it increases.

6.2 Computational results

This section, first discusses the effects of various fractional-order derivatives and compares the results with integer-order derivative. To approximate the solution of the fraction-order system (1.1), all computations are performed using Garrappa's MATLAB code "flmm2.m" [5] with algorithm as given below. Numerical simulations are performed using the implicit fractional linear multistep methods (FLMMs) of the second order.

Algorithm:

Step 1: Fix the initial values U_0. Model parameter values are used as in Case A.

Step 2: Set the end time T. Import the nonlinear function $F(U(t))$.

Step 3: Set up the Jacobian of $F(U(t))$. Fix fractional order derivative values.

Step 4: Solve the system by using the in-house MATLAB code flmm2.m.

Step 5: Repeat Step 4 for different values of α.

Step 6: Plot the output.

Step 7: End.

We consider the parameter values Case A as in the Section 6.1. We perform numerical simulations for the fractional derivatives $\alpha = 0.3, 0.5, 0.7, 0.9$ & 1. First, we calculate BRN R_0 using (4.2), we get $R_0 = 4.25 > 1$. Further, here $R_0 > 1$ and (5.2) is satisfied for the parameter values Case A.

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

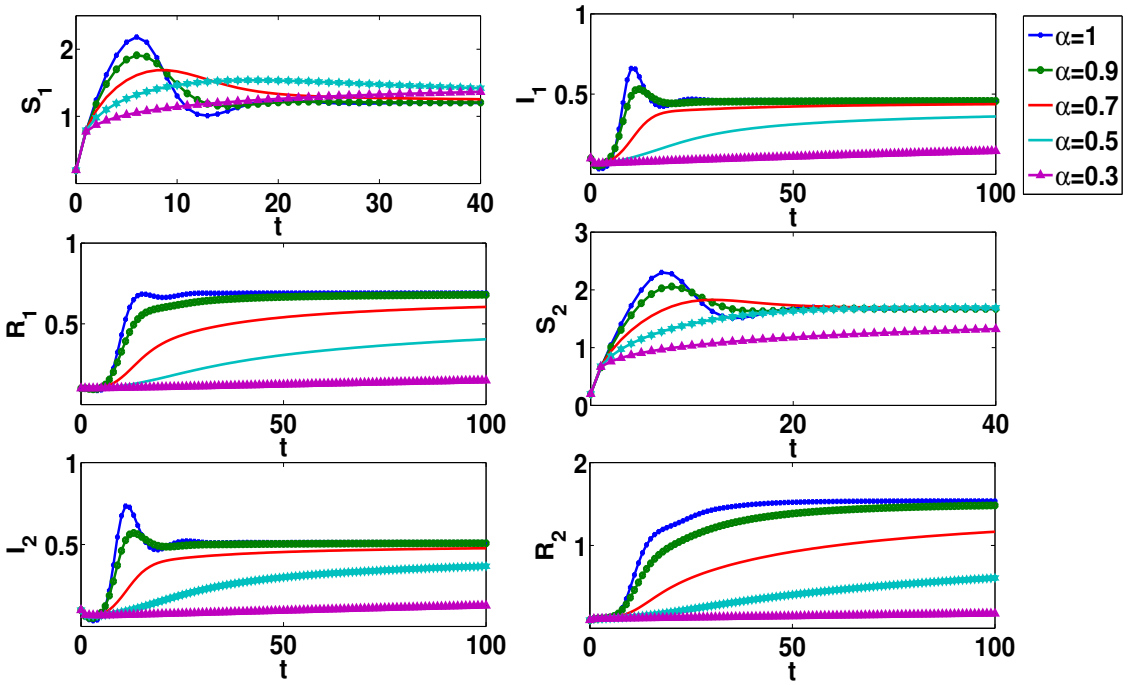

Figure 1: Plots represent the effects of various fractional order derivatives $\alpha = 0.3, 0.5, 0.7, 0.9$ & 1 and compare the results with integer order derivative of the model (1.1) with parameter values as in Case A. Finally, all the solutions parameters converging to an EEP $E_1 =$ (1.1552, 0.46896, 0.70344, 1.57049, 0.571584, 1.71475).

Therefore, the Theorem 5.1 guarantees that there exists at least one EEP. Next, we calculate the equilibrium points procedure mentioned as in Section 4 and 5. Then, we get four equilibrium points for the considered model (1.1). However, only $E_0 =$ (3.5, 0, 0, 5, 0, 0) and $E_1 =$ (1.1552, 0.46896, 0.70344, 1.57049, 0.571584, 1.71475) are in $\mathbb{R}_6^+$ and other two are complex numbers, so we omitted. Here $E_0 =$ (3.5, 0, 0, 5, 0, 0) is a DFEP. Next, $E_1 =$ (1.1552, 0.46896, 0.70344, 1.57049, 0.571584, 1.71475) is an EEP for (1.1) with Case A.

Here, R_0 is greater than 1 and the coefficients of characteristic equation $\lambda^6 + 1.7698\lambda^5 + 1.4965\lambda^4 + 0.6896\lambda^3 + 0.1949\lambda^2 + 0.0278\lambda^1 + 0.0014$ satisfy conditions (5.12) of Theorem 5.2. Therefore, we conclude that EEP $E_1 =$ (1.1552, 0.46896, 0.70344, 1.57049, 0.571584, 1.71475) is locally asymptotic stable for the model (1.1). Next, numerical simulations show the effects of various order of fractional derivatives depicted in Fig. 1. It is clearly shows that huge differences are there in the populations of each compartments S_i, I_i, R_i $i = 1, 2$. We noted that the dynamics of solutions of model (1.1) impacted significantly due to the fractional derivatives, see Fig. 1. Population in all the compartments S_i, I_i, R_i $i = 1, 2$ varying from the initial level when t increases and continues until the convergence to the equilibrium point. This behaviour observed for all $\alpha = 0.3, 0.5, 0.7, 0.9$ & 1. Therefore, from numerical results, we conclude that dynamics of SIR epidemic model with varying population changes concerning order of fractional derivatives. Now, we replace the transmission rate parameters S_1 to I_1 and S_2 to I_2 ξ_{ii} for $i = 1, 2$ respectively as

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220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

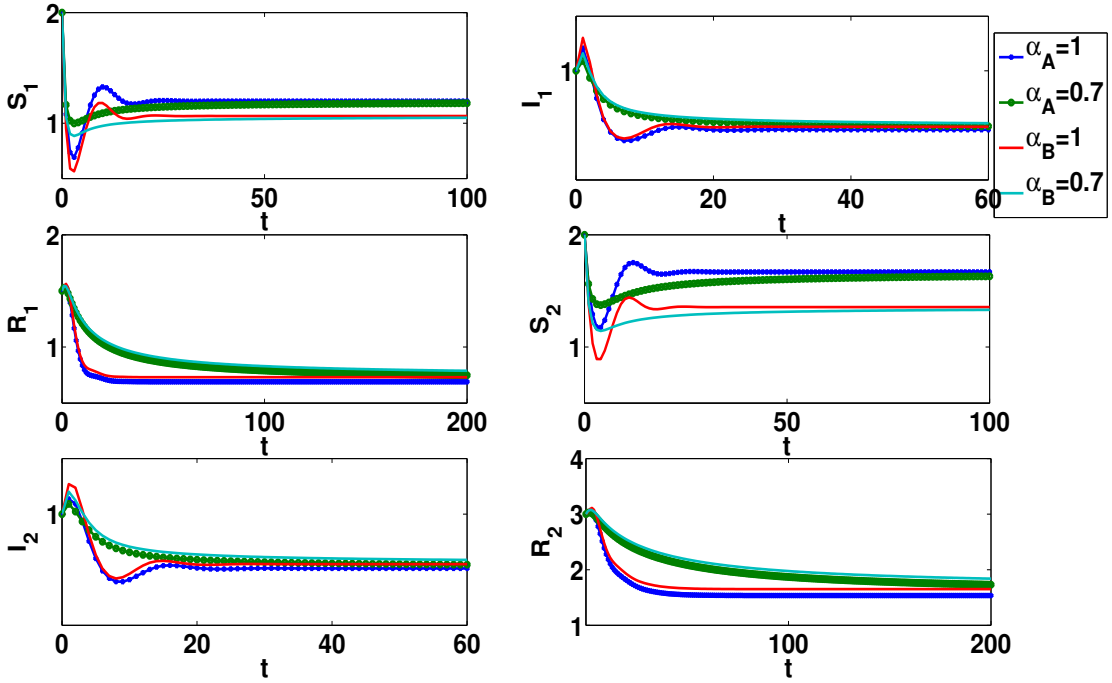

Figure 2: Plots represent the effects of transmission rate parameters of the model (1.1). Model parameter values are assumed as in Case B. Further, comparative results of Case B with Case A is also presented. Moreover, all the solutions parameters converging to an EEP $(1.1552, 0.46896, 0.70344, 1.57049, 0.571584, 1.71475)$ and $(1.02229, 0.495542, 0.743313, 1.25176, 0.624706, 1.87412)$ of Case A and Case B respectively.

$$\text{Case B:} \quad \begin{array}{lllllll} \sigma_1{=}\ 0.7, & \xi_{11}{=}\ 0.6, & \xi_{12}{=}\ 0.3, & \gamma_1{=}\ 0.2, & \nu_1{=}\ 0.5, & \rho_1{=}0.3, \\ \sigma_2{=}0.5, & \xi_{21}{=}\ 0.1, & \xi_{22}{=}\ 0.4, & \gamma_2{=}\ 0.1, & \nu_2{=}\ 0.2\ , & \rho_2{=}\ 0.3, \end{array}$$

In the sensitivity analysis of BRN, we discussed any change in the transmission parameter impact the value of R_0. Accordingly, for Case B, R_0 is calculated as $5.433 > 1$. Here, for Case B, $R_0 > 1$ and it satisfies the equation (5.1), then by the Theorem 5.1, there exists at least one EEP. Calculate the equilibrium as above, we get EEP a E_1=(1.02229, 0.495542, 0.743313,1.25176, 0.624706, 1.87412) for Case B. However, DFEP E_0 remains the same for Case B. Further, the characteristic equation for E_1 in Case B is $\lambda^6+1.7835\lambda^5+1.6467\lambda^4+0.8294\lambda^3+0.2596\lambda^2+0.0396\lambda^1+0.002$ and its coefficients satisfy the condition (5.12). Therefore, again by Theorem 5.2, we conclude that E_1 is locally asymptotic stable for Case B. Comparison of the EEP of both cases, we observed that the solution parameters S_i $i = 1, 2$ decreased and I_i, R_i $i = 1, 2$ increased for Case B than Case A. It has shown in Fig. 2.

Finally, we consider the following model parameters and discuss the numerical results briefly.

$$\text{Case C:} \quad \begin{array}{lllllll} \sigma_1{=}\ 0.7, & \xi_{11}{=}\ 0.5, & \xi_{12}{=}\ 0.3, & \gamma_1{=}\ 0.6, & \nu_1{=}\ 0.4, & \rho_1{=}0.2, \\ \sigma_2{=}0.5, & \xi_{21}{=}\ 0.1, & \xi_{22}{=}\ 0.3, & \gamma_2{=}\ 0.3, & \nu_2{=}\ 0.6\ , & \rho_2{=}\ 0.4, \end{array}$$

The BRN for Case C is R_0 is calculated as $0.8707 < 1$. Then we obtain the DFEP $E_0 = (1.16667, 0, 0, 1.66667, 0, 0)$. Further, parameter values of Case C satisfy (4.3) and (4.5). It is clear that the DFEP is locally and globally asymptotic stable. Since $R_0 < 1$, even for various fractional order derivatives, $\alpha = 1, 0.9, 0.4$, all solutions parameters, $S_i, I_i, R_i,\ i = 1, 2$ converges

6 Apr 2023 03:20:28 PDT
220129-Shangerganesh Version 2 - Submitted to Rocky Mountain J. Math.

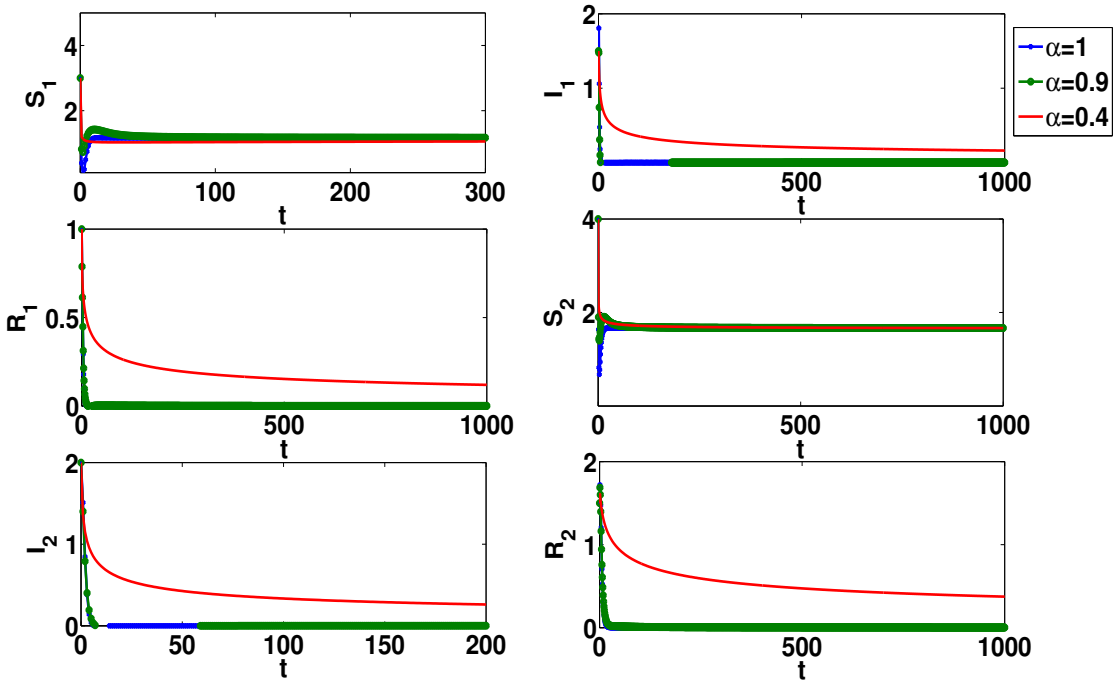

Figure 3: Plots represent the effects of various fractional order derivatives $\alpha = 0.4, 0.9$ & 1 and compare the results with integer order derivative of the model (1.1) with parameter values as in Case C. Finally, all the solutions parameters converging to a DFEP $E_0 = (1.16667, 0, 0, 1.66667, 0, 0)$.

to DFEP with different time(t), see Fig. 3.

Conclusion

We considered here a fractional-order SIR epidemic model with varying population sizes. We first studied the existence and uniqueness of solutions of the considered model. Then, we also proved that the solutions of the model are bounded. Further, we found a DFEP and then estimated the model's BRN. Next, we performed a stability analysis for the DFEP. We proved that the EEP exists and is locally asymptomatic stable using certain conditions. Finally, sensitivity analysis and numerical simulations are performed to validate the theoretical results. As a result of the studies mentioned above, it is simple to pinpoint the variables that are crucial for limiting the spread of infections in the varying population model. In addition, this research may be helpful in developing a disease control strategy to stop the infection from spreading throughout the system.

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