SOME RESULTS ON TOP LOCAL COHOMOLOGY MODULES

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Abstract. Let I be an ideal of a commutative Noetherian ring R. In this paper, we determine the set of all attached primes of the top local cohomology module of any finitely generated R-module M with respect to the ideal I of R in terms of certain elements of $\operatorname{Supp} M$. Then as a consequence of this result we show that for any pair of finitely generated R-modules M and N with $\operatorname{Supp} N \subseteq \operatorname{Supp} M$, if $\operatorname{cd}(I, M) = \operatorname{cd}(I, N) = c \geq 0$, then $\operatorname{Att}_R H^c_I(N) \subseteq \operatorname{Att}_R H^c_I(M)$ and $\operatorname{Rad}(\operatorname{Ann}_R H^c_I(M)) \subseteq \operatorname{Rad}(\operatorname{Ann}_R H^c_I(N))$. Furthermore, in the case that R is a local ring, we prove some similar results concerning the associated primes of Matlis dual functors of top local cohomology modules.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I be an ideal of R. The local cohomology modules $H^i_I(M)$, $i = 0, 1, 2, \ldots,$ of an R-module M with respect to I were introduced by Grothendieck, [9]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\bigcup_{n=1}^{\infty}(0 :_M I^n)$. There is a natural isomorphism:

$$H^i_I(M) \simeq \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [9] or [2] for more details about local cohomology.

For an R-module M, the *cohomological dimension of M with respect to I* is defined as

$$\operatorname{cd}(I, M) := \sup\{i \in \mathbb{N}_0 : H^i_I(M) \neq 0\},$$

with the usual convention that the supremum of the empty set is interpreted as $-\infty$. This notion has been studied by several authors (see [5, 7, 8, 10]).

A prime ideal P is said to be *attached* to M if $P = \operatorname{Ann}_R M/N$ for some submodule N of M, equivalently $P = \operatorname{Ann}_R M/PM$. If $(R, \mathfrak{m})$ is a Noetherian complete local ring, the attached primes of an Artinian module are precisely the associated primes of its Matlis dual. In the sequel we denote the set of all attached primes of M by $\operatorname{Att}_R M$. Note that all attached primes of a module M contain the annihilator of M.

One of the important problems in the study of local cohomology is finding the set of attached primes of $H^i_I(M)$. There are a lot of results in the literature concerning the

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attached primes of Artinian local cohomology modules, for example see [13] .Note that the local cohomology module $H^i_I(M)$, of a finitely generated module M is Artinian in some cases for example if $I = \mathfrak{m}$ or $i = \dim M$ for all ideal I of R. But the non-Artinian case is a source of many interesting open questions.

Atazadeh, Sedghi, and Naghipour in [1] have shown that for any finitely generated R-module M with $\operatorname{cd}(I, M) = c \geq 0$, the set of all attached primes of the top local cohomology module $H^c_I(M)$ is contained in the set $\{P \in \operatorname{Supp} M \ : \ \operatorname{cd}(I, R/P) = c\}$. Also, in that paper, they asked whether the converse holds: must every prime $P \in \operatorname{Supp} M$ satisfying $\operatorname{cd}(I, R/P) = c$ be attached to $H^c_I(M)$?

In this paper, we determine the set of all attached primes of the top local cohomology module $H^c_I(M)$ in terms of certain elements of $\operatorname{Supp} M$. More precisely, we shall establish the following theorem, which is a generalization of the main results of [1, 5, 6].

Theorem 2.2. *Let I be an ideal of a Noetherian ring R and M be a finitely generated R-module with $\operatorname{cd}(I, M) = c \geq 0$. Then*

$$\operatorname{Att}_R H^c_I(M) = \{P \in \operatorname{Supp} M \ : \ \operatorname{Ann}_R H^c_I(R/P) = P\}.$$

By using this result, we see that the example constructed by Datta, Switala, and Zhang in [4] is a counter-example to the mentioned question in [1]. More precisely, in [4] the authors have constructed an example as below:

Example of Datta, Switala, and Zhang. (See [4, Example 4.2]) Let $R = \mathbb{Z}[x_0, \ldots, x_5]$, I be an ideal of R generated by 10 monomials

$$\{x_0x_1x_2, x_0x_1x_3, x_0x_2x_4, x_0x_3x_5, x_0x_4x_5, x_1x_2x_5, x_1x_3x_4, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5\}$$

and $\mathfrak{m} = (x_0, \ldots, x_5)$. It is easy to see that $\operatorname{Ext}^4_R(R/I, R) \simeq R/\mathfrak{m}$, which is annihilated by 2. For all $t \geq 1$ and I_t be the ideal generated by the t-th powers of the displayed monomial generators of I. Since $\operatorname{Ext}^4_R(R/I, R) \simeq R/\mathfrak{m}$, it follows from the proof of [4, Proposition 4.1] that

$$\operatorname{Ext}^4_R(R/I_t, R) \simeq R/(2, x_0^t, \ldots, x_5^t)$$

for all $t \geq 1$. It follows from [11, Theorem 1] that the transition map

$$\operatorname{Ext}^4_R(R/I_t, R) \to \operatorname{Ext}^4_R(R/I_{t+1}, R)$$

is injective for all $t \geq 1$. Therefore $H^4_I(R) \neq 0$ and is supported only in the maximal ideal $\mathfrak{m}$. It follows from [4, Proposition 4.1] that $2 \cdot \operatorname{Ext}^4_R(R/I_t, R) = (0)$ for all $t \geq 1$; and hence $2 \cdot H^4_I(R) = (0)$. Now consider the $\mathfrak{m}$-adic completion $\hat{R}$ of R, a complete unramified regular local ring. We again write I for the ideal $I\hat{R} \subseteq \hat{R}$. (The ring $\hat{R}$ is isomorphic to $\mathbb{Z}_2[[x_0, \ldots, x_5]]$, where $\mathbb{Z}_2$ denotes the ring of 2-adic integers, and $I\hat{R}$ is generated by the same ten monomials.) Local cohomology commute with the flat base change $R \to \hat{R}$, so $H^4_I(\hat{R}) \neq (0)$ and $H^j_I(\hat{R}) = (0)$ for all $j \geq 4$. Consequently, $H^4_I(\hat{R})$ is a top local cohomology module that is annihilated by 2. In particular, $\operatorname{Ann}_{\hat{R}} H^4_I(\hat{R}) = (2)$ by [4, Theorem 3.1]. Therefore, the prime ideal zero of R doesn't belong to $\operatorname{Att}_R H^4_I(R)$,

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however $\operatorname{cd}(I, R/(0)) = \operatorname{cd}(I, R) = 4$.

Also, as a consequence of Theorem 2.2, we shall prove the following theorem.

Theorem 2.4. *Let I be an ideal of a Noetherian ring R. Assume that M and N are two finitely generated R-modules such that $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. If $\operatorname{cd}(I, M) = \operatorname{cd}(I, N) = c \geq 0$, then the following statements hold:*

i) $\operatorname{Att}_R H_I^c(N) \subseteq \operatorname{Att}_R H_I^c(M)$.
ii) $\operatorname{Rad}(\operatorname{Ann}_R H_I^c(M)) \subseteq \operatorname{Rad}(\operatorname{Ann}_R H_I^c(N))$.

Furthermore, in the case that R is a local ring, we determine the associated primes of the Matlis dual functor of top local cohomology module $H_I^c(M)$ in terms of certain elements of $\operatorname{Supp} M$. More precisely, we shall prove the following theorem.

Theorem 2.6. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module with $\operatorname{cd}(I, M) = c \geq 0$. Then*

$$\operatorname{Ass}_R D(H_I^c(M)) = \{P \in \operatorname{Supp} M \ : \ P \in \operatorname{Ass}_R D(H_I^c(R/P))\},$$

where $D(-) := \operatorname{Hom}_R(-, E)$ denotes the Matlis dual functor and $E := E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. In particular, $\operatorname{Ass}_R D(H_I^c(M)) \subseteq \operatorname{Att}_R H_I^c(M)$.

Also, as a consequence of Theorem 2.6, we will prove the following theorem.

Theorem 2.7 *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$. Assume that M and N are two finitely generated R-modules such that $\operatorname{Supp} N \subseteq \operatorname{Supp} M$ and $\operatorname{cd}(I, M) = \operatorname{cd}(I, N) = c \geq 0$. Then $\operatorname{Ass}_R D(H_I^c(N)) \subseteq \operatorname{Ass}_R D(H_I^c(M))$.*

For any R-module N, we use the notation $E_R(N)$ for the injective envelope of the R-module N. For any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Also, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [12] or [2].

2. The results

We start this section with the following auxiliary lemma that is used frequently throughout the paper.

Lemma 2.1. *Let I be an ideal of a Noetherian ring R. Then the following statements hold:*

i) *Assume that U and V are two finitely generated R-modules such that $\operatorname{Supp} U \subseteq \operatorname{Supp} V$. Then, $\operatorname{cd}(I, U) \leq \operatorname{cd}(I, V)$. In particular, if $\operatorname{Supp} U = \operatorname{Supp} V$, then $\operatorname{cd}(I, U) = \operatorname{cd}(I, V)$.*

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ii) *Suppose that M is a finitely generated R-module with* $\operatorname{cd}(I,M)=c\geq 0$ *and P is a prime ideal in* $\operatorname{Supp} M$. *Then,* $H^c_I(M)\otimes_R R/P\simeq H^c_I(M/PM)$.

iii) *For a finitely generated R-module M and an arbitrary R-module N,*

$$\operatorname{Ass}_R \operatorname{Hom}_R(M,N)=\operatorname{Ass}_R N\cap \operatorname{Supp} M.$$

iv) *For an arbitrary R-module M and for all integer $i\geq 0$,*

$$\operatorname{Ann}_R M\subseteq \operatorname{Ann}_R H^i_I(M).$$

Proof. (i) holds by [7, Theorem 2.2].

(ii) We can consider M as a module over R or as a module over R/P, and the independence theorem [2, Theorem 4.2.1] gives that the local cohomology will agree regardless of the base field. In particular, since $\operatorname{Supp} R/\operatorname{Ann}_R M=\operatorname{Supp} M$, by using (i) and [2, Theorem 4.2.1], we see that

$$\operatorname{cd}((I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M, R/\operatorname{Ann}_R M)=\operatorname{cd}(I,R/\operatorname{Ann}_R M)=\operatorname{cd}(I,M)=c.$$

Furthermore, by using [2, Exercise 6.1.8 and Theorem 4.2.1], one sees that

$$\begin{aligned}
H^c_I(M)\otimes_R R/P &\simeq H^c_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(M)\otimes_R R/P\\
&\simeq \big(H^c_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M)\otimes_{R/\operatorname{Ann}_R M} M\big)\otimes_R R/P\\
&\simeq H^c_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M)\otimes_{R/\operatorname{Ann}_R M}\big(M\otimes_R R/P\big)\\
&\simeq H^c_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M)\otimes_{R/\operatorname{Ann}_R M} M/PM\\
&\simeq H^c_{(I+\operatorname{Ann}_R M)/\operatorname{Ann}_R M}(M/PM)\\
&\simeq H^c_I(M/PM),
\end{aligned}$$

as required.

(iii) Follows from [3, Exercise 1.2.27].

(iv) It follows from this fact that the local cohomology functor $H^i_I(-)$ is R-linear. □

Now, we are ready to state and prove the most important result of this paper. in fact, in the following theorem we are able to determine the set of all attached primes of the top local cohomology module of any finitely generated R-module M with respect to any ideal I of a Noetherian ring R. This theorem is crucial for the proof of our other main results.

Theorem 2.2. *Let I be an ideal of a Noetherian ring R and M be a finitely generated R-module with* $\operatorname{cd}(I,M)=c\geq 0$. *Then*

$$\operatorname{Att}_R H^c_I(M)=\{P\in \operatorname{Supp} M\ :\ \operatorname{Ann}_R H^c_I(R/P)=P\}.$$

Proof. Assume that $\mathfrak{p}\in\{P\in \operatorname{Supp} M\ :\ \operatorname{Ann}_R H^c_I(R/P)=P\}$. Since

$$\operatorname{Ass}_R \operatorname{Hom}_R(M,R/\mathfrak{p})=\operatorname{Ass}_R R/\mathfrak{p}\cap \operatorname{Supp} M=\{\mathfrak{p}\}\neq\emptyset,$$

we see that $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$. Select an element $f \in \operatorname{Hom}_R(M, R/\mathfrak{p})$ in which $f \neq 0$ and put $J/\mathfrak{p} := \operatorname{im} f$. The following short exact sequence

$$0 \longrightarrow J/\mathfrak{p} \longrightarrow R/\mathfrak{p} \longrightarrow R/J \longrightarrow 0,$$

induces the following exact sequence

$$H^c_I(J/\mathfrak{p}) \longrightarrow H^c_I(R/\mathfrak{p}) \longrightarrow H^c_I(R/J),$$

which shows that

$$\big(\operatorname{Ann}_R H^c_I(J/\mathfrak{p})\big)\big(\operatorname{Ann}_R H^c_I(R/J)\big) \subseteq \operatorname{Ann}_R H^c_I(R/\mathfrak{p}) = \mathfrak{p}\,.$$

Since $J = \operatorname{Ann}_R R/J \subseteq \operatorname{Ann}_R H^c_I(R/J)$ and $J \not\subseteq \mathfrak{p}$, we see that $\operatorname{Ann}_R H^c_I(R/J) \not\subseteq \mathfrak{p}$. Therefore, $\operatorname{Ann}_R H^c_I(J/\mathfrak{p}) \subseteq \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal of R. It is easy to see that,

$$\mathfrak{p} = \operatorname{Ann}_R J/\mathfrak{p} \subseteq \operatorname{Ann}_R H^c_I(J/\mathfrak{p}).$$

Thus, $\operatorname{Ann}_R H^c_I(J/\mathfrak{p}) = \mathfrak{p}$. Since $\operatorname{Supp}\ker f \subseteq \operatorname{Supp} M$, by Lemma 2.1 we can deduce that $H^{c+1}_I(\ker f) = 0$. Consequently, the following short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow J/\mathfrak{p} \longrightarrow 0,$$

induces the following exact sequence

$$H^c_I(M) \longrightarrow H^c_I(J/\mathfrak{p}) \longrightarrow 0,$$

which shows that $\mathfrak{p} \in \operatorname{Att}_R H^c_I(M)$.

In order to establish the reverse inclusion, let $\mathfrak{q} \in \operatorname{Att}_R H^c_I(M)$. Then, it directly follows from the definition that $\operatorname{Ann}_R H^c_I(M) \otimes_R R/\mathfrak{q} = \mathfrak{q}$. Since $\operatorname{Ann}_R M \subseteq \operatorname{Ann}_R H^c_I(M) \subseteq \mathfrak{q}$, so $\mathfrak{q} \in \operatorname{Supp} M$. Therefore, by Lemma 2.1 we have

$$\mathfrak{q} = \operatorname{Ann}_R H^c_I(M) \otimes_R R/\mathfrak{q} = \operatorname{Ann}_R H^c_I(M/\mathfrak{q}M).$$

Select a short exact sequence

$$0 \longrightarrow L \longrightarrow \bigoplus_{i=1}^{k} R/\mathfrak{q} \longrightarrow M/\mathfrak{q}M \longrightarrow 0,$$

for some positive integer k. Since $\operatorname{Supp} L \subseteq \operatorname{Supp} R/\mathfrak{q} \subseteq \operatorname{Supp} M$, by using Lemma 2.1 we can deduce that $H^{c+1}_I(L) = 0$. Therefore, from the last short exact sequence and the fact that local cohomology commute with direct sum, we get the exact sequence

$$\bigoplus_{i=1}^{k} H^c_I(R/\mathfrak{q}) \longrightarrow H^c_I(M/\mathfrak{q}M) \longrightarrow 0,$$

which implies that

$$\mathfrak{q} = \operatorname{Ann}_R R/\mathfrak{q} \subseteq \operatorname{Ann}_R H^c_I(R/\mathfrak{q}) = \operatorname{Ann}_R \bigoplus_{i=1}^{k} H^c_I(R/\mathfrak{q}) \subseteq \operatorname{Ann}_R H^c_I(M/\mathfrak{q}M) = \mathfrak{q}\,.$$

Thus, $\operatorname{Ann}_R H^c_I(R/\mathfrak{q}) = \mathfrak{q}$. Hence,

$$\mathfrak{q} \in \{P \in \operatorname{Supp} M \;:\; \operatorname{Ann}_R H^c_I(R/P) = P\}.$$

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Therefore,

$$\operatorname{Att}_R H^c_I(M) = \{P \in \operatorname{Supp} M \ : \ \operatorname{Ann}_R H^c_I(R/P) = P\}.$$

□

In order to prove our second theorem we need the following well-known lemma.

Lemma 2.3. *Let R be a Noetherian ring and L be a non-zero R-module. Then*

$$\operatorname{Rad}(\operatorname{Ann}_R L) = \bigcap_{P \in \operatorname{Att}_R L} P.$$

Proof. See [1, Lemma 3.2]. □

The following theorem is the second main result of this paper.

Theorem 2.4. *Let I be an ideal of a Noetherian ring R. Assume that M and N are two finitely generated R-modules such that* $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. *If* $\operatorname{cd}(I, M) = \operatorname{cd}(I, N) = c \geq 0$*, then the following statements hold:*

i) $\operatorname{Att}_R H^c_I(N) \subseteq \operatorname{Att}_R H^c_I(M)$.
ii) $\operatorname{Rad}(\operatorname{Ann}_R H^c_I(M)) \subseteq \operatorname{Rad}(\operatorname{Ann}_R H^c_I(N))$.

Proof. (i) follows from Theorem 2.2 and (ii) follows from (i) together with Lemma 2.3. □

Corollary 2.5. *Let I be an ideal of a Noetherian ring R. Assume that M and N are two finitely generated R-modules such that* $\operatorname{Supp} M = \operatorname{Supp} N$.*If* $\operatorname{cd}(I, M) = \operatorname{cd}(I, N) = c \geq 0$, *then the following statements hold:*

i) $\operatorname{Att}_R H^c_I(M) = \operatorname{Att}_R H^c_I(N)$.
ii) $\operatorname{Rad}(\operatorname{Ann}_R H^c_I(M)) = \operatorname{Rad}(\operatorname{Ann}_R H^c_I(N))$.

Proof. Follow from Theorem 2.4. □

The following theorem is our next main result.

Theorem 2.6. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module with* $\operatorname{cd}(I, M) = c \geq 0$. *Then*

$$\operatorname{Ass}_R D(H^c_I(M)) = \{P \in \operatorname{Supp} M \ : \ P \in \operatorname{Ass}_R D(H^c_I(R/P))\},$$

where $D(-) := \operatorname{Hom}_R(-, E)$ *denotes the Matlis dual functor and* $E := E_R(R/\mathfrak{m})$ *is the injective hull of the residue field* $R/\mathfrak{m}$. *In particular, we will show that* $\operatorname{Ass}_R D(H^c_I(M)) \subseteq \operatorname{Att}_R H^c_I(M)$.

Proof. Suppose that $\mathfrak{p} \in \{P \in \operatorname{Supp} M \ : \ P \in \operatorname{Ass}_R D(H^c_I(R/P))\}$. Since

$$\operatorname{Ass}_R \operatorname{Hom}_R(M, R/\mathfrak{p}) = \operatorname{Ass}_R R/\mathfrak{p} \cap \operatorname{Supp} M = \{\mathfrak{p}\} \neq \emptyset,$$

we see that $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$. Select an element $f \in \operatorname{Hom}_R(M, R/\mathfrak{p})$ in which $f \neq 0$ and put $J/\mathfrak{p} := \operatorname{im} f$. Since $\operatorname{Supp} J/\mathfrak{p} = \operatorname{Supp} R/\mathfrak{p} \subseteq \operatorname{Supp} M$, by Lemma 2.1 one sees that $\operatorname{cd}(I, J/\mathfrak{p}) \leq \operatorname{cd}(I, M) = c$ and so $H_I^{c+1}(J/\mathfrak{p}) = 0$. Thus, the following short exact sequence

$$0 \longrightarrow J/\mathfrak{p} \longrightarrow R/\mathfrak{p} \longrightarrow R/J \longrightarrow 0,$$

induces the following exact sequence

$$H_I^c(J/\mathfrak{p}) \longrightarrow H_I^c(R/\mathfrak{p}) \longrightarrow H_I^c(R/J) \longrightarrow 0,$$

and from this exact sequence we obtain the exact sequence

$$0 \longrightarrow D(H_I^c(R/J)) \longrightarrow D(H_I^c(R/\mathfrak{p})) \longrightarrow D(H_I^c(J/\mathfrak{p})).$$

Hence, the assumption $\mathfrak{p} \in \operatorname{Ass}_R D(H_I^c(R/\mathfrak{p}))$ implies that $\mathfrak{p} \in \operatorname{Ass}_R D(H_I^c(R/J))$ or $\mathfrak{p} \in \operatorname{Ass}_R D(H_I^c(J/\mathfrak{p}))$. Since

$$\operatorname{Ass}_R D(H_I^c(R/J)) \subseteq V(\operatorname{Ann}_R D(H_I^c(R/J))) \subseteq V(\operatorname{Ann}_R R/J) = V(J),$$

and $\mathfrak{p} \notin V(J)$, it is concluded that $\mathfrak{p} \in \operatorname{Ass}_R D(H_I^c(J/\mathfrak{p}))$.

Since $\operatorname{Supp} \ker f \subseteq \operatorname{Supp} M$, by Lemma 2.1 we can deduce that $H_I^{c+1}(\ker f) = 0$. Consequently, the following short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow J/\mathfrak{p} \longrightarrow 0,$$

induces the following exact sequence

$$H_I^c(M) \longrightarrow H_I^c(J/\mathfrak{p}) \longrightarrow 0,$$

and from this exact sequence we obtain the exact sequence

$$0 \longrightarrow D(H_I^c(J/\mathfrak{p})) \longrightarrow D(H_I^c(M)),$$

which shows that $\mathfrak{p} \in \operatorname{Ass}_R D(H_I^c(M))$.

In order to establish the reverse inclusion, let $\mathfrak{q} \in \operatorname{Ass}_R D(H_I^c(M))$. Then we have the exact sequence

$$0 \longrightarrow R/\mathfrak{q} \longrightarrow D(H_I^c(M)).$$

By effecting the left exact functor $\operatorname{Hom}_R(R/\mathfrak{q}, -)$ to this exact sequence we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/\mathfrak{q}, R/\mathfrak{q}) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{q}, D(H_I^c(M)).$$

Since $\operatorname{Hom}_R(R/\mathfrak{q}, R/\mathfrak{q}) \simeq R/\mathfrak{q}$, thus $R/\mathfrak{q}$ can be embedded in the R-module

$$\operatorname{Hom}_R(R/\mathfrak{q}, D(H_I^c(M))).$$

Also we have

$$\operatorname{Ann}_R M \subseteq \operatorname{Ann}_R D(H_I^c(M)) \subseteq \mathfrak{q},$$

and hence $\mathfrak{q} \in \operatorname{Supp} M$. Furthermore, by using Lemma 2.1 we see that

$$\operatorname{Hom}_R(R/\mathfrak{q}, D(H_I^c(M))) \simeq D(H_I^c(M) \otimes_R R/\mathfrak{q}) \simeq D(H_I^c(M/\mathfrak{q}M)).$$

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Therefore, the R-module $R/\mathfrak{q}$ can be embedded in the R-module $D(H_I^c(M/\mathfrak{q}M))$, which means that $\mathfrak{q} \in \operatorname{Ass}_R D(H_I^c(M/\mathfrak{q}M))$. Select a short exact sequence

$$0 \longrightarrow L \longrightarrow \bigoplus_{i=1}^{k} R/\mathfrak{q} \longrightarrow M/\mathfrak{q}M \longrightarrow 0,$$

for some positive integer k. Since $\operatorname{Supp} L \subseteq \operatorname{Supp} R/\mathfrak{q} \subseteq \operatorname{Supp} M$, by using Lemma 2.1 we can deduce that $H_I^{c+1}(L) = 0$. Therefore, from the last short exact sequence we get the exact sequence

$$\bigoplus_{i=1}^{k} H_I^c(R/\mathfrak{q}) \longrightarrow H_I^c(M/\mathfrak{q}M) \longrightarrow 0,$$

and this exact sequence induces the following exact sequence

$$0 \longrightarrow D(H_I^c(M/\mathfrak{q}M)) \longrightarrow \bigoplus_{i=1}^{k} D(H_I^c(R/\mathfrak{q})),$$

which implies that

$$\mathfrak{q} \in \operatorname{Ass}_R \bigoplus_{i=1}^{k} D(H_I^c(R/\mathfrak{q})) = \operatorname{Ass}_R D(H_I^c(R/\mathfrak{q})).$$

Hence,

$$\mathfrak{q} \in \{P \in \operatorname{Supp} M \ : \ P \in \operatorname{Ass}_R D(H_I^c(R/P))\}.$$

Therefore,

$$\operatorname{Ass}_R D(H_I^c(M)) = \{P \in \operatorname{Supp} M \ : \ P \in \operatorname{Ass}_R D(H_I^c(R/P))\}.$$

Now, let $P \in \operatorname{Ass}_R D(H_I^c(M))$. Then we have $P \in \operatorname{Supp} M$ and $P \in \operatorname{Ass}_R D(H_I^c(R/P))$. Therefore,

$$P \subseteq \operatorname{Ann}_R H_I^c(R/P) \subseteq \operatorname{Ann}_R D(H_I^c(R/P)) \subseteq P,$$

which implies that $\operatorname{Ann}_R H_I^c(R/P) = P$. So, by Theorem 2.2 it is concluded that $P \in \operatorname{Att}_R H_I^c(M)$. Hence, $\operatorname{Ass}_R D(H_I^c(M)) \subseteq \operatorname{Att}_R H_I^c(M)$. □

The following theorem is the final main result of this paper.

Theorem 2.7. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$. Assume that M and N are two finitely generated R-modules such that $\operatorname{Supp} N \subseteq \operatorname{Supp} M$ and $\operatorname{cd}(I, M) = \operatorname{cd}(I, N) = c \geq 0$. Then $\operatorname{Ass}_R D(H_I^c(N)) \subseteq \operatorname{Ass}_R D(H_I^c(M))$.*

Proof. The assertion follows from Theorem 2.6. □

Corollary 2.8. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$. Assume that M and N are two finitely generated R-modules such that $\operatorname{Supp} M = \operatorname{Supp} N$ and $\operatorname{cd}(I, M) = c \geq 0$. Then $\operatorname{Ass}_R D(H_I^c(M)) = \operatorname{Ass}_R D(H_I^c(N))$.*

Proof. The assertion follows from Lemma 2.1 and Theorem 2.7. □

Proposition 2.9. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module with $\operatorname{cd}(I, M) = c \geq 0$. Then the sets $\operatorname{Att}_R H_I^c(M)$ and $\operatorname{Ass}_R D(H_I^c(M))$ have the same maximal elements with respect to inclusion.*

Proof. Suppose that P is a maximal element of $\operatorname{Att}_R H_I^c(M)$ with respect to inclusion. Then by Theorem 2.2 we see that $P \in \operatorname{Supp} M$ and $\operatorname{Ann}_R H_I^c(R/P) = P$. Therefore, by [2, Remark 10.2.2] we have $\operatorname{Ann}_R D(H_I^c(R/P)) = P$. We show that

$$\operatorname{Ass}_R D(H_I^c(R/P)) = \{P\}.$$

Since $D(H_I^c(R/P)) \neq 0$, it follows that $\operatorname{Ass}_R D(H_I^c(R/P)) \neq \emptyset$. Hence, it suffices to prove that $\operatorname{Ass}_R D(H_I^c(R/P)) \subseteq \{P\}$. Let $Q \in \operatorname{Ass}_R D(H_I^c(R/P))$. Clearly,

$$P = \operatorname{Ann}_R D(H_I^c(R/P)) \subseteq Q.$$

Thus, $Q \in V(P) = \operatorname{Supp} R/P \subseteq \operatorname{Supp} M$. By using the hypothesis $H_I^c(R/P) \neq 0$, and Lemma 2.1 it can bee seen that

$$\operatorname{cd}(I, R/P) = \operatorname{cd}(I, M) = c.$$

Therefore, by Lemma 2.1 we have $H_I^c(R/P) \otimes_R R/Q \simeq H_I^c(R/Q)$.

Since $Q \in \operatorname{Ass}_R D(H_I^c(R/P))$, it follows that $\operatorname{Ann}_R \operatorname{Hom}_R(R/Q, D(H_I^c(R/P))) = Q$. Also, it is clear that

$$\operatorname{Hom}_R(R/Q, D(H_I^c(R/P))) \simeq D(H_I^c(R/P) \otimes_R R/Q) \simeq D(H_I^c(R/Q)).$$

Therefore,

$$\operatorname{Ann}_R H_I^c(R/Q) = \operatorname{Ann}_R D(H_I^c(R/Q)) = \operatorname{Ann}_R \operatorname{Hom}_R(R/Q, D(H_I^c(R/P))) = Q.$$

So, by Theorem 2.2 we have $Q \in \operatorname{Att}_R H_I^c(M)$. Since $P \subseteq Q$ and P is a maximal element of $\operatorname{Att}_R H_I^c(M)$, it is concluded that $Q = P$. Thus, $\operatorname{Ass}_R D(H_I^c(R/P)) = \{P\}$ and hence by Theorem 2.6 we have $P \in \operatorname{Ass}_R D(H_I^c(M))$. Also, by Theorem 2.6 we see that $\operatorname{Ass}_R D(H_I^c(M)) \subseteq \operatorname{Att}_R H_I^c(M)$. Therefore, the sets $\operatorname{Att}_R H_I^c(M)$ and $\operatorname{Ass}_R D(H_I^c(M))$ have the same maximal elements. □

Note. Theorems 2.6, 2.7, Corollary 2.8 and Proposition 2.9 potentially have ramifications for Ext-modules whenever R is a complete Cohen-Macaulay ring, as

$$D(H_{\mathfrak{m}}^i(M)) \simeq \operatorname{Ext}_R^{dimR-i}(M, \omega_R).$$

where ω_R is the canonical module of R.

Proposition 2.10. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M be a finitely generated R-module with $\operatorname{cd}(I, M) = c \geq 0$. If the elements of $\operatorname{Att}_R H_I^c(M)$ are pairwise incomparable under inclusion then $\operatorname{Ass}_R D(H_I^c(M)) = \operatorname{Att}_R H_I^c(M)$.*

Proof. By Theorem 2.6 we know that $\operatorname{Ass}_R D(H_I^c(M)) \subseteq \operatorname{Att}_R H_I^c(M)$. Since the elements of $\operatorname{Att}_R H_I^c(M)$ are pairwise incomparable under inclusion it is clear that each element of $\operatorname{Att}_R H_I^c(M)$ is maximal with respect to inclusion. Therefore, Proposition 2.9 implies that $\operatorname{Att}_R H_I^c(M) \subseteq \operatorname{Ass}_R D(H_I^c(M))$, as required. □

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