Classifications of infinite direct sums of Banach spaces with applications to Fourier analysis on compact groups

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Abstract. In this paper, the notion of a direct sum of a family of Banach spaces is introduced and studied. Necessary and sufficient conditions are found that a Banach space can be regarded, in a unique way, as a direct sum of a family of its closed subspaces. A class of direct sums of Banach spaces, that many of the direct sums are in the form of a closed subspace of a member of this class, is introduced. As an application, the direct sums of trigonometric polynomials on a compact group G are introduced and classified. Furthermore, among other results, it is proved that the spaces $C(G)$ and $L^p(G)$ $(1 \leq p < \infty)$ are direct sums of trigonometric polynomials and can be regarded as closed subspaces of the members of that class of direct sums of Banach spaces introduced in this paper.

Introduction and preliminaries

The organization of this paper is as follows. In Section 1, the notion of a direct sum of Banach spaces is introduced, and a number of properties of this notion are given along with some examples. The following notations are needed. Let I be a nonempty index set, and $(X_i)_{i\in I}$ a family of Banach spaces. The product of $(X_i)_{i\in I}$ is denoted by $\prod_{i\in I} X_i$, and consists of all $\mathfrak{x} = (x_i)_{i\in I}$ for which $x_i \in X_i$ $(i \in I)$. For each $j \in I$, the j'th canonical projection $\pi_j : \prod_{i\in I} X_i \to X_j$ is defined by $\pi_j(\mathfrak{x}) = \mathfrak{x}_j$, where $\mathfrak{x} = (x_i)_{i\in I} \in \prod_{i\in I} X_i$ and $\mathfrak{x}_j = x_j$. The algebraic direct sum $\bigoplus_{i\in I} X_i$ of $(X_i)_{i\in I}$ is defined as the set of all $\mathfrak{x} \in \prod_{i\in I} X_i$ such that $\mathfrak{x}_i = 0$ for all but finitely many $i \in I$. If $j \in I$, then the appropriate copy of $x \in X_j$ in $\bigoplus_{i\in I} X_i$ is denoted by x^j, and defined by $(x^j)_j = x$ and $(x^j)_i = 0$ for $i \neq j$. The j'th canonical injection $\iota_j : X_j \to \bigoplus_{i\in I} X_i$ is defined by $\iota_j(x) = x^j$ $(x \in X_j)$. In the beginning of this section, a *direct sum of Banach spaces* $(X_i)_{i\in I}$ is defined as a subsapce of the product of $(X_i)_{i\in I}$ that contains the appropriate copy of each $x \in X_i$ $(i \in I)$, and under some norm is a Banach space with continuous coordinates π_i $(i \in I)$. The notion of direct sums of Banach spaces is also defined in Definition 2.1 of [7] for a countable family of Banach spaces, which in this paper is defined in a more general and comprehensive way for an arbitrary family of Banach spaces. This notion extends the notion of BK-space (which is, for example, studied in [2], and with the literature of this paper is a direct sum of countable copies of $\mathbb{C}$), and the notions of the ℓ_p-direct sums of $(X_i)_{i\in I}$ $(1 \leq p \leq \infty)$. Recall that $\ell_p(X_i)_{i\in I}$ (or simply $\ell_p(I)$, where $X_i = \mathbb{C}$ for all $i \in I$, and ℓ_p if furthermore $I = \mathbb{N}$) is the set of all $\mathfrak{x} \in \prod_{i\in I} X_i$ for which $\sum_{i\in I} \|\mathfrak{x}_i\|^p < \infty$ for $1 \leq p < \infty$, and $\sup_{i\in I} \|\mathfrak{x}_i\| < \infty$ for $p = \infty$. At the final of this section the concept of an internal direct sum is introduced. The Banach space X is called *an internal direct sum* of a family $(X_i)_{i\in I}$ of it's closed

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subspaces, if there exists a unique linear map $\mathcal{P} : X \to \prod_{i\in I} X_i$ that isometrically maps X onto a direct sum of $(X_i)_{i\in I}$, and $\mathcal{P}x_i = x_i^i$ for each $i \in I$ and $x_i \in X_i$. It is proved that X is an internal direct sum of $(X_i)_{i\in I}$, if and only if, the linear span of $\cup_{i\in I} X_i$ is dense in X and for each $i \in I$, there exists a bounded projection p_i on X (i.e. a bounded linear map that $p_i^2 = p_i$) with $p_i(X) = X_i$ $(i \in I)$ such that the family $(p_i)_{i\in I}$ is separating (i.e. $\cap_{i\in I} \ker p_i = 0$) and mutually orthogonal (i.e. $p_i p_j = 0$, where $i, j \in I$ and $i \neq j$).

In Section 2 a wide class of direct sums of Banach spaces is introduced. In this paper, for each finite subset F of I, let $\mathcal{P}_F := \sum_{i\in F} \iota_i \circ \pi_i$, i.e. for each $\mathfrak{x} \in \prod_{i\in I} X_i$, $(\mathcal{P}_F \mathfrak{x})_i = \mathfrak{x}_i$ for $i \in F$, otherwise $(\mathcal{P}_F \mathfrak{x})_i = 0$. It is proved that if Γ is a family of functions γ from $\prod_{i\in I} X_i$ to a Banach space X such that $\|\gamma(\mathfrak{x})\| = \|\gamma(\mathcal{P}_{F_\gamma}\mathfrak{x})\|$, for a finite subset F_γ of I and each $\mathfrak{x} \in \prod_{i\in I} X_i$, then under some conditions, $\mathfrak{b}(\Gamma) := \{\mathfrak{x} \in \prod_{i\in I} X_i : \sup_{\gamma\in\Gamma} \|\gamma(\mathfrak{x})\| < \infty\}$, and $\mathfrak{bc}(\Gamma)$ in the case that Γ is a net, is defined as the set of all $\mathfrak{x} \in \mathfrak{b}(\Gamma)$ such that $\lim_{\gamma\in\Gamma} \gamma(\mathfrak{x})$ exists, are direct sum of $(X_i)_{i\in I}$. It is shown that many of the known direct sums of Banach spaces are closed subspaces of a direct sum of the form $\mathfrak{b}(\Gamma)$. As an example of these direct sums, for a Banach sapce X and a family of nonzero elements $\mathsf{e} = (e_i)_{i\in I}$ of X, the concept of *the* (X, e)-*direct sum* of $(X_i)_{i\in I}$, that consisting of all $\mathfrak{x} \in \prod_{i\in i} X_i$ for which the series $\sum_{i\in I} \|\mathfrak{x}_i\| e_i$ is unconditionally partially bounded (i.e. $\sup_{F\in\mathcal{F}} \| \sum_{i\in F} \|\mathfrak{x}_i\| e_i\| < \infty$, where $\mathcal{F}$ is the family of all finite subsets of I) is introduced. Finally, an example of a direct sum of Banach spaces that is not a closed subspace of a direct sum of the form $\mathfrak{b}(\Gamma)$ is given.

Section 3 is devoted to applications to compact groups. The terminologies and notations of [5] are used here. Let G be a compact group with the dual object Σ. For each $\sigma \in \Sigma$, select a fixed member $U^{(\sigma)}$ of σ with representation space H_σ. Recall from [5] that the set of all finite linear combinations of functions of the form $x \mapsto \langle U_x^{(\sigma)} \xi, \eta\rangle$, where $\xi, \eta \in H_\sigma$, is denoted by $\mathfrak{T}_\sigma(G)$. Also, the linear span of $\cup_{\sigma\in\Sigma} \mathfrak{T}_\sigma(G)$ is denoted by $\mathfrak{T}(G)$, and functions in $\mathfrak{T}(G)$ are called *trigonometric polynomials* on G. An internal direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$ is called a *direct sums of trigonometric polynomials on G*. In this section direct sums of trigonometric polynomials are classified. It is shown that the Banach spaces $L^p(G)$ $(1 \le p < \infty)$ and $C(G)$ are direct sums of trigonometric polynomials, and can be regarded as direct sums of the form $\mathfrak{bc}(\Gamma)$, that introduced in Section 2.

1. Direct sum of Banach spaces

Throughout this paper, let I be a nonempty index set, and $(X_i)_{i\in I}$ a family of Banach spaces.

Definition 1.1. A subspace $\mathfrak{X}$ of $\prod_{i\in I} X_i$ that contains $\bigoplus_{i\in I} X_i$, is called a *direct sum of* $(X_i)_{i\in I}$, if there exists a complete norm on $\mathfrak{X}$ with continuous coordinates (i.e. the restrictions of the projections π_i $(i \in I)$ to $\mathfrak{X}$ is continuous).

Proposition 1.2. *Let X be a Banach space, $\mathcal{P} : X \to \prod_{i\in I} X_i$ a linear map such that the maps $p_i := \pi_i \circ \mathcal{P}$ $(i \in I)$ are continuous, and $\mathfrak{X}$ a direct sum of $(X_i)_{i\in I}$ under the norm $\|.\|_{\mathfrak{X}}$ such that $\mathcal{P}X \subseteq \mathfrak{X}$. Then,*

(i) $\|\mathfrak{x}\|_{\mathcal{P}} := \inf_{x\in\mathcal{P}^{-1}\mathfrak{x}} \|x\|_X$ $(\mathfrak{x} \in \mathcal{P}X)$ *is a well defined complete norm on $\mathcal{P}X$ for which the projections $\pi_i|_{\mathcal{P}X}$ $(i \in I)$ are continuous;*

(ii) *$\mathcal{P}X$ is s direct sum of $(X_i)_{i\in I}$ if and only if for each $i \in I$, $X_i = p_i(\cap_{j\neq i} \ker p_j)$;*

(iii) *for some $c > 0$, $\|\mathfrak{x}\|_{\mathfrak{X}} \le c\|\mathfrak{x}\|_{\mathcal{P}}$ $(\mathfrak{x} \in \mathcal{P}X)$;*

(iv) $\mathcal{P} : X \to \mathfrak{X}$ *is continuous.*

Proof. (i): If $x \in \overline{\ker \mathcal{P}}$, then for each $i \in I$, by continuity of p_i, $\pi_i(\mathcal{P}x) = p_i(x) \in p_i(\overline{\ker \mathcal{P}}) \subseteq \overline{p_i(\ker \mathcal{P})} = \{0\}$, that implies $\mathcal{P}x = 0$. Thus $\ker \mathcal{P}$ is a closed subspace of the Banach space X, and so $\frac{X}{\ker \mathcal{P}}$ is a Banach space with respect to the quotient norm $\|.\|_q$. But, the map $\mathcal{P}_0 : \frac{X}{\ker \mathcal{P}} \to \mathcal{P}X$ through $\mathcal{P}_0(x + \ker \mathcal{P}) = \mathcal{P}x$ $(x \in X)$ is a bijection. Thus $\mathcal{P}X$ is a Banach space with respect to the norm $\|\mathfrak{x}\|_{\mathcal{P}} := \|\mathcal{P}_0^{-1}\mathfrak{x}\|_q = \inf_{x \in \mathcal{P}^{-1}\mathfrak{x}} \|x\|_X$, where $\mathfrak{x} \in \mathcal{P}X$. Now let $i \in I$. Since for each $x \in X$ and $y \in \ker \mathcal{P}X$, $\pi_i(\mathcal{P}x) = p_i x = p_i(x + y)$, so $\|\pi_i(\mathcal{P}x)\| \le \|p_i\|\|\mathcal{P}X\|_{\mathcal{P}}$, that implies $\pi_i|_{\mathcal{P}X}$ is continuous.

(ii): Suppose $\mathcal{P}X$ is a direct sum of $(X_i)_{i \in I}$. Let $i \in I$ and $x_i \in X_i$. Since $x_i^i \in \mathcal{P}X$, so there exists $x \in X$ such that $\mathcal{P}x = x_i^i$, equivalently, $p_i(x) = x_i$ and $p_j(x) = 0$ $(j \neq i)$, that implies $x_i \in p_i\left(\cap_{j \neq i} \ker p_j\right)$. Thus $X_i = p_i\left(\cap_{j \neq i} \ker p_j\right)$.

Conversely, suppose for each $i \in I$, $X_i = p_i\left(\cap_{j \neq i} \ker p_j\right)$. Let $i \in I$ and $x_i \in X_i$. Since $x_i \in p_i\left(\cap_{j \neq i} \ker p_j\right)$, so there exists $x \in \cap_{j \neq i} \ker p_j$ such that $p_i(x) = x_i$. Thus $x_i^i = \mathcal{P}x \in \mathcal{P}X$. It follows that $\bigoplus_{i \in I} X_i \subseteq \mathcal{P}X$, that together (i) implies that $\mathcal{P}X$ is a direct sum of $(X_i)_{i \in I}$.

(iii): Let $(\mathfrak{x}_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{P}X$ which $\|.\|_{\mathcal{P}}$-converges to 0 and $\|.\|_{\mathfrak{X}}$ converges to a point $\mathfrak{x} \in \mathfrak{X}$. For each $i \in I$, by continuity of the maps $\pi_i|_{\mathfrak{X}}$ and $\pi_i|_{\mathcal{P}X}$ (by (i)),

$$\pi_i(\mathfrak{x}) = \pi_i(\|.\|_{\mathfrak{X}} - \lim_{n \to \infty} \mathfrak{x}_n) = \lim_{n \to \infty} \pi_i(\mathfrak{x}_n) = \pi_i(\|.\|_{\mathcal{P}} - \lim_{n \to \infty} \mathfrak{x}_n) = \pi_i(0) = 0,$$

and so $\mathfrak{x} = 0$. Thus, by Closed graph theorem, the inclusion map $\iota : \mathcal{P}X \to \mathfrak{X} : \mathfrak{x} \mapsto \mathfrak{x}$ is continuous. It completes the proof.

(iv): Note that for each $x \in X$, by (iii) and (i), $\|\mathcal{P}x\|_{\mathfrak{X}} \le c\|\mathcal{P}x\|_{\mathcal{P}} = c\|x\|_q \le c\|x\|_X$ □

The following result, as a direct consequence of the above corollary, shows that there is no ambiguity to define the norm of direct sums of Banach spaces, and each direct sum of $(X_i)_{i \in I}$ contains appropriate copies of X_i, where $i \in I$.

Corollary 1.3. *Let $\mathfrak{X}$ be a direct sum of Banach spaces $(X_i)_{i \in I}$. Then*
(i) *all norms that makes the space $\mathfrak{X}$ into a direct sum of $(X_i)_{i \in I}$, are equivalent;*
(ii) *the canonical injections $\iota_i : X_i \to \mathfrak{X}$ $(i \in I)$ are continuous.*

Proof. (i) is a direct consequence of Proposition 1.2(iv).

(ii) is a consequence of Proposition 1.2(iv), and the fact that for each $i, j \in I$ with $j \neq i$, $\pi_i \circ \iota_i$ is the identity map on X_i and $\pi_j \circ \iota_i = 0$. □

The following result shows that, in general, the product and algebraic direct sum of a family of Banach spaces are not direct sums of that family.

Proposition 1.4. *If there are infinitely many $i \in I$ with $X_i \neq 0$ and $\mathfrak{X}$ is a direct sum of Banach spaces $(X_i)_{i \in I}$, then $\mathfrak{X} \neq \bigoplus_{i \in I} X_i, \prod_{i \in I} X_i$.*

Proof. Choose a sequence $(i_n)_{n \in \mathbb{N}}$ of distinct elements of I with $X_{i_n} \neq 0$. Thus for each $n \in \mathbb{N}$, there exists $x_{i_n} \in X_{i_n}$ with $x_{i_n} \neq 0$. Suppose the norm $\|.\|$ makes $\mathfrak{X}$ into a direct sum. If $\mathfrak{x}_n := \frac{\iota_{i_n}(x_{i_n})}{\|\iota_{i_n}(x_{i_n})\|}$ $(n \in \mathbb{N})$, then the absolutely convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n} \mathfrak{x}_n$ converges to some $\mathfrak{a} \in \mathfrak{X}$ (by the completeness of $\mathfrak{X}$). For each $n \in \mathbb{N}$, the continuity of π_{i_n} implies that $\pi_{i_n}(\mathfrak{a}) = \frac{1}{2^n} \pi_{i_n}(\mathfrak{x}_n) \neq 0$, and so $\mathfrak{a} \notin \bigoplus_{i \in I} X_i$. Hence, $\bigoplus_{i \in I} X_i \neq \mathfrak{X}$. Now, suppose $\mathfrak{b}$ is an element of $\prod_{i \in I} X_i$ with

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$\mathfrak{b}_{i_n} = \frac{n\|\pi_{i_n}\| x_{i_n}}{\|x_{i_n}\|}$ $(n \in \mathbb{N})$. If $\mathfrak{b} \in \mathfrak{X}$, then for each $n \in \mathbb{N}$, by continuity of π_{i_n}, $n\|\pi_{i_n}\| = \|\mathfrak{b}_{i_n}\| = \|\pi_{i_n}(\mathfrak{b})\| \le \|\pi_{i_n}\|\|\mathfrak{b}\|$, and so $\|\mathfrak{b}\| \ge n$, that's a contradiction. Thus $\mathfrak{b} \notin \mathfrak{X}$, and so $\mathfrak{X} \neq \prod_{i\in I} X_i$. □

Remark 1.5. Suppose there are finitely many $i \in I$ with $X_i \neq 0$. It is easy to see that, if $\mathfrak{X}$ is a direct sum of Banach spaces $(X_i)_{i\in I}$, then $\mathfrak{X} = \prod_{i\in I} X_i = \bigoplus_{i\in I} X_i$ and it is a direct sum under the absolute norm $\|\mathfrak{x}\|_\infty := \sup_{1\le i\le m} \|\mathfrak{x}_i\|$ (see also Corollary 1.3(i)).

Example 1.6. (a) Since $c_{00} := \bigoplus_{n\in\mathbb{N}} \mathbb{C}$ has a countable basis, so by the Baer's category theorem it is not a Banach space under any norm. But, there exists a complete norm on $s := \mathbb{C}^{\mathbb{N}}$. To see this, note that $\dim s = \dim \ell_1$ (see for example Theorem I.1 of [8]). Thus, there exists a vector space isomorphism $I : s \to \ell_1$. Clearly s with respect to the norm $\|\mathfrak{x}\| := \|I(x)\|_1$ $(x \in s)$ is a Banach space.

(b) If B is a basis for ℓ_1 that contains $e_m = (\delta^n_m)_{n\in\mathbb{N}}$ $(m \in \mathbb{N})$, where δ^n_m is the Kronecker's delta symbol, and $e_0 = \sum_{n=1}^{\infty} \frac{e_n}{2^n}$, then $\bigoplus_{b\in B} \mathbb{C}$ with respect to the norm $\|\alpha\| := \sum_{b\in B} |\alpha_b|$ $(\alpha \in \bigoplus_{b\in B} \mathbb{C})$ is a Banach space that is isometrically isomorphic with ℓ_1. Since $\pi_{e_0}(e_0) = 1$ and $\sum_{n=1}^{\infty} \frac{1}{2^n} \pi_{e_0}(e_n) = 0$, so the projection π_{e_0} is not continuous.

The remainder of this section is devoted to internal direct sum that defined as the following.

Definition 1.7. Let X be a Banach space, and (X_i) a family of its closed subspaces. Then X is called *an internal direct sum* of $(X_i)_{i\in I}$, if there exists a unique linear map $\mathcal{P} : X \to \prod_{i\in I} X_i$ that maps X isometrically isomorphic onto a direct sum of $(X_i)_{i\in I}$ and $\mathcal{P}x_i = x^i_i$, where $i \in I$ and $x_i \in X_i$.

Lemma 1.8. *Let X be a Banach space, $(X_i)_{i\in I}$ a family of its closed subspaces, and $\mathcal{P}x = (p_i x)_{i\in I}$ $(x \in X)$ a linear map from X into $\prod_{i\in I} X_i$. The following assertions are equivalent:*

(i) *$\mathcal{P}$ maps X isometrically isomorphic onto a direct sum of $(X_i)_{i\in I}$ and $\mathcal{P}x_i = x^i_i$, where $i \in I$ and $x_i \in X_i$;*

(ii) *$(p_i)_{i\in I}$ is a family of separating mutually orthogonal bounded projections in X with $p_i(X) = X_i$ $(i \in I)$.*

Proof. (i)⇒(ii): Let $i, j \in I$. Since the i'th projection of $\mathcal{P}X$, that is denoted by π_i, is bounded, so $p_i = \pi_i \circ \mathcal{P}$ is bounded. If $i, j \in I$ and $x \in X$, then $p_j x \in X_j$, and so $p_i p_j(x) = \pi_i(\mathcal{P}p_j(x)) = \pi_i((p_j x)^j)$, that follows $p_i p_j = 0$ $(j \neq i)$ and $p_i^2 = p_i$. But, if $x_i \in X_i$, then $p_i x_i = \pi_i(\mathcal{P}x_i) = \pi_i(x^i_i) = x_i$, and so p_i is a projection onto X_i. Since $\mathcal{P}$ is injective and $\ker \mathcal{P} = \cap_{i\in I} \ker p_i$, so $\cap_{i\in I} \ker p_i = \{0\}$, i.e. $(p_i)_{i\in I}$ is a separating family.

(ii)⇒(i): Let $i \in I$ and $x_i \in X_i$. Since $p_i x_i = x_i$ and $p_j x_i = 0$ for $j \neq i$, so $x_i \in p_i\left(\cap_{j\neq i} \ker p_j\right)$. Hence, by Proposition 1.2(ii), $\mathcal{P}X$ is a direct sum of $(X_i)_{i\in I}$. But, $(p_i)_{i\in I}$ is a separating family, and so $\ker \mathcal{P} = \{0\}$. Now, by Proposition 1.2(i), $\mathcal{P}$ maps X isometrically isomorphism onto $\mathcal{P}X$ that equipped with the norm $\|.\|_{\mathcal{P}}$. □

Proposition 1.9. *Let X be a Banach space, and $(X_i)_{i\in I}$ a family of its closed subspaces such that there exists a family $(p_i)_{i\in I}$ of separating mutually orthogonal bounded projections in X with $p_i(X) = X_i$ $(i \in I)$. Then X is an internal direct sum of $(X_i)_{i\in I}$, if and only if, the linear span of $\cup_{i\in I} X_i$ is dense in X. Furthermore,* *the linear span of* $\cup_{i\in I}X_i$ *is equal to* X*, if and only if, there are finitely many* $i \in I$ *with* $X_i \neq 0$.

Proof. Let $\mathcal{P}x := (p_i x)_{i\in I}$ $(x \in X)$ and $I_0 = \{i \in I : X_i \neq 0\}$.

Suppose the linear span of $\cup_{i\in I}X_i$ is dense in X. Let $\mathcal{Q}x = (q_i x)_{i\in I}$ $(x \in X)$ be a linear map from X into $\prod_{i\in I} X_i$ that maps X isometrically isomorphic onto a direct sum of $(X_i)_{i\in I}$ and $\mathcal{Q}x_i = x_i^i$, where $i \in I$ and $x_i \in X_i$. Let $i, j \in I$ and $j \neq i$. Then, by Lemma 1.8, for each $x_i \in X_i$, $q_i x_i = x_i = p_i x_i$, and for each $x_j \in X_j$, $q_i x_j = q_i q_j x_j = 0 = p_i p_j x_j = p_i x_j$. It, together the continuity of p_i and q_i and the fact that $\cup_{i\in I}X_i$ is dense in X, implies $q_i = p_i$. Hence $\mathcal{Q} = \mathcal{P}$. Now, by Lemma 1.8 and Definition 1.7, X is an internal direct sum of $(X_i)_{i\in I}$.

Conversely, suppose X is an internal direct sum of $(X_i)_{i\in I}$. Firstly, suppose I_0 is infinite. By Proposition 1.4 there exists $\mathfrak{a} \in \prod_{i\in i} X_i$ that $\mathfrak{a} \notin \mathcal{P}X$. Suppose $f \in \mathfrak{X}^*$ and $f(\cup_{i\in I}X_i) = 0$. Let $\mathcal{Q} = (q_i)_{i\in I}$ be the map from X into $\prod_{i\in I} X_i$ given by $\mathcal{Q}x = \mathcal{P}x + f(x)\mathfrak{a}$ $(x \in X)$. Let $i, j \in I$ and $j \neq i$. Then, $q_i x = p_i x + f(x)\mathfrak{a}_i$ $(x \in X)$. Thus, by the properties of f, q_i is continuous and $q_i x_i = p_i x_i + f(x_i)\mathfrak{a}_i = x_i$ for all $x_i \in X_i$, that implies q_i is a bounded projection with $q_i(X) = X_i$. Also, if $x \in X$, then $q_j x \in X_j$, and so $p_i q_j x = p_i(p_j q_j x) = 0$, that together the fact $f(\cup_{i\in I}X_i) = 0$, implies $q_i q_j(x) = p_i(q_j x) + f(q_j x)\mathfrak{a}_i = 0$. On the other hand, if $x \in \ker \mathcal{Q}$, then $\mathcal{P}x + f(x)\mathfrak{a} = 0$. But, $\mathfrak{a}$ doesn't belong to the vector space $\mathcal{P}X$, and so $\mathcal{P}x = 0$, that implies $x = 0$. By Lemma 1.8 and uniqueness of $\mathcal{P}$, $\mathcal{Q} = \mathcal{P}$, and so $f = 0$. Hence by Hahn-Banach Theorem, the linear span of $\cup_{i\in I}X_i$ is dense in X.

Now, suppose I_0 is finite. Then $\mathcal{P}X \subseteq \prod_{i\in I} X_i = \bigoplus_{i\in I} X_i$, and so by injectivity of $\mathcal{P}$, X is equal to the linear span of $\cup_{i\in I}X_i$.

Finally, if X is equal to the linear span of $\cup_{i\in I}X_i$, then $\bigoplus_{i\in I} X_i = \mathcal{P}X$. But, $\mathcal{P}X$ is a direct sum of $(X_i)_{i\in I}$, so by Proposition 1.4, I_0 is finite. □

2. A class of direct sums of Banach spaces

In this section, a class of direct sums of Banach spaces is introduced, which, as mentioned in the rest of the paper, many direct sums are in the form of a closed subspace of a member of this class. Recall that $(X_i)_{i\in I}$ is a family of Banach spaces.

Definition 2.1. Let X be a Banach space, and Γ a family of functions $\gamma : \prod_{i\in I} X_i \to X$ such that $\|\gamma(\mathfrak{x})\| = \|\gamma(\mathcal{P}_{F_\gamma}\mathfrak{x})\|$, for a finite subset F_γ of I and each $\mathfrak{x} \in \prod_{i\in I} X_i$. Then $\mathfrak{b}(\Gamma)$ is defined as the set of all $\mathfrak{x} \in \prod_{i\in I} X_i$ for which $\|\mathfrak{x}\|_\Gamma := \sup_{\gamma\in\Gamma} \|\gamma(\mathfrak{x})\| < \infty$, and if Γ is also a net, then $\mathfrak{bc}(\Gamma)$ is defined as the set of all $\mathfrak{x} \in \mathfrak{b}(\Gamma)$ such that $\lim_{\gamma\in\Gamma} \gamma(\mathfrak{x})$ exists.

Theorem 2.2. *Let* Γ *be a family of functions* γ *from* $\prod_{i\in I} X_i$ *into a Banach space* X *such that* $\|\gamma(\mathfrak{x})\| = \|\gamma(\mathcal{P}_{F_\gamma}\mathfrak{x})\|$*, for a finite subset* F_γ *of* I *and each* $\mathfrak{x} \in \prod_{i\in I} X_i$*. If*

(a) *for each* $\gamma \in \Gamma$, $q_\gamma(\mathfrak{x}) := \|\gamma(\mathfrak{x})\|$ $(\mathfrak{x} \in \prod_{i\in I} X_i)$ *is a seminorm,*

(b) *for each* $\gamma \in \Gamma$ *and* $i \in I$, $q_\gamma \circ \iota_i$ $(i \in I)$ *is lower semicontinuous,*

(c) *for each* $i \in I$ *and* $x_i \in X_i$, $\|\iota_i(x_i)\|_\Gamma < \infty$,

(d) *for each* $i \in I$*, there exists* $\alpha_i > 0$ *such that for each* $\mathfrak{x} \in \prod_{i\in I} X_i$, $\|\mathfrak{x}_i\| \leq \alpha_i \|\mathfrak{x}\|_\Gamma$,

then $\mathfrak{b}(\Gamma)$ *is a direct sum of* $(X_i)_{i\in I}$*. Furthermore, if* Γ *is a net of linear maps, then* $\mathfrak{bc}(\Gamma)$ *is a closed subspace of* $\mathfrak{b}(\Gamma)$*, and is a direct sum of* $(X_i)_{i\in I}$ *if* $\lim_{\gamma\in\Gamma}(\gamma\circ\iota_i)(x_i)$ *exists for each* $i \in I$ *and* $x_i \in X_i$.

Proof. By (a), $\|.\|_\Gamma$ is a seminorm on $\mathfrak{b}(\Gamma)$. Since by (d), for each $\mathfrak{x} \in \mathfrak{b}(\Gamma)$ and $i \in I$, $\|\pi_i(\mathfrak{x})\| \leq \alpha_i \|\mathfrak{x}\|_\Gamma$, so $(\mathfrak{b}(\Gamma)\|.\|_\Gamma)$ not only is a normed space, but also its projections are continuous. By (c), $\bigoplus_{i\in I} X_i \subseteq \mathfrak{b}(\Gamma)$. Thus, $\mathfrak{b}(\Gamma)$ is a direct sum of $(X_i)_{i\in I}$, provided that the completeness of $\|.\|_\Gamma$ is proved. To see this, firstly note that by (a) and (b), for each $i \in I$ and $\gamma \in \Gamma$, $q_\gamma \circ \iota_i$ is a lower semicontinuous seminorm. It, together (c), Banach-Steinhauss Theorem (Theorem11 on Page 122 of [9]), and the definition of $\|.\|_\Gamma$, implies that for some $\beta_i \geq 0$, $\|\iota_i(x_i)\|_\Gamma \leq \beta_i \|x_i\|$ $(x_i \in X_i)$. Now, suppose $(\mathfrak{a}_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathfrak{b}(\Gamma)$. For each $i \in I$, $(\pi_i(\mathfrak{a}_n))_{n=1}^\infty$ is a Cauchy sequence in X_i (by the continuity of π_i), and so converges to some $a_i \in X_i$. Let $\mathfrak{a} = (a_i)_{i\in I}$. Thus, for each $\gamma \in \Gamma$ and $n \in \mathbb{N}$ (note that $q_\gamma = q_\gamma \circ \mathcal{P}_{F_\gamma}$),

$$\begin{aligned} q_\gamma(\mathfrak{a}_n - \mathfrak{a}) &\leq \underline{\lim}_{m\to\infty} \left(q_\gamma(\mathfrak{a}_n - \mathfrak{a}_m) + q_\gamma(\mathfrak{a}_m - \mathfrak{a})\right) \\ &= \underline{\lim}_{m\to\infty} \left(q_\gamma(\mathfrak{a}_n - \mathfrak{a}_m) + q_\gamma\left(\mathcal{P}_{F_\gamma}(\mathfrak{a}_m - \mathfrak{a})\right)\right) \\ &\leq \underline{\lim}_{m\to\infty} \left(\|\mathfrak{a}_n - \mathfrak{a}_m\|_\Gamma + \Big\| \sum_{i\in F_\gamma} \iota_i(\pi_i(\mathfrak{a}_m) - a_i)\Big\|_\Gamma\right) \\ &\leq \underline{\lim}_{m\to\infty} \|\mathfrak{a}_n - \mathfrak{a}_m\|_\Gamma + \overline{\lim}_{m\to\infty} \sum_{i\in F_\gamma} \beta_i \|\pi_i(\mathfrak{a}_m) - a_i\| \\ &= \underline{\lim}_{m\to\infty} \|\mathfrak{a}_n - \mathfrak{a}_m\|_\Gamma, \end{aligned}$$

and so for each $n \in \mathbb{N}$, $\|\mathfrak{a}_n - \mathfrak{a}\|_\Gamma \leq \underline{\lim}_{m\to\infty} \|\mathfrak{a}_n - \mathfrak{a}_m\|_\Gamma$. It, together the Cauchyness of $(\mathfrak{a}_n)_{n=1}^\infty$, implies that $\mathfrak{a} \in \mathfrak{b}(\Gamma)$ and $(\mathfrak{a}_n)_{n=1}^\infty$ converges to $\mathfrak{a}$.

Finally, let Γ be a net of linear maps. Suppose $(\mathfrak{x}_n)_{n=1}^\infty$ is a sequence in $\mathfrak{bc}(\Gamma)$ that converges to some $\mathfrak{x} \in \mathfrak{b}(\Gamma)$. For $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\|\mathfrak{x}_{n_\epsilon} - \mathfrak{x}\|_\Gamma < \frac{1}{3}\epsilon$. Thus, for each $\gamma_1, \gamma_2 \in \Gamma$,

$$\begin{aligned} \|\gamma_1(\mathfrak{x}) - \gamma_2(\mathfrak{x})\| &\leq \|\gamma_1(\mathfrak{x}) - \gamma_1(\mathfrak{x}_{n_\epsilon})\| + \|\gamma_1(\mathfrak{x}_{n_\epsilon}) - \gamma_2(\mathfrak{x}_{n_\epsilon})\| + \|\gamma_2(\mathfrak{x}_{n_\epsilon}) - \gamma_2(\mathfrak{x})\| \\ &\leq 2\|\mathfrak{x}_{n_\epsilon} - \mathfrak{x}\|_\Gamma + \|\gamma_1(\mathfrak{x}_{n_\epsilon}) - \gamma_2(\mathfrak{x}_{n_\epsilon})\| < \frac{2}{3}\epsilon + \|\gamma_1(\mathfrak{x}_{n_\epsilon}) - \gamma_2(\mathfrak{x}_{n_\epsilon})\|, \end{aligned}$$

that together the convergence of $(\gamma(x_{n_\epsilon}))_{\gamma\in\Gamma}$, implies that $(\gamma(\mathfrak{x}))_{\gamma\in\Gamma}$ is a Cauchy net in the Banach space X, and so is convergent (see Proposition 2.1.49 of [10]). But, $\mathfrak{x} \in \mathfrak{b}(\Gamma)$, and so $\mathfrak{x} \in \mathfrak{bc}(\Gamma)$. It follows that $\mathfrak{bc}(\Gamma)$ is a closed subspace of $\mathfrak{b}(\Gamma)$. □

In the rest of this section, let $\mathcal{F}$ be the net of all finite subsets of I with the inclusion order. In the following example, the notion of ℓ_p-sums of Banach spaces is extended.

Example 2.3. (a) Let X be a Banach space, $\mathsf{e} = (e_i)_{i\in I}$ be a family of nonzero elements of X, and Γ the family of all functions $\gamma_F(\mathfrak{x}) = \sum_{i\in F} \|x_i\| e_i$ $(\mathfrak{x} \in \prod_{i\in I} X_i)$, where $F \in \mathcal{F}$. Clearly, $q_{\gamma_F} \circ \iota_i$ is continuous and $q_{\gamma_F} = q_{\gamma_F} \circ \mathcal{P}_F$. Let $i \in I$. For each $x_i \in X_i$, $\|\iota_i(x_i)\|_\Gamma = \|x_i\|\|e_i\|$, and for each $\mathfrak{x} \in \prod_{i\in I} X_i$, $\|\mathfrak{x}_i\| \leq \frac{1}{\|e_i\|}\|\mathfrak{x}\|_\Gamma$. Hence by Theorem 2.2, $\mathfrak{b}(\Gamma)$ is a direct sum of $(X_i)_{i\in I}$. In this case, $\mathfrak{b}(\Gamma)$ is called *the* (X, e)-*direct sum* of $(X_i)_{i\in I}$, denoted by $(X, \mathsf{e}) - \oplus_{i\in I} X_i$, and the norm $\|.\|_\Gamma$ denoted by $\|.\|_{(X,\mathsf{e})}$.

(b) Let $e_j := (\delta_i^j)_{i\in I}$, where $j \in I$ and δ_i^j is the Kronecker's delta symbol. If $\mathsf{e} = (e_i)_{i\in I}$, and $1 \leq p \leq \infty$, then $(\ell_p(I), \mathsf{e}) - \bigoplus_{i\in I} X_i = \ell_p(X_i)_{i\in I}$.

(c) An unusual example of a (X,e)-direct sum of $(X_i)_{i\in I}$ is now given. If $\mathsf{e}=(1)_{i\in I}$, then one can prove easily that $(\mathbb{C},\mathsf{e})-\bigoplus_{i\in I}X_i=\ell_1(X_i)_{i\in I}$.

(d) Let $\mathrm{P}=(\mathrm{p}_F)_{F\in\mathcal{F}}$, where $\mathrm{p}_F:\prod_{i\in I}X_i\to\ell^\infty(X_i)_{i\in I}$ is given by $\mathrm{p}_F(\mathfrak{x}):=\mathcal{P}_F\mathfrak{x}$ for each $F\in\mathcal{F}$ and $\mathfrak{x}\in\prod_{i\in I}X_i$. By Theorem 2.2, $\mathfrak{bc}(\mathrm{P})$ is a direct sum of $(X_i)_{i\in I}$. Clearly, $\mathfrak{x}\in\mathfrak{bc}(\mathrm{P})$, if and only if, $(\mathcal{P}_F\mathfrak{x})_{F\in\mathcal{F}}$ is a Cauchy net in $\ell^\infty(X_i)_{i\in I}$ that is equivalent with $\{i\in I:\|\mathfrak{x}_i\|>\epsilon\}\in\mathcal{F}$ for all $\epsilon>0$. Recall that in this case, $\mathfrak{bc}(\mathrm{P})$ is called the c_0-direct sums of $(X_i)_{i\in I}$, and denoted by $c_0(X_i)_{i\in I}$.

Example 2.4. Let X be a Banach space.

(a) Let $(X_i)_{i\in I}$ be a family of closed subspaces of X. The space $ucs(X_i)_{i\in I}$ consists of $\mathfrak{x}\in\prod_{i\in I}X_i$ for which the series $\sum_{i\in I}\mathfrak{x}_i$ is unconditionally convergent (i.e. the net $(s_F\mathfrak{x})_{F\in\mathcal{F}}$ is convergent in X, where $s_F\mathfrak{x}:=\sum_{i\in F}\mathfrak{x}_i$ $(F\in\mathcal{F})$). If $\mathfrak{x}\in ucs(X_i)_{i\in I}$, then there exists $F_0\in\mathcal{F}$ such that for all $F\in\mathcal{F}$, $\|s_{F\cup F_0}\mathfrak{x}-s_{F_0}\mathfrak{x}\|<1$. But, $s_F\mathfrak{x}=(s_{F\cup F_0}\mathfrak{x}-s_{F_0}\mathfrak{x})+s_{F\cap F_0}\mathfrak{x}$ for all $F\in\mathcal{F}$. Hence, $\sup_{F\in\mathcal{F}}\|s_F\mathfrak{x}\|<1+\sum_{i\in F_0}\|\mathfrak{x}_i\|<\infty$. It follows that $ucs(X_i)_{i\in I}=\mathfrak{bc}((s_F)_{F\in\mathcal{F}})$, and so by Theorem 2.2, $ucs(X_i)_{i\in I}$ is a direct sum of $(X_i)_{i\in I}$ under the norm $\|\mathfrak{x}\|_{ucs}:=\sup_{F\in\mathcal{F}}\|s_F\mathfrak{x}\|$.

(b) Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of closed subspaces of X. The space $cs(X_i)_{i\in\mathbb{N}}$ consists of $\mathfrak{x}\in\prod_{i\in\mathbb{N}}X_i$ for which the series $\sum_{i=1}^\infty\mathfrak{x}_i$ is convergent. Clearly, $cs(X_i)_{i\in\mathbb{N}}=\mathfrak{bc}((s_i)_{i\in\mathbb{N}})$, where $s_ix:=\sum_{j=1}^i\mathfrak{x}_j$ $(i\in\mathbb{N})$, and so by Theorem 2.2, $cs(X_i)_{i\in\mathbb{N}}$ is a direct sum of $(X_i)_{i\in\mathbb{N}}$ under the norm $\|\mathfrak{x}\|_{cs}:=\sup_{i\in\mathbb{N}}\|s_i\mathfrak{x}\|$.

In the following, an example of a direct sum that can not be expressed as a closed subspace of a direct sum in the form of $\mathfrak{b}(\Gamma)$ is given.

Example 2.5. Let $\mathfrak{X}$ be the set of all $x\in\ell_\infty$, for which $\lim_{j\to\infty}\mathfrak{x}_{(2j-1)2^{i-1}}$ exists for all $i\in\mathbb{N}$, and $\sum_{i=1}^\infty\left|\lim_{j\to\infty}\mathfrak{x}_{(2j-1)2^{i-1}}\right|<\infty$. It is easy to see that $\mathfrak{X}$ is a direct sum of countable copies of $\mathbb{C}$ under the norm $\|\mathfrak{x}\|=\|\mathfrak{x}\|_\infty+\sum_{i=1}^\infty\left|\lim_{j\to\infty}\mathfrak{x}_{(2j-1)2^{i-1}}\right|$ $(\mathfrak{x}\in\mathfrak{X})$. The space $\mathfrak{X}$ is not a closed subspace of a direct sum of the form $\mathfrak{b}(\Gamma)$ that introduced in Definition 2.1. Suppose to the contrary, $\mathfrak{X}$ is a closed subspace of $\mathfrak{b}(\Gamma)$, where Γ satisfies the conditions of Definition 2.1. Thus, there exists $c_1,c_2>0$ such that $c_1\|\mathfrak{x}\|_\Gamma\le\|\mathfrak{x}\|\le c_2\|\mathfrak{x}\|_\Gamma$ $(\mathfrak{x}\in\mathfrak{X})$. Let $m\in\mathbb{N}$, and $\mathfrak{x}(m)$ is the sequence given by $\mathfrak{x}(m)_{(2j-1)2^{i-1}}:=1$, for $1\le i\le m$ and $j\in\mathbb{N}$, otherwise $\mathfrak{x}(m)_{(2j-1)2^{i-1}}:=0$. Then for each $m\in\mathbb{N}$,

$$\begin{aligned}1+m &= \|\mathfrak{x}(m)\|\le c_2\|\mathfrak{x}(m)\|_\Gamma=c_2\sup_{\gamma\in\Gamma}q_\gamma(\mathfrak{x}(m))=c_2\sup_{\gamma\in\Gamma}q_\gamma(\mathcal{P}_{F_\gamma}(\mathfrak{x}(m)))\\ &\le c_2\sup_{\gamma\in\Gamma}\|\mathcal{P}_{F_\gamma}(\mathfrak{x}(m))\|_\Gamma\le\frac{c_2}{c_1}\sup_{\gamma\in\Gamma}\|\mathcal{P}_{F_\gamma}(\mathfrak{x}(m))\|=\frac{c_2}{c_1},\end{aligned}$$

that's a contradiction.

3. Applications to compact groups

Throughout this section let G be a compact group with the normalized Haar measure λ and the dual object Σ. For each $\sigma\in\Sigma$, select a fixed member $U^{(\sigma)}$ of σ with representation space H_σ. Recall that for each $\sigma\in\Sigma$, $d_\sigma=\dim H_\sigma<\infty$ (Theorem 22.13 of [4]). Thus for each $\sigma\in\Sigma$, $\mathfrak{T}_\sigma(G)$ is finite dimensional, and so is a closed subspace of each normed space X that contains $\mathfrak{T}_\sigma(G)$ as a subspace.

Definition 3.1. An internal direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$ that is also a subspace of $M(G)$ is called *a direct sum of trigonometric polynomials on G.*

For classifying the direct sums of trigonometric polynomials the following definition is needed.

Definition 3.2. The map $\mathcal{F} : M(G) \to \prod_{\sigma\in\Sigma} \mathfrak{T}_\sigma(G)$ is defined by $\mathcal{F}\mu := (\mathcal{F}_\sigma\mu)_{\sigma\in\Sigma}$, where $\mathcal{F}_\sigma\mu := \mu * u_\sigma$ and $u_\sigma(x) := d_\sigma \mathrm{tr}(U_x^{(\sigma)})$ $(x \in G)$ for all $\sigma \in \Sigma$.

By Definition 34.2, Remark 34.3 Lemma 34.1 of [5], it is easy to see that $\mathcal{F}_\sigma\mu(x) = d_\sigma \mathrm{tr}(A_\sigma U_x^{(\sigma)})$ $(x \in G)$, where A_σ is the σ's Fourier coefficient operator of μ that defined by $A_\sigma = \int_G U_{x^{-1}}^{(\sigma)} d\mu(x)$. Note that the formal expression $\sum_{\sigma\in\Sigma} \mathcal{F}_\sigma\mu$ is the Fourier series of μ

Proposition 3.3. *Let $(X, \|.\|_X)$ be a Banach space which is also a subspace of $M(G)$. Then, $\mathcal{F}$ maps X isometrically isomorphic onto a direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$ if and only if $\mathfrak{T}(G) \subseteq X$ and there exists a positive constant c such that $\|\mu\| \le c\|\mu\|_X$ $(\mu \in X)$.*

Proof. Suppose there exists a positive constant c such that for each $\mu \in X$, $\|\mu\| \le c\|\mu\|_X$. On one hand, by Lemma 34.1(iv) of [5] all Fourier operators of u_σ is 0 excepts the σ's Fourier operator that is equal to I_{d_σ}. Hence, by Remark 34.3(c) of [5] and Definition 3.2, $(\mathcal{F}_\sigma)_{\sigma\in\Sigma}$ is a family of mutually orthogonal projections with $\mathcal{F}_\sigma(M(G)) = \mathfrak{T}_\sigma(G)$ $(\sigma \in \Sigma)$, and also is separating by Remark 34.3(b) of [5]. On the other hand, for each $\sigma \in \Sigma$, there exists $c_\sigma > 0$ such that $\|t\|_X \le c_\sigma\|t\|_1$ for all $t \in \mathfrak{T}_\sigma(G)$ (note that $\mathfrak{T}_\sigma(G)$ is finite dimensional, and so all norms on it is equivalent), so by Theorem 20.12 of [4] for each $\mu \in X$, $\|\mathcal{F}_\sigma\mu\|_X \le c_\sigma\|\mathcal{F}_\sigma\mu\|_1 \le c_\sigma\|u_\sigma\|_1\|\mu\| \le cc_\sigma\|u_\sigma\|_1\|\mu\|_X$, that implies $\mathcal{F}_\sigma$ is continuous. Hence by Proposition 1.8, $\mathcal{F}$ maps X isometrically isomorphic onto a direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$.

Conversely, suppose $\mathcal{F}$ maps X isometrically isomorphic onto a direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$. Since for each $\sigma \in \Sigma$, $\mathcal{F}_\sigma = \pi_\sigma \circ \mathcal{F}$, where π_σ is the σ's projection of $\mathcal{F}X$, so $\mathcal{F}_\sigma$ is continuous. Hence by Proposition 1.2(i), $\|\mathcal{F}\mu\|_{\mathcal{F}} = \|\mu\|_X$ $(\mu \in X)$. But, by the first paragraph of the proof, $\mathcal{F}M(G)$ is a direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$ under the norm $\|\mathcal{F}\mu\| = \|\mu\|$ $(\mu \in M(G))$. Now, by Proposition 1.2(iii) there exists $c > 0$ such that for each $\mu \in X$, $\|\mathcal{F}\mu\| \le c\|\mathcal{F}\mu\|_{\mathcal{F}}$, and so $\|\mu\| \le c\|\mu\|_X$. . □

Example 3.4. Let G be an infinite compact group. Then $M(G)$ is infinite dimensional, and so by Theorem 4.2 of [1], there exists a complete norm $\|.\|'$ on $M(G)$ that is not equivalent to $\|.\|_1$. Thus by Proposition 3.3, $\mathcal{F}$ does not map $X = (M(G), \|.\|')$ isometrically isomorphic onto a direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$.

Corollary 3.5. *The Banach spaces $C(G)$ and $L^p(G)$ $(1 \le p < \infty)$ are direct sums of trigonometric polynomials on G.*

Proof. For each $1 \le p \le \infty$ and $f \in L^p(G)$, $\|f\|_1 \le \|f\|_p$, and $\mathfrak{T}(G)$ is dense in $C(G)$ and $L^p(G)$ $(1 \le p < \infty)$ (see for example Page 110 of [3]). By using Proposition 3.3, Lemma 1.8, and Proposition 1.9, the proof is completed. □

In the remainder of this section, it is proved that L^p-spaces on G is of the form $\mathfrak{b}(\Gamma)$ or $\mathfrak{bc}(\Gamma)$ that introduced in Definition 2.1. In the proof of the following proposition, the fact that $\|f * \mu\|_p, \|\mu * f\|_p \le \|f\|_p\|\mu\|$ for each $\mu \in M(G)$ and $f \in L^p(G)$ (Theorem 20.12 of [4]) is used frequently.

Proposition 3.6. *Let $1 \le p \le \infty$, (h_α) be a net in $\mathfrak{T}(G)$, and $\mathrm{H}_p := (\mathrm{h}_\alpha^p)_\alpha$, where h_α^p maps $\mathfrak{t} \in \prod_{\sigma\in\Sigma} \mathfrak{T}_\sigma(G)$ to $\sum_{\sigma\in\Sigma} h_\alpha * \mathfrak{t}_\sigma \in L^p(G)$ for each α. If $\lim_\alpha \|h_\alpha * u_\sigma - u_\sigma\|_p = 0$, then*

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(i) $\mathfrak{b}(\mathrm{H}_p) \subseteq \mathcal{F}L^p(G)$ *for* $1 < p \leq \infty$, *and* $\mathfrak{b}(\mathrm{H}_1) \subseteq \mathcal{F}M(G)$.

(ii) *if* $\sup_\alpha \|h_\alpha * u_\sigma\|_p < \infty$ $(\sigma \in \Sigma)$, *then* $\mathfrak{b}(\mathrm{H}_p)$ *and* $\mathfrak{bc}(\mathrm{H}_p)$ *are direct sums of* $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$,

(iii) *if* $\sup_\alpha \|h_\alpha\|_1 < \infty$, *then* $\mathfrak{b}(\mathrm{H}_p) = \mathfrak{bc}(\mathrm{H}_p) = \mathcal{F}L^p(G)$ $(1 < p < \infty)$, $\mathfrak{b}(\mathrm{H}_1) = \mathcal{F}M(G)$, $\mathfrak{bc}(\mathrm{H}_1) = \mathcal{F}L^1(G)$, $\mathfrak{b}(\mathrm{H}_\infty) = \mathcal{F}L^\infty(G)$, *and* $\mathfrak{bc}(\mathrm{H}_\infty) = \mathcal{F}C(G)$.

Proof. (i): Suppose $1 < p \leq \infty$ and $\mathfrak{t} \in \mathfrak{b}(\mathrm{H}_p)$. Then, $(\mathrm{h}_{\alpha,p}(\mathfrak{t}))_\alpha$ is a $\|.\|_p$-bounded net in $\mathfrak{T}(G) \subseteq L^p(G) = L^q(G)^*$, where $\frac{1}{p} + \frac{1}{q} = 1$, and so by Banach-Alaoglu Theorem, it has a subnet $(\mathrm{h}^p_\beta(\mathfrak{t}))_\beta$ that weak*-converges to some $f \in L^p(G) = L^q(G)^*$. Let $\sigma \in \Sigma$. By a simple calculation, one can proved that the net $(\mathrm{h}^p_\beta(\mathfrak{t}) * u_\sigma)_\beta$ is weak*-convergent to $f * u_\sigma = \mathcal{F}_\sigma f$. It, together the facts that $(\mathrm{h}^p_\beta(\mathfrak{t}) * u_\sigma)_\beta$ is a net in the finite dimensional space $\mathfrak{T}_\sigma(G)$ and on a finite dimensional space all Hausdorff vector topologies are equivalent, implies that $(\mathrm{h}^p_\beta(\mathfrak{t}) * u_\sigma)_\beta$ is $\|.\|_p$-convergent to $f * u_\sigma = \mathcal{F}_\sigma f$. On the other hand, $\lim_\beta \|h_\beta * u_\sigma - u_\sigma\|_p = 0$ and $u_\sigma * \mathfrak{t}_\sigma = \mathfrak{t}_\sigma$. It follows that

$$\begin{aligned}\mathcal{F}_\sigma f &= \lim_\beta \mathrm{h}^p_\beta(\mathfrak{t}) * u_\sigma = \lim_\beta \sum_{\eta\in\Sigma}((h_\beta * \mathfrak{t}_\eta) * u_\sigma) = \lim_\beta h_\beta * \mathfrak{t}_\sigma \\ &= \lim_\beta h_\beta * (u_\sigma * \mathfrak{t}_\sigma) = \lim_\beta (h_\beta * u_\sigma) * \mathfrak{t}_\sigma = u_\sigma * \mathfrak{t}_\sigma = \mathfrak{t}_\sigma.\end{aligned}$$

Hence, $\mathfrak{t} = \mathcal{F}f \in \mathcal{F}L^p(G)$. Thus, $\mathfrak{b}(\mathrm{H}_p) \subseteq \mathcal{F}L^p(G)$. A similar method yields $\mathfrak{b}(\mathrm{H}_1) \subseteq \mathcal{F}M(G)$ (note that $\mathfrak{T}(G) \subseteq M(G) = L^1(G)^*$).

(ii): Let $\sigma \in \Sigma$ and $t_\sigma \in \mathfrak{T}_\sigma(G)$. Then,

$$\sup_\alpha \|h_\alpha * t_\sigma\|_p = \sup_\alpha \|h_\alpha * (u_\sigma * t_\sigma)\|_p \leq \sup_\alpha \|h_\alpha * u_\sigma\|_p \|t_\sigma\|_1 < \infty,$$

and $t_\sigma = u_\sigma * t_\sigma = \lim_\alpha (h_\alpha * u_\sigma) * t_\sigma = \lim_\alpha h_\alpha * t_\sigma$, that implies not only $\iota_\sigma(t_\sigma) \in \mathfrak{bc}(\mathrm{H}_p)$, but also for each $\mathfrak{t} \in \prod_{\sigma\in\Sigma} \mathfrak{T}_\sigma(G)$,

$$\begin{aligned}\|\mathfrak{t}_\sigma\|_p &= \lim_\alpha \|h_\alpha * \mathfrak{t}_\sigma\|_p = \lim_\alpha \left\| \sum_{\eta\in\Sigma} h_\alpha * (\mathfrak{t}_\eta * u_\sigma) \right\|_p \\ &= \lim_\alpha \|\mathrm{h}^p_\alpha(\mathfrak{t}) * u_\sigma\|_p \leq \|u_\sigma\|_1 \sup_\alpha \|\mathrm{h}^p_\alpha(\mathfrak{t})\|_p \leq \|u_\sigma\|_1 \|\mathfrak{t}\|_{\mathrm{H}_p}.\end{aligned}$$

Hence by using Theorem 2.2, the proof is completed.

(iii): Suppose $1 < p < \infty$, $f \in L^p(G)$ and $\mathfrak{t} = \mathcal{F}f$. For each $\epsilon > 0$, there exists $t_\epsilon \in \mathfrak{T}(G)$ such that $\|f - t_\epsilon\|_p < \epsilon_1$, where $\epsilon_1 = \frac{\epsilon}{\sup_\alpha \|h_\alpha\|_1 + 1}$ (see also Corollary 3.5). Since for each $\sigma \in \Sigma$, $\sup_\alpha \|h_\alpha * u_\sigma\|_p \leq \|u_\sigma\|_p \sup_\alpha \|h_\alpha\|_1 < \infty$, so by (ii), $\mathfrak{bc}(\mathrm{H}_p)$ is a direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$. But, $(u_\sigma * t_\epsilon)_{\sigma\in\Sigma} \in \bigoplus_{\sigma\in\Sigma} \mathfrak{T}_\sigma(G)$. Thus $(u_\sigma * t_\epsilon)_{\sigma\in\Sigma} \in \mathfrak{bc}(\mathrm{H}_p)$, and so $\lim_\alpha \|h_\alpha * t_\epsilon - t_\epsilon\|_p = 0$. It follows that,

$$\begin{aligned}\overline{\lim_\alpha} \|\mathrm{h}^p_\alpha(\mathfrak{t}) - f\|_p &= \overline{\lim_\alpha} \|\mathrm{h}^p_\alpha(\mathcal{F}f) - f\|_p = \overline{\lim_\alpha} \|h_\alpha * f - f\|_p \\ &\leq \overline{\lim_\alpha}(\|h_\alpha * f - h_\alpha * t_\epsilon\|_p + \|h_\alpha * t_\epsilon - t_\epsilon\|_p + \|t_\epsilon - f\|_p) \\ &= \overline{\lim_\alpha}(\|h_\alpha * f - h_\alpha * t_\epsilon\|_p + \|t_\epsilon - f\|_p) \\ &\leq (\sup_\alpha \|h_\alpha\|_1)\|f - t_\epsilon\|_p + \|t_\epsilon - f\|_p < \epsilon,\end{aligned}$$

that implies $\lim_\alpha \|\mathrm{h}_{\alpha,p}(\mathfrak{t}) - f\|_p = 0$, and so $\mathfrak{t} \in \mathfrak{bc}(\mathrm{H}_p) \subseteq \mathfrak{b}(\mathrm{H}_p)$. It together (i) implies that $\mathfrak{b}(\mathrm{H}_p) = \mathfrak{bc}(\mathrm{H}_p) = \mathcal{F}L^p(G)$ $(1 < p < \infty)$.

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Let $p = 1$. If $\mu \in M(G)$, then $\sup_\alpha \|\mathrm{h}^1_\alpha(\mathcal{F}\mu)\| = \sup_\alpha \|h_\alpha * \mu\|_1 \leq (\sup_\alpha \|h_\alpha\|_1)\|\mu\| < \infty$, and so $\mathcal{F}\mu \in \mathfrak{b}(\mathrm{H}_1)$. Hence by (i), $\mathfrak{b}(\mathrm{H}_1) = \mathcal{F}M(G)$. Since $\mathfrak{T}(G)$ is $\|.\|_1$-dense in $L^1(G)$, so $\mathfrak{bc}(\mathrm{H}_1) \subseteq \mathcal{F}L^1(G)$. Applying a method exactly as the previous paragraph yields $\mathcal{F}L^1(G) \subseteq \mathfrak{bc}(\mathrm{H}_1)$, and so $\mathfrak{bc}(\mathrm{H}_1) = \mathcal{F}L^1(G)$. Exactly the same proof, shows that $\mathfrak{b}(\mathrm{H}_\infty) = \mathcal{F}L^\infty(G)$, and $\mathfrak{bc}(\mathrm{H}_\infty) = \mathcal{F}C(G)$ (note that $\mathfrak{T}(G)$ is $\|.\|_\infty$-dense in $C(G)$). □

Example 3.7. Let $\mathbb{T}$ be the multiplicative group of all complex numbers with absolute value 1. Then, $\Sigma := \{e_m : m \in \mathbb{Z}\}$, where $e_m(z) = z^m$ for $m \in \mathbb{Z}$ and $z \in \mathbb{T}$. Suppose for each $n \in \mathbb{N}$, $h_n = D_n$, where D_n is the Dirchlet kernel (i.e. $D_n = \sum_{m=-n}^{n} e_m$). Let $\mathrm{H}_p := (h_n^p)_{n\in\mathbb{N}}$, where $1 \leq p \leq \infty$. Clearly, if $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $n \geq |m|$, then $h_n * e_m = e_m$. Thus, by Proposition 3.6(ii), $\mathfrak{b}(\mathrm{H}_1)$ is a direct sum of $(\mathfrak{T}_{e_m}(\mathbb{T}))_{m\in\mathbb{Z}}$, and by Proposition 3.6(i), $\mathfrak{b}(\mathrm{H}_1) \subseteq \mathcal{F}M(\mathbb{T})$. But, $\mathfrak{b}(\mathrm{H}_1) \neq \mathcal{F}M(\mathbb{T})$. To see this, note that if $\mathfrak{b}(\mathrm{H}_1) = \mathcal{F}M(\mathbb{T})$, then $\mathcal{F}L^1(\mathbb{T}) \subseteq \mathfrak{b}(\mathrm{H}_1)$ and so $\sup_{n\in\mathbb{N}} \|D_n * f\|_1 < \infty$ for all $f \in L^1(\mathbb{T})$. It, together Banach Steinhauss' Theorem and the last paragraph on Page 56 of [6], implies that $\sup_{n\in\mathbb{N}} \|D_n\|_1 < \infty$, that's a contradiction (see also Exersice 1 on Page 59 of [6]). By a similar method, it is shown that $\mathfrak{b}(\mathrm{H}_\infty)$ is a direct sum of $(\mathfrak{T}_{e_m}(\mathbb{T}))_{m\in\mathbb{Z}}$, $\mathcal{F}C(\mathbb{T}) \subsetneqq \mathfrak{b}(\mathrm{H}_\infty)$, and $\mathfrak{b}(\mathrm{H}_\infty) \subsetneqq \mathcal{F}L^\infty(\mathbb{T})$. Also, by Theorem 1.5 of [6], for each $1 < p < \infty$, $\mathcal{F}L^p(\mathbb{T}) \subseteq \mathfrak{bc}(\mathrm{H}_p)$ (note that by Corollary 1.9 of [6], for each $f \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$, $\mathrm{h}_{n,p}(\mathcal{F}f) = S_n f$, where $S_n f$ is the n'th partial sum of the Fourier series of f), and so by Proposition 3.6(ii), $\mathfrak{b}(\mathrm{H}_p) = \mathfrak{bc}(\mathrm{H}_p) = \mathcal{F}L^p(\mathbb{T})$.

Corollary 3.8. *Let X be any of spaces $L^p(G)$ $(1 \leq p \leq \infty)$, $C(G)$ and $M(G)$. Then $\mathcal{F}X$, as a direct sum of $(\mathfrak{T}_\sigma(G))_{\sigma\in\Sigma}$, is a closed subspace of a direct sum of the form $\mathfrak{b}(\Gamma)$ that is introduced in Definition 2.1.*

Proof. By Theorem 28.53 of [5], there exists a net $(h_\alpha)_\alpha$ in $\mathfrak{T}(G)$ such that for each α, $\|h_\alpha\|_1 = 1$ and $\lim_\alpha \|h_\alpha * f - f\|_1 = 0$, where $f \in L^1(G)$. Thus, if $\sigma \in \Sigma$, then $(h_\alpha * u_\sigma)_\alpha$ is a net in $\mathfrak{T}_\sigma(G)$ that $\|.\|_1$-converges to u_σ, and so $\|.\|_p$-converges to u_σ for each $1 \leq p \leq \infty$ (note that $\mathfrak{T}_\sigma(G)$ is finite dimensional). Using Proposition 3.6(iii) completes the proof. □

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