

A generalization of Sturm's comparison theorem

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Abstract. The Riccati equation method is used to establish a new comparison theorem for systems of two linear first order ordinary differential equation. This result is based on the concept of "null-classes", and is a generalization of Sturm's comparison theorem.

Key words: Sturm's comparison theorem, Sturm's majorant (minorant), linear systems, Riccati equation, null-elements, null-classes.

1. Introduction. Let $p_j(t)$, $q_j(t)$, $j = 1, 2$ be real-valued continuous functions on the interval $[a, b]$. Consider the second order linear ordinary differential equations

$$(p_j(t)\phi')' + q_j(t)\phi = 0, \quad j = 1, 2. \quad (1.1_j)$$

Definition 1.1. Eq. (1.1₂) is called a Sturm majorant for Eq. (1.1₁) on $[a, b]$ if

$$p_1(t) \geq p_2(t) > 0 \quad \text{and} \quad q_1(t) \leq q_2(t), \quad t \in [a, b]. \quad (1.2)$$

If in addition $q_1(t) < q_2(t)$ or

$$p_1(t) > p_2(t) > 0 \quad \text{and} \quad q_2(t) \neq 0$$

for some $t \in [a, b]$, then Eq. (1.1₂) is called a strict Sturm majorant for Eq. (1.1₁) on $[a, b]$. The Sturm's famous comparison theorem states (see [1], p. 334)

Theorem 1.1 (Sturm). Let (1.1₂) be a Sturm majorant for (1.1₁) and let $\phi = \phi_1(t) \neq 0$ be a solution of Eq. (1.1₁) having exactly n ($n \geq 1$) zeroes $t = t_1 < t_2 < \dots < t_n$ on $(t_0, t^0]$. Let $\phi = \phi_2(t) \neq 0$ be a solution of Eq. (1.1₂) satisfying

$$\frac{p_1(t)\phi_1'(t)}{\phi_1(t)} \geq \frac{p_2(t)\phi_2'(t)}{\phi_2(t)} \quad (1.3)$$

at $t = t_0$ (the expression on the right [or left] of (1.3) is considered to be $+\infty$ if $\phi_2(t_0) = 0$ [or $\phi_1(t_0) = 0$]; in particular, (1.3) holds at $t = t_0$ if $\phi_1(t_0) = 0$). Then $\phi_2(t)$ has at least n

zeroes on $(t_0, t_n]$. Furthermore $\phi_2(t)$ has at least n zeroes on (t_0, t^0) if either the inequality (1.3) holds at $t = t_0$ or (1.1₂) is a strict Sturm majorant for Eq. (1.1₁) on $[t_0, t_n]$.

Note that if $p_j(t) \neq 0$, $j = 1, 2$, $t \in [a, b]$, then the equations (1.1_j), $j = 1, 2$ are equivalent (reducible) to the systems

$$\begin{cases} \phi' = \frac{1}{p_j(t)}\psi \\ \psi' = -q_j(t)\phi, \quad t \in [a, b]. \end{cases} \quad (1.4_j)$$

$j = 1, 2$. They are particular cases of more general systems

$$\begin{cases} \phi' = f_j(t)\psi, \\ \psi' = -g_j(t)\phi, \quad t \in [a, b], \end{cases} \quad (1.5_j)$$

$j = 1, 2$, where $f_j(t)$, $g_j(t)$, $j = 1, 2$ are real-valued continuous functions on $[a, b]$. On the other hand obviously the systems (1.5_j), $j = 1, 2$ are reducible to the systems (1.1_j), $j = 1, 2$ respectively if in particular $f_j(t) \neq 0$, $t \in [a, b]$, $j = 1, 2$. In this case we can reformulate Theorem 1.1 for the systems (1.5_j), $j = 1, 2$ as follows.

Definition 1.2. *The system (1.5₂) is called a Shturm majorant for the system (1.5₁) on $[a, b]$, if*

$$f_2(t) \geq f_1(t) > 0, \quad \text{and} \quad g_2(t) \geq g_1(t), \quad t \in [a, b]. \quad (1.6)$$

If in addition $g_2(t) > g_1(t)$ or

$$f_2(t) > f_1(t) > 0 \quad \text{and} \quad g_2(t) \neq 0$$

for some $t \in [a, b]$, then the system (1.5₂) is called a strict Sturm majorant for the system (1.5₁) on $[a, b]$.

Theorem 1.2. *Let the system (1.5₂) be a Sturm majorant for the system (1.5₁) and let $(\phi_1(t), \psi_1(t))$ be a solution of (1.5₁) such that $\phi_1(t)$ has exactly n ($n \geq 1$) zeroes $t = t_1 < \dots < t_n$ on $(t_0, t^0]$. Let $(\phi_2(t), \psi_2(t))$ be a solution of (1.5₂) such that*

$$\frac{\psi_1(t_0)}{\phi_1(t_0)} \geq \frac{\psi_2(t_0)}{\phi_2(t_0)} \quad (1.7)$$

(the expression in the right [or left] of (1.7) is considered to be $+\infty$, if $\phi_2(t_0) = 0$ [or $\phi_2(t_0) = 0$]; in particular (1.7) holds if $\phi_1(t_0) = 0$). Then $\phi_2(t)$ has at least n zeroes on $(t_0, t_n]$. Furthermore $\phi_2(t)$ has at least n zeroes on (t_0, t^0) if either inequality (1.7) holds or (1.5₂) is a strict Sturm majorant for (1.5₁) on $[t_0, t_n]$.

Here arises the following question. Does Theorem 1.2 remain valid if we weaken the conditions (1.6) up to the following ones?

$$f_2(t) \geq f_1(t) \geq 0 \quad \text{and} \quad g_2(t) \geq g_1(t), \quad t \in [a, b]. \quad (1.8)$$

Before answering this question let us make some observations. Let $p(t)$, $q(t)$, $r(t)$, $P(t)$, $Q(t)$, $R(t)$ and $S(t)$ be real valued continuous functions on $[t_0, +\infty)$ and let $p(t) > 0$, $t \geq t_0$. Consider the general second order linear equation

$$(p(t)\phi')' + q(t)\phi' + r(t)\phi = 0, \quad t \geq t_0, \quad (1.9)$$

and the general linear homogeneous system of two equations

$$\begin{cases} \phi' = P(t)\phi + Q(t)\psi; \\ \psi' = R(t)\phi + S(t)\psi, \quad t \geq t_0. \end{cases} \quad (1.10)$$

The substitution $p(t)\phi' = \psi$ in (1.9) reduces it to the system

$$\begin{cases} \phi' = \frac{1}{p(t)}\psi, \\ \psi' = -r(t)\phi - \frac{q(t)}{p(t)}\psi, \quad t \geq t_0 \end{cases}, \quad (1.11)$$

which is a particular case of the system (1.10), and the substitutions

$$\phi \rightarrow \phi \exp\left\{\int_{t_0}^t P(\tau)d\tau\right\}, \quad \psi \rightarrow \psi \exp\left\{\int_{t_0}^t S(\tau)d\tau\right\}, \quad (1.12)$$

in (1.10) reduces it to its simpler form

$$\begin{cases} \phi' = \tilde{Q}(t)\psi, \\ \psi' = \tilde{R}(t)\phi, \quad t \geq t_0, \end{cases} \quad (1.13)$$

where $\tilde{Q}(t) \equiv \exp\left\{\int_{t_0}^t [S(\tau) - P(\tau)]d\tau\right\}$, $\tilde{R}(t) \equiv \exp\left\{\int_{t_0}^t [P(\tau) - S(\tau)]d\tau\right\}$, $t \geq t_0$.

Therefore Eq. (1.9) is reducible to its simpler form

$$(\tilde{p}(t)\phi')' + \tilde{q}(t)\phi = 0, \quad t \geq t_0. \quad (1.14)$$

Further, for brevity, by a zero of any solution $(\phi(t), \psi(t))$ of the system (1.10) we mean any zero of the function $\phi(t)$.

Definition 1.3. *The system (1.10) is called oscillatory if every its solution has arbitrarily large zeroes.*

Definition 1.4. *The system (1.10) is called non-oscillatory if for its arbitrary nontrivial solution $(\phi(t), \psi(t))$ there exists $T = T_{\phi, \psi} \geq t_0$ such that $\phi(t) \neq 0$, $t \geq T$.*

Let us compare the system (1.10) with Eq. (1.9) (or, which is the same, with the system (1.11)) from the point of view of mutual displacement and richness (the quantity and density on any finite interval) of zeroes of their solutions. From Theorem 1.1 it follows that the classic definitions of oscillation and non-oscillation of Eq. (1.9) are equivalent to Definitions 1.3 and 1.4 respectively. From Theorem 1.1 it follows also that the following statements are valid (due to mentioned above connections between equations (1.9) and (1.14) and the systems (1.11), (1.13))

- 1) The system (1.11) is either oscillatory or non-oscillatory,
- 2) Between two arbitrary two zeroes of any nontrivial solution $(\phi(t), \psi(t))$ of the system (1.11) there exists at least one zero of any solution of that system, linearly independent of $(\phi(t), \psi(t))$.

Note that the statement 1) does not hold for the general system (1.10) (see [2]). Therefore, there is no comparison theorem for general systems (1.10), similar to Theorem 1.1 (otherwise the statement 1) will be true for the general system (1.10)). The following example shows that assertion 2) is also false for the general system (1.10).

Example 1.1. *Set:* $f(t) \equiv \begin{cases} 0, & t \in [0, \pi], \\ \sin^2 t, & t \in [\pi, 2\pi], \end{cases} \quad g(t) \equiv 1, \quad t \in [0, 2\pi].$ *Consider the system*

$$\begin{cases} \phi' = f(t)\psi, \\ \psi' = g(t)\phi, \quad t \in [0, 2\pi]. \end{cases} \quad (1.15)$$

It is not difficult to verify that for the non trivial solution $(\phi(t), \psi(t))$ of this system with $\phi(0) = 0$, $\psi(0) = 1$ the function $\phi(t)$ is identically zero on $[0, \pi]$, whereas for all solutions $(\phi_1(t), \psi_1(t))$ of this system with $\phi_1(0) \neq 0$ the function $\phi_1(t)$ does not vanish on $[0, \pi]$. Therefore between two zeroes $t_1 = 0$ and $t_2 = \pi$ of the solution $(\phi(t), \psi(t))$ there is no zero of $(\phi(t), \psi(t))$.

Thus the statement 2) for the system (1.15) does not hold. This means that the answer on the posed above question is negative. Moreover, this example shows that some solutions of the general system (1.10) may have infinitely (in the case of the system (1.15) continuum) number of zeroes on any finite interval, but some others only finite ones,

whereas it is well known that any solution of Eq. (1.9) (hence of the system (1.11)) can have at most finite number of zeroes on any finite interval. However, despite this, the posed question can be answered positively if in the formulation of Theorem 1.2 the word "zeros" we replace by the word "zero-classes" (a concept, which is introduced in [3] and has the property, that for every solution $(\phi(t), \psi(t))$ of the system (1.5_j) the function $\phi(t)$ ($\psi(t)$) has no more than finite number of null-classes on $[a, b]$, see below) and the ordering symbol $<$ is replaced by another symbol \prec , ordering the null-classes (see below). It turns out that for any solution $(\phi(t), \psi(t))$ of the system (1.4_j), each zero-class of the function $\phi(t)$ coincides with one of its zeros and each zero of $\phi(t)$ is its a zero-class. Therefore, the solution of the above question with the indicated changes: "zeroes" \rightarrow "null-classes", $< \rightarrow \prec$ gives a generalization of Sturm's theorem.

In this paper we use the Riccati equation method to obtain a generalization of Sturm's comparison theorem (Theorem 1.1), which is based on the concept of "null-classes".

2. Auxiliary propositions. Along with the systems (1.5_j), $j = 1, 2$ consider the Riccati equations

$$y' + f_j(t)y^2 + g_j(t) = 0, \quad t \in [a, b], \quad j = 1, 2, \quad (2.1_j)$$

and the differential inequalities

$$\eta' + f_j(t)\eta^2 + g_j(t) \geq 0, \quad t \in [a, b], \quad j = 1, 2. \quad (2.2_j)$$

Remark 2.1. Every solution of Eq. (2.1₂) on an interval $[t_1, t_2)$ ($\subset [a, b]$) is also a solution of the inequality (2.2₂) on $[t_1, t_2)$,

Remark 2.2. If $f_1(t) \geq 0$, $t \in [t_1, t_2)$, then for every $\lambda \in (-\infty, +\infty)$ the function $\eta_\lambda(t) \equiv \lambda - \int_{t_1}^t g_1(\tau) d\tau$, $t \in [t_1, t_2)$ is a solution to the inequality (2.2₁) on $[t_1, t_2)$.

The following comparison theorem plays a crucial role in the proof of the main result.

Theorem 2.1. Let $y_2(t)$ be a solution of Eq. (2.1₂) on $[t_0, \tau_0)$ ($\subset [a, b]$) and let $\eta_1(t)$ and $\eta_2(t)$ be solutions of the inequalities (2.2₁) and (2.2₂) respectively on $[t_0, \tau_0)$, moreover suppose that $y_2(t_0) \leq \eta_j(t_0)$, $j = 1, 2$. In addition let the following conditions be satisfied

$f_1(t) \geq 0$, $t \in [t_0, \tau_0)$; $y_{(0)} - y_2(t_0) + \int_{t_0}^t \exp\left\{ \int_{t_0}^\tau f_1(s)(\eta_1(s) + \eta_2(s)) ds \right\} \left[(f_2(\tau) - f_1(\tau))y_2^2(\tau) + g_2(\tau) - g_1(\tau) \right] d\tau \geq 0$, $t \in [t_0, \tau_0)$, for some $y_{(0)} \in [y_2(t_0), \eta_1(t_0)]$. Then Eq. (2.1₁) has a solution $y_1(t)$ with the initial condition $y_1(t_0) \geq y_{(0)}$, on $[t_0, \tau_0)$; moreover $y_1(t) \geq y_2(t)$, $t \in [t_0, \tau_0)$.

See the proof in [4].

Besides of Theorem 2.1 for the proof of the main result we need also in the following three lemmas.

Lemma 2.1. Let $f_1(t) \geq 0$, $t \in [t_0, \tau_0)$, and let (t_1, t_2) be the maximum existence interval for a solution $y(t)$ of Eq. (2.1₁), where $t_0 < t_1 < t_2 < \tau_0$. Then

$$\lim_{t \rightarrow t_2 - 0} y(t) = -\infty, \quad \lim_{t \rightarrow t_1 + 0} y(t) = +\infty. \quad (2.3)$$

See the proof in [4].

Remark 2.3. The first equality of (2.3) remains valid also in the case when (t_1, t_2) is not the maximum existence interval, but $y(t)$ is not continuable to the right from the point t_2 .

Lemma 2.2. Let $f_1(t) \geq 0$, $t \in [t_0, \tau)$, and let $(t_{1k}, t_{2k}) (\subset [t_0, \tau_0))$ be the maximum existence interval for a solution $y_k(t)$ of Eq. (2.1₁), $k = 1, 2$. In addition let $y_1(t_1) > y_2(t_1)$ for some $t_1 \in (t_{11}, t_{21}) \cap (t_{12}, t_{22})$. Then $t_{11} > t_{12}$ and $t_{21} > t_{22}$.

See the proof in [4].

Lemma 2.3. Let $f_1(t) \geq 0$, $t \in [t_0, \tau_0)$, let $y_0(t)$ be a solution of Eq. (2.1₁) on $[t_0, \tau_0)$, and let $\eta_0(t)$ be a solution of the inequality (2.2₁) on $[t_0, \tau_0)$; moreover let $y_0(t_0) \leq \eta_0(t_0)$. Then $y_0(t) \leq \eta_0(t)$, $t \in [t_0, \tau_0)$.

See the proof in [4].

3. Main result. On the set $2^{\mathbb{R}}$ of subsets of real numbers \mathbb{R} define the order relation \prec , assuming $x \prec y$ if and only if for every $t_x \in x \in 2^{\mathbb{R}}$, $t_y \in y \in 2^{\mathbb{R}}$ the inequality $t_x < t_y$ is valid. Let $(\phi(t), \psi(t))$ be a nontrivial solution of the system (1.5₁). Since $\phi(t)$ is a continuous function on $[a, b]$ that zeroes form a closed set.

Definition 3.1. A connected component of zeroes of the function $\phi(t)$ of a solution $(\phi(t), \psi(t))$ of the system (1.5₁) is called a null-element of $\phi(t)$.

Let $z(t)$ be a solution of Eq. (2.1₁) with $z(a) = i$. Then $z(t)$ exists on $[a, b]$ and $y(t) \equiv \text{Im } z(t) > 0$, $t \in [a, b]$ and

$$\phi(t) = \frac{\mu}{\sqrt{y(t)}} \sin \left(\int_a^t f_1(\tau) y(\tau) d\tau + \theta \right), \quad t \in [a, b], \quad (3.1)$$

where μ and θ are some real constants (see [3]). Let $n(\phi)$ be a null-element of the function $\phi(t)$. By (3.1)

$$\int_a^t f_1(\tau) y(\tau) d\tau + \theta = \pi k_0, \quad t \in N(\phi), \quad k_0 \in \mathbb{Z}. \quad (3.2)$$

Hereafter by $[t_1, t_2]$ we mean the set of all points of \mathbb{R} lying between t_1 and t_2 , including themselves.

Definition 3.2. Null-elements $N_1(\phi)$ and $N_2(\phi)$ of the function $\phi(t)$ are called congenerous, if for every $t_j \in N(\phi)$, $j = 1, 2$ the inequality

$$\left| \int_{t_1}^t f_1(\tau)y(\tau)d\tau \right| < \pi, \quad t \in [t_1, t_2]$$

is valid.

It was shown in [3] that the congeniality relation between null-elements $N_1(\phi)$ and $N_2(\phi)$ is an equivalence relation.

Definition 3.3. An equivalence class of congenerous null-elements $N(\phi)$ of the function of a solution $(\phi(t), \psi(t))$ of the system (1.5₁) is called a null-class of $\phi(t)$.

Remark 3.1. It follows from (3.1) and (3.2) that if $f_1(t) \geq 0$, $t \in [a, b]$, then every null-class consists of only one null-element.

It was shown in [3] that for every solution $(\phi(t), \psi(t))$ of the system (1.5₁) the function $\phi(t)$ has a finite number of null-classes, and these null-classes are linearly ordered by \prec .

Let $(\phi_j(t), \psi_j(t))$ be a solution of the system (1.5_{*j*}), $j = 1, 2$, and let $N_1(\phi_2) = [t_1, \tau_1] \prec N_2(\phi_2) = [t_2, \tau_2] \prec \dots \prec N_n(\phi_2) = [t_n, \tau_n]$ be all of the null-classes of the function $\phi_2(t)$ on $[t_0, t^0]$, $t^0 \in (t_0, \tau_0)$.

Definition 3.4. The quarter $(\phi_1, \psi_1, f_1, g_1)$ is called a majorant of the quarter $(\phi_2, \psi_2, f_2, g_2)$ on $[t_0, t^0]$ if the following conditions are satisfied

1. $\psi_1(t_0)/\phi_1(t_0) \leq \psi_2(t_0)/\phi_2(t_0)$ (The expression on the left hand [respectively on the right hand] side of the inequality is considered to be equal $+\infty$ if $\phi_1(t_0) = 0$ [respectively, if $\phi_2(t_0) = 0$] in particular this is in the case if $\phi_2(t_0) = 0$);

2. $f_1(t) \geq f_2(t) \geq 0$, $t \in [t_0, t^0]$;

3. there exists $\xi_k \in (\tau_k, t_{k+1})$ ($k = 0, \dots, n - 1$) such that

(3₁) $g_1(t) \geq g_2(t)$, $t \in [\tau_k, \xi_k]$ ($k = 0, \dots, n - 1$)

(3₂) any solutions $\eta_{jk}(t)$ of the inequalities (2.1_{*j*}) ($j = 1, 2$) on $[\xi_k, t_{k+1}]$ such that $\eta_{jk}(\xi_k) > \psi_2(\xi_k)/\phi_2(\xi_k)$ (such solutions always exist by condition 2 and Remark 2.2) satisfy the inequalities

$$\int_{\xi_k}^t \exp \left\{ \int_{\xi_k}^{\tau} f_2(s) [\eta_{1,k}(s) + \eta_{2,k}(s)] ds \right\} [g_1(\tau) - g_2(\tau)] d\tau \geq 0, \quad t \in [\xi_k, t_{k+1}], \quad k = 0, \dots, n - 1.$$

In addition, suppose that $f_2(t) > 0$, $t \in [t_0, t^0]$ and either the strict inequality takes place in condition 1 or at least one of the following conditions is satisfied:

1'. $f_1(t') > f_2(t')$ and $g_1(t') \neq 0$ for some $t' \in [t_0, t^0]$,

2'. $g_1(t') > g_2(t')$ for some $t' \in \bigcup_{k=1}^{n-1} [\tau_k, \xi_k]$,

3'. $\int_{\xi_k}^{t_{k+1}} \exp\left\{\int_{\xi_k}^{\tau} f_2(s)[\eta_{1k}(s)+\eta_{2k}(s)] ds\right\} [g_1(\tau)-g_2(\tau)] d\tau > 0$ for some $k \in \{0, \dots, n-1\}$.

Then the quarter $(\phi_1, \psi_1, f_1, g_1)$ is called a strict majorant of the quarter $(\phi_2, \psi_2, f_2, g_2)$ on $[t_0, t^0]$.

Theorem 3.1 (main result). Let $(\phi_2(t), \psi_2(t))$ be a solution of the system (1.5₂), let $N_1(\phi_2) = [t_1, \tau_1] \prec \dots \prec N_n(\phi_2) = [t_n, \tau_n]$ be all of the null-classes of $\phi_2(t)$ on $[t_0, t^0]$ ($t^0 \in (t_0, \tau_0)$) and let $(\phi_1(t), \psi_1(t))$ be a solution of the system (1.5₁). If $(\phi_1, \psi_1, f_1, g_1)$ is a majorant for $(\phi_2, \psi_2, f_2, g_2)$ on $[t_0, t^0]$, then the function $\phi_1(t)$ has at least one null-class in $(\tau_k, \tau_{k+1}]$ for each $k = 0, \dots, n-1$. In addition, if $(\phi_1, \psi_1, f_1, g_1)$ is a strict majorant for $(\phi_2, \psi_2, f_2, g_2)$ on $[t_0, t^0]$, then the function $\phi_1(t)$ has at least n null-classes on (t_0, t^0) .

Proof. Suppose that the function $\phi_1(t)$ has no zeroes on $[\tau_{k_0}, t_{k_0+1}]$ for some k_0 .

Then the function $\psi_1(t)/\phi_1(t)$ exists on $(\tau_{k_0}, \tilde{\tau}_{k_0+1}]$, for some $\tilde{\tau}_{k_0+1} > t_{k_0+1}$ and is a solution of Eq. (2.1₁) there. At first consider the case when $k_0 \neq 0$. In this case (τ_{k_0}, t_{k_0+1}) is the maximal existence interval for the solution $y_1(t) \equiv \psi_2(t)/\phi_2(t)$ of Eq. (2.1₂). Let $\eta_{j,k_0}(t)$ ($j = 1, 2$) be solutions of the respective inequalities (2.2_j) on $[\xi_{k_0}, t_{k_0+1}]$ with the initial conditions $\eta_{j,k_0}(\xi_{k_0}) > y_2(\xi_{k_0})$, $j = 1, 2$ (in virtue of condition 1 and Remark 2.2 these solutions exist always). Let $\tilde{y}_2(t)$ be a solution of Eq. (2.1₂) such that $y_2(\xi_{k_0}) < \tilde{y}_2(\xi_{k_0}) \leq \min_{j=1,2} \{\eta_{j,k_0}(\xi_{k_0})\}$ and let (t_{k_0}, t_{k_0+1}) be the maximum existence interval for $\tilde{y}_2(t)$.

By Lemma 2.2 it follows from the inequality $y_2(\xi_{k_0}) < \tilde{y}_2(\xi_{k_0})$ that $\tau_{k_0} < \tilde{t}_{k_0}$ and $t_{k_0+1} < \tilde{t}_{k_0+1}$. We assume that $\tilde{y}_2(\xi_{k_0})$ is close enough to $y_2(\xi_{k_0})$ to ensure that $\tilde{t}_{k_0} \in (t_{k_0}, \xi_{k_0})$ and $\tilde{t}_{k_0+1} \in (t_{k_0}, \tilde{\tau}_{k_0+1})$. Since \tilde{t}_{k_0} is the left endpoint of the maximum existence interval of $\tilde{y}_2(t)$ by Lemma 2.1 we have $\tilde{y}_2(\tilde{t}_{k_0} + 0) = +\infty$ and since $\tilde{y}_2(\tilde{t}_{k_0}) < +\infty$ [because of the inclusion $\tilde{t}_{k_0} \in (\tau_{k_0}, \xi_{k_0}]$] we have

$$y_1(\zeta_{k_0}) \leq \tilde{y}_2(\zeta_{k_0}), \quad (3.5)$$

for some $\zeta_{k_0} \in (\tilde{t}_{k_0}, \xi_{k_0})$. Let $(\phi_1, \psi_1, f_1, g_1)$ be a majorant for $(\phi_2, \psi_2, f_2, g_2)$. Then by virtue of Theorem 2.1 it follows from (3.5) that

$$y_1(\xi_{k_0}) \leq \tilde{y}_2(\xi_{k_0}) \quad (3.6)$$

and since $\tilde{y}_2(\xi_{k_0}) \leq \min_{j=1,2} \{\eta_{j,k_0}(\xi_{k_0})\}$, we have

$$y_1(\xi_{k_0}) \leq \eta_{1,k_0}(\xi_{k_0}), \quad \tilde{y}_2(\xi_{k_0}) \leq \eta_{2,k_0}(\xi_{k_0}).$$

By Lemma 2.3 from here it follows that

$$y_1(t) \leq \eta_{1,k_0}(t), \quad \tilde{y}_2(t) \leq \eta_{2,k_0}(t) \quad t \in [\xi_{k_0}, \tilde{t}_{k_0+1}).$$

Therefore $\tilde{y}_2(\tilde{t}_{k_0+1} - 0) \geq y_1(\tilde{t}_{k_0+1}) > -\infty$. Then by virtue of Lemma 2.1 $(\tilde{t}_{k_0}, \tilde{t}_{k_0+1})$ is not the maximum existence interval for $\tilde{y}_2(t)$. The obtained contradiction shows that $\phi_1(t)$ has at least one zero l_0 on $(\tau_{k_0}, t_{k_0=1}]$, which belongs to a null-element of the function $\phi_1(t)$. Hence, according to Remark 3.1 l_0 belongs to a null-class $N(\phi_1)$ of the function $\phi_1(t)$. By (3.1) from condition 2 it follows that if $N(\phi_1) \cap N_{k_0}(\phi_2) \neq \emptyset$. Then $N(\phi_1) \subset N_{k_0}(\phi_2)$, and, therefore, $N(\phi_1) \subset (\tau_{k_0}, \tau_{k_0+1}]$. Now consider the case when $k_0 = 0$. If $\phi_2(t_0) = 0$, then the existence of a null-class of $\phi_1(t)$ in $(t_0, t_1]$ can be proved by analogy with the proof in the preceding case. Suppose $\phi_2(t_0) \neq 0$. Then by condition 1 we have also $\phi_1(t_0) \neq 0$. Show that $\phi_1(t)$ has at least one zero on $(t_0, t_1]$. Suppose $\phi_1(t) \neq 0$, $t \in (t_0, t_1]$. Then $y_1(t) \equiv \psi_1(t)/\phi_1(t)$ is defined at least on $[t_0, t_1]$ and is a solution of Eq. (2.1₁) on that interval. The function $y_2(t) \equiv \psi_2(t)/\phi_2(t)$ is a solution of Eq. (2.1₂) on $[t_0, t_1]$, in virtue of Remark 2.3 $y_2(t_1 - 0) = -\infty$. But on the other hand since $y_2(t_0) \geq y_1(t_0)$ and $(\phi_1, \psi_1, f_1, g_1)$ is a majorant for $(\phi_2, \psi_2, f_2, g_2)$ it follows from Theorem 2.1 that $y_2(t_1 - 0) \geq y_1(t_1) > -\infty$. The obtained contradiction shows that $\phi_1(t)$ has at least one zero on $(t_0, t_1]$. Then $\phi_1(t)$ has at least one null-class in $[t_0, \tau_1]$. The first part of the theorem is proved. The second (last) part of the theorem can be proved by analogy of the proof of the second part of Theorem 4.1 from [3], as far as the strict majorant condition implies the reducibility of the systems (1.5_j), $j = 1, 2$ to the second order linear ordinary differential equations like (1.1_j), $j = 1, 2$ respectively. The theorem is proved.

Definition 3.5. *The system (1.5₁) is called a majorant of the system (1.5₂) on $[t_0, t^0]$ ($\subset [a, b]$) if the following conditions are satisfied*

- 1.) $f_1(t) \geq f_2(t) \geq 0$, $t \in [t_0, t^0]$,
- 2.) $g_1(t) \geq g_2(t)$, $t \in [t_0, t^0]$.

In addition suppose that $f_2(t) > 0$, $t \in [t_0, t^0]$ and at least one of the following conditions is satisfied

- 1') $f_1(t') > f_2(t')$ and $g_1(t') \neq 0$ for some $t' \in [t_0, t^0]$;
- 2') $g_1(t') > g_2(t')$ for some $t' \in [t_0, t^0]$.

Then the system (1.5₁) is called a strict majorant of the system (1.5₂).

From Theorem 3.1 we immediately get.

Corollary 3.1. *Let the system (1.5₁) be a majorant for the system (1.5₂). Let $(\phi_2(t), \psi_2(t))$ be a nontrivial solution of the system (1.5₂) and let $\phi_2(t)$ have exactly $n(\geq 1)$ null-classes $N_1(\phi_2) \prec \dots \prec N_n(\phi_2)$ on $[t_0, t^0]$. Let $(\phi_1(t), \psi_1(t))$ be a nontrivial solution of the system (1.5₁) satisfying*

$$\frac{\psi_2(t_0)}{\phi_2(t_0)} \geq \frac{\psi_1(t_0)}{\phi_1(t_0)} \quad (3.7)$$

(The expression on the right [or left] of (3.7) is considered to be $+\infty$, if $\phi_1(t_0) = 0$ [or $\phi_2(t_0) = 0$]; in particular, (3.7) holds if $\phi_2(t_0) = 0$). Then $\phi_1(t)$ has at least n null-classes

in (t_0, t_n) , where $N_n(\phi_2) = [t_n, \tau_n]$, if either the strict inequality (3.7) holds or (1.5₁) is a strict majorant for (1.5₂) on $[t_0, t_n]$. □

Remark 3.2. Corollary 3.1 is a generalization of Theorem 1.1.

Remark 3.3. It may happen that the null classes of the components $\phi(t)$ of the solutions $(\phi(t), \psi(t))$ of the systems (1.5_k), $k = 1, 2$ consist of only one element (one point). This can happen, for example, in the case when the functions $f_k(t)$, $k = 1, 2$ have only finite number of zeroes on $[t_0, t^0]$. For this case Corollary 3.1 provides satisfiability of Theorem 1.2 with replacement (1.6) by (1.8), i. e. Theorem 1.2 with mentioned replacement remains valid under the mild additional restriction that the solutions of the systems (1.5_k), $k = 1, 2$ have no more than finite number of zeroes on $[t_0, t^0]$.

Definition 3.6. The system (1.10) is called strict oscillatory if for every its nontrivial solution $(\phi(t), \psi(t))$ the function $\phi(t)$ has infinite number of null-classes, otherwise it is called strict non-Oscillatory.

Due to the connection (1.12) between the system (1.10) with the system (1.13) from Theorem 3.1 we obtain immediately the following generalization of statement 1).

Corollary 3.2. Assume $Q(t)$ does not change sign on $[T, +\infty)$ for some $T \geq t_0$. Then the system (1.10) is either strict oscillatory or strict non-oscillatory.

Remark 3.4. From the strict oscillation (strict non-oscillation) of the system (1.10) it follows its oscillation (non-oscillation), but the opposite is not true (see [3]).

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