THE HEAT TRANSFORM ON THE COMPLEX PLANE

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ABSTRACT. The heat transform H_t, for positive time $t > 0$, is the convolution on the complex plane with the heat kernel. In the field of analytic function spaces and related operator theory, H_t coincides with the Berezin transform for the Fock space F_t^2 induced by the Gaussian measure $e^{-|z|^2/t}\,dA(z)$.

In this paper we study fixed-points of H_t and the limit behavior of $H_t f$ as $t \to 0^+$. Fixed-points of H_t are shown to be closely related to eigenfunctions of the Laplacian corresponding to certain special eigenvalues, while the limit behavior of $H_t f$ as $t \to 0^+$ depends on certain continuity and oscillation properties of f.

The paper is expository, although it contains a few new results. In particular, the main results about fixed-points of H_t are known, but we present a completely new proof here.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane and

$$\Delta = \frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

be the complex Laplacian on $\mathbb{C}$. For a bounded continuous function f on $\mathbb{C}$, the solution of the initial-value problem

$$\Delta u(z,t) = \frac{\partial u(z,t)}{\partial t}, \qquad \lim_{t\to 0^+} u(z,t) = f(z), \tag{1}$$

for the classical heat equation is given by

$$u(z,t) = \frac{1}{\pi t}\int_{\mathbb{C}} f(w) e^{-|z-w|^2/t}\,dA(w),$$

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where dA is ordinary area measure on $\mathbb{C}$. See [17]. We will write

$$H_t f(z) = \frac{1}{\pi t} \int_{\mathbb{C}} f(w) e^{-|z-w|^2/t} \, dA(w) \tag{2}$$

and call $H_t f$ the heat transform of f. Obviously, $H_t f$ is well defined for functions f much more general than the bounded continuous ones.

We focus on two problems in this paper:

(i) fixed-points of the heat transform H_t, which are closely related to eigenfunctions of the Laplacian in general and harmonic functions in particular.
(ii) limit behavior of $H_t f$ as $t \to 0^+$, which is closely related to the initial value problem in (1) for the heat equation on the complex plane.

The main results about fixed-points are summarized as follows.

Theorem A. *Let f be a non-zero Lebesgue measurable function on $\mathbb{C}$ such that the involved heat transforms are well defined.*

(a) *If $\Delta f = \lambda f$ for some $\lambda \in \mathbb{C}$, then $H_t f = f$ for some $t > 0$ if and only if $\lambda t = 2n\pi i$ for some integer n.*
(b) *$H_t f = f$ for all $t > 0$ if and only if f is harmonic on $\mathbb{C}$.*

Part (a) above can be found in [3, 15]. We give another completely different proof here. Since the heat transform H_t is linear, we see that if $f = \sum f_k$ is any finite sum, where f_k is an eigenfunction of Δ corresponding to the eigenvalue $2k\pi i/t$, then $H_t f = f$. This is also true for infinite sums under certain assumptions on convergence. Consequently, for any given $t > 0$, the heat transform H_t fixes many more functions than the harmonic ones. Earlier examples of non-harmonic fixed-points of H_t can be found in [15, 17].

We will see in the next section that the heat transform H_t is simply the Berezin transform for the Fock space F^2_α when $\alpha = 1/t$. Our approach to fixed-points for H_t is motivated by the by-now-classical result about the Berezin transform for the ordinary Bergman space A^2 on the unit disk. In that context, the fixed-points of the Berezin transform are exactly the harmonic functions (subject to a natural integrability condition). See [1]. Fixed-points of the Berezin transform for the Bergman space of the open unit ball in $\mathbb{C}^n$ are also studied in [1], where it was shown that the dimension n plays a very critical role. Fixed-points for the Fock-Berezin tranform are also studied in [14].

We need to use several function spaces in the study of limit behavior for $H_t f$ as $t \to 0^+$. First, as usual, $L^\infty(\mathbb{C})$ is the space of all bounded

Lebesgue measurable functions on $\mathbb{C}$ equipped with the sup-norm

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in \mathbb{C}\}.$$

Let $\mathcal{BC}$ be the space of all continuous functions in $L^\infty(\mathbb{C})$. The space of all uniformly continuous functions on $\mathbb{C}$ will be denoted by $\mathcal{UC}$. Deferring the definition of $\mathcal{BMO}^p$ to Section 5, we can now state our second main result, subject to some obvious integrability condition that will be defined at the end of the next section.

Theorem B. *Let f be a Lebesgue measurable function f on $\mathbb{C}$. Then*

(a) *$H_t f(z) \to f(z)$ uniformly on compacta as $t \to 0^+$ if and only if f is continuous on $\mathbb{C}$.*
(b) *f belongs to $\mathcal{UC}$ if and only if $f \in \mathcal{BMO}^1$ and $H_t f(z) \to f(z)$ uniformly on $\mathbb{C}$ as $t \to 0^+$.*
(c) *If f is subharmonic, then $H_t f(z) \to f(z)$ uniformly on $\mathbb{C}$ as $t \to 0^+$ if and only if $M_r f(z) \to f(z)$ uniformly on $\mathbb{C}$ as $r \to 0^+$ and $H_s f - f \in \mathcal{BC}$ for some $s > 0$, where $M_r f(z)$ is the mean of f over the circle $z + re^{i\theta}$, $0 \le \theta \le 2\pi$.*

The limit behavior of $H_t f$ as $t \to 0$ has been studied by several authors. See [4, 5, 6, 7] for example. Note that part (a) above removes the boundedness requirement of f in Theorem 3.14 of [17].

We thank the referee for a very careful reading of the manuscript and making several suggestions on how to improve the presentation. In particular, the current simple proof of Theorem 10 was based on the referee's suggestion to make use of the fact that the function $(t, z) \mapsto H_t f(z)$ solves the heat equation.

2. Preliminaries on Fock spaces

In this paper we study the heat transform as the Berezin transform for the Fock space. Recall that for any fixed positive parameter α the Fock space F^2_α is defined by

$$F^2_\alpha = L^2(\mathbb{C}, d\lambda_\alpha) \cap H(\mathbb{C}),$$

where $H(\mathbb{C})$ is the space of all entire functions on $\mathbb{C}$ and

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}\, dA(z)$$

is the so-called Gaussian measure.

Each F^2_α is a reproducing kernel Hilbert space. The inner product in F^2_α will be written as

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)}\, d\lambda_\alpha(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)}\, e^{-\alpha|z|^2}\, dA(z).$$

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The reproducing kernel of F^2_α is given by $K(z,w) = e^{\alpha z\overline{w}}$. Thus we have the reproducing formula

$$f(z) = \frac{\alpha}{\pi}\int_{\mathbb{C}} f(w)e^{\alpha z\overline{w}}e^{-\alpha|w|^2}\,dA(w) \tag{3}$$

for all $f \in F^2_\alpha$ and $z \in \mathbb{C}$. Since this is a pointwise identity, a limit argument shows that the formula in (3) holds for many more entire functions than those in F^2_α. Also, if we extend α to the right-half of the complex plane, an argument using the identity theorem shows that the formula in (3) also holds when α is a complex number with positive real part and f is an entire function subject to an obvious integrability condition (which will be made precise later). This technical improvement of the reproducing formula, namely, to allow α to be complex, will be used in Section 5.

The normalized reproducing kernel of F^2_α at $z \in \mathbb{C}$ is the following unit vector k_z in F^2_α:

$$k_z(w) = K(w,z)/\sqrt{K(z,z)} = e^{\alpha w\overline{z} - \alpha|z|^2/2}.$$

For a reasonably good function f on $\mathbb{C}$, the F^2_α-Berezin transform of f is the function $B_\alpha f$ defined by

$$\begin{aligned} B_\alpha f(z) &= \langle f k_z, k_z\rangle_\alpha \\ &= \frac{\alpha}{\pi}\int_{\mathbb{C}} f(w)|k_z(w)|^2 e^{-\alpha|w|^2}\,dA(w) \\ &= \frac{\alpha}{\pi}\int_{\mathbb{C}} f(w)e^{-\alpha|z-w|^2}\,dA(w) \\ &= \frac{\alpha}{\pi}\int_{\mathbb{C}} f(z\pm w)e^{-\alpha|w|^2}\,dA(w). \end{aligned}$$

We see that the heat transform $H_t f$ is simply the Berezin transform $B_\alpha f$ for $\alpha = 1/t$. Each of the notations $H_t f$ and $B_\alpha f$ has its own advantages and disadvantages. For example, the heat transforms H_t have the semi-group property: $H_s H_t = H_{s+t}$, which is very helpful in many applications. See [13, 17]. But the appearance of the time parameter t as a reciprocal in the definition of H_t sometimes complicates computations, especially when we need to take the derivative with respect to the weight parameter.

The Berezin transform has become an indispensable tool in the study of operators on function spaces. Initiated by Berezin [8] in connection with quantum mechanics and then studied systematically in [8]-[12] for a number of reproducing Hilbert spaces, the Berezin transform has proven especially successful in the study of Toeplitz operators, Hankel operators, and composition operators on Bergman and Fock spaces. See [16, 17].

For convenience and for consistency with existing notion in the literature, we will write $d\mu_t = d\lambda_{1/t}$. Thus

$$d\mu_t(z) = \frac{1}{\pi t} e^{-|z|^2/t}\, dA(z)$$

for any $t > 0$. We say that a function f on $\mathbb{C}$ satisfies condition (I_t) if

$$\int_{\mathbb{C}} |f(z)e^{az}|\, d\mu_t(z) < \infty$$

for all $a \in \mathbb{C}$. It is clear that if f satisfies condition (I_{t_0}) for some $t_0 > 0$, then the heat transform $H_t f$ in (2) is a well-defined function on $\mathbb{C}$ for any $t \in (0, t_0]$. Similarly, if $\alpha_0 = 1/t_0$ and f is an entire function on $\mathbb{C}$ satisfying condition I_{t_0}, then the reproducing formula in (3) holds for all $z \in \mathbb{C}$ and all *complex* α with $\mathrm{Re}\, \alpha \geq \alpha_0$.

3. Eigenvalues and eigenfunctions of the Laplacian

In this section we study eigenvalues and eigenvectors of the complex Laplace operator Δ on $\mathbb{C}$. All functions in this section are assumed to be second-order differentiable on $\mathbb{C}$, so that we can freely apply the Laplacian to them. We begin with the following elementary example.

Example 1. *Suppose $f(z) = e^{az+b\bar{z}}$, where a and b are complex constants. Then $\Delta f = \lambda f$, where $\lambda = ab$.*

Thus every complex number λ is an eigenvalue of the Laplacian. Furthermore, the eigenspace E_λ corresponding to λ is infinite dimensional, because we can write λ as $\lambda = ab$ in infinitely many different ways.

For any Lebesgue measurable function f on $\mathbb{C}$ we define another function Rf on $\mathbb{C}$ as follows:

$$Rf(z) = \frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta})\, d\theta.$$

The function Rf is called the radialization of f. It is clear that Rf is radial, that is, $Rf(z) = Rf(|z|)$ for all $z \in \mathbb{C}$.

Lemma 2. *The operators Δ and R commute, that is, $\Delta(Rf) = R(\Delta f)$.*

Proof. This follows from differentiating inside the integral sign. □

Consider the example $f(z) = e^{az+b\bar{z}}$ again. We have

$$Rf(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{aze^{i\theta}} e^{b\bar{z}e^{-i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{(az)^n(b\overline{z})^m}{n!\,m!}\,e^{i(n-m)\theta}\right]d\theta$$
$$= \sum_{n=0}^\infty \frac{(ab)^n}{(n!)^2}\,|z|^{2n}.$$

Lemma 3. *For any complex number λ the function*

$$g_\lambda(z) = \sum_{n=0}^\infty \frac{\lambda^n}{(n!)^2}|z|^{2n}$$

is radial and satisfies $\Delta g_\lambda = \lambda g_\lambda$.

Proof. It is clear that g_λ is radial. The identity $\Delta g_\lambda = \lambda g_\lambda$ follows easily from a direct calculation. □

Recall that for any complex λ we use E_λ to denote the eigenspace of Δ corresponding to the eigenvalue λ. Thus each g_λ is a radial function in E_λ. The next proposition shows that g_λ is the only radial function in E_λ up to constant multiples.

Proposition 4. *If $g \in E_\lambda$ is radial, then $g = g(0)g_\lambda$.*

Proof. Since g is radial, we think of g as a function defined on $[0,+\infty)$. Define another function f on $[0,+\infty)$ by $f(x) = g(\sqrt{x})$. We have

$$g(z) = f(|z|^2) = f(z\overline{z})$$

and

$$\Delta g(z) = f'(|z|^2) + |z|^2 f''(|z|^2).$$

Thus $\Delta g = \lambda g$ on $\mathbb{C}$ implies that

$$xf''(x) + f'(x) = \lambda f(x), \qquad x \in (0,+\infty). \tag{4}$$

Let X_λ be the solution space of the second-order homogeneous linear ordinary differential equation in (4). Then X_λ is a two-dimensional vector space. By Lemma 3, we know that

$$f_1(x) := g_\lambda(\sqrt{x})$$

is a nonzero solution of (4). We will find another independent solution of (4) using the standard method of "reduction of order" in the theory of differential equations.

Thus we look for a solution of (4) on $(0,+\infty)$ in the form of

$$f_2(x) := u(x)f_1(x),$$

where

$$xu''f_1 + (2xf_1' + f_1)u' = 0.$$

Rewrite this as

$$\frac{u''}{u'} = -\frac{2xf_1' + f_1}{xf_1} = -\frac{2f_1'}{f_1} - \frac{1}{x},$$

and integrate both sides. We obtain

$$\ln(u') = -2\ln f_1 - \ln x = \ln \frac{1}{xf_1^2}.$$

Here we are ignoring the integration constant since we are only looking for one other non-trivial solution of (4). It follows that

$$u'(x) = \frac{1}{xf_1^2(x)}, \qquad x \in (0, +\infty).$$

Integrating one more time and ignoring the integration constant again, we get

$$u(x) = \int_1^x \frac{dt}{tf_1^2(t)}$$

and

$$f_2(x) = f_1(x) \int_1^x \frac{dt}{tf_1^2(t)}, \qquad x \in (0, +\infty).$$

In particular, the function $u(x)$ is well-defined on $(0, +\infty)$. Let

$$\overline{f_1^2} = u_1 + iv_1,$$

where u_1 and v_1 are real fucntions defined on $[0, \infty)$. Since f_1 is continuous on $[0, \infty)$ with $f_1(0) = 1$, we have $u_1 \to 1$ and $v_1 \to 0$ as $x \to 0^+$. Thus there exists $0 < \delta_0 < 1$ such that $1/2 < u_1(x) < 3/2$ for all $x \in (0, \delta_0)$. It follows from the continuity of f_1 on $[\delta_0, 1]$ that there exists $C_{\lambda,\delta_0} > 0$ such that

$$\left| \int_{\delta_0}^1 \frac{u_1(t)}{t|f_1(t)|^4}\, dt \right| \le C_{\lambda,\delta_0}.$$

Therefore,

$$\begin{aligned} \left| \int_0^1 \frac{dt}{tf_1^2(t)} \right| &\ge \left| \int_0^1 \frac{u_1(t)}{t|f_1(t)|^4}\, dt \right| \\ &\ge \left| \int_0^{\delta_0} \frac{u_1(t)}{t|f_1(t)|^4}\, dt \right| - \left| \int_{\delta_0}^1 \frac{u_1(t)}{t|f_1(t)|^4}\, dt \right| \\ &\ge \frac{1}{2} \int_0^{\delta_0} \frac{dt}{t|f_1(t)|^4} - C_{\lambda,\delta_0}. \end{aligned}$$

Since

$$\int_0^{\delta_0} \frac{dt}{t|f_1(t)|^4} = +\infty,$$

this shows that

$$\int_0^1 \frac{dt}{tf_1^2(t)} = \infty.$$

Consequently,

$$\lim_{x\to 0^+} f_2(x) = \lim_{x\to 0^+} f_1(x) \int_1^x \frac{dt}{tf_1^2(t)} = \infty.$$

It is clear that f_1 and f_2 are two linearly independent functions in X_λ. Thus they form a basis for X_λ and we can write

$$g(\sqrt{x}) = C_1 f_1(x) + C_2 f_2(x)$$

on $(0, +\infty)$ for two constants C_1 and C_2. Since $g(\sqrt{x})$ and $f_1(x)$ are both bounded near $x = 0$ and $f_2(x)$ is unbounded near $x = 0$, we must have $C_2 = 0$. This shows that

$$g(x) = C_1 f_1(x^2) = C_1 g_\lambda(x), \qquad x \in (0, +\infty).$$

Let $x \to 0^+$ and use $g_\lambda(0) = 1$. We arrive at $g(0) = C_1$ and so $g = g(0)g_\lambda$. This completes the proof of the proposition. □

Proposition 5. *Suppose $\Delta f = \lambda f$ and $f_z(w) = f(z + w)$. Then*

$$R(f_z)(w) = f(z)g_\lambda(w)$$

for all z and w in $\mathbb{C}$.

Proof. It is clear that $\Delta(f_z) = (\Delta f)_z$, that is, the Laplace operator commutes with translations. Thus the condition $\Delta f = \lambda f$ implies $\Delta(f_z) = \lambda f_z$ for all $z \in \mathbb{C}$. By Lemma 2, we have $\Delta[R(f_z)] = \lambda R(f_z)$. The function $R(f_z)$ is radial, so by Proposition 4, we must have $R(f_z) = f(z)g_\lambda$ for all $z \in \mathbb{C}$. □

4. Fixed-points of the heat transform

In this section we study fixed-points of the heat tranforms H_t. The main result is the following, which was stated as part (a) of Theorem A in the introduction. This result can be found in [3, 15], but the proof here is new.

Theorem 6. *Suppose $f \neq 0$ satisfies condition (I_t) and $\Delta f = \lambda f$ for some $\lambda \in \mathbb{C}$. Then $H_t f = f$ if and only if $\lambda t = 2n\pi i$ for some integer n.*

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Proof. Since $\Delta f = \lambda f$, it follows from Proposition 5 that

$$\frac{1}{2\pi}\int_0^{2\pi} f(z+we^{i\theta})\,d\theta = f(z)g_\lambda(w)$$

for all z and w in $\mathbb{C}$. In particular,

$$\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta = f(z)\sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)^2}r^{2n}$$

for all $z \in \mathbb{C}$ and $r \geq 0$. Multiply both sides by $(r/t)e^{-r^2/t}$ and then integrate with respect to r from 0 to $+\infty$. We obtain

$$\frac{1}{2}H_t f(z) = f(z)\sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)^2 t}\int_0^{\infty} r^{2n+1}e^{-r^2/t}\,dr.$$

It follows from two changes of variables that

$$\begin{aligned} H_t f(z) &= f(z)\sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)^2 t}\int_0^{\infty} r^n e^{-r/t}\,dr \\ &= f(z)\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{(n!)^2}\int_0^{\infty} r^n e^{-r}\,dr \\ &= f(z)\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!} \\ &= e^{\lambda t} f(z). \end{aligned}$$

Since f is not identically zero, we see that $H_t f = f$ if and only if $\lambda t = 2n\pi i$ for some integer n. □

In the remainder of this section we will study several special classes of functions and characterize the fixed-points of the heat transform for these functions.

Proposition 7. *Suppose $f(z) = e^{a\overline{z}}h(z)$, where a is a constant and h is an entire function satisfying condition (I_t). Then $H_t f = f$ if and only if h is periodic with period ta.*

Proof. It is clear that h satisfies condition (I_t) if and only if f does. It is also clear that the result is true for $a = 0$. For $a \neq 0$, it follows from the reproducing property (3) that

$$\begin{aligned} H_t f(z) - f(z) &= \int_{\mathbb{C}} f(z+w)\,d\mu_t(w) - f(z) \\ &= \int_{\mathbb{C}} e^{a(\overline{z}+\overline{w})}h(z+w)\,d\mu_t(w) - e^{a\overline{z}}h(z) \end{aligned}$$

$$= e^{a\overline{z}}\left[\int_{\mathbb{C}} h(z+w)e^{(ta)\overline{w}/t}\,d\mu_t(w) - h(z)\right]$$

$$= e^{a\overline{z}}[h(z+ta) - h(z)]$$

for each $z \in \mathbb{C}$, which proves the desired result. □

The result above is a complement to Theorem 6. In fact, if

$$h(z) = e^{2n\pi iz/ta},$$

where $n \in \mathbb{Z}$, then h is a periodic entire function with period ta and by Example 1 the function $f(z) = e^{a\overline{z}}h(z)$ satisfies $\Delta f = (2n\pi i/t)f$ and $H_t f = f$. More can be said about functions of the form $f(z) = e^{a\overline{z}}h(z)$.

Proposition 8. *Suppose $f(z) = e^{a\overline{z}}h(z)$, where a is a nonzero constant and h is an entire function. Then $\Delta f = \lambda f$ if and only if $h(z) = Ce^{\lambda z/a}$ for some constant C.*

Proof. It is clear that $\Delta f = \lambda f$ if and only if $ae^{a\overline{z}}h'(z) = \lambda e^{a\overline{z}}h(z)$ for each $z \in \mathbb{C}$, that is, $h' = (\lambda/a)h$, which is equivalent to $h(z) = Ce^{\lambda z/a}$. □

Remark 9. *Since the fixed-point set of H_t is a vector space, it follows from the analysis above that any function of the form*

$$f(z) = \int_{\mathbb{C}} \exp\left(a\overline{z} + \frac{2n(a)\pi iz}{at}\right) d\mu(a)$$

is a fixed-point of H_t, where $n(a)$ is any integer-valued function of $a \in \mathbb{C}$ and μ is any complex Borel measure such that the integral above converges. Since the fixed-point set of H_t is closed under complex conjugation, by taking the complex conjugate, any function of the form

$$f(z) = \int_{\mathbb{C}} \exp\left(\overline{a}z - \frac{2n(a)\pi i\overline{z}}{\overline{a}t}\right) d\mu(a)$$

is also a fixed-point of H_t. Many more fixed-points of H_t can be constructed from the results obtained so far in this section.

By Theorem 6, eigenfunctions of the complex Laplacian $\Delta = \partial^2/\partial z\partial\overline{z}$ corresponding to eigenvalues of the form $\lambda = 2n\pi i/t$, where n is any integer, are all fixed-points of the heat transform H_t. When $n = 0$, the resulting functions are exactly the harmonic ones on the complex plane.

The next result, which is likely known to experts, shows that harmonic functions are very special in the study of fixed-points for the heat transform, although they are not the only fixed-points for any particular heat transform H_t.

Theorem 10. *Suppose f satisfies condition (I_t) for all $t > 0$. Then f is harmonic if and only if $H_t f = f$ for every $t > 0$.*

Proof. It is well known that if f is harmonic then $H_t f = f$ for all $t > 0$.

On the other hand, if $H_t f = f$ for all $t > 0$, then by the fact that the function $(t, z) \mapsto H_t f(z)$ solves the heat equation, we have

$$\Delta f(z) = \Delta H_t f(z) = \frac{\partial H_t f(z)}{\partial t} = \frac{\partial f(z)}{\partial t} = 0,$$

so f is harmonic. □

Part of Theorem 10 can be strengthened as follows: if the set

$$\{t > 0 : H_t f = f\}$$

has a positive accumulation point, then f must be harmonic on $\mathbb{C}$. To see this, suppose that there exists a sequence $\{\alpha_n\}$ of positive numbers such that $\alpha_n \to \alpha_0$, $\alpha_0 > 0$, as $n \to \infty$, and that $B_{\alpha_n} f = f$ for all $n \geq 1$. It follows from the fact that $B_\alpha f$ is continuous in α on $(0, \infty)$ that $B_{\alpha_0} f = f$. Skipping some routine details, we have

$$\begin{aligned} \Delta f(z) &= \Delta B_{\alpha_0} f(z) \\ &= -\alpha_0 B_{\alpha_0} f(z) + \frac{\alpha_0^3}{\pi} \int_{\mathbb{C}} f(w)|z-w|^2 e^{-\alpha_0|z-w|^2}\, dA(w) \\ &= -\alpha_0 B_{\alpha_0} f(z) - \alpha_0^3 \left.\frac{d[B_\alpha f(z)/\alpha]}{d\alpha}\right|_{\alpha=\alpha_0} \\ &= -\alpha_0 B_{\alpha_0} f(z) - \alpha_0^3 \lim_{n\to\infty} \frac{[B_{\alpha_n} f(z)/\alpha_n] - [B_{\alpha_0} f(z)/\alpha_0]}{\alpha_n - \alpha_0} \\ &= -\alpha_0 f(z) - \alpha_0^3 f(z) \lim_{n\to\infty} \frac{(1/\alpha_n) - (1/\alpha_0)}{\alpha_n - \alpha_0} \\ &= -\alpha_0 f(z) + \alpha_0 f(z) \\ &= 0 \end{aligned}$$

for all $z \in \mathbb{C}$. Therefore, f is harmonic on $\mathbb{C}$.

5. Recovery of the initial heat distribution

Recall that $H_t f(z) = u(z, t)$ is the solution of the initial-value problem (1) for the heat equation on the complex plane. Here $u(z, t)$ is the temperature at location $z \in \mathbb{C}$ and time $t > 0$. In this section we wish to recover the original temperature distribution from $u(z, t)$, namely, we want to consider the limit behavior of $H_t f(z)$ as $t \to 0$. We will consider pointwise convergence, uniform convergence, and uniform convergence on compact sets.

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If $f \in \mathcal{B}C$, then it is easy to show that $H_tf(z) \to f(z)$ pointwise as $t \to 0^+$. See [17, Theorem 3.14], where the proof actually shows that the convergence is uniform on compact subsets of $\mathbb{C}$. On the other hand, for $f \in \mathcal{B}C$, the convergence of H_tf to f may not be uniform, as the following example shows.

Example 11. *For the function $f(z) = e^{i|z|^2}$, which is clearly in $\mathcal{B}C$, we have $\|H_tf - f\|_\infty \not\to 0$ as $t \to 0^+$.*

Proof. We will find the heat transform H_tf in closed form.

$$\begin{aligned} H_tf(z) &= \frac{1}{\pi t}\int_{\mathbb{C}} e^{i|z+w|^2} e^{-|w|^2/t}\, dA(w) \\ &= \frac{1}{\pi t}\, e^{i|z|^2}\int_{\mathbb{C}} e^{i\overline{z}w} e^{iz\overline{w}} e^{-(1/t-i)|w|^2}\, dA(w). \end{aligned}$$

Let $\alpha = (1/t) - i$. Then

$$H_tf(z) = \frac{e^{i|z|^2}}{\alpha t}\,\frac{\alpha}{\pi}\int_{\mathbb{C}} e^{iw\overline{z}} e^{\alpha(iz/\alpha)\overline{w}} e^{-\alpha|w|^2}\, dA(w).$$

The function $g(w) = e^{iw\overline{z}}$ is entire and satisfies condition (I_t) for any $t > 0$. It follows from the reproducing formula in (3), where α is allowed to be complex, that

$$H_tf(z) = \frac{e^{i|z|^2}}{1 - it} e^{i(iz/\alpha)\overline{z}} = \frac{1}{1 - it} e^{i|z|^2/(1-it)}.$$

Therefore,

$$|H_tf(z)| = \frac{1}{\sqrt{1+t^2}} e^{-t|z|^2/(1+t^2)}. \tag{5}$$

If $H_tf(z) \to f(z)$ uniformly on $\mathbb{C}$ as $t \to 0^+$, there would exist a positive number δ such that $|H_tf(z)| > 1/2$ for all $z \in \mathbb{C}$ and $t \in (0, \delta)$, because $|f(z)| = 1$ for all z. This is clearly not possible, as (5) shows that for any given $t > 0$ we actually have $|H_tf(z)| \to 0$ as $|z| \to \infty$. □

Part (a) of the result below shows that the boundedness assumption on f in (3.11) of [17] can be dropped.

Theorem 12. *Suppose f satisfies condition (I_{t_0}) for some $t_0 > 0$. Then the following two conditions are equivalent:*

(a) *$H_tf(z) \to f(z)$ uniformly on compacta as $t \to 0^+$.*
(b) *f is continuous on $\mathbb{C}$.*

Proof. If f satisfies condition (I_{t_0}), then the heat transform $H_tf(z)$ is well defined for all $t \in (0, t_0)$ and $z \in \mathbb{C}$. It is also clear that the function H_tf is continuous on $\mathbb{C}$ for each $t \in (0, t_0)$. Thus (a) implies (b).

To show that (b) implies (a), we assume that f is continuous on $\mathbb{C}$. We fix an $R > 0$ and proceed to show that $H_t f(z) \to f(z)$ uniformly for $|z| \le R$ as $t \to 0^+$.

Given any positive ε, by the uniform continuity of f on $|z| \le R + 1$, we can find some $\delta \in (0, 1)$ such that $|f(z + w) - f(z)| < \varepsilon$ for all z and w satisfying $|z| \le R$ and $|w| < \delta$. Write

$$\begin{aligned} H_t f(z) - f(z) &= \int_{\mathbb{C}} [f(z + w) - f(z)]\, d\mu_t(w) \\ &= \int_{|w|<\delta} + \int_{|w|\ge\delta} =: I_1(z) + I_2(z). \end{aligned} \tag{6}$$

We have

$$|I_1(z)| \le \int_{|w|<\delta} |f(z + w) - f(z)|\, d\mu_t(w) \le \varepsilon \int_{\mathbb{C}} d\mu_t(w) = \varepsilon \tag{7}$$

for all $|z| \le R$ and $t \in (0, t_0)$.

Let $C_1 = \sup\{|f(z)| : |z| \le R\}$. Then for all $t \in (0, t_0)$ we have

$$\begin{aligned} |I_2(z)| &\le \frac{1}{\pi t} \int_{|w|\ge\delta} |f(z + w) - f(z)| e^{-|w|^2/t}\, dA(w) \\ &\le \frac{1}{\pi t} \int_{|w|\ge\delta} |f(z + w)| e^{-|w|^2/t}\, dA(w) + \frac{C_1}{\pi t} \int_{|w|\ge\delta} e^{-|w|^2/t}\, dA(w) \\ &= \frac{t_0}{t} \int_{|w|\ge\delta} |f(z + w)| e^{-|w|^2(1/t-1/t_0)}\, d\mu_{t_0}(w) + C_1 e^{-\delta^2/t} \\ &\le \frac{t_0}{t} e^{-\delta^2(1/t-1/t_0)} \int_{\mathbb{C}} |f(z + w)|\, d\mu_{t_0}(w) + C_1 e^{-\delta^2/t}. \end{aligned}$$

By condition (I_{t_0}) and the fact that $H_{t_0}(|f|)$ is continuous on $\mathbb{C}$, there exists an other positive constant C_2 such that

$$\int_{\mathbb{C}} |f(z + w)|\, d\mu_{t_0}(\omega) = H_{t_0}(|f|)(z) \le C_2$$

for all $|z| \le R$. Thus

$$|I_2(z)| \le \frac{C_2 t_0}{t}\, e^{-\delta^2(1/t-1/t_0)} + C_1 e^{-\delta^2/t}$$

for all $|z| \le R$ and $t \in (0, t_0)$, which clearly shows that $I_2(z) \to 0$ uniformly for $|z| \le R$ as $t \to 0^+$. So we can find some $\sigma \in (0, \delta)$ such that $|I_2(z)| < \varepsilon$ for all $|z| \le R$ and $t \in (0, \sigma)$. This along with (6) and (7) gives $|H_t f(z) - f(z)| \le 2\varepsilon$ for all $|z| \le R$ and $t \in (0, \sigma)$. Therefore, $H_t f(z) \to f(z)$ uniformly for $|z| \le R$ as $t \to 0^+$, which completes the proof of the theorem. □

For any $p \geq 1$ we let $\mathcal{BMO}^p$ denote the space of all Lebesgue measurable functions f on $\mathbb{C}$ such that

$$\sup_{z\in\mathbb{C}} \frac{1}{\pi}\int_{D(z)} |f(w)-\widehat{f}(z)|^p\, dA(w) < \infty,$$

where

$$\widehat{f}(z) = \frac{1}{\pi}\int_{D(z)} f(w)\, dA(w)$$

is the mean of f over the Euclidean disk

$$D(z) = \{w \in \mathbb{C} : |z-w| < 1\}.$$

It is well known that $\mathcal{BMO}^p$ is a Banach space. Furthermore, for any fixed $\alpha > 0$, we have $f \in \mathcal{BMO}^p$ if and only if

$$\sup_{z\in\mathbb{C}} \frac{\alpha}{\pi}\int_{\mathbb{C}} |f(w+z)-B_\alpha f(z)|^p\, e^{-\alpha|w|^2}\, dA(w) < \infty.$$

See [17] for more details.

We will need the following version of Theorem 3.35 in [17] for the case $p = 1$.

Theorem 13. *Suppose $\alpha > 0$ and $f \in \mathcal{BMO}^1$. Then there exists a constant $C_\alpha > 0$ such that*

$$|B_\alpha f(z) - B_\alpha f(w)| \leq C_\alpha \|f\|_{\mathcal{BMO}^1} |z-w|$$

for all $z, w \in \mathbb{C}$, where

$$\|f\|_{\mathcal{BMO}^1} = \sup_{z\in\mathbb{C}} \frac{\alpha}{2\pi}\int_{\mathbb{C}} |f(w+z)-B_{\alpha/2}f(z)|\, e^{-\alpha|z|^2/2}\, dA(w).$$

Proof. Fix any $z \in \mathbb{C}$ and fix any directional parameter θ. Let $\gamma(t) = z + te^{i\theta}$, where $t \in [0,\infty)$. By the proof of Theorem 3.35 in [17], we have

$$\frac{d}{dt}B_\alpha f(\gamma(t)) = -\frac{2\alpha^2}{\pi}\int_{\mathbb{C}} f(\zeta)e^{-\alpha|\gamma(t)-\zeta|^2}\, \mathrm{Re}\,[\gamma'(t)(\overline{\gamma(t)}-\overline{\zeta})]\, dA(\zeta),$$

and

$$\frac{\alpha}{\pi}\int_{\mathbb{C}} e^{-\alpha|\gamma(t)-\zeta|^2}\, \mathrm{Re}\,[\gamma'(t)(\overline{\gamma(t)}-\overline{\zeta})]\, dA(\zeta) = 0.$$

Thus $\frac{d}{dt}B_\alpha f(\gamma(t))$ is equal to

$$-\frac{2\alpha^2}{\pi}\int_{\mathbb{C}} [f(\zeta)-B_{\alpha/2}f(\gamma(t))]e^{-\alpha|\gamma(t)-\zeta|^2}\, \mathrm{Re}\,[\gamma'(t)(\overline{\gamma(t)}-\overline{\zeta})]\, dA(\zeta).$$

It is clear that the function

$$g(z) = |z|e^{-\frac{\alpha}{2}|z|^2}$$

is bounded on $\mathbb{C}$ and

$$\left|\mathrm{Re}\,[\gamma'(t)(\overline{\gamma(t)}-\overline{\zeta})]\right| \le |\gamma(t)-\zeta|$$

for all $t > 0$ and $\zeta \in \mathbb{C}$. Thus there exists a constant $C_\alpha > 0$ such that

$$\left|\mathrm{Re}\,[\gamma'(t)(\overline{\gamma(t)}-\overline{\zeta})]\right| e^{-\frac{\alpha}{2}|\gamma(t)-\zeta|^2} \le C_\alpha$$

for all $t > 0$ and $\zeta \in \mathbb{C}$. Therefore,

$$\left|\frac{d}{dt}B_\alpha f(\gamma(t))\right| \le 4\alpha C_\alpha \|f\|_{\mathcal{BMO}^1}$$

for all $t > 0$. Integrating with respect to t, we obtain

$$|B_\alpha f(z) - B_\alpha f(w)| \le 4\alpha C_\alpha \|f\|_{\mathcal{BMO}^1} |z-w|$$

for all $z, w \in \mathbb{C}$. □

Theorem 13 was first proved for the case $p = 2$ in [2]. As a consequence of Theorem 13, we obtain the following result about *uniform recovery* of the initial heat distribution from the solution $H_t f(z)$ for the heat equation.

Theorem 14. *A function f on $\mathbb{C}$ belongs to $\mathcal{UC}$ if and only if $f \in \mathcal{BMO}^1$ and $\|H_t f - f\|_\infty \to 0$ as $t \to 0^+$.*

Proof. If $f \in \mathcal{UC}$, then it is easy to see that $f \in \mathcal{BMO}^1$. Also, by Proposition 3.2 of [4], we have $\|H_t f - f\|_\infty \to 0$ as $t \to 0^+$.

On the other hand, if $f \in \mathcal{BMO}^1$, then by Theorem 13, $H_t f \in \mathcal{UC}$ for every $t > 0$. It is clear that $\mathcal{UC}$ is closed in $L^\infty(\mathbb{C})$. Thus $\|H_t f - f\|_\infty \to 0$ as $t \to 0^+$ implies that $f \in \mathcal{UC}$. □

Remark 15. *It is also true that, for any fixed $p \ge 1$, a function f belongs to $\mathcal{UC}$ if and only if $f \in \mathcal{BMO}^p$ and $\|H_t f - f\|_\infty \to 0$ as $t \to 0^+$. The proof for $p > 1$ is the same, with Theorem 13 replaced by Theorem 3.35 in* [17]. *The point here is that Theorem 3.35 of* [17] *was only proved for $p > 1$, and we wish to record the result for $p = 1$ in this paper, just in case it is needed in the future.*

Finally in this section we consider the uniform recovery of the initial heat distribution in the case of subharmonic functions. This will be based on an elementary averaging function and a certain difference function. Thus for any $r > 0$ and $t > 0$ we define

$$M_r f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\, d\theta, \qquad z \in \mathbb{C},$$

and

$$D_{r,t}f(z) = \int_{|w|<r} [f(z+w) - f(z)]\, d\mu_t(w), \qquad z \in \mathbb{C}.$$

Lemma 16. *Suppose $t > 0$ and f satisfies condition (I_{2t}). If f is real-valued and subharmonic, then*

$$0 \le H_t(M_r f)(z) - f(z) \le \frac{e^{r^2/t}}{\pi t} \int_{\mathbb{C}} [f(z+w) - f(z)]\, e^{-|w|^2/(2t)}\, dA(w)$$

for all $r > 0$ and all $z \in \mathbb{C}$.

Proof. Apply Fubini's theorem (several times), make a change of variables, and then use polar coordinates. We have

$$\begin{aligned}
&H_t(M_r f)(z) - f(z) \\
&= \frac{1}{\pi t} \int_{\mathbb{C}} [M_r f(z+w) - f(z)] e^{-|w|^2/t}\, dA(w) \\
&= \frac{1}{\pi t} \int_{\mathbb{C}} [f(z+w) - f(z)]\, dA(w)\, \frac{1}{2\pi} \int_0^{2\pi} e^{-|w - re^{i\theta}|^2/t}\, d\theta \\
&= \frac{e^{-r^2/t}}{\pi t} \int_{\mathbb{C}} [f(z+w) - f(z)] e^{-|w|^2/t} \left[\sum_{n=0}^{\infty} \frac{(r^2|w|^2/t^2)^n}{(n!)^2}\right] dA(w) \\
&= \frac{2e^{-r^2/t}}{t} \int_0^{\infty} \rho e^{-\rho^2/t} \left[\sum_{n=0}^{\infty} \frac{(r^2\rho^2/t^2)^n}{(n!)^2}\right] \frac{d\rho}{2\pi} \int_0^{2\pi} [f(z + \rho e^{i\theta}) - f(z)]\, d\theta.
\end{aligned}$$

Since f is subharmonic,

$$\frac{1}{2\pi} \int_0^{2\pi} [f(z + \rho e^{i\theta}) - f(z)]\, d\theta \ge 0$$

for all $\rho \ge 0$ and $z \in \mathbb{C}$. This shows that

$$H_t(M_r f)(z) - f(z) \ge 0$$

for all $r > 0$ and $z \in \mathbb{C}$.

It is obvious that

$$\sum_{n=0}^{\infty} a_n b_n \le \sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n$$

for any two sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers. It follows that

$$\sum_{n=0}^{\infty} \frac{(r^2\rho^2/t^2)^n}{(n!)^2} \le \sum_{n=0}^{\infty} \frac{(2r^2/t)^n}{n!} \sum_{n=0}^{\infty} \frac{(\rho^2/2t)^n}{n!} = e^{2r^2/t} e^{\rho^2/(2t)}.$$

Combining this with the calculation in the previous paragraph, we obtain

$$\begin{aligned} H_t(M_rf)(z) - f(z) &\le \frac{e^{r^2/t}}{\pi t}\int_0^\infty \rho e^{-\rho^2/(2t)}\,d\rho\int_0^{2\pi}[f(z+\rho e^{i\theta})-f(z)]\,d\theta \\ &= \frac{e^{r^2/t}}{\pi t}\int_{\mathbb{C}}[f(z+w)-f(z)]e^{-|w|^2/(2t)}\,dA(w). \end{aligned}$$

This completes the proof of the lemma. □

Theorem 17. *Suppose f is a real-valued subharmonic function on $\mathbb{C}$. If f satisfies condition (I_{t_0}) for some $t_0 > 0$, then the following conditions are equivalent:*

(a) $\|H_tf - f\|_\infty \to 0$ *as* $t \to 0^+$.
(b) $H_sf - f \in \mathcal{BC}$ *for some* $s > 0$ *and* $\|M_rf - f\|_\infty \to 0$ *as* $r \to 0^+$.
(c) $H_sf - f \in \mathcal{BC}$ *for some* $s > 0$ *and* $\sup_{t>0}\|D_{r,t}f\|_\infty \to 0$ *as* $r \to 0^+$.

Proof. We are going to show that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

First assume that (a) holds. Since each H_tf is continuous on $\mathbb{C}$ and the uniform limit of continuous functions is still continuous, we see that $H_tf - f \in \mathcal{BC}$ for all sufficiently small positive t. By the triangle inequality, we have

$$|M_rf(z) - f(z)| \le |M_rf(z) - H_t(M_rf)(z)| + |H_t(M_rf)(z) - f(z)| \quad (8)$$

for all $r > 0$, $t \in (0, t_0/2)$, and $z \in \mathbb{C}$. It follows from Fubini't theorem that $H_t(M_rf) = M_r(H_tf)$. Thus

$$\begin{aligned} |M_rf(z) - H_t(M_rf)| &= |M_rf(z) - M_r(H_tf)(z)| \\ &= |M_r(f - H_tf)(z)| \\ &\le \|f - H_tf\|_\infty. \end{aligned} \quad (9)$$

On the other hand, by Lemma 16, we have

$$\begin{aligned} |H_t(M_rf)(z) - f(z)| &\le \frac{e^{r^2/t}}{\pi t}\int_{\mathbb{C}}[f(z+w)-f(z)]e^{-|w|^2/(2t)}\,dA(w) \\ &= 2e^{r^2/t}\,[H_{2t}f(z) - f(z)] \\ &\le 2e^{r^2/t}\|H_{2t}f - f\|_\infty. \end{aligned} \quad (10)$$

Combining (8), (9), and (10), we obtain

$$|M_rf(z) - f(z)| \le \|H_tf - f\|_\infty + 2e^{r^2/t}\|H_{2t}f - f\|_\infty.$$

Take the supremum over $z \in \mathbb{C}$ and let $r \to 0^+$. The result is that

$$\limsup_{r\to 0^+}\|M_rf - f\|_\infty \le \|H_tf - f\|_\infty + 2\|H_{2t}f - f\|_\infty$$

for all $t > 0$. Let $t \to 0$ and use condition (a). We obtain

$$\lim_{r\to 0^+} \|M_r f - f\|_\infty = 0,$$

which proves that (a) implies (b).

Next we assume that condition (b) holds. By polar coordinates,

$$\begin{aligned} D_{r,t}f(z) &= \int_{|w|<r} [f(z+w) - f(z)]\, d\mu_t(w) \\ &= \frac{2}{t}\int_0^r \rho e^{-\rho^2/t}\, d\rho\, \frac{1}{2\pi}\int_0^{2\pi} [f(z+\rho e^{i\theta}) - f(z)]\, d\theta. \end{aligned}$$

Since f is subharmonic, we have

$$0 \le D_{r,t}f(z) \le \frac{2}{t}\int_0^r \rho e^{-\rho^2/t} \|M_r f - f\|_\infty\, d\rho \le \|M_r f - f\|_\infty.$$

Take the supremum over z and t. We obtain

$$\sup_{t>0} \|D_{r,t}f\|_\infty \le \|M_r f - f\|_\infty,$$

which shows that (b) implies (c).

Finally we assume that condition (c) holds. By the sub-mean value property for subharmonic functions,

$$\begin{aligned} 0 &\le H_t f(z) - f(z) \\ &= \frac{1}{\pi t}\int_{\mathbb{C}} [f(z+w) - f(z)] e^{-|w|^2/t}\, dA(w) \\ &= \frac{1}{\pi t}\int_{|w|<r} [f(z+w) - f(z)] e^{-|w|^2/t}\, dA(w) \\ &\qquad + \frac{1}{\pi t}\int_{|w|>r} [f(z+w) - f(z)] e^{-|w|^2/t}\, dA(w) \\ &\le D_{r,t}f(z) + \frac{e^{-r^2/(2t)}}{\pi t}\int_{\mathbb{C}} [f(z+w) - f(z)] e^{-|w|^2/(2t)}\, dA(w) \\ &\le D_{r,t}f(z) + \frac{s e^{-r^2/(2t)}}{t}[H_s f(z) - f(z)] \end{aligned}$$

for all $z \in \mathbb{C}$, $r > 0$, and $t \in (0, s/2)$. This implies that

$$\|H_t f - f\|_\infty \le \sup_{t>0} \|D_{r,t}f\|_\infty + \frac{s e^{-r^2/(2t)}}{t}\, \|H_s f - f\|_\infty$$

for all $r > 0$ and $t \in (0, s/2)$. Given any $\varepsilon > 0$ we use condition (c) to find some $r_0 > 0$ such that $\sup_{t>0} \|D_{r_0,t}\|_\infty < \varepsilon$. Then

$$\|H_t f - f\|_\infty \le \varepsilon + \frac{se^{-r_0^2/(2t)}}{t} \|H_s f - f\|_\infty$$

for all $t \in (0, s/2)$. Letting $t \to 0^+$ gives

$$\limsup_{t\to 0^+} \|H_t f - f\|_\infty \le \varepsilon.$$

Since ε is arbitrary, we must have $\|H_t f - f\|_\infty \to 0$ as $t \to 0^+$. This shows that (c) implies (a). The proof of the theorem is now complete. □

We illustrate Theorem 17 above with an example.

Example 18. *Let $f(z) = |z|^p$, $0 \le p < \infty$. Then*

$$\lim_{t\to 0^+} \|H_t f - f\|_\infty = 0$$

if and only if $0 \le p \le 2$.

Proof. For $z = |z|e^{it}$ and $r > 0$, we have

$$\begin{aligned} M_r f(z) - f(z) &= \frac{1}{2\pi} \int_0^{2\pi} |z + re^{i\theta}|^p \, d\theta - |z|^p \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[|z|^2 + r^2 + 2|z|r\cos(\theta - t)\right]^{\frac{p}{2}} \, d\theta - |z|^p \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[|z|^2 + r^2 + 2|z|r\cos\theta\right]^{\frac{p}{2}} \, d\theta - |z|^p. \end{aligned}$$

If $0 \le p \le 2$, it follows from Hölder's inequality (see [16, Theorem 3.2]) that

$$\begin{aligned} 0 &\le M_r f(z) - f(z) \\ &\le \left[\frac{1}{2\pi} \int_0^{2\pi} (|z|^2 + r^2 + 2|z|r\cos\theta)\, d\theta\right]^{\frac{p}{2}} - |z|^p \\ &= (|z|^2 + r^2)^{\frac{p}{2}} - |z|^p. \end{aligned}$$

By the discrete version of Hölder's inequality (see [16, Theorem 3.3]),

$$0 \le M_r f(z) - f(z) \le |z|^p + r^p - |z|^p = r^p$$

for all $r > 0$ and $z \in \mathbb{C}$. This shows that $\|M_r f - f\|_\infty \to 0$ as $r \to 0^+$. Also, by polar coordinates and the estimate above,

$$0 \le H_t f(z) - f(z) = \int_{\mathbb{C}} [f(z + w) - f(z)]\, d\mu_t(w)$$

$$
\begin{aligned}
&= \frac{1}{\pi t}\int_0^\infty re^{-r^2/t}\,dr\int_0^{2\pi}[f(z+re^{i\theta})-f(z)]\,d\theta \\
&= \frac{2}{t}\int_0^\infty re^{-r^2/t}[M_rf(z)-f(z)]\,dr \\
&\le \frac{2}{t}\int_0^\infty r^{p+1}e^{-r^2/t}\,dr<\infty
\end{aligned}
$$

for all $z\in\mathbb{C}$ and $t>0$. Thus $H_tf-f\in\mathcal{BC}$ for every $t>0$. By Theorem 17, we have $\|H_tf-f\|_\infty\to 0$ as $t\to 0^+$.

On the other hand, if $p>2$, it follows from (the most common version of) Hölder inequality that

$$
\begin{aligned}
M_rf(z)-f(z) &= \frac{1}{2\pi}\int_0^{2\pi}\left(|z|^2+r^2+2|z|r\cos\theta\right)^{\frac{p}{2}}\,d\theta-|z|^p \\
&\ge \left[\frac{1}{2\pi}\int_0^{2\pi}(|z|^2+r^2+2|z|r\cos\theta)\,d\theta\right]^{\frac{p}{2}}-|z|^p \\
&= (|z|^2+r^2)^{\frac{p}{2}}-|z|^p \\
&= |z|^p\left[\left(1+\frac{r^2}{|z|^2}\right)^{\frac{p}{2}}-1\right].
\end{aligned}
$$

By elementary calculus, $(1+x)^\alpha\ge 1+\alpha x$ for all $x\ge 0$ and $\alpha\ge 1$. Thus

$$
M_rf(z)-f(z)\ge |z|^p\,\frac{p}{2}\,\frac{r^2}{|z|^2}=\frac{p}{2}\,|z|^{p-2}r^2
$$

for all $z\in\mathbb{C}$ and $r>0$. This clearly implies that $M_rf(z)$ does not converge to $f(z)$ uniformly on $\mathbb{C}$ as $r\to 0^+$. By Theorem 17, $\|H_tf-f\|_\infty\not\to 0$ as $r\to 0^+$. □

It is easy to see that every function $f\in\mathcal{UC}$ satisfies conditions (b) and (c) in Theorem 17. Thus Theorem 17 also holds for functions of the form $f=f_1+f_2$, where f_1 is subharmonic and f_2 is uniformly continuous on $\mathbb{C}$.

References

[1] P. Ahern, M. Flores, and W. Rudin, An invariant volume-mean-value property, *J. Funct. Anal.* **111** (1993), 380-397.

[2] W. Bauer, Mean oscillation and Hankel operators on the Segal-Bargmann space, *Integral Equations Operator Theory* **52** (2005), 1-15.

[3] W. Bauer, B. Choe, and H. Koo, Commuting Toeplitz operators with pluriharmonic symbols on the Fock space, *J. Funct. Anal.* **268** (2015), 3017-3060.

[4] W. Bauer and L.A. Coburn, Heat flow, weighted Bergman spaces and real analytic Lipschitz approximation, *J. Reine Angew. Math.* **703** (2015), 225-246.
[5] W. Bauer and L.A. Coburn, Toeplitz operators with uniformly continuous symbols, *Integral Equations Operator Theory* **83** (2015), 24-34.
[6] W. Bauer and L.A. Coburn, Uniformly continuous functions and quantization on the Fock space, *Bol. Soc. Mat. Mexicana* **22** (2016), 669-677.
[7] W. Bauer, L.A. Coburn, and R. Hagger, Toeplitz quantization on Fock space, *J. Funct. Anal.* **274** (2018), 3531-3551.
[8] F.A. Berezin, Covariant and contravariant symbols of operators, *Math. USSR-Izv.* **6** (1972), 1117-1151.
[9] F.A. Berezin, Quantization. *Math. USSR-Izv.* **8** (1974), 1109-1163.
[10] F.A. Berezin, Quantization in complex symmetric spaces, *Math. USSR-Izv.* **9** (1975), 341-379.
[11] F.A. Berezin, General concept of quantization, *Comm. Math. Phys.* **40** (1975), 153-174.
[12] F.A. Berezin, The relation between co- and contra-variant symbols of operators on classical complex symmetric spaces, *Soviet Math. Dokl.* **19** (1978), 786-789.
[13] C. Berger and L.A. Coburn, Heat flow and Berezin-Toeplitz estimates, *Amer. J. Math.* **116** (1994), 563-590.
[14] I. Casseli, Fixed points of the Berezin tranform of multidimensional polyanalytic Fock spaces, *J. Math. Anal. Appl.* **481** (2020), no.2, 123479, 15 pp.
[15] M. Englis, Functions invariant under the Berezin transform, *J. Funct. Anal.* **121** (1994), 233-254.
[16] K. Zhu, *Operator Theory in Function Spaces*, 2nd edition, American Mathematical Society, Providence, RI, 2007.
[17] K. Zhu, *Analysis on Fock Spaces*, GTM **263**, Springer Verlag, New York, 2012.

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