WEIGHTED COMPOSITION–DIFFERENTIATION OPERATOR OF ORDER n ON THE HARDY AND WEIGHTED BERGMAN SPACES

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Abstract. In this paper, we consider the weighted composition–differentiation operator $D_{\psi_n,\varphi_n,n}$ on the Hardy and weighted Bergman spaces. We describe the spectrum and the spectral radius of an operator $D_{\psi_n,\varphi_n,n}$. Also the lower estimate and the upper estimate on the norm of the weighted composition–differentiation operator on the Hardy space H^2 are obtained. Furthermore, we determine the norm of some composition–differentiation operators $D_{\varphi,n}$ on the Hardy space H^2.

1. Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane. The *Hardy space* H^2 is the set of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|=\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta\right)^{1/2}<\infty.$$

For $-1<\alpha<\infty$, the *weighted Bergman space* A^2_α is the space of all analytic functions f on $\mathbb{D}$ so that

$$\|f\|=\left(\int_{\mathbb{D}}|f(z)|^2\,(\alpha+1)(1-|z|^2)^\alpha dA(z)\right)^{1/2}<\infty,$$

where dA is the normalized area measure on $\mathbb{D}$. The case when $\alpha=0$, usually denoted A^2, is called the (unweighted) Bergman space. Throughout this paper, we will write $\mathcal{H}_\alpha$ to denote the Hardy space H^2 for $\alpha=-1$ or the weighted Bergman space A^2_α for $\alpha>-1$.

The weighted Bergman spaces and the Hardy space are *reproducing kernel* Hilbert spaces. For every $w\in\mathbb{D}$ and each non-negative integer n, let $K^{[n]}_{w,\alpha}$ denote the unique function in $\mathcal{H}_\alpha$ that $\langle f,K^{[n]}_{w,\alpha}\rangle=f^{(n)}(w)$ for each $f\in\mathcal{H}_\alpha$, where $f^{(n)}$ is the nth derivative of f (note that $f^{(0)}=f$); for convenience, we use the notation $K_{w,\alpha}$ when $n=0$. The function $K^{[n]}_{w,\alpha}$ is called the *reproducing kernel function.* The reproducing kernel functions for evaluation at w are given by $K_{w,\alpha}(z)=1/(1-\overline{w}z)^{\alpha+2}$ and

$$K^{[n]}_{w,\alpha}(z)=\frac{(\alpha+2)...(\alpha+n+1)z^n}{(1-\overline{w}z)^{n+\alpha+2}}$$

for $z,w\in\mathbb{D}$ and $n>1$.

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For an operator T on $\mathcal{H}_\alpha$, we write $\|T\|_\alpha$ to denote the norm of T acting on $\mathcal{H}_\alpha$. Through this paper, the spectrum of T, the point spectrum of T, and the spectral radius of T are denoted by $\sigma_\alpha(T)$, $\sigma_{p,\alpha}(T)$, and $r_\alpha(T)$, respectively.

We write H^∞ to denote the space of all bounded analytic functions on $\mathbb{D}$, with $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$.

We say that an operator T on a Hilbert space H is *hyponormal* if $T^*T - TT^* \geq 0$, or equivalently if $\|T^*f\| \leq \|Tf\|$ for all $f \in H$. Moreover, the operator T is said to be *cohyponormal* if T^* is hyponormal. Let P denote the projection of $L^2(\partial\mathbb{D})$ onto H^2. For each $b \in L^2(\partial\mathbb{D})$, we define the *Toeplitz operator* T_b on H^2 by $T_b(f) = P(bf)$. For φ an analytic self-map of $\mathbb{D}$, let C_φ be the *composition operator* such that $C_\varphi(f) = f \circ \varphi$ for any $f \in \mathcal{H}_\alpha$. The composition operator C_φ acts boundedly for every φ, with

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{(\alpha+2)/2} \leq \|C_\varphi\|_\alpha \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{(\alpha+2)/2}. \tag{1.1}$$

(See [2, Corollary 3.7] and [10, Lemma 2.3].) Let ψ be an analytic function on $\mathbb{D}$ and φ be an analytic self-map of $\mathbb{D}$. The *weighted composition operator* $C_{\psi,\varphi}$ is defined by $C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi)$ for $f \in \mathcal{H}_\alpha$.

Although for each positive integer n, the *differentiation operator* $D_n(f) = f^{(n)}$ is unbounded on $\mathcal{H}_\alpha$ (note that $\lim_{m\to\infty} \|D_n(z^m)\|/\|z^m\| = \infty$), there are some analytic maps $\varphi : \mathbb{D} \to \mathbb{D}$ such that the operator $C_\varphi D_n$ is bounded. The bounded and compact operators $C_\varphi D_n$ on $\mathcal{H}_\alpha$ were determined in [6], [8], [9] and [11]. Recently the author and Hammond [3] obtained the adjoint, norm, and spectrum of some operators $C_\varphi D_1$ on the Hardy space. For an analytic self-map φ of $\mathbb{D}$ and a positive integer n, the *composition–differentiation operator* on $\mathcal{H}_\alpha$ is defined by the rule $D_{\varphi,n}(f) = f^{(n)} \circ \varphi$; for convenience, we use the notation D_φ when $n = 1$. The operator D_φ is guaranteed to be bounded (and in fact compact) on $\mathcal{H}_\alpha$ if $\|\varphi\|_\infty < 1$ and is guaranteed to be unbounded if φ has finite angular derivative at any point in $\partial\mathbb{D}$ (see [9] and [11]). For an analytic function ψ on $\mathbb{D}$, the *weighted composition–differentiation operator* $D_{\psi,\varphi,n}$ on $\mathcal{H}_\alpha$ is defined

$$D_{\psi,\varphi,n}f(z) = \psi(z)f^{(n)}(\varphi(z)).$$

Some properties of weighted composition–differentiation operators were considered in [4] and [5].

In the last two years, the weighted composition–differentiation operator has received a lot of attention from authors. In this paper, we determine the spectrum of a compact operator $D_{\psi_n,\varphi_n,n}$ when the fixed point w of φ_n is inside the open unit disk and the function ψ_n has a zero at w of order at least n (Theorem 2.4). The spectral radius of a class of compact weighted composition–differentiation operators is obtained (Theorem 2.5). Then for the compact operator D_φ, we find the spectrum of this operator whenever $\varphi'(w) = 0$ that $w \in \mathbb{D}$ is a fixed point of φ (Corollaries 2.6 and 2.7). In addition, we find the lower estimate and the upper estimate for $\|D_{\psi,\varphi,n}\|_{-1}$ (Propositions 3.2 and 3.6). Moreover, the norm of a composition–differenetiation operator $D_{\varphi,n}$, acting on the Hardy space H^2, is determined in the case where $\varphi(z) = bz$ for some complex number b that $|b| < 1$ (Theorem 3.5).

2. Spectral Properties

To find the spectrum of $D_{\psi_n,\varphi_n,n}$ we need to obtain an invariant subspace of $D^*_{\psi_n,\varphi_n,n}$. To do this, we consider the action of the adjoint of the operator $D_{\psi_n,\varphi_n,n}$ on the reproducing kernel functions.

Lemma 2.1. *Let m be a non-negative integer. Suppose that $D_{\psi_n,\varphi_n,n}$ is a bounded operator on $\mathcal{H}_\alpha$ and the fixed point w of φ_n is inside the open unit disk. Assume that the function ψ_n has a zero at w of order at least n.*

(i) If $m > n$, then

$$D^*_{\psi_n,\varphi_n,n}K^{[m]}_{w,\alpha} = \sum_{i=n}^{m-1} \overline{\beta_{i-n}(w)}K^{[i]}_{w,\alpha} + \binom{m}{n}\overline{\psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{m-n}}K^{[m]}_{w,\alpha};$$

(ii) if $m = n$, then

$$D^*_{\psi_n,\varphi_n,n}K^{[m]}_{w,\alpha} = \overline{\psi_n^{(n)}(w)}K^{[n]}_{w,\alpha};$$

(iii) if $m < n$, then

$$D^*_{\psi_n,\varphi_n,n}K^{[m]}_{w,\alpha} = 0,$$

where the functions β_j's consist of some products of the derivatives of ψ_n and φ_n.

Proof. Let f be an arbitrary function in $\mathcal{H}_\alpha$. Let $m < n$. Since ψ_n has a zero at w of order at least n, we have

$$\begin{aligned}\langle f, D^*_{\psi_n,\varphi_n,n}K^{[m]}_{w,\alpha}\rangle &= \big(\psi_n\cdot\big(f^{(n)}\circ\varphi_n\big)\big)^{(m)}(w)\\ &= \sum_{i=0}^{m}\binom{m}{i}\psi_n^{(m-i)}(w)\big(f^{(n)}\circ\varphi_n\big)^{(i)}(w)\\ &= 0.\end{aligned}$$

It shows that $D^*_{\psi_n,\varphi_n,n}K^{[m]}_{w,\alpha} = 0$.

Now assume that $m \geq n$. We obtain

$$\begin{aligned}\langle f, D^*_{\psi_n,\varphi_n,n}K^{[m]}_{w,\alpha}\rangle &= \sum_{i=0}^{m}\binom{m}{i}\psi_n^{(m-i)}(w)\big(f^{(n)}\circ\varphi_n\big)^{(i)}(w)\\ &= \sum_{i=0}^{m-n}\binom{m}{i}\psi_n^{(m-i)}(w)\big(f^{(n)}\circ\varphi_n\big)^{(i)}(w)\\ &\quad+ \sum_{i=m-n+1}^{m}\binom{m}{i}\psi_n^{(m-i)}(w)\big(f^{(n)}\circ\varphi_n\big)^{(i)}(w)\\ &= \sum_{i=0}^{m-n}\binom{m}{i}\psi_n^{(m-i)}(w)\big(f^{(n)}\circ\varphi_n\big)^{(i)}(w). \qquad (2.1)\end{aligned}$$

If $m > n$, then by (2.1), we get

$$\langle f, D^*_{\psi_n,\varphi_n,n}K^{[m]}_{w,\alpha}\rangle = \langle f, \sum_{i=0}^{m-n-1}\overline{\beta_i(w)}K^{[i+n]}_{w,\alpha} + \binom{m}{m-n}\overline{\psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{m-n}}K^{[m]}_{w,\alpha}\rangle,$$

so

$$D^*_{\psi_n,\varphi_n,n} K^{[m]}_{w,\alpha} = \sum_{i=n}^{m-1} \overline{\beta_{i-n}(w)} K^{[i]}_{w,\alpha} + \binom{m}{n} \overline{\psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{m-n}} K^{[m]}_{w,\alpha}.$$

If $m = n$, then by (2.1), we see that

$$\langle f, D^*_{\psi_n,\varphi_n,n} K^{[m]}_{w,\alpha}\rangle = \psi_n^{(n)}(w) f^{(n)}(w) = \langle f, \overline{\psi_n^{(n)}(w)} K^{[n]}_{w,\alpha}\rangle.$$

Hence the result follows. □

In the next proposition, we identify all possible eigenvalues of $D_{\psi_n,\varphi_n,n}$.

Proposition 2.2. *Suppose that $D_{\psi_n,\varphi_n,n}$ is a bounded operator on $\mathcal{H}_\alpha$ and the fixed point w of φ_n is inside the open unit disk. If the function ψ_n has a zero at w of order at least n, then*

$$\{0\} \bigcup \left\{ \binom{l}{n} \psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{l-n} : l \in \mathbb{N}_{\geq n} \right\}$$

contains the point spectrum of $D_{\psi_n,\varphi_n,n}$.

Proof. Let λ be an arbitrary eigenvalue for $D_{\psi_n,\varphi_n,n}$ with corresponding eigenvector f. Note that

$$\lambda f(z) = \psi_n(z) f^{(n)}\big(\varphi_n(z)\big) \tag{2.2}$$

for each $z \in \mathbb{D}$. If $f(w) \neq 0$, then $\lambda = 0$. Let f have a zero at w of order $l \geq 1$. Differentiate (2.2) l times and evaluate it at the point $z = w$ to obtain

$$\lambda f^{(l)}(w) \;=\; \sum_{j=0}^{l} \binom{l}{j} \psi_n^{(l-j)}(w)(f^{(n)} \circ \varphi_n)^{(j)}(w). \tag{2.3}$$

First assume that $l < n$. Since ψ_n has a zero at w of order at least n, we have $\lambda = 0$ by (2.3).

Now assume that $l \geq n$. Then $\psi_n^{(l-j)}(w) = 0$ for each $j > l - n$. Hence (2.3) implies that

$$\lambda f^{(l)}(w) = \sum_{j=0}^{l-n} \binom{l}{j} \psi_n^{(l-j)}(w)\big(f^{(n)} \circ \varphi_n\big)^{(j)}(w)$$

and so

$$\lambda f^{(l)}(w) = \binom{l}{l-n} \psi_n^{(n)}(w) f^{(l)}(w)\big(\varphi_n'(w)\big)^{l-n}.$$

(Note that in case of $\varphi_n'(w) = 0$ and $l = n$, we set $\big(\varphi_n'(w)\big)^{l-n} = 1$.) Therefore, in this case, any eigenvalue must have the form

$$\binom{l}{n} \psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{l-n}$$

for a natural number l with $l \geq n$. □

Proposition 2.3. *Suppose that the hypotheses of Proposition 2.2 hold. Then the point spectrum of $D^*_{\psi_n,\varphi_n,n}$ contains*

$$\{0\} \bigcup \left\{ \binom{l}{n} \overline{\psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{l-n}} : l \in \mathbb{N}_{\geq n} \right\}.$$

Proof. Let l be a positive integer with $l \geq n$ and K_l denote the span of $\{K_{w,\alpha}, K^{[1]}_{w,\alpha}, ..., K^{[l]}_{w,\alpha}\}$. Note that this spanning set is linearly independent and so is a basis. Let A_l be the matrix of the operator $D^*_{\psi_n,\varphi_n,n}$ restricted to K_l with respect to this basis. We infer from Lemma 2.1 that

$$A_l = \begin{bmatrix} 0_{n,n} & * & \dots & * \\ 0 & \overline{\psi_n^{(n)}(w)} & \dots & * \\ 0 & 0 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \binom{l}{n}\overline{\psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{l-n}} \end{bmatrix},$$

where $0_{n,n}$ is the zero matrix of dimension $n \times n$. Then A_l is an upper triangular matrix. Since the subspace K_l is finite dimensional, it is closed and so the space $\mathcal{H}_\alpha$ can be decomposed as $\mathcal{H}_\alpha = K_l \oplus K_l^\perp$. Then the block matrix of $D^*_{\psi_n,\varphi_n,n}$ with respect to the above decomposition must be of the form

$$\begin{bmatrix} A_l & C_l \\ 0 & El \end{bmatrix}$$

(note that K_l is invariant under $D^*_{\psi_n,\varphi_n,n}$ by Lemma 2.1 and so the lower left corner of the above matrix is 0). Since the spectrum of $D^*_{\psi_n,\varphi_n,n}$ is the union of the spectrum of A_l and the spectrum of E_l (see [2, p. 270]), we conclude that $\left\{\binom{t}{n}\overline{\psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{t-n}} : t \in \mathbb{N} \text{ and } n \leq t \leq l\right\}$ is the subset of $\sigma_{p,\alpha}\big(D^*_{\psi_n,\varphi_n,n}\big)$. Since l is arbitrary, the result follows. □

In the following theorem, we characterize the spectrum of an operator $D_{\psi,\varphi,n}$ under the conditions of Proposition 2.2. The spectrum of an operator $D_{\psi,\varphi,n}$ which was obtained in [5, Theorem 3.1] is an example for Theorem 2.4.

Theorem 2.4. *Suppose that the hypotheses of Proposition 2.2 hold. If $D_{\psi_n,\varphi_n,n}$ is compact on $\mathcal{H}_\alpha$, then*

$$\sigma_\alpha(D_{\psi_n,\varphi_n,n}) = \{0\} \bigcup \left\{\binom{l}{n}\psi_n^{(n)}(w)\big(\varphi_n'(w)\big)^{l-n} : l \in \mathbb{N}_{\geq n}\right\}.$$

In particular, if $\psi_n^{(n)}(w) = 0$, then the operator $D_{\psi_n,\varphi_n,n}$ is quasinilpotent; that is, its spectrum is $\{0\}$.

In the next theorem, we obtain the spectral radius of a compact operator $D_{\psi,\varphi,n}$.

Theorem 2.5. *Suppose that $D_{\psi,\varphi,n}$ is a compact operator on $\mathcal{H}_\alpha$. Assume that the fixed point w of φ is inside the open unit disk and the function ψ has a zero at w of order n. Then*

$$r_\alpha\big(D_{\psi,\varphi,n}\big) = \binom{\left\lfloor \frac{n}{1-|\varphi'(w)|}\right\rfloor}{n} \big|\psi^{(n)}(w)\big|\big|\varphi'(w)\big|^{\left\lfloor \frac{n}{1-|\varphi'(w)|}\right\rfloor - n},$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function.

Proof. Theorem 2.4 implies that

$$\sigma_\alpha\big(D_{\psi,\varphi,n}\big) = \left\{\binom{l}{n}\psi^{(n)}(w)\big(\varphi'(w)\big)^{l-n} : l \in \mathbb{N}_{\geq n}\right\}$$

and so

$$r_\alpha\big(D_{\psi,\varphi,n}\big) = \sup\left\{\binom{l}{n}\big|\psi^{(n)}(w)\big|\big|\varphi'(w)\big|^{l-n} : l \in \mathbb{N}_{\geq n}\right\}.$$

If $\varphi'(w) = 0$, then $r_\alpha\big(D_{\psi,\varphi,n}\big) = \big|\psi^{(n)}(w)\big|$. Now suppose that $\varphi'(w) \neq 0$. Let the function $h(x) = x(x-1)...(x-n+1)\big|\varphi'(w)\big|^{x-n}$ on $[n, +\infty)$. Since $\big|\varphi'(w)\big| < 1$ (see the Grand Iteration Theorem), we conclude that $\lim_{x\to\infty} h(x) = 0$. Then h is a bounded function on $[n, +\infty)$ and so it obtains an absolute maximum point. If $h'(t) = 0$ for some $t \in [n, +\infty)$, then $g(t) = -\ln\big|\varphi'(w)\big|$, where $g(x) = \frac{1}{x} + \frac{1}{x-1} + ... + \frac{1}{x-n+1}$ for each $x \in [n, +\infty)$. We can easily see that g' is strictly decreasing and so the function h has at most one local extremum on $[n, +\infty)$, which must be its absolute maximum (note that if $h'(t) \neq 0$ for all t, then h has an absolute maximum of $n!$ at n). Therefore, for obtaining $r_\alpha\big(D_{\psi,\varphi,n}\big)$, we must find the greatest natural number l such that $l \geq n$ and

$$(l-1)...(l-n)\big|\varphi'(w)\big|^{l-n-1} \leq l...(l-n+1)\big|\varphi'(w)\big|^{l-n}$$

or equivalently $l \leq \frac{n}{1-\big|\varphi'(w)\big|}$ (note that if $n! = n(n-1)...1 \cdot \big|\varphi'(w)\big|^{n-n} > l...(l-n+1)\big|\varphi'(w)\big|^{l-n}$ for each $l > n$, then we have $n! > (n+1)!\big|\varphi'(w)\big|$. It shows that $n < \frac{n}{1-\big|\varphi'(w)\big|} < n+1$ and so $\big\lfloor \frac{n}{1-|\varphi'(w)|}\big\rfloor = n$). Thus the quantity $\binom{l}{n}\big|\varphi'(w)\big|^{l-n}$ is maximized when $l = \left\lfloor \frac{n}{1-\big|\varphi'(w)\big|}\right\rfloor$, so the conclusion follows. □

In the following two corollaries, we find the spectrum of the compact operator D_φ, whenever $\varphi'(w) = 0$ that $w \in \mathbb{D}$ is a fixed point of φ.

Corollary 2.6. *Suppose that D_φ is compact on $\mathcal{H}_\alpha$. Assume that $w \in \mathbb{D}$ is a fixed point of φ and $\varphi'(w) = \varphi''(w) = 0$. Then D_φ is quasinilpotent.*

Proof. Suppose that D_φ is compact on $\mathcal{H}_\alpha$. Then $D_{\varphi'\circ\varphi,\varphi_2,2} = D_\varphi D_\varphi$ is compact. Let $w \in \mathbb{D}$ be a fixed point of φ. If λ is an eigenvalue for D_φ corresponding to the eigenvector f, then λ^2 is an eigenvector for $D_{\varphi'\circ\varphi,\varphi_2,2}$ corresponding to eigenvector f. Since $\varphi'(w) = \varphi''(w) = 0$, Theorem 2.4 dictates that $D_{\varphi'\circ\varphi,\varphi_2,2}$ is quasinilpotent. Hence D_φ is quasinilpotent. □

Corollary 2.7. *Suppose that D_φ is compact on $\mathcal{H}_\alpha$. Assume that $w \in \mathbb{D}$ is a fixed point of φ, $\varphi'(w) = 0$, and $\varphi''(w) \neq 0$. Then*

$$\sigma_\alpha(D_\varphi) = \{0, \varphi''(w)\}.$$

Proof. Suppose that $\varphi'(w) = 0$ and $\varphi''(w) \neq 0$. Then $D_\varphi^* K_{w,\alpha}^{[2]} = \overline{\varphi''(w)} K_{w,\alpha}^{[2]}$ by [4, Lemma 1]. Hence $\varphi''(w)$ is an eigenvalue for D_φ and so D_φ is not quasinilpotent. We can see that $D_{\varphi'\circ\varphi,\varphi_2,2} = D_\varphi D_\varphi$ is compact. Using Theorem 2.4 for $D_{\varphi'\circ\varphi,\varphi_2,2}$ shows that $\sigma_\alpha\big(D_{\varphi'\circ\varphi,\varphi_2,2}\big) = \{0, \big(\varphi''(w)\big)^2\}$, so

$$\{0, \varphi''(w)\} \subseteq \sigma_\alpha(D_\varphi) \subseteq \{0, \varphi''(w), -\varphi''(w)\}.$$

Suppose that $-\varphi''(w)$ is an eigenvalue for D_φ with corresponding eigenvector f. Note that

$$f'(\varphi(z)) = -\varphi''(w) f(z). \tag{2.4}$$

Differentiate both sides of (2.4) to obtain

$$f^{(2)}(\varphi(z))\varphi'(z) = -\varphi''(w)f'(z). \tag{2.5}$$

Indeed, letting $z = w$ in (2.5) gives $f'(w) = 0$, so putting $z = w$ in (2.4) shows that $f(w) = 0$. Now by differentiating both sides of (2.5), we have

$$f^{(3)}(\varphi(z))\big(\varphi'(z)\big)^2 + f^{(2)}(\varphi(z))\varphi''(z) = -\varphi''(w)f''(z). \tag{2.6}$$

Hence letting $z = w$ in (2.6) implies that $f^{(2)}(w) = 0$. It is not hard to see that $f^{(3)}(w) = 0$ by differentiating both sides of (2.6) and letting $z = w$. Now assume that f has a zero at w of order m, where $m > 2$. Then $f(w) = f^{(1)}(w) = f^{(2)}(w) = \dots = f^{(m-1)}(w) = 0$. Differentiate both sides of (2.5) $m-1$ times to see that

$$\sum_{k=0}^{m-1} \binom{m-1}{k} \big(f'' \circ \varphi\big)^{(m-1-k)}(w)\big(\varphi'\big)^{(k)}(w) = -\varphi''(w)f^{(m)}(w). \tag{2.7}$$

Since $\varphi'(w) = 0$ and f has a zero at w of order m, we can see that

$$\begin{aligned}\sum_{k=0}^{m-1} \binom{m-1}{k} \big(f'' \circ \varphi\big)^{(m-1-k)}(w)\big(\varphi'\big)^{(k)}(w) &= \big(f'' \circ \varphi\big)^{(m-1)}(w)\varphi'(w) \\ &+ (m-1)\big(f'' \circ \varphi\big)^{(m-2)}(w)\varphi''(w) \\ &+ \sum_{k=2}^{m-1} \binom{m-1}{k} \big(f'' \circ \varphi\big)^{(m-1-k)}(w)\big(\varphi'\big)^{(k)}(w) \\ &= (m-1)\big(f'' \circ \varphi\big)^{(m-2)}(w)\varphi''(w). \qquad (2.8)\end{aligned}$$

Since $m > 2$, we have $\big(f''\circ\varphi\big)^{(m-2)}(w) = f^{(m)}(w)\big(\varphi'(w)\big)^{m-2} + \sum_{k=0}^{m-1} f^{(k)}(w)g_k(w)$, where g_k's are functions which consist of various products of the derivatives of φ. It follows that $\big(f'' \circ \varphi\big)^{(m-2)}(w) = 0$ because $\varphi'(w) = 0$ and f has a zero of order m at w. Then $\sum_{k=0}^{m-1} \binom{m-1}{k} \big(f'' \circ \varphi\big)^{(m-1-k)}(w)\big(\varphi'\big)^{(k)}(w) = 0$ by (2.8) and so (2.7) implies that $f^{(m)}(w) = 0$ which is a contradiction. Then

$$\sigma_\alpha(D_\varphi) = \{0, \varphi''(w)\}.$$

□

We can see that [3, Example 6] is an example for Corollary 2.7.

Remark 2.8. Assume that $\varphi \equiv a$, where a is constant with $|a| < 1$ so that $D_{\psi,\varphi,n}$ is bounded on $\mathcal{H}_\alpha$. Since $\|\varphi\|_\infty < 1$, the operator $D_{\psi,\varphi,n}$ is compact (see [8] and [11]). The spectra of some of such operators $D_{\psi,\varphi,n}$ were found in [5, Theorem 3.2] and [5, Theorem 3.3], but by the same idea which was stated in the proof of [5, Theorem 3.2], we can easily see that for these operators, we obtain

$$\sigma_\alpha\big(D_{\psi,\varphi,n}\big) := \begin{cases} \{0\} \cup \{\psi^{(n)}(a)\}, & \psi^{(n)}(a) \neq 0, \\ \{0\}, & \psi^{(n)}(a) = 0; \end{cases}$$

moreover, if $\psi^{(n)}(a) \neq 0$, then ψ is an eigenvector for $D_{\psi,\varphi,n}$ with corresponding eigenvalue $\psi^{(n)}(a)$.

Example 2.9. Suppose that $\varphi(z) = w + c_2(z-w)^2 + ... + c_n(z-w)^n$, where $n \geq 2$, $w \in \mathbb{D}$ and $c_2, ..., c_n$ are constant with $|w| + |c_2|\big(1+|w|\big)^2 + ... + |c_n|\big(1+|w|\big)^n < 1$.
(i) If $c_i = 0$ for each $i \geq 2$, then $\sigma_\alpha(D_\varphi) = \{0\}$ by Remark 2.8.
(ii) If $c_2 = 0$ and there is an integer $i > 2$ such that $c_i \neq 0$, then Corollary 2.6 implies that $\sigma_\alpha(D_\varphi) = \{0\}$.
(iii) Assume that $c_2 \neq 0$. Invoking Corollary 2.7, we see that $\sigma_\alpha(D_\varphi) = \{0, 2c_2\}$.

3. Norms

We begin this section with an example which is a starting point for estimating a lower bound for $\|D_{\psi,\varphi,n}\|_{-1}$.

Example 3.1. Suppose that $\varphi(z) = bz^3 + az^2$ with $\frac{1}{2} < |a| < 1$ and $|a| + |b| < 1$. We can see that $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(0) = 2a$. By Corollary 2.7, we have $r_\alpha(D_\varphi) = 2|a|$ and so $\|D_\varphi\|_\alpha \geq 2|a| > 1$. Compare $2|a|$ with the lower bound for $\|D_\varphi\|_{-1}$ which was found in [3, Proposition 4] (note that [3, Proposition 4] implies that $\|D_\varphi\|_{-1} \geq 1$).

The preceding example leads to obtain the lower estimate on the norm of $D_{\psi,\varphi,n}$ on the Hardy space by using the spectrum of a weighted composition–differentiation operator which was obtained in Proposition 2.3.

Proposition 3.2. *Suppose that $D_{\psi,\varphi,n}$ is a bounded operator on H^2. Assume that the fixed point w of φ is inside the open unit disk.*

(i) If $\varphi'(w) \neq 0$, then

$$\|D_{\psi,\varphi,n}\|_{-1} \geq |\phi^{(n)}(w)| \binom{\lfloor \frac{n}{1-|\varphi'(w)|} \rfloor}{n} |\varphi'(w)|^{\lfloor \frac{n}{1-|\varphi'(w)|} \rfloor - n};$$

(ii) if $\varphi'(w) = 0$, then

$$\|D_{\psi,\varphi,n}\|_{-1} \geq |\phi^{(n)}(w)|;$$

(iii) if $\varphi'(w) = 0$, $\psi''(w) = 0$ and $n = 1$, then

$$\|D_{\psi,\varphi,1}\|_{-1} \geq \max\Big\{|\phi'(w)|, |\psi(w)\varphi''(w)|\Big\},$$

where

$$\phi(z) := \begin{cases} \psi(z), & \psi^{(0)}(w) = ... = \psi^{(n-1)}(w) = 0, \\ \psi(z)\big(\frac{w-z}{1-\overline{w}z}\big)^{n-m}, & \psi^{(0)}(w) = ... = \psi^{(m-1)}(w) = 0,\ \psi^{(m)}(w) \neq 0 \text{ and } 1 \leq m < n, \\ \psi(z)\big(\frac{w-z}{1-\overline{w}z}\big)^{n}, & \psi(w) \neq 0. \end{cases}$$

Proof. First suppose that $\psi^{(0)}(w) = ... = \psi^{(n-1)}(w) = 0$. Proposition 2.3 and the idea which was used in the proof of Theorem 2.5 imply that

$$\|D_{\psi,\varphi,n}\|_{-1} \geq |\psi^{(n)}(w)| \binom{\lfloor \frac{n}{1-|\varphi'(w)|} \rfloor}{n} |\varphi'(w)|^{\lfloor \frac{n}{1-|\varphi'(w)|} \rfloor - n}. \tag{3.1}$$

(Note that in case of $\varphi'(w) = 0$, we set $|\varphi'(w)|^{\lfloor \frac{n}{1-|\varphi'(w)|} \rfloor - n} = 1$.)

Now assume that $\psi(z) = (w-z)^m g(z)$, where $1 \leq m < n$ and $g(w) \neq 0$. Let $\phi(z) = \psi(z)\big(\frac{w-z}{1-\overline{w}z}\big)^{n-m}$. Since $T_{\frac{w-z}{1-\overline{w}z}}$ is an isometry on H^2 and the nth derivative of $\psi(z)\left(\frac{w-z}{1-\overline{w}z}\right)^{n-m}$ at the point w is $\frac{(-1)^n n! g(w)}{\left(1-|w|^2\right)^{n-m}}$, by replacing ϕ with ψ in (3.1), we obtain

$$\left\|D_{\psi,\varphi,n}\right\|_{-1} = \left\|D_{\phi,\varphi,n}\right\|_{-1} \geq \frac{n!|g(w)|}{\left(1-|w|^2\right)^{n-m}} \binom{\left\lfloor \frac{n}{1-|\varphi'(w)|} \right\rfloor}{n} \left|\varphi'(w)\right|^{\left\lfloor \frac{n}{1-|\varphi'(w)|} \right\rfloor - n}.$$

Now suppose that $\psi(w) \neq 0$ and $\phi(z) = \psi(z)\left(\frac{w-z}{1-\overline{w}z}\right)^n$. By replacing ϕ with ψ in (3.1), we have

$$\left\|D_{\psi,\varphi,n}\right\|_{-1} = \left\|D_{\phi,\varphi,n}\right\|_{-1} \geq \frac{n!|\psi(w)|}{\left(1-|w|^2\right)^{n}} \binom{\left\lfloor \frac{n}{1-|\varphi'(w)|} \right\rfloor}{n} \left|\varphi'(w)\right|^{\left\lfloor \frac{n}{1-|\varphi'(w)|} \right\rfloor - n}.$$

Note that if $\varphi'(w) = 0$ and $\psi''(w) = 0$, then $D^*_{\psi,\varphi,1} K^{[2]}_{w,-1} = \overline{\psi(w)\varphi''(w)} K^{[2]}_{w,-1}$ by [4, Lemma 1]. Therefore, we conclude that $\left\|D_{\psi,\varphi,1}\right\|_{-1} \geq \left|\psi(w)\varphi''(w)\right|$. Hence the result follows. □

In the next example, we show that for some operators D_φ, Proposition 3.2 is more useful than [3, Proposition 4] for estimating the lower bound for $\left\|D_\varphi\right\|_{-1}$.

Example 3.3. Suppose that $\varphi(z) = az^n + bz$, where $\frac{1}{2} < |b| < 1 - |a|$ and n is a positive integer that $n \geq 2$. Proposition 3.2 implies that

$$\left\|D_\varphi\right\|_{-1} \geq \left\lfloor \frac{1}{1-|b|} \right\rfloor |b|^{\lfloor 1/(1-|b|) \rfloor - 1} > 1$$

and so this lower bound is greater than the lower bound for $\left\|D_\varphi\right\|_{-1}$ which was estimated in [3, Proposition 4].

In the following proposition, we obtain $\|D_{\psi,\varphi,n}\|_\alpha$, when $D_{\psi,\varphi,n}$ is a cohyponormal operator which satisfies the hypotheses of Proposition 2.2.

Proposition 3.4. *Suppose that ψ is not identically zero and φ is a nonconstant analytic self-map of $\mathbb{D}$ so that $D_{\psi,\varphi,n}$ is bounded on $\mathcal{H}_\alpha$. Assume that $w \in \mathbb{D}$ is the fixed point of φ and ψ has a zero at w of order at least n. Then $D_{\psi,\varphi,n}$ is cohyponormal on $\mathcal{H}_\alpha$ if and only if $\psi(z) = az^n$ and $\varphi(z) = bz$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{D} \setminus \{0\}$; moreover, in this case*

$$\left\|D_{\psi,\varphi,n}\right\|_\alpha = n!|a| \binom{\left\lfloor \frac{n}{1-|b|} \right\rfloor}{n} |b|^{\left\lfloor \frac{n}{1-|b|} \right\rfloor - n}.$$

Proof. Suppose that $D_{\psi,\varphi,n}$ is cohyponormal. Lemma 2.1 shows that $D^*_{\psi,\varphi,n} K_{w,\alpha} = 0$. Hence $K_{w,\alpha}$ is an eigenvector for $D^*_{\psi,\varphi,n}$ corresponding to eigenvalue 0. Since $D_{\psi,\varphi,n}$ is cohyponormal, we conclude that $D_{\psi,\varphi,n} K_{w,\alpha}(z) = \frac{(\alpha+2)\ldots(\alpha+n+1)\overline{w}^n \psi(z)}{(1-\overline{w}\varphi(z))^{\alpha+2+n}} = 0$ and so $w = 0$. Lemma 2.1 implies that

$$D^*_{\psi,\varphi,n} K^{[n]}_{0,\alpha}(z) = \overline{\psi^{(n)}(0)} K^{[n]}_{0,\alpha}(z) = \overline{\psi^{(n)}(0)} (\alpha+2)\ldots(\alpha+n+1) z^n.$$

Since $D_{\psi,\varphi,n}$ is cohyponormal, it follows that

$$D_{\psi,\varphi,n} K^{[n]}_{0,\alpha}(z) = \psi^{(n)}(0)(\alpha+2)\ldots(\alpha+n+1) z^n.$$

Because $D_{\psi,\varphi,n} K^{[n]}_{0,\alpha} = n!(\alpha+2)\ldots(\alpha+n+1)\psi$, we conclude that $\psi(z) = \frac{\psi^{(n)}(0)}{n!} z^n$, where $\psi^{(n)}(0) \neq 0$ (note that ψ is not identically zero). Then $\psi^{(m)}(0) = 0$ for each $m \neq n$. Hence Lemma 2.1 shows that

$$D^*_{\psi,\varphi,n} K^{[n+1]}_{0,\alpha} = (n+1)\overline{\psi^{(n)}(0)\varphi'(0)} K^{[n+1]}_{0,\alpha}.$$

9 Jan 2022 06:24:14 PST
220109-Fatehi Version 1 - Submitted to Rocky Mountain J. Math.

Therefore, we have

$$D_{\psi,\varphi,n}K_{0,\alpha}^{[n+1]} = (n+1)\psi^{(n)}(0)\varphi'(0)K_{0,\alpha}^{[n+1]}. \tag{3.2}$$

On the other hand, we obtain

$$\begin{aligned} D_{\psi,\varphi,n}K_{0,\alpha}^{[n+1]}(z) &= (n+1)!(\alpha+2)...(\alpha+n+2)\psi(z)\varphi(z) \\ &= (n+1)!(\alpha+2)...(\alpha+n+2)\frac{\psi^{(n)}(0)}{n!}z^n\varphi(z) \qquad (3.3) \end{aligned}$$

for each $z \in \mathbb{D}$. Since $D_{\psi,\varphi,n}$ is cohyponormal and bounded, (3.2) and (3.3) imply that $\varphi(z) = \varphi'(0)z$, where $|\varphi'(0)| < 1$.

Conversely is obvious by [7, Proposition 3.2] (note that an analogue of [7, Proposition 3.2] holds in H^2 by the similar idea).

Due to the cohyponormality of $D_{\psi,\varphi,n}$, invoking Theorem 2.5, it follows that

$$\left\|D_{\psi,\varphi,n}\right\|_\alpha = r_\alpha(D_{\psi,\varphi,n}) = n!|a|\binom{\lfloor \frac{n}{1-|b|}\rfloor}{n}|b|^{\lfloor \frac{n}{1-|b|}\rfloor - n}.$$

□

From now on we consider φ with $\|\varphi\|_\infty < 1$; this assumption guarantees that $D_{\varphi,n}$ is bounded on H^2 (see [8] and [9]). In Theorem 3.5, we extend [3, Theorem 2].

Theorem 3.5. *If $\varphi(z) = bz$ for some $b \in \mathbb{D}\setminus\{0\}$, then*

$$\left\|D_{\varphi,n}\right\|_{-1} = n!\binom{\lfloor \frac{n}{1-|b|}\rfloor}{n}|b|^{\lfloor \frac{n}{1-|b|}\rfloor - n}. \tag{3.4}$$

Proof. The result follows immediately from Proposition 3.4 and the fact that T_{z^n} is an isometry on H^2. □

In view of Theorem 3.5, we can see that $\left\|D_{\varphi,n}\right\|_{-1} = n!$ for $0 < |b| \le \frac{1}{n+1}$ and $\left\|D_{\varphi,n}\right\|_{-1} > n!$ for $\frac{1}{n+1} < |b| < 1$. Since C_{z^k} is an isometry on H^2, (3.4) holds for $\varphi(z) = bz^k$ where k is a positive integer.

In the next proposition, we estimate an upper bound for $\left\|D_{\psi,\varphi,n}\right\|_{-1}$.

Proposition 3.6. *If φ is a nonconstant analytic self-map of $\mathbb{D}$ with $\|\varphi\|_\infty < 1$ and the function ψ belongs to H^∞, then*

$$\left\|D_{\psi,\varphi,n}\right\|_{-1} \le n!\|\psi\|_\infty\sqrt{\frac{b+|\varphi(0)|}{b-|\varphi(0)|}}\binom{\lfloor \frac{n}{1-b}\rfloor}{n}b^{\lfloor \frac{n}{1-b}\rfloor - n}$$

whenever $\|\varphi\|_\infty \le b < 1$. In particular, $\left\|D_{\varphi,n}\right\|_{-1} = n!$ whenever both $\|\varphi\|_\infty \le \frac{1}{n+1}$ and $\varphi(0) = 0$.

Proof. Suppose that $\|\varphi\|_\infty \le b < 1$ and $\psi \in H^\infty$. We define $\varphi_b = (1/b)\varphi$ and $\rho(z) = bz$ (see [3, p. 2898]). Since $\left\|D_{\psi,\varphi,n}\right\|_{-1} \le \|\psi\|_\infty\|C_{\varphi_b}\|_{-1}\|D_{\rho,n}\|_{-1}$, we can see that

$$\left\|D_{\psi,\varphi,n}\right\|_{-1} \le n!\|\psi\|_\infty\sqrt{\frac{b+|\varphi(0)|}{b-|\varphi(0)|}}\binom{\lfloor \frac{n}{1-b}\rfloor}{n}b^{\lfloor \frac{n}{1-b}\rfloor - n} \tag{3.5}$$

by Theorem 3.5 and (1.1). Now suppose that $\|\varphi\|_\infty \le \frac{1}{n+1}$ and $\varphi(0) = 0$. By the Cauchy–Bunyakowsky–Schwarz Inequality, we have $|\varphi'(0)| \le \|\varphi\|\left\|K_{0,-1}^{[1]}\right\| \le$

$\|\varphi\|_\infty \leq \frac{1}{n+1}$. Consequently $\left\|D_{\varphi,n}\right\|_{-1} \geq n!$ by Proposition 3.2. On the other hand, (3.5) implies that $\left\|D_{\varphi,n}\right\|_{-1} \leq n!$. Therefore $\left\|D_{\varphi,n}\right\|_{-1} = n!$. □

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