Higher connected stable ranks and their rational variants of AF algebras

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Abstract

We study the $k(\geq 1)$-connected stable rank and the k-homotopy stabilization rank ([8]) and their rational homotopy variants of AF algebras. We prove that, for each odd integer k, the rational k-connected stable rank (the rational k-homotopy stabilization rank resp.) of an AF algebra is equal to the k-connected stable rank (the k-homotopy stabilization rank resp.) and also characterize the condition that the (rational) k-connected stable rank of an AF algebra A is at most m in terms of the Bratteli diagram of A. These ranks of AF algebras for even integer k are also studied. They are k-connected stable rank-counterparts of the (rational) K-stabilty theorem for AF algebras due to Seth and Vaidyanathan [13]. Our proof applies the proof scheme and the results of [13].

1 Introduction and Main theorem

The notion of the connected stable rank $\mathrm{csr}(A)$ of a C^*- algebra A was introduced by Rieffel and was applied to compute the homotopy groups of the unitaries of noncommutative tori [11], [12]. Its higher dimensional analogue $\mathrm{csr}_k(A)$, the k-connected stable rank, and its variant $\mathrm{hsr}_k(A)$, the k-homotopy stabilization rank, were introduced and studied by Nica in [8], [9], and more recently by Vaidyanathan and Nirbhay-Vaidyanathan in [16] and [10]. When A is unital, they are defined as follows. For an integer $n \geq 1$, let

$$\begin{aligned}
\mathrm{csr}_k(A) &= \min\{m \mid \mathrm{Lg}_n(A) \text{ is } k\text{-connected for each } n \geq m\},\\
\mathrm{hsr}_k(A) &= \min\{m \mid \text{the homomorphism } \pi_j(\mathrm{GL}_n(A)) \to \pi_j(\mathrm{GL}_{n+1}(A))\\
&\qquad \text{induced by the canonical inclusion is an isomorphism}\\
&\qquad \text{for each } j = 0, \dots, k \text{ and each } n \geq m\},
\end{aligned}$$

where $\mathrm{Lg}_n(A)$ denotes the subset of the n-fold prodcut A^n of A defined by $\mathrm{Lg}_n(A) := \{(a_1, \dots, a_n) \in A^n \mid \text{there exists } (b_1, \dots, b_n) \in A^n \text{ such that } \sum_{i=1}^n b_i a_i = 1\}$. In particular $\mathrm{csr}(A) = \mathrm{csr}_0(A)$. See the next section for the definition for a general (not necessarily unital) C^*-algebra. A closely related notion is that of K-stability introduced by Thomsen [15]: a C^*-algebra A is said to be *K-stable* if $\mathrm{hsr}_k(A) = 1$ for for each $k \geq 0$, in

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other words (for unital A), for each $n \geq 1$, the canonical inclusion $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A)$ induces an isomorphism between the k-th homomotopy groups of these groups. The irrational rotation algebra, Cuntz algebras and infinite dimensional simple AF algebras are such examples. For each such algebra we have $\mathrm{csr}_k(A) \leq 2$ and the homotopy group $\pi_k(\mathrm{GL}_n(A))$ is isomorphic to the K-group $K_i(A)$ with $i + k \equiv 1 \pmod 2$ for each $n \geq 1$ and each $k \geq 0$.

These notions have rational homotopy counterpart: the rational K-stability has been introduced and studied in [2] and [13] and also the rational k-connected stable rank $\mathrm{csr}_k^{\mathbb{Q}}(A)$ of a Banach algebra A was defined and studied for commutative C^*-algebras in [4]. In view of the fact that the structure of the rational homotopy groups of unitary groups is much simpler than that of their homotopy groups, the following theorem is somewhat surprising:

Theorem 1.1. *[13] Let A is an AF algebra. The following conditions are equivalent.*

(1) A is K-stable.

(2) A is rationally K-stable.

(3) If B is a finite dimensional C^*-algebra, then every $*$-homomorphism $A \to B$ of A to B is trivial.

The present paper proves a k-connected stable rank/k-homotopy stablization rank-counterpart of the above theorem by applying the proof scheme and the results of the paper [13].

In order to state main results, we first notice that every AF algebra has the connected stable rank one: $\mathrm{csr}_0(A) = 1$ for each AF algebra A([9, Example 11.4]). The complex matrix algebra of size ν is denoted by $M_\nu(\mathbb{C})$. Every finite dimensional C*-algebra F is isomorphic to the direct sum of finitely many matrix algebras. Let $\mathcal{N}(F)$ be the multiset of the sizes of the matrix algebras that form the direct summands of F so that:

$$F \cong \oplus_{\nu \in \mathcal{N}(F)} M_\nu(\mathbb{C}).$$

The first main result is on the above ranks for an odd integer k. It will be shown in Theorem 2.15 that $\mathrm{csr}_k(A) \leq \mathrm{hsr}_k(A) + 1 \leq \lceil \frac{k+1}{2} \rceil + 1$. The undefined notation is explained in the next section.

Main Theorem 1. *Let A be an AF algebra. For an odd integer $k \geq 1$ and an integer m with $1 \leq m \leq \frac{k+1}{2}$, the following conditions are equivalent.*

(1) $\mathrm{hsr}_k(A) \leq m$.

(1a) $\mathrm{hsr}_k^{\mathbb{Q}}(A) \leq m$.

(2) $\mathrm{csr}_k(A) \leq m + 1$.

(2a) $\mathrm{csr}_k^{\mathbb{Q}}(A) \leq m + 1$.

(3) There exists an inductive sequence $\{A_p, \varphi_{qp} : A_p \to A_q\}$ of finite dimensional C^*-algebras and injective $*$-homomorphisms with $A = \varinjlim(A_p, \varphi_{qp})$ such that $\nu \geq \frac{k+1}{2m}$ for each $\nu \in \mathcal{N}(A_p)$ and each $p \geq 1$.

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(4) For each integer $\nu \geq 1$ with $\nu < \frac{k+1}{2m}$, every $*$-homomorphism $A \to M_\nu(\mathbb{C})$ is trivial.

The second main theorem deals with the k-homotopy stabilization rank for an even integer k. It will be shown in Theorem 2.15 that $\mathrm{csr}_k^{\mathbb{Q}}(A) = \mathrm{csr}_{k-1}^{\mathbb{Q}}(A)$ and $\mathrm{csr}_k^{\mathbb{Q}}(A) = \mathrm{csr}_{k-1}^{\mathbb{Q}}(A)$, hence the estimate of these rational ranks reduces to that of the ranks for an odd integer. Also, by [8, Proposition 32], $\mathrm{csr}_k(A) \leq \mathrm{hsr}_k(A) + 1$ and equality does not hold for general AF algebra A (see Theorem 2.15 for $k = 2$). According to a table in [5, p.254-255], $\pi_{2n+j}(U(n)) \neq 0$ for each nonnegative integer $j \leq 9$, where $U(n)$ is the group of unitaries of $M_n(\mathbb{C})$. It follows from this that every even integer k with $4 \leq k \leq 12$ satisfies the hypothesis $(*)$ of the next theorem.

Main Theorem 2. *Let A be an AF algebra and let $k \geq 4$ be an even integer and let m be an integer such that $1 \leq m \leq \frac{k}{2} + 1$. Assume that*

$$(*) \quad \pi_k(U(n)) \neq 0 \textit{ for each } n \textit{ with } 2 \leq n \leq \frac{k}{2}.$$

Then the following conditions are equivalent.

(1) $\mathrm{hsr}_k(A) \leq m$.

(2)(2.1) $\pi_j(\mathrm{GL}_n(A)) = 0$ *for each $n \geq m$ and for each even integer $j \leq k$, and*

(2.2) the canonical inclusion induces an isomorphism $\pi_j(\mathrm{GL}_n(A)) \to \pi_j(\mathrm{GL}_{n+1}(A))$ for each $n \geq m$ and for each odd integer $j < k$.

(3) There exists an inductive sequence (A_p, φ_{qp}) of finite dimensional C^*-algebras and injective $*$-homomorphisms with $A = \varinjlim(A_p, \varphi_{qp})$ such that $\nu \geq \frac{k+2}{2m}$ for each $\nu \in \mathcal{N}(A_p)$.

(4) For each integer $\nu \geq 1$ with $\nu < \frac{k+2}{2m}$, every $*$-homomorphism $A \to M_\nu(\mathbb{C})$ is trivial.

2 Preliminaries and auxiliary results

This section recalls some definitions and proves some auxiliary results.

2.1 The k-connected stable rank and the k-homotopy stabilization rank of a C^*-algebra

In order to deal with a (not necessarily unital) C^*-algebra A, we follow [15] and [6] to introduce the spaces $\mathrm{GL}_m^+(A)$, $\mathrm{Lg}_m^+(A)$ and $\mathrm{Lc}_m(A)$. The unitization of A is denoted by A^+ with the unit 1_{A^+}. The C^*-algebra of all $m \times m$ matrices with entries in A is denoted by $M_m(A)(\cong M_m(\mathbb{C}) \otimes A)$. When the algebra A has the unit, the identity matrix is denoted by E_m.

For two elements $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ of the m-fold product A^m of A, let

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{m} a_i b_i. \tag{2.1}$$

The column vector ${}^t(0, \ldots, 0, 1_{A^+}) \in (A^+)^m$ is denoted by $\mathbf{e}_m$.

Definition 2.1. *Let A be a C^*-algebra and let $m \geq 1$ be an integer. We make the following definitions.*

(1) Assume that A is unital.

(1.1) $\mathrm{GL}_m(A)$ *denotes the group of all invertible elements of* $M_m(A)$.

(1.2) $\mathrm{Lg}_m(A) := \{\mathbf{x} \in A^m \mid$ *there exists* $\mathbf{b} \in A^m$ *such that* $\langle \mathbf{b}, \mathbf{x} \rangle = 1\}$.

(2) For a (not-necessarily unital) C^*-algebra A, let:.

(2.1) $$\mathrm{GL}_m^+(A) := \{a \in \mathrm{GL}_m(A^+) \mid a - E_m \in M_m(A)\} = (E_m + M_m(A)) \cap \mathrm{GL}_m(A^+),$$

(2.2) $$\mathrm{Lg}_m^+(A) = \{\mathbf{x} \in \mathrm{Lg}_m(A^+) \mid \mathbf{x} - \mathbf{e}_m \in A^m\} = (\mathbf{e}_m + A^m) \cap \mathrm{Lg}_m(A^+).$$

(3) $\mathrm{Lc}_m(A) = \mathrm{GL}_m^+(A) \cdot \mathbf{e}_m = \{a \cdot \mathbf{e}_m \mid a \in \mathrm{GL}_m^+(A)\}$.

If A is unital, the isomorphism $A^+ \cong A \oplus \mathbb{C}$ induces isomorphisms $M_n(A^+) \cong M_n(A) \oplus M_n(\mathbb{C})$, $(A^+)^n \cong A^n \oplus \mathbb{C}^n$, which imply an isomorphism and a homeomorphism

$$\mathrm{GL}_n^+(A) \cong \mathrm{GL}_n(A), \ \mathrm{Lg}_n(A) \approx \mathrm{Lg}_n^+(A)$$

(see [6, Proposition 2.11]).

Remark 2.2. *(1) Since every finite dimensional C^*-algebra F is unital, we have $\mathrm{GL}_n^+(F) \cong \mathrm{GL}_n(F)$, $\mathrm{Lg}_n(F) \approx \mathrm{Lg}_n^+(F)$ by the above isomorphism/homeomorphism.*

(2) Under the notation of [15, Definition 1.1], we have $\mathrm{GL}_m^+(A) = E_m + \mathbf{gl}(M_m(A))$.

(3) The canonical inclusion $\iota_m : \mathrm{GL}_m^+(A) \hookrightarrow \mathrm{GL}_{m+1}^+(A)$ is defined by:

$$\iota_m(x) = \mathrm{diag}(x, 1_{A^+}), \quad x \in \mathrm{GL}_m^+(A). \tag{2.2}$$

(4) In the literature, $\mathrm{Lg}_m^+(A)$ is simply denoted by $\mathrm{Lg}_m(A)$ ([6], [11], [15] etc.). Here we use the above symbol to avoid confusion.

The following theorem plays the fundamental role.

Theorem 2.3. *([15, Corollary 3.5, Lemma 3.7], [6, Lemma 2..7, Theorem 2.8]) Let A be a (not necessarily unital) C^*-algebra.*

(1) Let $p_A : \mathrm{GL}_m^+(A) \to \mathrm{Lc}_m(A)$ be the surjection defined by $p_A(a) = a \cdot \mathbf{e}_m$, $a \in \mathrm{GL}_m^+(A)$. Then p_A is a locally trivial bundle with the fiber

$$\mathrm{TL}_m(A) = \left\{ \begin{pmatrix} x & 0 \\ c & 1_{A^+} \end{pmatrix} \mid x \in \mathrm{GL}_{m-1}^+(A), \ c \in A^{m-1} \right\}$$

and $\mathrm{TL}_m(A)$ is homotopy equivalent to $\mathrm{GL}_{m-1}^+(A)$. Hence there exists an exact sequence induced by the fibration

$$\mathrm{GL}_m^+(A) \to \mathrm{GL}_{m+1}^+(A) \to \mathrm{Lc}_{m+1}(A)$$

as follows:

$$
\begin{aligned}
&\cdots \to \pi_k(\mathrm{GL}_m^+(A)) \to \pi_k(\mathrm{GL}_{m+1}^+(A)) \to \pi_k(\mathrm{Lc}_{m+1}(A)) \to \\
&\to \pi_{k-1}(\mathrm{GL}_m^+(A)) \to \pi_{k-1}(\mathrm{GL}_{m+1}^+(A)) \to \pi_{k-1}(\mathrm{Lc}_{m+1}(A)) \to \cdots \\
&\to \pi_0(\mathrm{GL}_m^+(A)) \to \pi_0(\mathrm{GL}_{m+1}^+(A)) \to \pi_0(\mathrm{Lc}_{m+1}(A)) \to 0
\end{aligned} \tag{2.3}
$$

where $\mathrm{GL}_m^+(A)$, $\mathrm{GL}_{m+1}^+(A)$ *and* $\mathrm{Lg}_{m+1}^+(A)$ *have the base points* E_m, E_{m+1} *and* $\mathbf{e}_{m+1}$ *respectively.*

(2) *Let* $\mathrm{Lc}_m(A)_0$ *and* $\mathrm{Lg}_m^+(A)_0$ *be the components of* $\mathrm{Lc}_m(A)$ *and* $\mathrm{Lg}_m^+(A)$ *respectively containing* $\mathbf{e}_m$, *and also* $\mathrm{GL}_n^+(A)_0$ *be the component of* $\mathrm{GL}_n^+(A)$ *containing* E_m. *Then we have* $\mathrm{Lc}_m(A) \subset \mathrm{Lg}_m^+(A)$ *and* $\mathrm{Lc}_m(A)_0 = \mathrm{Lg}_m^+(A)_0 = \mathrm{GL}_m^+(A)_0 \cdot \mathbf{e}_m$. *In particular, if* $\mathrm{Lg}_m^+(A)$ *is connected, then we have the equality* $\mathrm{Lg}_m^+(A) = \mathrm{Lc}_m(A)$.

Every homomorphism $\varphi : A \to B$ between C^*-algebras A and B naturally extends to a unital homomorphism of the unitizations that is also denoted by $\varphi : A^+ \to B^+$ for simplicity. Likewise, the induced homomorphism $\mathrm{GL}_m^+(A) \to \mathrm{GL}_m^+(B)$ and the induced map $\mathrm{Lg}_m^+(A) \to \mathrm{Lg}_m^+(B)$ are denoted by the same symbol φ. For a continuous map $f : X \to Y$ between topological spaces X and Y, $f_\sharp : \pi_j(X) \to \pi_j(Y)$ denotes the induced homomorphism between the j-th homotopy groups. A topological space Z is said to be *k-connected* if $\pi_j(Z) = 0$ for each $j = 0, \ldots, k$, where $\pi_0(Z) = 0$ means that Z is path connected.

Definition 2.4. *([8], [9], [15]) Let A be a (not necessarily unital) C^*-algebra.*

(1) *For an integer $k \geq 0$, the k-th connected stable rank* $\mathrm{csr}_k(A)$ *is defined as follws:*

$$\mathrm{csr}_k(A) = \min\{m \mid \mathrm{Lg}_n^+(A) \textit{ is } k\textit{-connected for each } n \geq m\}.$$

Also let $\mathrm{csr}(A) := \mathrm{csr}_0(A)$, *called the connected stable rank of A.*

(2) *For an integer $k \geq 0$, let* $\iota_n : \mathrm{GL}_n^+(A) \hookrightarrow \mathrm{GL}_{n+1}^+(A)$ *be the canonical inclusion given by (2.2). Then the k-th homotopy stabilization rank* $\mathrm{hsr}_k(A)$ *is defined as follows:*

$$
\begin{aligned}
\mathrm{hsr}_k(A) = \min\{m \mid\ & \textit{the inclusion } \iota_n \textit{ induces an isomorphism} \\
& (\iota_n)_\sharp : \pi_j(\mathrm{GL}_n^+(A)) \to \pi_j(\mathrm{GL}_{n+1}^+(A)) \textit{ for each } n \geq m \textit{ and} \\
& \textit{for each } \ j = 0, \ldots, k\}.
\end{aligned}
$$

(3) *The algebra A is said to be K-stable if* $\mathrm{hsr}_k(A) = 1$ *for each $k \geq 0$, that is, the canonical inclusion* $\iota_n : \mathrm{GL}_n^+(A) \hookrightarrow \mathrm{GL}_{n+1}^+(A)$ *induces an isomorphism* $\pi_k(\mathrm{GL}_n^+(A)) \to \pi_k(\mathrm{GL}_{n+1}^+(A))$ *for each $n \geq 1$ and for each $k \geq 0$.*

In what follows, the tensor product $\pi_j(X) \otimes \mathbb{Q}$ of the j-th homotopy group of a space X with the rational $\mathbb{Q}$ is denoted by $\pi_j(X)_\mathbb{Q}$, where for $j = 1$, $\pi_1(X)$ is assumed to be abelian. Recall that the fundamental group of every topological group, in particular, of $\mathrm{GL}_m^+(A)$, is abelian. The homomorphism $\pi_j(X)_\mathbb{Q} \to \pi_j(Y)_\mathbb{Q}$ induced by a continuous map $f : X \to Y$ is denoted by $f_\sharp$.

Definition 2.5. *For a C^*-algebra A such that $\pi_1(Lg_n^+(A))$ is abelian for each $n \geq 1$, we make the following definition.*

(1) *For an integer $k \geq 1$, let*

$$\mathrm{csr}_k^{\mathbb{Q}}(A) = \min\{m \mid \mathrm{Lg}_n^+(A) \text{ is connected and } \pi_j(\mathrm{Lg}_n^+(A))_{\mathbb{Q}} = 0 \\ \text{for each } j = 0, \ldots, k \text{ and for each } n \geq m\}.$$

(2) *For an integer $k \geq 0$, let*

$$\mathrm{hsr}_k^{\mathbb{Q}}(A) = \min\{m \mid (\iota_n)_\sharp : \pi_j(\mathrm{GL}_n^+(A))_{\mathbb{Q}} \to \pi_j(\mathrm{GL}_{n+1}^+(A))_{\mathbb{Q}} \text{ is an} \\ \text{isomorphism for each } n \geq m \text{ and } j = 0, \ldots, k\}.$$

(3) *A is said to be rationally K-stable if $\mathrm{hsr}_k^{\mathbb{Q}}(A) = 1$ for each $k \geq 0$, that is, the canonical inclusion $\iota_n : \mathrm{GL}_n^+(A) \hookrightarrow \mathrm{GL}_{n+1}^+(A)$ induces an isomorphism $\pi_k(\mathrm{GL}_n^+(A))_{\mathbb{Q}} \to \pi_k(\mathrm{GL}_{n+1}^+(A))_{\mathbb{Q}}$ for each $n \geq 1$ and for each $k \geq 0$.*

Applying the exact sequence (2.3) of Theorem 2.3, we see

Corollary 2.6. *Let A be a C^*-algebra such that $\mathrm{Lg}_n^+(A)$ is connected for each $n \geq 1$. Then A is K-stable if and only if $\mathrm{csr}_k(A) \leq 2$ for each $k \geq 1$. Assume further that $\pi_1(\mathrm{Lg}_n(A))$ is abelian for each $n \geq 1$. Then A is rationally K-stable if and only if $\mathrm{csr}_k^{\mathbb{Q}}(A) \leq 2$ for each $k \geq 1$.*

2.2 Finite dimensional algebras and AF algebras

For an inductive sequence $(A_p, \varphi_p : A_p \to A_{p+1})$ of C^*-algebras and $*$-homomorphisms, let $\varinjlim(A_p, \varphi_p)$ be the C^*-algebra inductive limit of the sequence. The canonical homomorphism $A_p \to \varinjlim(A_p, \varphi_p)$ is denoted by $\varphi_{\infty p} : A_p \to \varinjlim(A_p, \varphi_p)$. For $q \geq p+1$, $\varphi_{qp} : A_p \to A_q$ denotes the composition

$$\varphi_{qp} = \varphi_{q-1} \circ \cdots \circ \varphi_p : A_p \to A_{p+1} \to \cdots \to A_{q-1} \to A_q.$$

An *AF-algebra* A is the limit $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$ of an inductive sequence of finite dimensional C^*-algebras A_p. Each A_p is isomorphic to the direct sum of finitely many matrix algebras $A_p \cong \oplus_\nu M_\nu(\mathbb{C})$. Since $\{\varphi_{\infty p}(A_p) \mid p \geq 1\}$ forms an increasing sequence of finite dimensional subalgebras of A with $\cup_{p=1}^\infty \varphi_{\infty p}(A_p)$ being dense in A, we may assume without loss of generality that each φ_p is injective. The next theorem provides basic information on $*$-homomorphisms of matrix algebras.

Theorem 2.7. *([1, Chap.III, Corollary 1.2]) Let $\varphi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a $*$-homomorphism.*

(1) *If $m > n$, then $\varphi = 0$.*

(2) *If $m \leq n$, then there exists a $r \geq 0$ with $rm \leq n$ such that φ is unitarily equivalent to the homomorphism $\rho_r : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ defined by*

$$\rho_r(x) = \mathrm{diag}(\underbrace{x, \ldots, x}_{r}, 0, \ldots, 0). \tag{2.4}$$

For an inductive sequence $(A_p, \varphi_p : A_p \to A_{p+1})$ of finite dimensional C^*-algebras and $*$-homomorphism, let $\mathcal{D}((A_p, \varphi_p))$ be the Bratteli diagram ([1, Chap.III]): the set of nodes is the pairs

$$\{(p, \nu) \mid \nu \in \mathcal{N}(A_p), p \geq 1\}.$$

For two nodes (p, μ) and $(p+1, \nu)$, there exist r arrows from (p, μ) to $(p+1, \nu)$ if and only if the homomorphism

$$\varphi_p^{\nu\mu} := \mathrm{proj}_\nu \circ \varphi \circ \mathrm{incl}_\mu : M_\mu(\mathbb{C}) \hookrightarrow A_p \to A_{p+1} \to M_\nu(\mathbb{C}) \tag{2.5}$$

(incl_μ and proj_ν denote the canonical inclusion and the canonical projection respectively) is unitarily equivalent to ρ_r given in (2.4). If there is at least one arrow from (p, μ) to $(p+1, \nu)$, then we write $(p, \mu) \searrow (p+1, \nu)$.

Definition 2.8. *[13, Definition 3.5] Let $(A_p, \varphi_p : A_p \to A_{p+1})$ be an inductive sequence of finite dimensional C^*-algebras with the Bratteli diagram $\mathcal{D}((A_p, \varphi_p))$. For an integer $M \in \mathbb{Z}_{\geq 1}$ and $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ with $M \leq N$ and an integer $\mu \geq 1$, a sequence $\Lambda = (p, \nu_p)_{M \leq p \leq N}$ is called a μ-chain if*

(1) $\nu_p \in \mathcal{N}(A_p)$ for each p with $M \leq p \leq N$.

(2) $(p, \nu_p) \searrow (p+1, \nu_{p+1})$ in $\mathcal{D}((A_p, \varphi_p))$ for each p with $M \leq p \leq N-1$.

(3) If $(p, \nu) \searrow (p+1, \nu_{p+1})$ in $\mathcal{D}((A_p, \varphi_p))$, then $\nu = \nu_p$.

(4) For each p with $M \leq p \leq N$, we have $\nu_p = \mu$.

When $N = \infty$, we call Λ an infinite μ-chain.

Proposition 2.9. *([13, Lemma 3.6]) Assume that the Bratteli diagram $\mathcal{D}((A_p, \varphi_p))$ associated with an AF algebra $A = \varinjlim(A_p, \varphi_p)$ contains an infinite μ-chain. Then there exists a non-trivial homomorphism $\varphi : A \to M_\mu(\mathbb{C})$.*

Next we recall a couple of information on the homotopy groups of $\mathrm{GL}_n(M_\nu(\mathbb{C}))$. First notice that $\mathrm{GL}_n(\mathbb{C})$ is connected for each $n \geq 1$ and observe an isomorphism

$$\mathrm{GL}_n(M_\nu(\mathbb{C})) \cong \mathrm{GL}_{n\nu}(\mathbb{C})$$

for each $n, \nu \geq 1$.

Theorem 2.10. *([7, Chap.II, Corollary 3.17], [13, Example 1.6, Lemma 2.1, Lemma 2.2, Lemma 2.3]) Let $k \geq 1$ and $n \geq 1$. Let $\iota_n : \mathrm{GL}_n(M_\nu(\mathbb{C})) \to \mathrm{GL}_{n+1}(M_\nu(\mathbb{C}))$ be the canonical inclusion given in (2.2).*

(1) For each k with $k \leq 2n\nu - 1$, we have an isomorphism

$$\pi_k(\mathrm{GL}_n(M_\nu(\mathbb{C}))) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

(2) We have an isomorphism

$$\pi_k(\mathrm{GL}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \cong \begin{cases} \mathbb{Q} & \text{if } k \text{ is odd and } k \leq 2n\nu - 1, \\ 0 & \text{otherwise} \end{cases}$$

(3) *If $k \leq 2n\nu - 1$, then the induced homomorphism $(\iota_n)_\sharp : \pi_k(\mathrm{GL}_n(M_\nu(\mathbb{C}))) \to \pi_k(\mathrm{GL}_{n+1}(M_\nu(\mathbb{C})))$ is an isomorphism. The same holds for the homomorphism $(\iota_n)_\sharp : \pi_k(\mathrm{GL}_n(M_\nu(\mathbb{C})))_\mathbb{Q} \to \pi_k(\mathrm{GL}_{n+1}(M_\nu(\mathbb{C})))_\mathbb{Q}$.*

(4) *Let k be an odd integer such that $k \leq 2m - 1$. Let $\varphi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a $*$-homomorphism which is unitarily equivalent to the homomorphism $\rho_r : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ given in (2.4) for some $r \geq 0$. Then φ induces the homomorphism $\varphi_\sharp : \pi_k(\mathrm{GL}_m(\mathbb{C}))_\mathbb{Q} \to \pi_k(\mathrm{GL}_n(\mathbb{C}))_\mathbb{Q}$ given by*

$$\pi_k(\varphi)(\alpha) = r\alpha, \quad \alpha \in \pi_k(M_m(\mathbb{C}))_\mathbb{Q} \cong \mathbb{Q}.$$

2.3 Some auxiliary results

The next result follows from [3] and [15], and is mentioned in [13, Proposition 1.4].

Theorem 2.11. *[13, Proposition 1.4]. Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$ be the limit of an inductive sequence (A_p, φ_p) of C^*-algebras and $*$-homomorphisms. Then for each $j \geq 0$ and $n \geq 1$, the homomorphism $\varphi_{\infty p} : A_p \to A$ induces isomorphisms:*

$$\begin{aligned} &\varinjlim\nolimits_p (\varphi_{\infty p})_\sharp : \varinjlim\nolimits_p \pi_j(\mathrm{GL}_n^+(A_p)) \to \pi_j(\mathrm{GL}_n^+(A)), \\ &\varinjlim\nolimits_p (\varphi_{\infty p})_\sharp : \varinjlim\nolimits_p \pi_j(\mathrm{GL}_n^+(A_p))_\mathbb{Q} \to \pi_j(\mathrm{GL}_n^+(A))_\mathbb{Q}. \end{aligned}$$

We have the following analogue of the above for the space $\mathrm{Lg}_n^+(A)$.

Theorem 2.12. *Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$ be the limit of an inductive sequence (A_p, φ_p) of C^*-algebras and injective $*$-homomorphisms. Then for each $j \geq 0$ and $n \geq 1$, we have isomorphisms induced by the homomorphism $\varphi_{\infty p} : A_p \to A$:*

$$\begin{aligned} &\varinjlim\nolimits_p (\varphi_{\infty p})_\sharp : \varinjlim\nolimits_p \pi_j(\mathrm{Lg}_n^+(A_p)) \to \pi_j(\mathrm{Lg}_n^+(A)), \\ &\varinjlim\nolimits_p (\varphi_{\infty p})_\sharp : \varinjlim\nolimits_p \pi_j(\mathrm{Lg}_n^+(A_p))_\mathbb{Q} \to \pi_j(\mathrm{Lg}_n^+(A))_\mathbb{Q}. \end{aligned}$$

Since the homomorphisms $\varphi_p : A_p \to A_{p+1}$ and $\varphi_{\infty p} : A_p \to A$ naturally extends to unital homomorphisms $A_p^+ \to A_{p+1}^+$ and $A_p^+ \to A^+$, we may assume without loss of generality that φ_p and $\varphi_{\infty p}$ are unital. Our proof of the above theorem applies the next lemma.

Lemma 2.13. *Let (K, L) be a pair of compact polyhedra such that L is a subcomplex of K, and let $p \geq 1$ be an integer. Also let $(f : K \to A, g : L \to A_p)$ be a pair of continuous maps such that $f|L = \varphi_{\infty p} \circ g$. Then for each $\varepsilon > 0$, there exist an index $q \geq p$ and a continuous map $f_q : K \to A_q$ such that $\|f - \varphi_{\infty q} \circ f_q\|_\infty < \varepsilon$ and $f_q|L = \varphi_{qp} \circ g$.*

Proof. Let $d = \dim K$ and take a sufficiently fine triangulation of K such that L is a subcomplex with respect to the triangulation, and for each simplex σ of K, we have

$$\mathrm{diam} f(\sigma) := \sup\{\|f(x) - f(y)\| \mid x, y \in \sigma\} < \frac{\varepsilon}{3^{2d+1}}. \tag{2.6}$$

For $i = 0, \ldots, d$, let $K^{(i)}$ be the union of L with all simplices of K of dimension at most i.

For each vertex v of K, there exist an index $q_v \geq p$ and a point $\gamma(v) \in A_{q_v}$ such that $\|\varphi_{\infty q_v}(\gamma(v)) - f(v)\| < \frac{\varepsilon}{3^{2d+1}}$. We further make a choice that $q_v = p$ and $\gamma(v) = g(v)$ whenever $v \in L$. Let $q = \max\{q_v \mid v \text{ is a vertex of } K\}$ and let $f^{(0)}(v) = \varphi_{qq_v}(\gamma(v))$. The map $f^{(0)}$ naturally extends to a map on $K^{(0)}$ by defining $f^{(0)}|L = g$. Then we see

$$\|\varphi_{\infty q} \circ f^{(0)} - f|K^{(0)}\| < \frac{\varepsilon}{3^{2d+1}}. \tag{2.7}$$

For each simplex σ and for two vertices v, w of σ, we see from (2.6) and (2.7) that

$$\begin{aligned}\|\varphi_{\infty q} f^{(0)}(v) - \varphi_{\infty q} f^{(0)}(w)\| &\leq \|\varphi_{\infty q} f^{(0)}(v) - f(v)\| + \|f(v) - f(w)\| + \\ &\quad + \|f(w) - \varphi_{\infty q} f^{(0)}(w)\| \\ &< \frac{\varepsilon}{3^{2d}}\end{aligned}$$

and hence

$$\operatorname{diam}(\varphi_{\infty q} f^{(0)}(\sigma \cap K^{(0)})) < \frac{\varepsilon}{3^{2d}}$$

for each simplex σ of K.

Starting with the above $f^{(0)} : K^{(0)} \to A_q$, we inductively define a sequence of maps $\{f^{(i)} : K^{(i)} \to A_q \mid i = 0, \ldots, d\}$ such that

(i) $\|f|K^{(i)} - \varphi_{\infty q} \circ f^{(i)}\|_\infty < \frac{\varepsilon}{3^{2(d-i)}}$ and $\operatorname{diam}(\varphi_{\infty q} f(\sigma \cap K^{(i)})) < \frac{\varepsilon}{3^{2(d-i)}}$ for each simplex σ of K,

(ii) $f^{(i)}|L = \varphi_{qp} \circ g$, and

(iii) $f^{(i+1)}|K^{(i)} = f^{(i)}$.

Then $f_q := f^{(d)}$ is the required map.

Assume that $f^{(i)}$ has been defined and take an $(i+1)$-simplex σ of K not in L. The map $f^{(i)}|\partial\sigma : \partial\sigma \to A_q$ satisfies $\operatorname{diam}(\varphi_{\infty q} f^{(i)}(\partial\sigma)) < \frac{3\varepsilon}{3^{2(d-i)}}$. Using the convexity of the Banach space A_q, we obtain an extension $\bar{g}_\sigma : \sigma \to \operatorname{conv}(f^{(i)}(\partial\sigma))$ of $f^{(i)}|\partial\sigma$, where $\operatorname{conv}(f^{(i)}(\partial\sigma))$ denotes the convex hull of $f^{(i)}(\partial\sigma)$. Then we have

$$\begin{aligned}\operatorname{diam}(\varphi_{\infty q}\bar{g}_\sigma(\sigma)) &\leq \operatorname{diam}(\operatorname{conv}(\varphi_{\infty q} f^{(i)}(\partial\sigma))) = \operatorname{diam}(\varphi_{\infty q} f^{(i)}(\partial\sigma))) \\ &\leq \operatorname{diam}(f^{(i)}(\partial\sigma)) < \frac{3\varepsilon}{3^{2(d-i)}}.\end{aligned}$$

Also we see

$$\begin{aligned}\|f|\sigma - \varphi_{\infty q} \circ \bar{g}_\sigma\|_\infty &\leq \operatorname{diam} f(\sigma) + \|f|\partial\sigma - \varphi_{\infty q} \circ \bar{g}_\sigma|\partial\sigma\|_\infty + \operatorname{diam}(\varphi_{\infty q}\bar{g}_\sigma(\sigma)) \\ &\leq (\frac{1}{3^{2d}} + \frac{1}{3^{2(d-i)}} + \frac{3}{3^{2(d-i)}})\varepsilon < \frac{\varepsilon}{3^{2(d-i-1)}}.\end{aligned}$$

Repeating the above process to each $(i+1)$-simplex of K not in L and then extending the resulting map to $K^{(i+1)}$ by defining $\varphi_{qp} \circ g$ on L, we obtain the map $f^{(i+1)} : K^{(i+1)} \to A_q$ that is an extension of $f^{(i)}$.

This completes the inductive step and completes the proof.

□

Proof. (of Proposition 2.12) Fix $j = 0, \dots, k$ and $n \geq 1$, and let $\alpha : S^j \to \mathrm{Lg}_n^+(A)$ be a continuous map. There exists a continuous map $\beta : S^j \to (A^+)^n$ such that $\langle \beta, \alpha \rangle \equiv 1$. Recall that $\mathrm{Lg}_n^+(A)$ is an open set of A^n. This together with the compactness of $\alpha(S^j)$ allows us to take a small $\varepsilon > 0$ such that

(i) if a map $f : S^j \to \mathrm{Lg}_n^+(A)$ satisfies $\|f - \alpha\|_\infty < \varepsilon$, then f and α are homotopic in $\mathrm{Lg}_n^+(A)$, and

(ii) if two maps $\alpha', \beta' : S^j \to (A^+)^n$ satisfy $\|\alpha' - \alpha\|_\infty < \varepsilon$ and $\|\beta' - \beta\|_\infty < \varepsilon$, then $\|\langle \beta', \alpha' \rangle - 1_{A^+}\|_\infty < 1$.

Apply Lemma 2.13 to $\alpha - \mathbf{e}_n : S^j \to A^n$ and β in order to find continuous maps $\alpha_p, \beta_p : S^j \to (A_p^+)^n$ such that

(iii) $\alpha_p(x) - \mathbf{e}_n \in (A_p)^n$ for each $x \in S^j$, and

(iv) $\|\alpha - \varphi_{\infty p} \circ \alpha_p\|_\infty < \varepsilon$ and $\|\beta - \varphi_{\infty p} \circ \beta_p\|_\infty < \varepsilon$.

Since $\varphi_{\infty q}$ is injective and unital, we see from (ii), (iii) and (iv),

$$\|\langle \beta_p(x), \alpha_p(x) \rangle - 1_{A_p^+}\| = \|\langle \varphi_{\infty p}(\beta_p(x)), \varphi_{\infty p}(\alpha_p(x)) \rangle - 1_{A^+}\| < 1 \text{ for each } x \in S^j.$$

Thus $\langle \beta_p(x), \alpha_p(x) \rangle$ is invertible in $(A_p)^+$ for each $x \in S^j$. Define $\gamma_p : S^j \to (A_p^+)^n$ by

$$\gamma_p(x) = \langle \beta_p(x), \alpha_p(x) \rangle^{-1} \beta_p(x), \quad x \in S^j.$$

We see $\langle \gamma_p, \alpha_p \rangle \equiv 1$ and hence obtain $\alpha_p(S^j) \subset \mathrm{Lg}_n^+(A_p)$. Also by (i) $\varphi_{\infty p} \circ \alpha_p$ is homotopic to α. This proves that the homomorphism $\varinjlim_p (\varphi_{\infty p})_\sharp : \varinjlim_p \pi_j(\mathrm{Lg}_n^+(A_p)) \to \pi_j(\mathrm{Lg}_n^+(A))$ is surjective.

To prove that $\varinjlim_p (\varphi_{\infty p})_\sharp$ is injective, take a continuous map $\alpha_p : S^j \to \mathrm{Lg}_n^+(A_p)$ and assume that $\varphi_{\infty p} \circ \alpha_p$ admits a continuous extension $\bar{\alpha} : D^{j+1} \to \mathrm{Lg}_n^+(A)$. There exists a continuous map $\beta : D^{j+1} \to (A^+)^n$ such that $\langle \beta, \bar{\alpha} \rangle \equiv 1$. Apply Lemma 2.13 to obtain an index $q \geq p$ and continuous maps $\hat{\alpha}_q : D^{j+1} \to (A_q^+)^n$ and $\beta_q : D^{j+1} \to (A_q^+)^n$ such that $\|\alpha - \varphi_{\infty q} \circ \hat{\alpha}_q\|_\infty < \varepsilon$, $\|\beta - \varphi_{\infty q} \circ \hat{\beta}_q\|_\infty < \varepsilon$, and $\hat{\alpha}_q | S^j = \varphi_{qp} \circ \alpha_p$. Since $\varphi_{\infty q}$ is injective, we see from (ii)

$$\|\langle \beta_q, \hat{\alpha}_q \rangle - 1_{A_q^+}\|_\infty = \|\langle \varphi_{\infty q} \circ \beta_q, \varphi_{\infty q} \circ \hat{\alpha}_q \rangle - 1_{A^+}\|_\infty < 1$$

and hence $\langle \beta_q(x), \hat{\alpha}_q(x) \rangle$ is invertible in A_q^+ for each $x \in D^{j+1}$. By the same argument as in the previous paragraph, we see that $\hat{\alpha}_q(D^{j+1}) \subset \mathrm{Lg}_n^+(A_q)$. Thus $\hat{\alpha}_q$ is an extension of $\varphi_{qp} \circ \alpha_p : S^j \to \mathrm{Lg}_n^+(A_q)$ to $D^{j+1} \to \mathrm{Lg}_n^+(A_q)$. This proves the injectivity of $\varinjlim_p (\varphi_{\infty p})_\sharp$. ∎

Theorem 2.15 gives basic information on $\mathrm{csr}_k(A)$ and $\mathrm{hsr}_k(A)$ for an AF algebra A. The proof applies the next lemma. Recall from Remark 2.2 (1) that $\mathrm{Lg}_n^+(F) \approx \mathrm{Lg}_n(F)$ for each finite dimensional C^*-algebra F .

Lemma 2.14. *(1) For each integer $\nu \geq 1$, we have the following:*

(1-1) *For each odd integer k, we have*

$$\pi_k(\mathrm{Lc}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \cong \begin{cases} \pi_k(\mathrm{GL}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \cong \mathbb{Q} & \text{if } 2(n-1)\nu - 1 < k \leq 2n\nu - 1, \\ 0 & \text{if } k \leq 2(n-1)\nu - 1 \text{ or } k > 2n\nu - 1. \end{cases}$$

(1-2) *For each even integer k, we have $\pi_k(\mathrm{Lc}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} = 0$.*

(2) *Let A be an AF algebra.*

(2-1) *We have the equality $\mathrm{Lg}_n^+(A) = \mathrm{Lc}_n(A)$ for each $n \geq 1$.*

(2-2) *The fundamental group $\pi_1(\mathrm{Lg}_n^+(A))$ is abelian for each $n \geq 1$.*

(2-3) *For each even integer k and for each integer $n \geq 1$, we have $\pi_k(\mathrm{GL}_n^+(A))_{\mathbb{Q}} = \pi_k(\mathrm{Lg}_n^+(A))_{\mathbb{Q}} = 0$.*

Proof. (1). For an odd integer k, we consider the exact sequence (2.3) for $M_\nu(\mathbb{C})$, being rationalized:

$$\begin{aligned} &\pi_{k+1}(\mathrm{GL}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \pi_{k+1}(\mathrm{Lc}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \\ &\to \pi_k(\mathrm{GL}_{n-1}(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \pi_k(\mathrm{GL}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \pi_k(\mathrm{Lc}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \\ &\to \pi_{k-1}(\mathrm{GL}_{n-1}(M_\nu(\mathbb{C})))_{\mathbb{Q}} \end{aligned}$$

By Theorem 2.10 (2), we have $\pi_{k+1}(\mathrm{GL}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} = \pi_{k-1}(\mathrm{GL}_n^+(M_\nu(\mathbb{C})))_{\mathbb{Q}} = 0$. Since the homomorphism $\pi_j(\mathrm{GL}_{n-1}(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \pi_j(\mathrm{GL}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}}$ is injective for each j, we see that $\pi_{k+1}(\mathrm{Lc}_n(M_\nu(\mathbb{C}))_{\mathbb{Q}} = 0$. This proves (1-2) and also reduces the above sequence to

$$0 \to \pi_k(\mathrm{GL}_{n-1}(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \pi_k(\mathrm{GL}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to \pi_k(\mathrm{Lc}_n(M_\nu(\mathbb{C})))_{\mathbb{Q}} \to 0,$$

from which (1-1) follows with the help of Theorem 2.10 (2).

(2). (2-1) follows from [9, Example 11.4] and Theorem 2.3 (2). For the proof of (2-2), we note that, by the exact sequence (2.3)

$$\pi_1(\mathrm{GL}_n(M_\nu(\mathbb{C}))) \xrightarrow{p_\sharp} \pi_1(\mathrm{Lc}_n(M_\nu(\mathbb{C}))) \longrightarrow \pi_0(\mathrm{GL}_n(M_\nu(\mathbb{C})))$$

and the connectedness of $\mathrm{GL}_n(M_\nu(\mathbb{C}))$, $p_\sharp$ is surjective. Hence $\pi_1(\mathrm{Lc}_n(M_\nu(\mathbb{C})))$ is abelian. Hence for each finite dimensional algebra $F \cong \oplus_\nu M_\nu(\mathbb{C})$, we see that $\pi_1(\mathrm{Lg}_n(F)) \cong \oplus_\nu \pi_1(\mathrm{Lg}_n(M_\nu(\mathbb{C}))) = \oplus_\nu \pi_1(\mathrm{Lc}_n(M_\nu(\mathbb{C})))$ is an abelian group. By Theorem 2.12, we see that $\pi_1(\mathrm{Lg}_n^+(A))$ is abelian.

For (2-3), first we observe that, for each finite dimensional C^*-algebra F, $\pi_k(\mathrm{GL}_n(F))_{\mathbb{Q}} = 0$ and also $\pi_k(\mathrm{Lg}_n^+(F))_{\mathbb{Q}} = \pi_k(\mathrm{Lc}_n(F))_{\mathbb{Q}} = 0$ by Theorem 2.10 (2). By Thereom 2.12 and Theorem 2.12, we obtain $\pi_k(\mathrm{GL}_n^+(A))_{\mathbb{Q}} = \varinjlim_p \pi_k(\mathrm{GL}_n^+(A_p))_{\mathbb{Q}} = 0$ and $\pi_k(\mathrm{Lg}_n^+(A))_{\mathbb{Q}} = \varinjlim_p \pi_k(\mathrm{Lg}_n^+(A_p))_{\mathbb{Q}} = 0$. $\square$

Theorem 2.15. *Let A be an AF algebra and let $k(\geq 0)$ be an integer.*

(1) $\mathrm{csr}_k(A) \leq \mathrm{hsr}_k(A) + 1 \leq \mathrm{csr}_{k+1}(A)$.

(2) $\mathrm{hsr}_k(A) \leq \lceil \frac{k+1}{2} \rceil$.

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(3) For each odd integer k, we have $\mathrm{csr}_k^{\mathbb{Q}}(A) = \mathrm{hsr}_k^{\mathbb{Q}}(A) + 1$.

(4) For each even integer $k \geq 2$, *we have* $\mathrm{csr}_k^{\mathbb{Q}}(A) = \mathrm{csr}_{k-1}^{\mathbb{Q}}(A)$ *and* $\mathrm{hsr}_k^{\mathbb{Q}}(A) = \mathrm{hsr}_{k-1}^{\mathbb{Q}}(A)$.

(5) $\mathrm{hsr}_1(A) = \mathrm{hsr}_2(A) = 1$ *and* $\mathrm{csr}_1(A) = \mathrm{csr}_2(A) \leq 2$.

Proof. (1) is a consequence of [9, Proposition 32] and $\mathrm{csr}(A) = 1$ ([9, Example 11.4]).

For (2), let k be an odd integer, $m = \frac{k+1}{2}$ and take an arbitrary $n \geq m$. For $A = \varinjlim A_p$ with $A_p \cong \oplus_{\nu \in \mathcal{N}(A_p)} M_\nu(\mathbb{C})$, we see the canonical inclusion $\iota_n : \mathrm{GL}_n(A_p) \to \mathrm{GL}_{n+1}(A_p)$ induces an isomorphism $(\iota_n)_\sharp : \pi_j(\mathrm{GL}_n(A_p)) \to \pi_j(\mathrm{GL}_{n+1}(A_p))$ for each $j = 0, \ldots, k$, because $k = 2m - 1 \leq 2n\nu - 1$ for each $\nu \in \mathcal{N}(A_p)$. By Theorem 2.11 and the commutativity of the diagram:

$$\begin{tikzcd}
\mathrm{GL}_n(A_p) \arrow[r, "\varphi_{\infty p}"] \arrow[d, "\iota_n"'] & \mathrm{GL}_n^+(A) \arrow[d, "\iota_n^\infty"] \\
\mathrm{GL}_{n+1}(A_p) \arrow[r, "\varphi_{\infty p}"'] & \mathrm{GL}_{n+1}^+(A)
\end{tikzcd}$$

we see that the homomorphism $(\iota_n^\infty)_\sharp : \pi_j(\mathrm{GL}_n(A)) \to \pi_j(\mathrm{GL}_{n+1}(A))$ is an isomorphism for each $j = 0, \ldots, k$. This proves the inequality $\mathrm{hsr}_k(A) \leq \frac{k+1}{2}$. For an even integer k, we repeat the above for $m = \frac{k}{2} + 1$ to obtain the inequality.

(3) For an odd integer k, we use the exact sequence (2.3), being rationalized. For each odd integer j with $1 \leq j \leq k$, and for $n \geq 1$, the sequence reduces to:

$$0 \longrightarrow \pi_j(\mathrm{GL}_n^+(A))_{\mathbb{Q}} \longrightarrow \pi_j(\mathrm{GL}_{n+1}^+(A))_{\mathbb{Q}} \longrightarrow \pi_j(\mathrm{Lc}_{n+1}(A))_{\mathbb{Q}} \longrightarrow 0$$

due to Lemma 2.14. Hence $\pi_j(\mathrm{GL}_n^+(A))_{\mathbb{Q}} \to \pi_j(\mathrm{GL}_{n+1}^+(A))_{\mathbb{Q}}$ is an isomorphism if and only if $\pi_j(\mathrm{Lc}_{n+1}(A))_{\mathbb{Q}} = 0$. Also, since $\mathrm{csr}(A) = \mathrm{csr}_0(A) = 1$, that is, $\mathrm{Lg}_n^+(A)$ is connected for each $n \geq 1$, we obtain from Theorem 2.3 (2) that $\mathrm{Lc}_n(A) = \mathrm{Lg}_n^+(A)$. From these, we have the desired equality.

(4) For an even integer k, we have $\pi_k(\mathrm{GL}_n^+(A))_{\mathbb{Q}} = \pi_k(\mathrm{Lg}_n^+(A))_{\mathbb{Q}} = 0$ for each $n \geq 1$ by Lemma 2.14. These directly imply the desired equalities.

(5) Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$, where each A_p is a finite dimensional C^*-algebra and φ_p is an injective $*$- homomorphism. For each $n, \nu \geq 1$, we have $\pi_2(\mathrm{GL}_n(M_\nu(\mathbb{C}))) = 0$ and the canonical inclusion (2.2) induces an isomorphism $\pi_1(\mathrm{GL}_n(M_\nu(\mathbb{C}))) \to \pi_1(\mathrm{GL}_{n+1}(M_\nu(\mathbb{C})))$. This implies

$$\pi_1(\mathrm{Lc}_m(M_\nu(\mathbb{C}))) = 0 \quad i = 1, 2$$

for each $m \geq 2$. Also $\pi_2(\mathrm{Lc}_1(M_\nu(\mathbb{C}))) \cong \pi_2(S^{2\nu-1}) = 0$ for each $\nu \geq 1$.

Thus we have $\pi_1(\mathrm{Lc}_n(A_p)) = 0$ for each $n \geq 2$ and $\pi_2(\mathrm{Lc}_n(A_p)) = 0$ for each $n \geq 1$. By Theorem 2.12 and the connectedness of $\mathrm{Lg}_n^+(A)$ together with Theorem 2.3, we have $\pi_1(\mathrm{Lg}_n^+(A)) = 0$ for each $n \geq 2$ and $\pi_2(\mathrm{Lg}_n^+(A)) = 0$ for each $n \geq 1$. Thus we have $\mathrm{csr}_1(A) = \mathrm{csr}_2(A) \leq 2$.

The same argument with the help of Theorem 2.11 is carried out to conclude $\mathrm{hsr}_1(A) = \mathrm{hsr}_2(A) = 1$.

$\square$

3 Proof of Main Theorems

3.1 Proof of Main Theorem 1

Given an odd integer $k \geq 1$ and an integer m with $1 \leq m \leq \frac{k+1}{2}$, let $j_0 = \frac{k+1}{2} - m$. We define subsets $N^0, N^1, \ldots, N^{j_0}$ of $\mathbb{Z}_{\geq 1}$ as follows:

$$\begin{aligned}
N^0 &= \{\nu \in \mathbb{Z}_{\geq 1} \mid k \leq 2m\nu - 1\}, \\
N^1 &= \{\nu \in \mathbb{Z}_{\geq 1} \mid 2m\nu - 1 < k \leq 2(m+1)\nu - 1\}, \\
&\cdots \\
N^i &= \{\nu \in \mathbb{Z}_{\geq 1} \mid 2(m+i-1)\nu - 1 < k \leq 2(m+i)\nu - 1\}, \\
&\cdots \\
N^{j_0} &= \{\nu \in \mathbb{Z}_{\geq 1} \mid 2(m+j_0-1)\nu - 1 = (k-1)\nu - 1 < k \leq \\
&\qquad\qquad \leq 2(m+j_0)\nu - 1 = (k+1)\nu - 1\}.
\end{aligned}$$

It is straightforward to verify that $\cup_{i=0}^{j_0} N^i = \mathbb{Z}_{\geq 1}$. For a finite dimensional algebra F, let $\mathcal{E}(F) = \mathcal{N}(F) \cap N^0$ and for each $i = 1, \ldots, j_0$, let $\mathcal{Z}^i(F) = \mathcal{N}(F) \cap N^i$ and also $\mathcal{Z}(F) = \cup_{i=1}^{j_0} \mathcal{Z}^i(F)$. Further we define $E(F) = \oplus_{\nu \in \mathcal{E}(F)} M_\nu(\mathbb{C})$ and $Z(F) = \oplus_{\nu \in \mathcal{Z}(F)} M_\nu(\mathbb{C})$ so that $F \cong E(F) \oplus Z(F)$.

For the proof of Main Theorem 1, it is convenient to introduce an auxiliary statement that will turn out to be equivalent to all other statements of the theorem:

(5) Let $A = \varinjlim(A_p, \varphi_{qp} : A_p \to A_q)$, where each A_p is a finite dimensional C^*-algebra and φ_{qp} is an injective $*$-homomorphism. For each p, there exists $q \geq p$ such that $\varphi_{qp}(A_p) \subset \oplus_{\mu \in \mathcal{N}(A_q),\ 2m\mu - 1 \geq k} M_\mu(\mathbb{C})$.

Proof. The following is our scheme of the proof.

(i) (1) $\Rightarrow$ (1a), (2) $\Rightarrow$ (2a) and (1a) $\Leftrightarrow$ (2a).

(ii) (2a) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

(iii) (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1)

(i): The implications (1)$\Rightarrow$ (1a) and (2)$\Rightarrow$ (2a) are straightforward and the equivalence (1a)$\Leftrightarrow$(2a) is a direct consequence of Proposition 2.15 (3).

(ii) (2a) $\Rightarrow$ (5). Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$, where each A_p is a finite dimensional algebra and φ_p is an injective $*$-homomorphism. In what follows, the matrix algebra $M_\nu(\mathbb{C})$ is simply denoted by M_ν.

Step.1. By Lemma 2.14 (1) and the definition of $\mathcal{Z}^1(A_p)$, we have

$$\pi_k(\mathrm{Lc}_{m+1}(A_p))_{\mathbb{Q}} \cong \oplus_{\nu \in \mathcal{Z}^1(A_p)} \pi_k(\mathrm{GL}_{m+1}(M_\nu))_{\mathbb{Q}}. \tag{3.1}$$

By Theorem 2.12 and the assumption (2a), we have $\varinjlim_p \pi_k(\mathrm{Lg}_{m+1}(A_p))_{\mathbb{Q}} \cong \pi_k(\mathrm{Lg}^+_{m+1}(A))_{\mathbb{Q}} = 0$. Fix $p \geq 1$. Since $\pi_k(\mathrm{Lg}^+_{m+1}(A_p))_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space, there exists $q_1 \geq p$ such that

$$(\varphi_{q_1\ p})_\sharp(\pi_k(\mathrm{Lg}_{m+1}(A_p))_{\mathbb{Q}}) = 0. \tag{3.2}$$

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Step.2. We examine the restriction $\varphi_{q_1\ p}| \oplus_{\nu\in\mathcal{Z}^1(A_p)} M_\nu$. Take an arbitrary $\nu \in \mathcal{Z}^1(A_p)$. First observe from Theorem 2.7 (1) that

$$\varphi_{q_1\ p}(M_\nu) \subset \oplus_{\mu\in\mathcal{N}(A_{q_1}),\ \mu\geq\nu} M_\mu.$$

For $\mu \in \mathcal{Z}^1(A_{q_\nu})$ with $\mu \geq \nu$, let $\varphi^{\mu\nu}_{q_1\ p} : M_\nu \to M_\mu$ be the homomorphism defined by

$$\varphi^{\mu\nu}_{q_1\ p} = \mathrm{proj}_\mu \circ \varphi_{q_1\ p} \circ \mathrm{incl}_\nu : M_\nu \hookrightarrow A_p \to A_{q_1} \to M_\mu$$

where incl_ν denotes the canonical inclusion and proj_μ denotes the canonical projection. Noticing that $\mathrm{Lg}_{m+1}(M_\nu) = \mathrm{Lc}_{m+1}(M_\nu)$, we have the following commutative diagram:

$$\begin{tikzcd}
 & \pi_k(\mathrm{GL}_m(M_\mu))_\mathbb{Q} \arrow[d] \\
\pi_k(\mathrm{GL}_{m+1}(M_\nu))_\mathbb{Q} \arrow[r, "(\varphi^{\mu\nu}_{q_1 p})_\sharp"] \arrow[d] & \pi_k(\mathrm{GL}_{m+1}(M_\mu))_\mathbb{Q} \arrow[d, "p_\sharp"] \\
\pi_k(\mathrm{Lg}_{m+1}(M_\nu))_\mathbb{Q} \arrow[r, "0"] & \pi_k(\mathrm{Lg}_{m+1}(M_\mu))_\mathbb{Q} \arrow[d] \\
 & 0
\end{tikzcd}$$

where the vertial sequences are parts of the exact sequence (2.3), being rationalized. The homomorphism in the third row is trivial because of (3.2). Since $\mu \in \mathcal{Z}^1(A_p)$, we have $2m\mu - 1 < k$, which implies $\pi_k(\mathrm{GL}_m(M_\mu))_\mathbb{Q} = 0$ and thus the homomorphism $p_\sharp$ is an isomorphism. Thus we see that

$$(\varphi^{\mu\nu}_{q_1\ p})_\sharp = 0. \tag{3.3}$$

Suppose that there exists $\mu \in \mathcal{Z}^1(A_{q_1})$ with $\mu \geq \nu$ such that $\varphi^{\mu\nu}_{q_1\ p}$ is non-trivial. Then the homomorphism $\varphi^{\mu\nu}_{q_1\ p}$ is unitarily equivalent to ρ_r of (2.4) with $r \geq 1$. Since $k \leq 2(m+1)\nu - 1 \leq 2(m+1)\mu - 1$, we conclude from Theorem 2.10 (3) that the homomorphism

$$(\varphi^{\mu\nu}_{q_1\ p})_\sharp : \pi_k(\mathrm{GL}_{m+1}(M_\nu))_\mathbb{Q} \to \pi_k(\mathrm{GL}_{m+1}(M_\mu))_\mathbb{Q}$$

is an isomorphism on the nonzero group $\pi_k(\mathrm{GL}_{m+1}(M_\nu))_\mathbb{Q}$, a contradiction to (3.3). Hence we conclude $\varphi^{\mu\nu}_{q_1\ p} = 0$ for each such μ and obtain the inclusion

$$\varphi_{q_1\ p}(Z^1(A_p)) \subset E(A_{q_1}).$$

Step 3. We repeat the argument of Step 2 by replacing $\pi_k(\mathrm{GL}_{m+1}(M_\nu))_\mathbb{Q}$ with $\pi_k(\mathrm{GL}_{m+2}(M_\nu))_\mathbb{Q}$, and $\pi_k(\mathrm{Lc}_{m+1}(M_\nu))_\mathbb{Q}$ with $\pi_k(\mathrm{Lc}_{m+2}(M_\nu))_\mathbb{Q}$ respectively. Then we find $q_2^0 \geq q_1$ such that

$$\varphi_{q_2^0\ q_1}(Z^2(A_p)) \subset E(A_{q_2^0}) \oplus Z^1(A_{q_2^0}).$$

Another application of Step 2 yields $q_2 \geq q_2^0$ such that $\varphi_{q_2\ q_2^0}(Z^1(A_{q_2^0})) \subset E(A_{q_2})$, which results in:

$$\varphi_{q_2\ p}(Z^1(A_p) \oplus Z^2(A_p)) \subset E(A_{q_2}).$$

We inductively repeat the above process to find $q := q_{j_0}$ such that

$$\varphi_{q\ p}(Z(A_p)) = \varphi_{q\ p}(\oplus_{i=1}^{j_0} Z^i(A_p)) \subset E(A_q).$$

Since $\varphi_{q\ p}(E(A_p)) \subset E(A_q)$, we obtain the desired inclusion $\varphi_{q\ p}(A_p) \subset E(A_q)$. This proves the implication (2a)$\Rightarrow$ (5).

(5)$\Rightarrow$ (3). We follow the last part of the proof of [13, Lemma 3.7]. Take an inductive sequence $(A_p, \varphi_p : A_p \to A_{p+1})$ with $A = \varinjlim(A_p, \varphi_p)$ and each φ_p is an injective $*$-homomorophism of finite dimensional C^*-algebras. By the assumption (5), and by taking a subsequence if necessary, we may assume that

$$\varphi_p(Z(A_p)) \subset E(A_{p+1}). \tag{3.4}$$

Define $\psi_p : E(A_{p+1}) \to E(A_{p+2})$ by

$$\psi_p = \mathrm{proj}_{E(A_{p+2})} \circ \varphi_{p+1} \circ \iota_{E(A_{p+1})} : E(A_{p+1}) \hookrightarrow A_{p+1} \to A_{p+2} \to E(A_{p+2}),$$

where $\mathrm{incl}_{E(A_{p+1})}$ and $\mathrm{proj}_{E(A_{p+2})}$ denote the canonical inclusion and the canonical projection respectively. We see from (3.4) that

$$\varphi_p = \iota_{E(A_{p+1})} \circ (\mathrm{proj}_{E(A_{p+1})} \circ \varphi_p). \tag{3.5}$$

Let $\xi_p := \mathrm{proj}_{E(A_{p+1})} \circ \varphi_p : A_p \to E(A_{p+1})$. As in [13, Lemma 3.7], (3.5) shows that the induced limit homomorphism $\varinjlim \xi_p : A \to \varinjlim(E(A_{p+1}), \psi_p)$ is an isomorphism. Since φ_{p+1} is injective and $\varphi_{p+1}(E(A_{p+1})) \subset E(A_{p+2})$, we see directly that ψ_p is injective. Therefore $(E(A_{p+1}), \psi_p)$ is the sequence required in (3).

(3) $\Rightarrow$ (4). Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$ with $A_p = E(A_p)$. Let $B = M_\nu(\mathbb{C})$ with $2m\nu - 1 < k$. Given a $*$-homomorphism $\varphi : A \to B$, fix an index p and take a direct summand M_μ of A_p. Since $2m\nu - 1 < k \le 2m\mu - 1$, we see that $\nu < \mu$ and hence $\varphi \circ \varphi_{\infty p} = 0$ by Theorem 2.7 (1). Since $\cup_p \varphi_{\infty p}(A_p)$ is dense in A, we have the conclusion (4).

(4)$\Rightarrow$ (5). Here we follow the proofs of [13, Lemma 3.6, Lemma 3.7]. Assume that $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$, where A_p is finite dimensional and φ_p is injective. Suppose that there exists an infinite sequence (p_i) such that $\varphi_{p_{i+1}\ p_i}(Z(A_{p_i})) \not\subset E(A_{p_i})$ for each $i = 1, 2, \ldots$. Without loss of generality we may assume that $p_1 = 1$. Since $\varphi(E(A)) \subset E(B)$ for each $*$-homomorphism $\varphi : A \to B$ of finite dimensional C^*-algebras A and B, we see $\varphi_{qp}(Z(A_p)) \not\subset E(A_q)$ for each p, q with $q \ge p$. This implies that the Bratteli diagram $\mathcal{D}((A_p, \varphi_p))$ associated with (A_p, φ_p) contains an infinite sequence

$$(1, \nu_1) \searrow (2, \nu_2) \searrow \cdots \searrow (p, \nu_p) \searrow (p+1, \nu_{p+1}) \searrow \cdots \tag{3.6}$$

where $\nu_p \in \mathcal{Z}(A_p)$ for each p. Since $\nu_p \le \frac{k}{2m}$ for each p, there exists an index $\nu \le \frac{k}{2m}$ such that $\nu_p = \nu$ for infinitely many p. By taking a subsequence, we may assume that $\nu_p = \nu$ for each $p \ge 1$.

Fix an arbitrary $p \geq 2$ and suppose that there exists a node $(p-1, \mu) \neq (p-1, \nu_{p-1})$ such that $(p-1, \mu) \searrow (p, \nu_p)$ in $\mathcal{D}((A_p, \varphi_p))$. This means that there exists a nontrivial $*$-homomorphism $\psi : M_\mu \oplus M_\nu \to M_\nu$. By Proposition 2.7, ψ is unitarily equivalent to a homomorphism given by

$$(x_\mu, x_\nu) \mapsto \mathrm{diag}(\underbrace{x_\mu, \ldots, x_\mu}_{r_\mu}, \underbrace{x_\nu, \ldots, x_\nu}_{r_\nu}, 0, \ldots, 0), \quad (x_\mu, x_\nu) \in M_\mu \oplus M_\nu$$

where $r_\mu, r_\nu \geq 1$. However this is impossible from the dimensional reason and we conclude that there are no such $(p-1, \mu)$.

This proves that the above sequence is an infinite ν-chain (Definition 2.8). By Propostion 2.9 there exists a non-trivial homomorphism $A \to M_\nu$, a contradition to (4).

(iii) (3)$\Rightarrow$ (2) and (3) $\Rightarrow$ (1). Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$ so that $A_p = E(A_p)$, and hence $k \leq 2m\nu - 1$ for each $\nu \in \mathcal{N}(A_p)$ and for each $p \geq 1$. Take an arbitrary $n \geq m$ and consider a part of the exact sequence (2.3):

$$\begin{aligned}&\pi_k(\mathrm{GL}_n^+(M_\nu)) \to \pi_k(\mathrm{GL}_{n+1}^+(M_\nu)) \to \pi_k(\mathrm{Lc}_{n+1}(M_\nu)) \to \\ &\to \pi_{k-1}(\mathrm{GL}_n^+(M_\nu)).\end{aligned}$$

Since $k-1 < k \leq 2m\nu-1 \leq 2n\nu-1$, the homomorphism $\pi_k(\mathrm{GL}_n^+(M_\nu)) \to \pi_k(\mathrm{GL}_{n+1}^+(M_\nu))$ is an isomorphism and $\pi_{k-1}(\mathrm{GL}_n^+(M_\nu)) = 0$. Thus we have $\pi_k(\mathrm{Lc}_{n+1}(M_\nu)) = 0$ and hence $\pi_k(\mathrm{Lc}_{n+1}(A_p)) = \oplus_\nu \pi_k(\mathrm{Lc}_{n+1}(M_\nu)) = 0$. By Theorem 2.11 and $\mathrm{Lg}_{n+1}^+(A) = \mathrm{Lc}_{n+1}(A)$(Theorem 2.3 (2)), we obtain $\mathrm{csr}_k(A) \leq m+1$ as required in (2). The same argument yields the implication (3) $\Rightarrow$ (1).

This completes the proof of Main Theorem 1.

□

3.2 Proof of Main Theorem 2

The structure of the proof is the same as that of Main Theorem 1. Again it is convenient to introduce an auxiliary statement that will turn out to be equivalent to all other statements of the theorem.

(5) Let $A = \varinjlim(A_p, \varphi_{qp} : A_p \to A_q)$, where each A_p is a finite dimensional C^*-algebra and φ_{qp} is an injective $*$-homomorphism. For each p, there exists $q \geq p$ such that $\varphi_{qp}(A_p) \subset \oplus_{\mu \in \mathcal{N}(A_q),\ 2m\mu-2 \geq k} M_\mu(\mathbb{C})$.

Proof. Let $k \geq 4$ be an even integer satisfying $(*)$ and let m be an integer with $1 \leq m \leq \frac{k}{2} + 1$. As before, the matrix algebra $M_\nu(\mathbb{C})$ is simply denoted by M_ν. We prove the following implications.

(i) (1)$\Rightarrow$ (5).

(ii) (5) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

(iii) (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

The implications in (ii) are proved in exactly the same way as those in Main Theorem 1 by simply replacing "$2m\nu - 1$" with "$2m\nu - 2$."

(iii): (3) $\Rightarrow$ (2). Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$ such that $2m\nu - 2 \geq k$ for each $\nu \in \mathcal{N}(A_p)$ and each $p \geq 1$. For each even integer j with $j \leq k$ and each integer $n \geq m$, we have $2n\nu - 2 \geq 2m\nu - 2 \geq k \geq j$ and hence

$$\pi_j(\mathrm{GL}_n(A_p)) = \oplus_{\nu \in \mathcal{N}(A_p)} \pi_j(\mathrm{GL}_n(M_\nu)) = 0.$$

Hence by Theorem 2.11, we have $\pi_j(\mathrm{GL}_n(A)) = 0$. By the same way with the help of Theorem 2.11, we can prove that the induced homomorphism $\pi_j(\mathrm{GL}_n(A)) \to \pi_j(\mathrm{GL}_{n+1}(A))$ is an isomorphism for each odd integer $j < k$. This proves (2).

(2) $\Rightarrow$ (1) follows directly from the definition.

Thus it remains to prove the implication (1)$\Rightarrow$ (5). The proof has the same structure as that of the the proof (2a)$\Rightarrow$ (5) of Main Theorem 1. Let $A = \varinjlim(A_p, \varphi_p : A_p \to A_{p+1})$, where each A_p is a finite dimensional C^*-algebra and φ_p is an injective $*$-homomorphism. Given an even integer $k \geq 1$ and an integer m with $1 \leq m \leq \frac{k}{2} + 1$, we let $j_0 = \frac{k}{2} + 1 - m$. This time we define subsets $\bar{N}^0, \bar{N}^1, \ldots, \bar{N}^{j_0}$ of $\mathbb{Z}_{\geq 1}$ as follows:

$$\begin{aligned}
\bar{N}^0 &= \{\nu \in \mathbb{Z}_{\geq 1} \mid k \leq 2m\nu - 2\}, \\
\bar{N}^1 &= \{\nu \in \mathbb{Z}_{\geq 1} \mid 2m\nu - 2 < k \leq 2(m+1)\nu - 2\}, \\
&\cdots \\
\bar{N}^i &= \{\nu \in \mathbb{Z}_{\geq 1} \mid 2(m+i-1)\nu - 2 < k \leq 2(m+i)\nu - 2\}, \\
&\cdots \\
\bar{N}^{j_0} &= \{\nu \in \mathbb{Z}_{\geq 1} \mid 2(m+j_0-1)\nu - 2 = k\nu - 2 < k \leq \\
&\qquad \leq 2(m+j_0)\nu - 2 = (k+2)\nu - 2\}.
\end{aligned}$$

We have $\cup_{i=0}^{j_0} \bar{N}^i = \mathbb{Z}_{\geq 1}$. Also for a finite dimensional C^*-algeba F, let $\bar{\mathcal{E}}(F) = \mathcal{N}(F) \cap \bar{N}^0$ and for $i = 1, \ldots, j_0$, let $\bar{\mathcal{Z}}^i(F) = \mathcal{N}(F) \cap \bar{N}^i$ and also $\bar{\mathcal{Z}}(F) = \cup_{i=1}^{j_0} \bar{\mathcal{Z}}^i(F)$. Further we define $\bar{E}(F) = \oplus_{\nu \in \bar{\mathcal{E}}(F)} M_\nu(\mathbb{C})$ and $\bar{Z}(F) = \oplus_{\nu \in \bar{\mathcal{Z}}(F)} M_\nu(\mathbb{C})$ so that $F = \bar{E}(F) \oplus \bar{Z}(F)$.

Step.1. Fix an index $\nu \in \bar{\mathcal{Z}}^1(A_p)$ and consider the following diagram:

$$\begin{tikzcd}
\mathrm{GL}_m(M_\nu) \arrow[r, "\mathrm{incl}_\nu"] \arrow[d] & \mathrm{GL}_m(A_p) \arrow[r, "\varphi_{\infty p}"] \arrow[d] & \mathrm{GL}_m(A) \arrow[d, "\iota_m"] \\
\mathrm{GL}_{m+1}(M_\nu) \arrow[r] & \mathrm{GL}_{m+1}(A_p) \arrow[r] & \mathrm{GL}_{m+1}(A)
\end{tikzcd}$$

where incl_ν denotes the homomorphism induced by the inclusion $M_\nu \to A_p$. The above diagram induces a commutative diagram in the k-th homotopy groups. Since $m \geq 2$, $2m\nu - 2 < k$ and k is even, we have $2 \leq m\nu \leq k/2$. By the hypothesis $(*)$ we see that $\pi_k(\mathrm{GL}_m(M_\nu)) \cong \pi_k(U(m\nu)) \neq 0$. On the other hand, since $k \leq 2(m+1)\nu - 2$, we have $\pi_k(\mathrm{GL}_{m+1}(M_\nu)) = 0$. By the assumption (1), the homomorphism $(\iota_m)_\sharp : \pi_k(\mathrm{GL}_m(A)) \to \pi_k(\mathrm{GL}_{m+1}(A))$ is an isomorphism. Thus we see $(\varphi_{\infty p})_\sharp \circ (\mathrm{incl}_\nu)_\sharp = 0 : \pi_k(\mathrm{GL}_m(M_\nu)) \to$

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$\pi_k(\mathrm{GL}_m(A))$. Since $\pi_k(\mathrm{GL}_m(M_\nu))$ is finitely generated, we can apply Theorem 2.11 to find $q \geq p$ such that

$$(\varphi_{q\ p})_\sharp \circ (\mathrm{incl}_\nu)_\sharp = 0. \tag{3.7}$$

Step.2. For an index $\mu \in \mathcal{N}(A_q)$, consider

$$\varphi_{q\ p}^{\mu\nu} := \mathrm{incl}_\nu \circ \varphi_{q_\nu\ p} \circ \mathrm{proj}_\mu : M_\nu \to M_\mu.$$

By Theorem 2.7, $\mu \geq \nu$ whenever $\varphi_{q\ p}^{\mu\nu}$ is non-trivial.

Assume that $\nu \in \mathcal{N}(A_q)$ and suppose that $\varphi_{qp}^{\mu\nu}$ is non-trivial. Then it is unitarily equivalent to id_{M_ν} and hence the induced homomorphism

$$(\varphi_{q\ p}^{\mu\nu})_\sharp = \mathrm{id} : \pi_k(\mathrm{GL}_m(M_\nu)) \to \pi_k(\mathrm{GL}_m(M_\nu))$$

is an isomorphism. However, since $\pi_k(\mathrm{GL}_m(M_\nu)) \neq 0$, this contradicts (3.7) and we conclude that $\varphi_{q\ p}^{\mu\nu} = 0$ and in particular we have

$$\varphi_{q\ p}(M_\nu) \subset \oplus_{\mu \in \mathcal{N}(A_q),\ \mu \geq \nu+1} M_\mu. \tag{3.8}$$

If there exists $\mu \in \bar{\mathcal{Z}}^1(A_q)$ with $\mu \geq \nu + 1$ such that $\varphi_{q\ p}^{\mu\nu}$ is non-trivial, then enumerate all such μ's as $\mu_1, \ldots, \mu_\ell$. Since $\pi_k(\mathrm{GL}_m(M_{\mu_t})) \neq 0$, we can repeat the above process to each μ_t, with (3.7), to find $q_1, \ldots, q_\ell$ such that

$$\varphi_{q_t\ q}(M_{\mu_t}) \subset \oplus_{\lambda \in \mathcal{N}(A_{q_t}), \lambda \geq \mu_t + 1} M_\lambda$$

and let $q(1) = \max\{q_1, \ldots, q_t\}$. Then we see

$$\varphi_{q(1)\ p}(M_\nu) \subset \oplus_{\mu \in \mathcal{N}(A_{q(1)}), \mu \geq \nu+2} M_\mu.$$

This process can be repeated until we find $q_\nu \geq q(1)$ such that

$$\varphi_{q_\nu\ p}(M_\nu) \subset \oplus_{\mu \in \bar{\mathcal{E}}(A_{q_\nu})} M_\mu = \bar{E}(A_{q_\nu}).$$

We repeat the above to each $\nu \in \bar{\mathcal{Z}}^1(A_p)$ and let $q_1 = \max\{q_\nu \mid \nu \in \bar{\mathcal{Z}}^1(A_p)\}$. Then we have

$$\varphi_{q_1\ p}(\bar{Z}^1(A_p)) \subset \bar{E}(A_{q_1}).$$

Step 3. Now we follow Step 3 of the proof of (2a)$\Rightarrow$ (5) in Main Theorem 1: we take $\nu \in \mathcal{Z}^2(A_p)$ and repeat Step 2 above by replacing $\pi_k(\mathrm{GL}_m(M_\nu))$ and $\pi_k(\mathrm{GL}_{m+1}(M_\nu))$ with $\pi_k(\mathrm{GL}_{m+1}(M_\nu))$ and $\pi_k(\mathrm{GL}_{m+2}(M_\nu))$ respectively. Then we find $q_2^0 \geq q_1$ such that

$$\varphi_{q_2^0\ q_1}(\bar{Z}^2(A_p)) \subset \bar{E}(A_{q_2^0}) \oplus \bar{Z}^1(A_{q_2^0}).$$

Another application of the previous step yields $q_2 \geq q_2^0$ such that $\varphi_{q_2\ q_2^0}(\bar{Z}^1(A_{q_2^0})) \subset \bar{E}(A_{q_2})$, which results in:

$$\varphi_{q_2\ p}(\bar{Z}^1(A_p) \oplus \bar{Z}^2(A_p)) \subset \bar{E}(A_{q_2}).$$

We inductively proceed to find $q := q_{j_0}$ such that

$$\varphi_{q\ p}(\bar{Z}(A_p)) = \varphi_{q\ p}(\oplus_{i=1}^{j_0} \bar{Z}^i(A_p)) \subset \bar{E}(A_q)$$

as desired. This proves the conclusion (5).

This completes the proof of Main Theorem 2. $\square$

The author is grateful to the referee for pointing out that [14, Theorem 5.7] is relevant to the statement (1) below.

Example 3.1. *Let A be an AF algebra with a Bratteli diagram $\mathcal{D} = \{(p,\nu) \mid \nu \in \mathcal{N}(A_p),\ p \geq 1\}$, where the nodes at the level p is the set $\{(p,\nu) \mid \nu \in \mathcal{N}(A_p)\}$.*

(1) (cf. [13, Theorem 3.8]) Let $m_p = \min_{\nu \in \mathcal{N}(A_p)} \nu$. Then $\lim_{p\to\infty} m_p = \infty$ if and only if $\mathrm{hsr}_k(A) = \mathrm{csr}_k(A) = 1$ for each $k \geq 1$.

(2) If $\mathcal{D}$ contains an infinite ν-chain (Definition 2.8), then

$$\mathrm{hsr}_k(A) \geq \begin{cases} \lceil \frac{k+1}{2\nu} \rceil & \text{if } k \text{ is odd.} \\ \lceil \frac{k+2}{2\nu} \rceil & \text{if } k \text{ is even and satisfies the condition } (*). \end{cases}$$

Proof. (1): If $\lim_{p\to\infty} m_p = \infty$, then the equalities $\mathrm{hsr}_k(A) = \mathrm{csr}_k(A) = 1$, for each $k \geq 1$, are verified straightforwardly. The converse implication follows from the proof of [14, Theorem 5.7].

For (2) assume that k is odd and suppose that $m := \mathrm{hsr}_k(A) < \frac{k+1}{2\nu}$. There exists a non-trivial homomorphism $A \to M_\nu(\mathbb{C})$ by Proposition 2.9. By Main Theorem 1, we have $2m\nu - 1 \geq k$, a contradiction. The same argument works for an even k satisfying the condition $(*)$. □

For example the Bratteli diagram of the GICAR-algebra A ([1, Example III.5.5]) contains an infinite 1-chain. By the above and Proposition 2.15, we see that $\mathrm{hsr}_k(A) = \frac{k+1}{2}$ for each odd integer k and $\mathrm{hsr}_k(A) = \frac{k}{2} + 1$ for each even integer $k \geq 4$ satisfying $(*)$.

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