Permutations and woven g-frames

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Abstract

Two g-frames $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ for the Hilbert space $\mathcal{U}$ are called woven if for each subset σ of $\mathcal{I}$ the weaving $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$. The aim of this paper is considering the reordered families of a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ with the various woven problems. First, we state some useful results for exact g-frames and excess of g-frames. Then for $\sigma \subset \mathcal{I}$ we consider the families of weavings $\{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ where π is a permutation function on $\mathcal{I}$ and obtain some new conclusions. At last, we give relations of reordered weavings and operators, especially g-frame operators.

Keywords: g-frame, woven g-frame, permutation function, excess of g-frame.

MSC Classification: 42C15 , 46B28

1 Introduction

A powerful tool in the study of many branches of mathematics and other sciences are frames that were introduced first in 1952 [11]. In 1980 Young [19], and after that in 1986 Daubechies, et al. [9] reintroduced frames in Hilbert spaces. A frame in a Hilbert space defined as follows:

Let H be a Hilbert space and let $\mathcal{I}$ be a countable index set. A sequence $\{\phi_i\}_{i\in\mathcal{I}}$ in H is called a frame for H if there exist positive numbers $A \leq B < \infty$

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such that

$$A\|x\|^2 \leq \sum_{i\in\mathcal{I}} |\langle x, \phi_i\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$. Some applications of frames in pure and applied mathematics, harmonic analysis, and even quantum communication can be found in [3–6, 8, 12, 14]. In 2016, Bemrose and et al. introduced a new concept in frame theory which is motivated by a problem in distributed signal processing, particularly in wireless sensor network, and is called woven frames [2]. Two frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\psi_i\}_{i\in\mathcal{I}}$ for the Hilbert space H are called woven if there exist universal positive finite bounds A and B such that for each $\sigma \subset \mathcal{I}$ we have

$$A\|x\|^2 \leq \sum_{i\in\sigma} |\langle x, \phi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle x, \psi_i\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$. After the woven frames were introduced, Hafshejani and Dehghan introduced P-woven frames. To study P-woven frames the interested reader can refer to [13]. Also, the concept of reordered weavings of a frame is considered in [1].

The subject which we study in this manuscript is related to woven g-frames [10]. A generalization of frames are g-frames which are defined by Sun in 2005 [16]. Sun in [16] introduced a type of frames that are called g-frames, and he showed that most generalizations of frames can be regarded as special cases of g-frames. For more details about g-frames we refer the reader to [15, 17, 18, 20]. Some problems such as the restrictions of hardware conditions, particularly in large wireless sensor network cause the network should be split into some sub-networks. Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ be g-frames for the space, so we can measure a signal $\mathcal{X}$ with either Λ_i or Γ_i. In this case a package of information is a set of numbers $\{\Lambda_i\mathcal{X}\}_{i\in\sigma} \cup \{\Gamma_i\mathcal{X}\}_{i\in\sigma^c}$ for some subset $\sigma \subset \mathcal{I}$. If $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is a g-frame for each $\sigma \subset \mathcal{I}$, the signal $\mathcal{X}$ can be obtained regardless of which measurement is taken [7].

In some applications, it is better to recover a signal $\mathcal{X}$ with the families of g-frames $\{\Lambda_i\}_{i\in\sigma}\cup\{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ where π is a permutation function on $\mathcal{I}$ and $\sigma \subset \mathcal{I}$. For example, some coefficients may be erased when a signal is transmitted or some coefficients are shifted together. Therefore, those g-frames are useful that are resistant to these events and do not show much change during signal reconstruction. This manuscript is devoted to the reordered families of a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ which are of the forms $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ where π is a permutation function on $\mathcal{I}$, and the authors are concentrated on the weavings $\{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$. The outline of the rest of the paper is organized as follows:

In section 2 we review the basic definitions of g-frames, and we give some results about g-frames that are used in other sections. In section 3 we focus on reordered weavings of a g-frame. At the first of section 3, to clarify the our motivation we present some examples and after that we solve some problems that are designed and are initiative. In section 4 we consider the woven g-frame operators and study some relations with reordered weavings of a g-frame.

2 Preliminaries

In this section, first we review the definition of a g-frame and other subjects that we need in this paper [16]. Then we study the exact g-frames and provide some requierd content.

For the Hilbert spaces $\mathcal{U}$ and $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, let $B(\mathcal{U},\mathcal{V}_i)$ be the Banach space of all bounded linear operators from $\mathcal{U}$ in to $\mathcal{V}_i$ and consider $\Lambda_i \in B(\mathcal{U},\mathcal{V}_i)$, $i\in\mathcal{I}$. The sequence $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ if there exist two positive constants A and B, that are called the lower and upper g-frame bounds, respectively such that:

$$A\|f\|^2 \le \sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad \forall f\in\mathcal{U}.$$

A g-frame is said to be a tight g-frame if $A = B$, and also it is said Parseval if $A = 1$. When the sequence $\{\mathcal{V}_i : i\in\mathcal{I}\}$ is clear, $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-frame for $\mathcal{U}$. Also it is called $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\mathcal{V}$ whenever $\mathcal{V}_i = \mathcal{V}$ for each $i\in\mathcal{I}$. A sequence $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-Bessel sequence with bound B if it satisfies in right hand side in the definition of a g-frame. If a g-frame ceases to be a g-frame whenever anyone of its elements is removed, it is called exact g-frame. A sequence $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called g-complete if $\{f\in\mathcal{U} : \ \Lambda_i f = 0, i\in\mathcal{I}\} = \{0\}$. If $\{\Lambda_i\}_{i\in\mathcal{I}}$ is g-complete and there are $0 < A \le B < \infty$ such that

$$A\sum_{i\in\mathcal{I}_1} \|g_i\|^2 \le \|\sum_{i\in\mathcal{I}_1} \Lambda_i^* g_i\|^2 \le B\sum_{i\in\mathcal{I}_1} \|g_i\|^2,$$

for any finite subsset $\mathcal{I}_1 \subset \mathcal{I}$ and $g_i \in \mathcal{V}_i$, $i\in\mathcal{I}_1$, then $\{\Lambda_i\}_{i\in\mathcal{I}}$ is called a g-Riesz basis for $\mathcal{U}$. Now, we are going to reviwe the g-frame operators.

The space $\left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2}$ is defined by

$$\left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2} = \left\{ \{f_i\}_{i\in\mathcal{I}} : f_i \in \mathcal{V}_i, \quad i\in\mathcal{I} \quad and \quad \sum_{i\in\mathcal{I}} \|f_i\|^2 < \infty \right\},$$

and has the inner product

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i\rangle.$$

It is clear that $\left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2}$ is a Hilbert space. By consider

$$\mathcal{V}_i' = (..., 0, 0, 0, \mathcal{V}_i, 0, 0, 0, ...),$$

without lose of generality we can assume that for each $i\in\mathcal{I}$, $\mathcal{V}_i$ is a subspace of $\left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2}$.

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For the g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ the well-defined operators T_Λ, T_Λ^* and S_Λ are considered as follows:

$$T_\Lambda : \mathcal{U} \to \left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2}, \quad T_\Lambda(f) = \{\Lambda_i f\}_{i\in\mathcal{I}}.$$
$$T_\Lambda^* : \left(\sum_{i\in\mathcal{I}} \oplus \mathcal{V}_i\right)_{l_2} \to \mathcal{U}, \quad T_\Lambda^*\left(\{f_i\}_{i\in\mathcal{I}}\right) = \sum_{i\in\mathcal{I}} \Lambda_i^* f_i.$$
$$S_\Lambda : \mathcal{U} \to \mathcal{U}, \qquad S_\Lambda f = T_\Lambda^* T_\Lambda f = \sum_{i\in\mathcal{I}} \Lambda_i^* \Lambda_i f.$$

Usually T_Λ, T_Λ^* and S_Λ are called analysis, synthesis and g-frame operators. It is well-known that S is bounded, invertible and positive. Also, the g-frame $\{\Lambda_i S^{-1}\}_{i\in\mathcal{I}}$ is called canonical dual g-frame of $\{\Lambda_i\}_{i\in\mathcal{I}}$.

2.1 Exact *g*-frames

In this subsection, we give some results that are useful in the rest of this paper. It is well-known that every g-Riesz basis is exact g-frame, but the converse is not true. Now, we state a theorem which is in fact about exact g-frames.

Theorem 1 *[16, Theorem 3.5] Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, and $\{\Theta_i\}_{i\in\mathcal{I}}$ be the canonical dual g-frame.*

(1) If there exists $g_0 \in \mathcal{V}_{i_0} \setminus 0$ such that $\Theta_{i_0}\Lambda_{i_0}^* g_0 = g_0$, then $\{\Lambda_i : i \in \mathcal{I}, i \neq i_0\}$ is not g-complete.

(2) If there exists $f_0 \in \mathcal{U} \setminus 0$ such that $\Lambda_{i_0}^*\Theta_{i_0} f_0 = f_0$, then $\{\Lambda_i : i \in \mathcal{I}, i \neq i_0\}$ is not g-complete.

(3) If $I - \Lambda_{i_0}\Theta_{i_0}^*$ or $I - \Theta_{i_0}\Lambda_{i_0}^*$ is bounded invertible on $\mathcal{V}_{i_0}$, then $\{\Lambda_i : i \in \mathcal{I}, i \neq i_0\}$ is a g-frame for $\mathcal{U}$.

The proof of the next proposition follows from Theorem 1, and the fact that, if $\dim\mathcal{V}_\mathrm{i} < \infty$, then

$$\begin{aligned}\ker(\mathrm{I}_{\mathcal{V}_\mathrm{i}} - \Theta_\mathrm{i}\Lambda_\mathrm{i}^*) = 0 &\Leftrightarrow \mathrm{range}(\mathrm{I}_{\mathcal{V}_\mathrm{i}} - \Theta_\mathrm{i}\Lambda_\mathrm{i}^*) = \mathcal{U} \\ &\Leftrightarrow \overline{\mathrm{range}}(I_{\mathcal{V}_i} - \Theta_i\Lambda_i^*) = \mathcal{U},\end{aligned}$$

where $I_{\mathcal{V}_i}$ is the identity operator on $\mathcal{V}_i$.

Proposition 2 *Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, $\{\Theta_i\}_{i\in\mathcal{I}}$ is the canonical g-dual frame for $\{\Lambda_i\}_{i\in\mathcal{I}}$, $i_0 \in \mathcal{I}$ is arbitrary and* $\dim\mathcal{V}_\mathrm{i} < \infty$ *for each $i \in \mathcal{I}$. The following statements are equivalent:*

(1) $\{\Lambda_i\}_{i\in\mathcal{I}}$ is an exact g-frame for $\mathcal{U}$.

(2) $I-\Theta_{i_0}\Lambda^*_{i_0}$ *is not an injective operator on* $\mathcal{V}_{i_0}$*.*

(3) $I-\Theta_{i_0}\Lambda^*_{i_0}$ *is not a surjective operator on* $\mathcal{V}_{i_0}$*.*

(4) $\{\Lambda_i : i\in\mathcal{I}, i\neq i_0\}$ *is not g-complete.*

In the next, we define the excess of a g-frame which play a basic role in this paper.

Definition 1 Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$. Consider Ψ as follows:

$$\Psi=\Big\{\mathcal{J}\subset\mathcal{I} : \{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}} \;\; is \;\; a \;\; g\text{-frame} \;\; \text{for} \;\; \mathcal{U}\Big\}.$$

Set $\kappa=\sup\{|\mathcal{J}| : \mathcal{J}\in\Psi\}$, where $|\mathcal{J}|$ is the cardinal number of $\mathcal{J}$. Now for each $\mathcal{J}_0\in\Psi$ with $\kappa=|\mathcal{J}_0|$, we say $\{\Lambda_i\}_{i\in\mathcal{J}_0}$ is excess of $\{\Lambda_i\}_{i\in\mathcal{I}}$.

A result about excess of g-frames and exact g-frames is given in following theorem.

Theorem 3 *Assume that* $\{\Lambda_i\}_{i\in\mathcal{I}}$ *is a g-frame for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$*,* $\{\Theta_i\}_{i\in\mathcal{I}}$ *is the canonical g-dual frame,* $\mathcal{J}\subset\mathcal{I}$ *and* $\dim\mathcal{V}_\mathrm{i}<\infty$ *for each* $i\in\mathcal{I}$*. The following statements are equivalent:*

(1) $I-\Theta_i\Lambda^*_i$ *is an invertible operator on* $\mathcal{V}_i$ *for all* $i\in\mathcal{J}$*.*

(2) $\{\Lambda_i : i\in\mathcal{I}\setminus\mathcal{J}\}$ *is a g-frame for* $\mathcal{U}$*.*

(3) $\{\Lambda_i\}_{i\in\mathcal{J}}$ *is a nonempty subset of* excess *of* $\{\Lambda_i\}_{i\in\mathcal{I}}$*.*

Proof If for $i_0\in\mathcal{J}$ the operator $I-\Theta_{i_0}\Lambda^*_{i_0}$ is invertible on $\mathcal{V}_{i_0}$, then by the use of Proposition 2, the sequence $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\{i_0\}}$ is a g-frame for $\mathcal{U}$. So **(1)** $\rightarrow$ **(2)** is proved. The proof of **(2)** $\rightarrow$ **(3)** is a result of Definition 1. Now, assume $I-\Theta_{i_0}\Lambda^*_{i_0}$ is not an invertible operator on $\mathcal{V}_{i_0}$ for some $i_0\in\mathcal{J}$. So Proposition 2 implies that, $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\{i_0\}}$ is not g-complete, and so is not a g-frame for $\mathcal{U}$. Thus, Λ_{i_0} is not in the excess of $\{\Lambda_i\}_{i\in\mathcal{I}}$, and this proves **(3)** $\rightarrow$ **(1)**. □

3 Weavings and permutations

In this section, we consider the relation of the families $\{\Lambda_i\}_{i\in\sigma}\cup\{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ of a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ and permutaion functions on $\mathcal{I}$. At first, we mention some examples that are motivations for us. Then we state and prove a theorem which gives an equivalance condition for the reordered weavings of a g-frame. Throughout the paper, reordered weavings of $\{\Lambda_i\}_{i\in\mathcal{I}}$ are families of the form $\{\Lambda_i\}_{i\in\sigma}\cup\{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ where $\sigma\subset\mathcal{I}$ and π is a permutation function on $\mathcal{I}$.

Example 1 Suppose $\{\Lambda_1,\Lambda_2,\dots,\Lambda_M\}$ and $\{\Gamma_1,\Gamma_2,\dots,\Gamma_M\}$ are g-Riesz bases for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i=1}^M$ such that

$$\Gamma_i = \begin{cases} \Lambda_{j_0} & i = i_0 \\ \Lambda_{i_0} & i = j_0 \\ \Lambda_i & i \neq i_0, j_0. \end{cases}$$

Then $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_M\}$ and $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_M\}$ are not woven.

Proof Set $\sigma = \{i_0\}$. The family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is not a g-frame for $\mathcal{U}$, because $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_M\}$ is a g-Riesz basis for $\mathcal{U}$ and Λ_{j_0} does not appear in $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$. Thus, the conclusion is desired. □

In the above example, $\Gamma_i = \Lambda_{\pi(i)}$ where π is a permutation function on $\{1, \ldots, M\}$ defined by

$$\pi(i) = \begin{cases} i_0, & i = j_0 \\ j_0, & i = i_0 \\ i, & i \neq i_0, j_0. \end{cases}$$

Example 2 Assume that $\{\Lambda_1, \Lambda_2, \Lambda_3, \ldots\}$ is a g-Riesz basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathbb{N}}$, and consider the family

$$\{\Gamma_1, \Gamma_2, \Gamma_3, \ldots\} = \{\Lambda_3, \Lambda_1, \Lambda_5, \Lambda_2, \Lambda_7, \Lambda_4, \Lambda_9, \Lambda_6, \ldots\},$$

for $\mathcal{U}$. Then for some $\sigma \subset \mathbb{N}$ the family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$, but for each $\sigma \subset \mathbb{N}$ the family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is not a g-Riesz basis for $\mathcal{U}$.

Proof Let $\sigma \subset \mathbb{N}$ with $\sigma \neq \emptyset, \mathbb{N}$. The following cases occure:

(1) If $\sigma = \{1, 3, 5, 7, \ldots\}$, then Λ_1 appear twice in $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$.

(2) When $\sigma \subsetneq \{1, 3, 5, 7, \ldots\}$, put $i_0 = \min\sigma$. Then Λ_{i_0} appear twice in $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$.

(3) If $\sigma = \{2, 4, 6, 8, \ldots\}$, then $\Lambda_1 \notin \{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$.

(4) When $\sigma \subsetneq \{2, 4, 6, 8, \ldots\}$, the proof is similar to the proof of case (2).

(5) If $\sigma \subsetneq \mathbb{N}$ contains both of even and odd numbers, by the proof of the above caces we can deduce that $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is not a g-Riesz basis for $\mathcal{U}$.

Thus in cases (1) and (2), $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$ but in all of the above cases $\{\Lambda_i\}_{i\in\sigma} \cup \{\Gamma_i\}_{i\in\sigma^c}$ is not a g-Riesz basis for $\mathcal{U}$, and the proof is finished. □

Example 3 Let $\{E_1, E_2, \ldots, E_r\}$ be a g-orthonormal basis for $\mathbb{C}^n$ with respect to $\mathbb{C}^m$ and $r > 1$. Define a g-frame $\{\Lambda_i\}_{i\in\mathbb{Z}}$ for $\mathbb{C}^n$ by

$$\Gamma_i = \begin{cases} 2^{-|\frac{k}{2}|} E_1, & i = rk \\ 2^{-|\frac{k}{2}|} E_2, & i = rk + 1 \\ \quad\vdots & \\ 2^{-|\frac{k}{2}|} E_r, & i = rk + r - 1, \end{cases}$$

where $k \in \mathbb{Z}$. Let $S_l(i) = i+l$ be the l-shift operator on $\mathbb{Z}$ for $l \in \mathbb{Z}$. Then $\{\Gamma_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_{S_l(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathbb{C}^n$ if and only if $l = rq$ for some $q \in \mathbb{Z}$.

Proof First assume that $\{\Gamma_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_{S_l(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathbb{C}^n$, where $S_l(i) = l + i$ for each $i \in \mathbb{Z}$. By the way of contradiction, let $l \neq rq$ for all $q \in \mathbb{Z}$. So there exist $k, s \in \mathbb{Z}$ with $1 \leq s \leq r-1$ such that $l = rk + s$. Put $\sigma = \{i \in \mathbb{Z} : i \neq rp, \quad \forall p \in \mathbb{Z}\}$. Thus, $S_l(i)$ is not a multiple of r for each $i \in \sigma^c$. Now $E_1 \notin \{\Gamma_i\}_{i\in\sigma} \cup \{\Gamma_{S_l(i)}\}_{i\in\sigma^c}$, and the proof of this case is finished. Conversely, assume that $l = rq$ for some $q \in \mathbb{Z}$. Let $\sigma \subset \mathbb{Z}$ be arbitrary. So $\{rk+s\}_{s=0}^{r-1} \subset \sigma \cup S_l(\sigma^c)$ for some $k \in \mathbb{Z}$, and hence

$$\left\{2^{-|\frac{k}{2}|}E_i\right\}_{i=1}^{r} \subset \{\Gamma_i\}_{i\in\sigma} \cup \{\Gamma_{S_l(i)}\}_{i\in\sigma^c}.$$

Therefore, the conclusion is desired. □

The previous examples are motivations for us to study the reordered weavings of a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$, becuase in all of them $\Gamma_i = \Lambda_{\pi(i)}$ where π is a permutation function. The next theorem gives a necessary and sufficient condition about this subject. In following, $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, and $\dim\mathcal{V}_{\rm i} < \infty$ for each $i \in \mathcal{I}$.

Theorem 4 *Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$. For $\mathcal{J} \subset \mathcal{I}$, the following statements are equivalent:*

(1) $\{\Lambda_i\}_{i\in\mathcal{I}}$ is not an exact g-frame for $\mathcal{U}$.

(2) There exists a set of permutation functions $\{\pi_j\}_{j\in\mathcal{J}}$ on $\mathcal{I}$ such that for each $j \in \mathcal{J}$, $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$.

Proof $(\mathbf{1}) \to (\mathbf{2})$: If $\{\Lambda_i\}_{i\in\mathcal{I}}$ is not an exact g-frame for $\mathcal{U}$, then by Theorem 3, $\{\Lambda_i : i \in \mathcal{I}\setminus\mathcal{J}\}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathcal{I}\setminus\mathcal{J}\}$. If $\Lambda_j = \Lambda_{i_0}$ for some $i_0 \in \mathcal{I}\setminus\mathcal{J}$, define a permutation function π_j on $\mathcal{I}$ as follows:

$$\pi_j(i) = \begin{cases} i_0, \ i = j \\ j, \ i = i_0 \\ i, \ i \neq j, i_0. \end{cases}$$

It is easy to see that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$. On the other hand if $\Lambda_j \neq \Lambda_i$ for all $i \in \mathcal{I}\setminus\mathcal{J}$, then $\Lambda_j^*(\mathcal{V}_j) \subseteq \overline{\mathbf{span}}\{\Lambda_i^*(\mathcal{V}_i) : i \in \mathcal{I}\setminus\mathcal{J}\}$ because of by Proposition 2, the set $\{\Lambda_i : i \in \mathcal{I}\setminus\mathcal{J}\}$ is g-complete in $\mathcal{U}$. Since $\dim\mathcal{V}_{\rm i}$ is finite for each $i \in \mathcal{I}$, there exist $\{{\rm i}_1, \ldots, {\rm i}_{\rm k}\} \subset \mathcal{I}\setminus\mathcal{J}$ such that $\Lambda_j^*(\mathcal{V}_j) \subseteq \mathbf{span}\{\Lambda_{\rm i}^*(\mathcal{V}_{\rm i}) : {\rm i} = {\rm i}_1, \ldots, {\rm i}_{\rm k}\}$. For any s with $1 \leq s \leq k$, one can define a permutation funcction π_j on $\mathcal{I}$ by

$$\pi_j(i) = \begin{cases} i_s, \ i = j \\ j, \ i = i_s \\ i, \ i \neq j, i_s. \end{cases}$$

Let $\sigma \subset \mathcal{I}$ be arbitrary. So $\{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)}\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ when $j, i_s \in \sigma$ or $j, i_s \in \sigma^c$, and in these cases the conclusion is desired. On

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the other hand assume that $j \notin \sigma$ and $i_s \in \sigma$. In this case $\Lambda_j \notin \{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)}\}_{i\in\sigma^c}$ but $\Lambda_{i_s} \in \{\Lambda_i\}_{i\in\sigma}$ and $\Lambda_{i_s} \in \{\Lambda_{\pi_j(i)}\}_{i\in\sigma^c}$. Thus,

$$\Lambda_{i_s}^*(\mathcal{V}_{i_s}) \subseteq \mathbf{span}\{\Lambda_{\mathrm{i}}^*(\mathcal{V}_{\mathrm{i}}) : \mathrm{i} = \mathrm{j}, \mathrm{i}_1, \ldots, \mathrm{i}_\mathrm{k}, \mathrm{i} \neq \mathrm{i}_\mathrm{s}\},$$

because $\Lambda_j^*(\mathcal{V}_j) \subseteq \mathbf{span}\{\Lambda_{\mathrm{i}}^*(\mathcal{V}_{\mathrm{i}}) : \mathrm{i} = \mathrm{i}_1, \ldots, \mathrm{i}_\mathrm{k}, \mathrm{i} \neq \mathrm{j}\}$. This implies that

$$\overline{\mathbf{span}}\left(\{\Lambda_i^*(\mathcal{V}_i)\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)}^*(\mathcal{V}_{\pi_j(i)})\}_{i\in\sigma^c}\right) = \mathcal{U}.$$

Therefore, $\{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)})\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ and the proof of this case is complete.

$(\mathbf{2}) \to (\mathbf{1})$: For each $j \in \mathcal{J}$ let $\pi_j \neq I_d$ be a permutation function on $\mathcal{I}$ such that the g-frames $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ are woven. So for some $r, s \in \mathcal{I}$ with $r \neq s$, $\pi_j(r) = s$. Now one can consider $\sigma \subset \mathcal{I}$ as $\sigma = \mathcal{I}\backslash\{r\}$. Since $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ are woven g-frames and

$$\{\Lambda_i\}_{i\in\mathcal{I}\backslash\{r\}} = \{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)}\}_{i\in\sigma^c},$$

then $\{\Lambda_i\}_{i\in\mathcal{I}}$ is not an exact g-frame. Thus by Proposition 2 the proof is finished. □

A helpful result is brought in the following corollary.

Corollary 1 Assume that $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$. The following statements are equivalent:

(1) There exists a permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$.

(2) The excess of $\{\Lambda_i\}_{i\in\mathcal{I}}$ is nonempty.

(3) There exists a proper subset $\mathcal{J}$ of $\mathcal{I}$ such that for each $j \in \mathcal{J}$ the operator $I - \Theta_j\Lambda_j^*$ is invertible on $\mathcal{V}_j$, where $\{\Theta_i\}_{i\in\mathcal{I}}$ is the canonical g-dual frame of $\{\Lambda_i\}_{i\in\mathcal{I}}$.

Proof By the use of Theorems 3 and 4, the proof is easily to seen. □

Example 4 In Examples 1 and 2, the g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ is exact. Thus by Corollary 1, there is no any permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ can be woven. But the excess of g-frame $\{\Gamma_i\}_{i\in\mathcal{I}}$ in example 3 is nonempty and so by Corollary 1, there exists a permutation function π on $\mathcal{I}$ such that $\{\Gamma_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven.

The following example shows the part (3) in Corollary 1 is beneficial.

Example 5 Let $\{E_i\}_{i=1}^s$ be a g-orthonormal basis for $B(\mathbb{C}^n, \mathbb{C}^m)$. Consider the family $\{\Gamma_i\}_{i=1}^{2s}$ by

$$\{E_1, E_1, E_2, E_2, \ldots, E_s, E_s\}.$$

For each i, the operator $I - \Theta_i\Gamma_i^*$ is invertible on $\mathbb{C}^m$, where $\{\Theta_i\}_{i=1}^{2s}$ is the canonical g-dual frame of $\{\Gamma_i\}_{i=1}^{2s}$. So by Proposition 2 and Corollary 1, there exists a permutation function π on $\{1, \ldots, 2s\}$ such that $\{\Gamma_i\}_{i=1}^{2s}$ and $\{\Gamma_{\pi(i)}\}_{i=1}^{2s}$ are woven g-frames.

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The next theorem, presents conditions on a g-frame $\{\Lambda_i\}_{i\in\mathcal{I}}$ such that the family $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ satisfies in Corollary 1, where $\mathcal{J}\subset\mathcal{I}$.

Theorem 5 *Suppose $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ and A, B are the lower and upper bounds. If there exists $\mathcal{J}\subset\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ is a g-frame and $\{\Lambda_i\}_{i\in\mathcal{J}}$ is a g-Bessel sequence with bound $0<D<A$, then there exists permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ are woven g-frames with lower and upper bounds $A-D$ and $2B$ respectively.*

Proof By the use of Corollary 1, there exists a permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$ with bounds A, B. Let $\sigma\subset\mathcal{I}\setminus\mathcal{J}$ be arbitrary. It is easy to see that the family $\{\Lambda_i\}_{i\in\sigma}\cup\{\Lambda_{\pi(i)}\}_{i\in\sigma^c}$ is a g-Bessel sequence for $\mathcal{U}$ with bound $2B$. On the other hand for each $f\in\mathcal{U}$ we have:

$$\begin{aligned}\sum_{i\in\sigma}\|\Lambda_i f\|^2+\sum_{i\in\sigma^c\cap(\mathcal{I}\setminus\mathcal{J})}\|\Lambda_{\pi(i)}f\|^2 &= \sum_{i\in\sigma\cup\mathcal{J}}\|\Lambda_i f\|^2-\sum_{i\in\mathcal{J}}\|\Lambda_i f\|^2\\ &+\sum_{i\in\sigma^c\cap(\mathcal{I}\setminus\mathcal{J})}\|\Lambda_{\pi(i)}f\|^2\\ &\geq (A-D)\|f\|^2.\end{aligned}$$

Thus, the conclusion is desired. □

At the end of this section, by using the subsets of a g-Bessel sequence which are g-frames, we give a proposition that tries to furnish the conditions of Corollary 1.

Proposition 6 *Suppose $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-Bessel sequence with bound B, and for some $\mathcal{J}\subset\mathcal{I}$, $\{\Lambda_i\}_{i\in\mathcal{J}}$ is a g-frame for $\mathcal{U}$ with lower bounds A. There exists a permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven g-frames with bounds $A, 2B$.*

Proof If $\{\Lambda_i\}_{i\in\mathcal{J}}$ is an exact g-frame, take $\pi=I_d$. Then we assume that $\{\Lambda_i\}_{i\in\mathcal{J}}$ is not an exact g-frame. So by Corollary 1, there exists a permutation function π on $\mathcal{J}$ such that $\{\Lambda_i\}_{i\in\mathcal{J}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{J}}$ are woven g-frames. Let $\sigma\subset\mathcal{I}$ be arbitrary. For each $f\in\mathcal{U}$

$$\sum_{i\in\sigma}\|\Lambda_i f\|^2+\sum_{i\in\sigma^c}\|\Lambda_{\pi(i)}f\|^2\leq 2B\|f\|^2.$$

On the other hand

$$\begin{aligned}\sum_{i\in\sigma}\|\Lambda_i f\|^2+\sum_{i\in\sigma^c}\|\Lambda_{\pi(i)}f\|^2 &\geq \sum_{i\in\sigma\cap\mathcal{J}}\|\Lambda_i f\|^2+\sum_{i\in\sigma^c\cap\mathcal{J}}\|\Lambda_{\pi(i)}f\|^2\\ &\geq A\|f\|^2.\end{aligned}$$

Therefore, the proof is complete. □

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4 Reordered weavings of a g-frame and operators

In this section, we consider the subjects which are relative to reordered weavings and g-frame operators. At first, we give some notations which are used in this section.

Notations. For each $j \in \{1, \ldots, m\}$ one can define the followings:

$$\left(l^2(\{\mathcal{V}_i\}_{i\in\mathcal{I}})\right)_j = \left\{ \{a_{ij}\}_{i\in\sigma_j} : a_{ij} \in \mathcal{V}_i, \sigma_j \subset \mathcal{I}, \sum_{i\in\sigma_j} \|a_{ij}\|^2 < \infty \right\}.$$

Also, we define the space:

$$\bigoplus_{j=1}^{m} \left(l^2(\{\mathcal{V}_i\}_{i\in\mathcal{I}})\right)_j = \left\{ \{a_{ij}\}_{i\in\mathcal{I}, j\in[m]} : \{a_{ij}\}_{i\in\mathcal{I}} \in l^2(\{\mathcal{V}_i\}_{i\in\mathcal{I}})_j, \forall j \in [m] \right\},$$

with the inner product

$$\left\langle \{a_{ij}\}_{i\in\mathcal{I}, j\in[m]}, \{b_{ij}\}_{i\in\mathcal{I}, j\in[m]} \right\rangle = \sum_{j\in[m]} \sum_{i\in\mathcal{I}} \langle a_{ij}, b_{ij} \rangle,$$

where $[m] = \{1, 2, \ldots, m\}$.

The next proposition, with using operators provides conditions that the different reordered weavings of a g-frame can be woven.

Proposition 7 *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$. The followings are equivalent.*

(1) There exists a set of permutation functions $\{\pi_j\}_{j=1}^m$ on $\mathcal{I}$ such that the g-frames $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ are woven for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$.

(2) There exist a bounded linear operator $T : \bigoplus_{j=1}^m \left(l^2(\{\mathcal{V}_i\}_{i\in\mathcal{I}})\right)_j \longrightarrow \mathcal{U}$ and a positive number A such that $T(E_{ij}) = \Lambda_{\pi_j(i)}$ and $AI_{\mathcal{U}} \leq TT^*$, where $\{E_{ij}\}$ is the orthonormal basis for $\bigoplus_{j=1}^m \left(l^2(\{\mathcal{V}_i\}_{i\in\mathcal{I}})\right)_j$.

Proof For the proof of **(1)** $\rightarrow$ **(2)**, let $\{\sigma_j\}_{j=1}^m$ be a sequence of subsets of $\mathcal{I}$ such that $\bigcup_{j=1}^m \sigma_j = \mathcal{I}$. Without lose of generality we can assume that $\{\sigma_j\}_{j=1}^m$ is a partition of $\mathcal{I}$. Define the operator $T : \bigoplus_{j=1}^m \left(l^2(\{\mathcal{V}_i\}_{i\in\mathcal{I}})\right)_j \longrightarrow \mathcal{U}$ as follows:

$$T(\{a_{ij}\}_{j\in[m], i\in\sigma_j}) = \sum_{j\in[m]} \sum_{i\in\sigma_j} \Lambda^*_{\pi_j(i)}(a_{ij}).$$

It is easy to see that T is a well define, bounded and linear operator and $T(E_{ij}) = \Lambda_{\pi_j(i)}$. On the other hand $T^*f = \{\Lambda_{\pi_j(i)}f\}$ for each $f \in \mathcal{U}$. Now by using the assumption, let $A > 0$ be the universal lower g-frame bound for $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$, $j \in [m]$. So for each $f \in \mathcal{U}$ we have

$$\begin{aligned} A\|f\|^2 &= A\langle f, f\rangle \\ &\leq \sum_{j\in[m]}\sum_{i\in\sigma_j} \|\Lambda_{\pi_j(i)}f\|^2 \\ &= \sum_{j\in[m]}\sum_{i\in\sigma_j} \langle\Lambda^*_{\pi_j(i)}\Lambda_{\pi_j(i)}f, f\rangle \\ &= \langle TT^*f, f\rangle, \end{aligned}$$

which implies that $AI_{\mathcal{U}} \leq TT^*$. For the proof of **(2)** $\to$ **(1)**, let $\{\sigma_j\}_{j=1}^m$ be a partition of $\mathcal{I}$. Then

$$\begin{aligned} A\|f\|^2 &= A\langle f, f\rangle \\ &\leq \langle TT^*f, f\rangle \\ &= \sum_{j\in\mathcal{J}}\sum_{i\in[m]} \|\Lambda_{\pi_j(i)}f\|^2. \end{aligned}$$

So $A > 0$ is the universal lower g-frame bound for $\{\Lambda_{\pi_j(i)}\}_{i\in\sigma_j}$ for each $j \in [m]$. Now we have

$$\begin{aligned} \sum_{j\in[m]}\sum_{i\in\sigma_j} \|\Lambda_{\pi_j(i)}f\|^2 &= \|T^*f\|^2 \\ &\leq \|T^*\|^2\|f\|^2. \end{aligned}$$

Thus for all $j \in [m]$, $\|T^*\|^2$ is the universal upper g-frame bound for $\{\Lambda_{\pi_j(i)}\}_{i\in\sigma_j}$ and the proof is finished. □

The next theorem is about the canonical g-duals of the reordered weavings of a g-frame.

Theorem 8 *Suppose $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with frame bounds A, B and g-frame operator S. There exists a set of permutation functions $\{\pi_j\}_{j\in\mathcal{J}}$ on $\mathcal{I}$ such that for each $\sigma \subset \mathcal{I}$ the family $\{\Lambda_i S^{-1}\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)}S^{-1}\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$ with universal bounds $\frac{1}{2B}, \frac{1}{A}$ respectively.*

Proof If $\{\Lambda_i\}_{i\in\mathcal{I}}$ is an exact g-frame then by Corollary 1, there is no nontrivial permutation function π on $\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$. In this case we put $\pi = I_d$, and the conclusion is desired. Assume $\{\Lambda_i\}_{i\in\mathcal{I}}$ is not an exact g-frame. By Theorem 4, there exists a set of permutation functions $\{\pi_j\}_{j\in\mathcal{J}}$ on $\mathcal{I}$ such that for each $j \in \mathcal{J}$, $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$. On the other hand, the g-frame operator of $\{\Lambda_{\pi_j(i)}\}_{i\in\mathcal{I}}$ is S. Since S is a bounded operator on $\mathcal{U}$ with close range, so $\{\Lambda_{\pi_j(i)}S^{-1}\}_{i\in\mathcal{I}}$ is a g-frame for $\mathcal{U}$ with bounds $A\|S\|^{-2}$ and $B\|S^{-1}\|^2$ for each $j \in \mathcal{J}$. Now, since for each $\sigma \subset \mathcal{I}$ the family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)}\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$ with bounds A and $2B$, then

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$\{\Lambda_i S^{-1}\}_{i\in\sigma} \cup \{\Lambda_{\pi_j(i)} S^{-1}\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$ with bounds $\dfrac{1}{2B}$ and $\dfrac{1}{A}$, and the proof is finished. $\square$

In following proposition, T_Λ and T_Λ^* are the analysis and synthesis operators for the g-frame $\Lambda = \{\Lambda_i\}_{i\in\mathcal{I}}$. Also for $\sigma \subset \mathcal{I}$, T_Λ^σ and $T_\Lambda^{*\sigma}$ are the analysis and synthesis operators for the g-frame $\{\Lambda_i\}_{i\in\sigma} \cup \Lambda_{\pi(i)}\}_{i\in\sigma^c}$.

Proposition 9 *Assume $\Lambda = \{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-Bessel sequences for $\mathcal{U}$ with respect to $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ and bound B. Let π be a permutation function on $\mathcal{I}$ such that*

(1) $T_\Lambda^* T_{\Lambda_\pi} = I_{\mathcal{U}}$

(2) $T_\Lambda^{*\sigma} T_{\Lambda_\pi}^\sigma = T_{\Lambda_\pi}^{*\sigma} T_\Lambda^\sigma$,

where $\Lambda_\pi = \{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$. Then $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Lambda_{\pi(i)}\}_{i\in\mathcal{I}}$ are woven g-frames for $\mathcal{U}$.

Proof It is easily seen taht for all $\sigma \subset \mathcal{I}$ the families $\{\Lambda_i\}_{i\in\sigma} \cup \Lambda_{\pi(i)}\}_{i\in\sigma^c}$ are g-Bessel seqences for $\mathcal{U}$ with universal bound $2B$. Now let $f \in \mathcal{U}$ and let $\sigma \subset \mathcal{I}$ be arbitrary, so we have:

$$\begin{aligned}
\|f\|^4 &= \langle f, f\rangle^2 \\
&= \langle T_\Lambda^* T_{\Lambda_\pi} f, f\rangle^2 \\
&= \langle T_\Lambda^{*\sigma} T_{\Lambda_\pi}^\sigma f + T_\Lambda^{*\sigma^c} T_{\Lambda_\pi}^{\sigma^c} f, f\rangle^2 \\
&\leq 2|\langle T_\Lambda^{*\sigma} T_{\Lambda_\pi}^\sigma f, f\rangle|^2 + 2|\langle T_\Lambda^{*\sigma^c} T_{\Lambda_\pi}^{\sigma^c} f, f\rangle|^2 \\
&= 2|\langle \sum_{i\in\sigma} \Lambda_i^* \Lambda_{\pi(i)} f, f\rangle|^2 + 2|\langle \sum_{i\in\sigma^c} \Lambda_{\pi(i)}^* \Lambda_i f, f\rangle|^2 \\
&= 2|\sum_{i\in\sigma} \langle \Lambda_{\pi(i)} f, \Lambda_i f\rangle|^2 + 2|\sum_{i\in\sigma^c} \langle \Lambda_i f, \Lambda_{\pi(i)} f\rangle|^2 \\
&\leq 2\sum_{i\in\sigma} \|\Lambda_{\pi(i)} f\|^2 \sum_{i\in\sigma} \|\Lambda_i f\|^2 + 2\sum_{i\in\sigma^c} \|\Lambda_i f\|^2 \sum_{i\in\sigma^c} \|\Lambda_{\pi(i)} f\|^2 \\
&\leq 2B\|f\|^2 \sum_{i\in\sigma} \|\Lambda_i f\|^2 + 2B\|f\|^2 \sum_{i\in\sigma^c} \|\Lambda_{\pi(i)} f\|^2 \\
&\leq 2B\|f\|^2 \left(\sum_{i\in\sigma} \|\Lambda_i f\|^2 + \sum_{i\in\sigma^c} \|\Lambda_{\pi(i)} f\|^2 \right).
\end{aligned}$$

This implies that

$$\frac{1}{2B}\|f\|^2 \leq \left(\sum_{i\in\sigma} \|\Lambda_i f\|^2 + \sum_{i\in\sigma^c} \|\Lambda_{\pi i} f\|^2 \right).$$

Thus, $\dfrac{1}{2B}$ is the universal lower bound for the reordered weavings $\{\Lambda_i\}_{i\in\sigma} \cup \Lambda_{\pi(i)}\}_{i\in\sigma^c}$ and the proof is finished. $\square$

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References

[1] A. Askarizadeh, A. Ahmadi, Reordered Frames and Weavings. Bull. Iran. Math. Soc. 48,(2022), 41–51.

[2] T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers, and R. G. Lynch, Weaving frames, OAM, 10 , 4 (2016) 1093-1116.

[3] J. Benedetto, and S. Li, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal., 5 (1998), 389-427.

[4] J. Benedetto, and S. Li, Subband coding and noise reduction in frame multiresolution analysis, Proceeding of SPIE Conference on Mathematical Imaging, San Diego, (1994).

[5] B.G. Bodmann, and D.W. Kribs, V.I. paulsen, Decoherence-insensitive quantum communication by optimal C^*-encoding, IEEE Trans. Inf. Theory, 53 (2007) 4738-4749.

[6] O. Christensen, An introduction to frames and Riesz bases, Birkhäuser, Boston, Basel, Berlin, 2002.

[7] P. G. Casazza, G. Kutyniok, and S. Li, Fusion frames and distributed processing, Appl. Comput. Harmon. Anal., 25 (2008) pp. 114-132.

[8] C.E.d'Attellis, and E.M. Fernfedez-Berdaguer, Wavelet theory and harmonic analysis in applied Sciences, Birkhäuser, Boston - Basel - Berlin, 1997.

[9] I. Daubechies, A. Grossmann, and Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys., 27 (1986) 1271-1283.

[10] D. Li, J. Leng, T. Hunge, and X. Li, On weaving g-frames for Hilbert spaces, Complex Anal. Oper. Theory, 14(2) (2020), 1–25.

[11] R. Duffin, and A. Schaeffer, A class of nonharmonic Fourier series, Trans. Am. Math. Soc. , 72 (1952) 341-366.

[12] K. Gröchenig, Foundations of time-frequency analysis, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA, 2001.

[13] A. B. Hafshejani, and M. A. Dehghan, P-woven frames,J. Math. Anal. Appl., 479 (2019) 673-687.

[14] A. A. Hemmat, and J. P. Gabardo, Properties of oblique dual frames in shift-invariant systems, J. Math. Anal. Appls., 356 (2009) 346-354.

[15] A. Nejati, and A. Rahimi, G-frames and stability of g-frames in Hilbert spaces, Methods Funct. Anal. Topol., 14 (3) (2008), 271-286.

[16] W. Sun, g-frames and g-Riesz bases, J. Math. Anal. Appls., 322 (2006) 437-452.

[17] W. Sun, Stability of g-frames, J. Math. Anal. Appls., 326 (2007) 858-868.

[18] X. Guo, On Redundancy, dilations and canonical duals of g-frames in Hilbert spaces, Banach J. Math. Anal. , 9 (4) (2015) 81-99.

[19] R. Young, An introduction to nonharmonic Fourier series, Academic Press, New York, 1980.

[20] L. Zang, W. Sun, and D. Chen, Excess of a class of g-frames, J. Math. Anal. Appls., 352 (2009) 711-717.