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ARITHMETIC PROGRESSIONS OF b-NIVEN NUMBERS

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ABSTRACT. A positive integer is a *b-Niven number* (or *b-harshad number*) if it is a multiple of the sum of the digits of its base-b representation. For each base $b \geq 2$, the maximum length of a sequence of consecutive b-Niven numbers is known to be $2b$. In this article, we investigate the maximum lengths of arithmetic progressions of b-Niven numbers.

1. Introduction

Let $\mathbb{Z}^+$ and $\mathbb{Z}_{\geq 0}$ denote the sets of positive and nonnegative integers, respectively. For each $b \in \mathbb{Z}^+$ with $b \geq 2$, let $s_b : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the function mapping a positive integer to the sum of the digits in its base b expansion. More precisely, for $b \geq 2$,

$$s_b\left(\sum_{j=0}^{m} a_j b^j\right) = \sum_{j=0}^{m} a_j,$$

where $m \in \mathbb{Z}_{\geq 0}$ and $0 \leq a_j \leq b-1$ for each $0 \leq j \leq m$. We say that $n \in \mathbb{Z}^+$ is a *b-Niven number* (also known as a *b-harshad number*) if $s_b(n) \mid n$. When $b = 10$, n is called simply a *Niven number*.

In 1993, Cooper and Kennedy [1] showed that the length of a sequence of consecutive Niven numbers is at most 20. In 1994, Grundman [2] generalized this result and proved that, for each base $b \geq 2$, the length of a sequence of consecutive b-Niven numbers is at most $2b$. In this article, we investigate the more general question of the possible lengths of arithmetic progressions of b-Niven numbers. For ease in terminology, we use *d-AP* as a shorthand for "arithmetic progression with a common difference d," and *b-Niven d-AP* for "d-AP in which each term is a b-Niven number." Our main results concern the maximum length of a b-Niven d-AP for $d \geq 2$ and $b \geq 2d$.

In the following section, we give some initial results, including very general, though weak, upper and lower bounds for the maximum lengths of b-Niven d-APs. We present our main results in Section 3, giving upper bounds for the maximum lengths in order of increasing generality, but potentially less tight bounds. Finally, in Section 4, we restrict our study to the case $d = 2$, providing tighter bounds in most cases and determining the actual maximum lengths for some small values of b.

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2. Initial results

In this section, we provide upper and lower bounds for the maximum length of a b-Niven d-AP that apply to every case with $d \geq 2$ and $b \geq 2d$. We note that the upper bound given in Theorem 2.2 guarantees that a finite maximum length exists for each of these cases. For most of these, we improve upon this initial bound in later sections. In Theorem 2.3, we provide a lower bound for the maximum length (though lower bounds are not the focus of this paper). Finally, we give an auxiliary theorem that will be used in the following sections.

We begin with a lemma that simplifies many of the arguments throughout this paper.

Lemma 2.1. *Let $b \geq 2$. If x and y are two b-Niven numbers such that $s_b(x) = s_b(y)$, then $s_b(x) \mid (x-y)$.*

Proof. Since x and y are b-Niven numbers, $s_b(x) \mid x$ and $s_b(y) \mid y$. It follows from $s_b(x) = s_b(y)$ that $s_b(x) \mid y$. Thus $s_b(x) \mid (x-y)$. □

Theorem 2.2. *Let $b \geq 4$ and $d \geq 2$ be such that $b \geq 2d$. A b-Niven d-AP is of length at most*

$$\left\lceil \frac{b^{d+1} + (d-1)b}{d} \right\rceil.$$

Proof. For each $a \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq b-d-1$, let

$$x_{a,j} = ab^{d+1} + \sum_{i=2}^{d} (b-1)b^i + (b-d-1)b + (j+d)$$

and

$$y_{a,j} = ab^{d+1} + \sum_{i=1}^{d} (b-1)b^i + j.$$

Note that $y_{a,j} - x_{a,j} = (b-1)d$, so $x_{a,j}$ and $y_{a,j}$ lie in the same d-AP.

Suppose that $\mathscr{S}$ is a b-Niven d-AP of length greater than the bound given in the theorem. Then $\mathscr{S}$ contains at least one term of the form $x_{a,j}$. Let a be minimal such that $x_{a,j}$ is in $\mathscr{S}$ for some $0 \leq j \leq b-d-1$. By the minimality of a, the first term of $\mathscr{S}$ is at least $(a-1)b^{d+1} + \sum_{i=2}^{d}(b-1)b^i + (b-d)b$.

Since the length of $\mathscr{S}$ is greater than $(b^{d+1} + (d-1)b)/d$, the last term of $\mathscr{S}$ is at least $ab^{d+1} + \sum_{i=1}^{d}(b-1)b^i$. It follows that there exists at least one k with $0 \leq k \leq b-d-1$ such that $\mathscr{S}$ contains the terms $x_{a,k}$ and $y_{a,k}$. Applying Lemma 2.1 to these two numbers yields $(s_b(a) + (b-1)d + k) \mid (b-1)d$, implying that $s_b(a) + (b-1)d + k \leq (b-1)d$. Hence, $a = k = 0$, and so $x_{0,0}$ and $y_{0,0}$ are in $\mathscr{S}$, while $y_{0,d}$ is not. As a result, the length of $\mathscr{S}$ is at most $\lceil y_{0,0}/d \rceil$, which is strictly less than the bound given in the theorem, contradicting our supposition. □

In 1997, Wilson [4] proved that for each $b \geq 2$, the bound of $2b$ on the maximum length of a sequence of consecutive b-Niven numbers is sharp. This immediately leads to the following theorem, which provides a lower bound for the maximal length of a b-Niven d-AP.

Theorem 2.3. *Let $b \geq 2$ and $d \geq 2$. There exists a b-Niven d-AP of length $\lceil \frac{2b}{d} \rceil$.*

The next theorem, unlike others herein, imposes the restriction that the elements of the arithmetic progression are odd. With this and related restrictions, we determine the maximum length of such a

b-Niven d-AP and provide information about the terms of a sequence of this length. Theorem 2.4 is used in proving more general results in the rest of the paper.

Theorem 2.4. *Let $b \geq 2$ and $d \geq 2$ both be even with $b \geq d$. The maximum length of a b-Niven d-AP containing an odd term is exactly $\lceil \frac{2b}{d} \rceil$. Any b-Niven d-AP of length at least $\frac{b}{d}+1$ containing an odd term consists of numbers strictly between $mb^2 - b$ and $mb^2 + b$ for some $m \in \mathbb{Z}^+$.*

Proof. By Wilson [4], there exists a sequence of $2b$ consecutive b-Niven numbers. It follows easily that there exists a b-Niven d-AP of length $\lceil \frac{2b}{d} \rceil$ containing an odd term.

Let $\mathscr{S}$ be a b-Niven d-AP of length at least $\frac{b}{d}+1$ that contains an odd term. Since d is even, every term of $\mathscr{S}$ is odd, and since b is even, the last digit of every term is odd. Hence, if $ab+i$ is an element of $\mathscr{S}$ with $a \geq 0$ and $0 \leq i < b$, then $(s_b(a)+i) \mid (ab+i)$ implies that $s_b(a)$ is even.

Now, fix $c \in \mathbb{Z}^+$ and $0 < j < b$ such that $(c-1)b + j$ is the first term of $\mathscr{S}$. Since $\mathscr{S}$ is of length at least $\frac{b}{d}+1$, $\mathscr{S}$ contains a term of the form $cb + j'$ for some $0 < j' < b$. It follows that both $s_b(c-1)$ and $s_b(c)$ are even. Hence, c is a multiple of b. Further, since $c+1$ is not a multiple of b, each term of $\mathscr{S}$ is strictly less than $(c+1)b$. The theorem follows by letting $c = mb$. □

3. Main results

We now present our main results. Theorem 3.1 considers the case in which b is even and d is odd. Theorem 3.2 yields a less strong bound, but covers a much wider range of cases. Then, Theorems 3.4 and 3.5 focus on the cases in which $d \mid (b-1)$ and $d \mid b$, respectively, providing bounds that are particularly useful when Theorems 3.1 and 3.2 do not apply.

Theorem 3.1. *Let $b \geq 6$ be even and $d \geq 3$ be odd with $b \geq 2d$. A b-Niven d-AP is of length at most $2\lceil \frac{b}{d} \rceil + 1$. Any b-Niven d-AP of length at least $2\lceil \frac{b}{d} \rceil$ consists of numbers strictly between $mb^2 - b - d$ and $mb^2 + b + d$ for some $m \in \mathbb{Z}^+$.*

Furthermore:

(a) *if $d \mid (b-1)$, then a b-Niven d-AP is of length at most $2\lceil \frac{b}{d} \rceil$. Any b-Niven d-AP of this length consists of the numbers $mb^2 - b + 1, mb^2 - b + d + 1, \ldots, mb^2 + b + d - 1$ for some $m \in \mathbb{Z}^+$.*

(b) *if $d \mid (b-1)$, and if every factor f of $b-1$ such that $f > d$ satisfies $f \equiv -d \pmod 4$, then the maximum length of a b-Niven d-AP is exactly $2\lceil \frac{b}{d} \rceil - 1$. Any b-Niven d-AP of this length consists of numbers strictly between $mb^2 - b - d$ and $mb^2 + b + d$ for some $m \in \mathbb{Z}^+$.*

Proof. Let $\mathscr{S}$ be a b-Niven d-AP. Let x be the first odd term in $\mathscr{S}$, y be the last odd term in $\mathscr{S}$, and $\mathscr{T}$ be the b-Niven $2d$-AP starting with x and ending with y. By Theorem 2.4, $\mathscr{T}$ is of length at most $\lceil \frac{2b}{2d} \rceil = \lceil \frac{b}{d} \rceil$, and if $\mathscr{T}$ is of this length, then

$$mb^2 - b < x < y < mb^2 + b \tag{1}$$

for some $m \in \mathbb{Z}^+$. Noting that $\mathscr{S}$ has at most one term before x and one term after y, the main statement of the theorem follows.

Assume now that $d \mid (b-1)$ and that $\mathscr{S}$ is of length at least $2\lceil \frac{b}{d} \rceil$. The sequence $\mathscr{T}$ is of length at least $\lceil \frac{b}{d} \rceil$, which, combined with inequality (1), implies that $x = mb^2 - b + 1$ and $y = mb^2 + b - 1$. Suppose that $\mathscr{S}$ also contains the term $x - d = (m-1)b^2 + (b-2)b + (b-d+1)$. Note that $x + b -$

$1-d$ is in $\mathscr{S}$ and has the same base b digit sum as $x-d$. Applying Lemma 2.1 to $x+b-1-d$ and $x-d$ yields $(s_b(m-1)+2b-d-1) \mid (b-1)$, which is impossible since $b>d$. Hence, $\mathscr{S}$ does not contain $x-d$ and part (*a*) of the theorem follows.

Next, assume the hypotheses of part (*b*) of the theorem and that $\mathscr{S}$ is of length at least $2\left\lceil \frac{b}{d}\right\rceil - 1$. Since b is even, d is odd, and $d \mid (b-1)$,

$$\frac{b+d-1}{2d} - 1 = \frac{b-d-1}{2d} < \frac{b}{2d} < \frac{b+d-1}{2d} \in \mathbb{Z}$$

implies that

$$2\left\lceil \frac{b}{2d}\right\rceil = 2\left(\frac{b+d-1}{2d}\right) = \frac{b+d-1}{d} = \left\lceil \frac{b}{d}\right\rceil.$$

So $\mathscr{T}$ is of length at least

$$\frac{2\left\lceil \frac{b}{d}\right\rceil - 2}{2} = \left\lceil \frac{b}{d}\right\rceil - 1 = 2\left\lceil \frac{b}{2d}\right\rceil - 1 \geq \left\lceil \frac{b}{2d}\right\rceil + 1,$$

since $b \geq 2d$ and $d \nmid b$. Applying Theorem 2.4 with $2d$ in place of d we conclude that inequality (1) again holds.

Finally, suppose for a contradiction that $\mathscr{S}$ is of length at least $2\left\lceil \frac{b}{d}\right\rceil$. Then from part (*a*), $\mathscr{S}$ contains the term $y+d = mb^2+b+(d-1)$. Applying Lemma 2.1 to mb^2+d and $mb^2+b+(d-1)$ yields $(s_b(m)+d) \mid (b-1)$. Since $s_b(m)+d>d$, the hypothesis implies that $s_b(m)+d \equiv -d \pmod 4$. Thus, $s_b(mb^2+2d) = s_b(m)+2d \equiv 0 \pmod 4$. But mb^2+2d is a term in $\mathscr{S}$ and so $s_b(mb^2+2d) \mid (mb^2+2d)$, implying that $mb^2+2d \equiv 0 \pmod 4$. This is a contradiction, since b is even and d is odd. Thus, under the hypotheses of part (*b*), $\mathscr{S}$ cannot be of length $2\left\lceil \frac{b}{d}\right\rceil$. By Theorem 2.3, there exists a b-Niven d-AP of length $2\left\lceil \frac{b}{d}\right\rceil - 1$, implying that this bound is optimal. □

The next theorem is broadly applicable to achieve a better bound than the remaining theorems in this section. The key, however, is whether there exists a small value of n that satisfies the theorem's hypotheses.

Theorem 3.2. *Let $b \geq 4$, $d \geq 2$, and $n \geq 2$ be such that $b \geq nd$, $\gcd(n,d)=1$, and $n \nmid (b-1)$. A b-Niven d-AP is of length at most $\left\lceil \frac{2b}{d}\right\rceil + 2n - 2$. Any b-Niven d-AP of length at least $\left\lceil \frac{b}{d}\right\rceil + 2n - 1$ consists of numbers strictly between $mb^2-b-nd+d-1$ and $mb^2+b+nd-d$ for some $m \in \mathbb{Z}^+$.*

The proof of Theorem 3.2 uses the following lemma.

Lemma 3.3. *Let $b \geq 4$, $d \geq 2$, and $n \geq 2$ be such that $b \geq nd$, $\gcd(n,d)=1$, and $n \nmid (b-1)$. If a b-Niven d-AP has at least n terms less than ab and at least n terms greater than or equal to ab for some $a \in \mathbb{Z}^+$, then a is a multiple of b.*

Proof. Let $\mathscr{S}$ be a b-Niven d-AP as in the hypothesis of the lemma. Then $\mathscr{S}$ contains the terms $(a-1)b+(b-nd+i)$, $(a-1)b+(b-(n-1)d+i)$, $\ldots$, $(a-1)b+(b-d+i)$, $ab+i$, $ab+(d+i)$, $\ldots$, $ab+(n-1)d+i$ for some $0 \leq i \leq d-1$.

Suppose for a contradiction that $b \nmid a$. Then the corresponding digit sums of the above terms of $\mathscr{S}$ are $s_b(a)-1+b-nd+i$, $s_b(a)-1+b-(n-1)d+i$, $\ldots$, $s_b(a)-1+b-d+i$, $s_b(a)+i$, $s_b(a)+d+i$, $\ldots$, $s_b(a)+(n-1)d+i$, respectively. Since $\gcd(n,d)=1$, the first n terms of this digit sum sequence form a complete set of representatives of the congruence classes modulo n. Hence, there exists $1 \leq j \leq n$ such that $n \mid (s_b(a) - 1 + b - jd + i)$. Since the terms of $\mathscr{S}$ are b-Niven numbers, the corresponding number $(a-1)b + (b - jd + i)$ in $\mathscr{S}$ is also divisible by n. Similarly, by focusing on the last n terms of the digit sum sequence, we see that there exists $0 \leq j' \leq n-1$ such that $n \mid (s_b(a) + j'd + i)$ and so $n \mid (ab + j'd + i)$.

Combining these, we have that n is a factor of $(s_b(a) + j'd + i) - (s_b(a) - 1 + b - jd + i) = j'd + jd - (b-1)$ and $(ab + j'd + i) - ((a-1)b + (b - jd + i)) = j'd + jd$. Hence, $n \mid (b-1)$, a contradiction. $\square$

Proof of Theorem 3.2. Let $\mathscr{S}$ be a b-Niven d-AP of length at least $\left\lceil \frac{b}{d} \right\rceil + 2n - 1$. Let $a \in \mathbb{Z}^+$ be minimal such that at least n terms of $\mathscr{S}$ are less than ab. Then at most $n-1$ terms of $\mathscr{S}$ are less than $(a-1)b$. Together with the terms strictly between $(a-1)b - 1$ and ab, there are at most $\left\lceil \frac{b}{d} \right\rceil + n - 1$ terms less than ab. Consequently, at least n terms of $\mathscr{S}$ are greater than or equal to ab. Thus, Lemma 3.3 applies and $b \mid a$. Since $b \nmid (a+1)$, Lemma 3.3 further implies that at most $n-1$ terms of $\mathscr{S}$ are greater than or equal to $(a+1)b$. Noting that there are at most $\left\lceil \frac{2b}{d} \right\rceil$ terms strictly between $(a-1)b-1$ and $(a+1)b$, the bound on the length follows. The first term of $\mathscr{S}$ is at least $(a-1)b - d(n-1)$ and the last term is at most $(a+1)b - 1 + d(n-1)$. Letting $a = mb$ completes the proof. $\square$

The next two theorems give bounds under assumptions on the relationship between d and b. In Theorem 3.4, we require that d is a factor of $b-1$ and in Theorem 3.5, we require that d is a factor of b.

Theorem 3.4. *Let $b \geq 4$ and $d \geq 2$ be such that $b \geq 2d$ and $d \mid (b-1)$. A b-Niven d-AP is of length at most $b + 2\left(\frac{b-1}{d}\right)$. Any b-Niven d-AP of this length consists of numbers $mb^2 - b$, $mb^2 - b + d$, $\ldots$, $mb^2 + db + (b-d-2)$ for some $m \in \mathbb{Z}^+$.*

Proof. Let $\mathscr{S}$ be a b-Niven d-AP of length at least $b + 2\left(\frac{b-1}{d}\right)$. Since $d \mid (b-1)$, we have $\gcd(b,d) = 1$, and so each $0 \leq i \leq b-1$ occurs as the last digit of at least one term in $\mathscr{S}$. Let $a \geq 0$ be maximal such that $\mathscr{S}$ contains $(a+1)b + (b-2)$. Since $\mathscr{S}$ does not contain $(a+d+1)b + (b-2)$, the first term of $\mathscr{S}$ is at most $((a+d+1)b + (b-2)) - d\left(b + 2\left(\frac{b-1}{d}\right)\right) = ab$. Hence, $a > 0$ and $\mathscr{S}$ also contains $ab + (b-1)$.

If $b \nmid (a+1)$, then $s_b(a+1) = s_b(a) + 1$. Applying Lemma 2.1 to $ab + (b-1)$ and $(a+1)b + (b-2)$ yields $(s_b(a) + b - 1) \mid (b-1)$, which is impossible. Therefore, $b \mid (a+1)$, and so there exists $m \in \mathbb{Z}^+$ such that $a = mb - 1$.

Now suppose that $ab = (mb-1)b$ is not the first term of $\mathscr{S}$. Then $(mb-1)b - d = (m-1)b^2 + (b-2)b + (b-d)$ is in $\mathscr{S}$. Applying Lemma 2.1 to this and $(m-1)b^2 + (b-1)b + (b-d-1)$ gives us that $(s_b(m-1) + 2b - 2 - d) \mid (b-1)$, a contradiction, since $b \geq 2d$. Thus, the first term of $\mathscr{S}$ is $ab = mb^2 - b$, the last term is $(a+d+1)b + (b-d-2) = mb^2 + db + (b-d-2)$, and the length of $\mathscr{S}$ is exactly $b + 2\left(\frac{b-1}{d}\right)$. $\square$

Theorem 3.5. *Let $b \geq 4$ and $d \geq 2$ be such that $b \geq 2d$ and $d \mid b$. A b-Niven d-AP is of length at most $4b - \frac{2b}{d}$. Any b-Niven d-AP of this length consists of the numbers $mb^2 - (2d-1)b$, $mb^2 - (2d-1)b + d$, $\ldots$, $mb^2 + (2d-1)b - d$ for some $m \in \mathbb{Z}^+$.*

Lemma 3.6. *Let $b \geq 4$ and $d \geq 2$ be such that $b \geq 2d$ and $d \mid b$. If a b-Niven d-AP of length at least $2b - \frac{b}{d}$ contains a multiple of d but no multiple of b^2, then its first term is a multiple of b and its length is exactly $2b - \frac{b}{d}$.*

Proof. Let $\mathscr{S}$ be a b-Niven d-AP of length at least $2b - \frac{b}{d}$ that contains a multiple of d but no multiple of b^2. Note that every term of $\mathscr{S}$ is a multiple of d.

Suppose for a contradiction that the first term of $\mathscr{S}$ is not a multiple of b. Since $\mathscr{S}$ does not contain a multiple of b^2, the first term is of the form $ab^2 + ib + j$ for some $a \geq 0$, $0 \leq i \leq b - 2d$, and $d \leq j \leq b - d$ such that $d \mid j$. It follows that the last term of $\mathscr{S}$ is at least $(ab^2 + ib + j) + d\left(2b - \frac{b}{d} - 1\right) = ab^2 + (i + 2d - 1)b + (j - d)$.

Let $0 \leq k \leq d - 1$ be such that $d \mid (s_b(a) + i + k)$. If $k \neq 0$, let $k' = k - 1$; if $k = 0$, let $k' = 1$. Then $0 \leq k' \leq d - 1$ and $\gcd(d, s_b(a) + i + k') = 1$. Since $d \mid j$, we also have that $\gcd(d, s_b(a) + i + k' + j) = 1$.

Note that $\mathscr{S}$ contains both $ab^2 + (i + k')b + j$ and $ab^2 + (i + k' + d)b + (j - d)$, so by Lemma 2.1, $(s_b(a) + i + k' + j) \mid d(b - 1)$. Hence, $(s_b(a) + i + k' + j) \mid (b - 1)$, implying that $j \leq (b - 1) - (s_b(a) + i + k') \leq b - (s_b(a) + i + k)$. It follows that $ab^2 + (i + k)b + (b - (s_b(a) + i + k))$ is in $\mathscr{S}$ since it is a multiple of d between the first and the last terms of $\mathscr{S}$.

If $s_b(a) + i + k \neq 0$, then $s_b(ab^2 + (i + k)b + (b - (s_b(a) + i + k))) = b$, implying that $ab^2 + (i + k)b + (b - (s_b(a) + i + k))$ is a multiple of b, a contradiction. If $s_b(a) + i + k = 0$, then $a = i = 0$, implying that $db + (b - d)$ is in $\mathscr{S}$. This again is impossible since $s_b(db + (b - d)) = b$ and b does not divide $db + (b - d)$. Therefore, the first term of $\mathscr{S}$ is a multiple of b, as desired.

Suppose now that the length of $\mathscr{S}$ is greater than $2b - \frac{b}{d}$. Let $\mathscr{S}'$ be the sequence obtained by removing the first term of $\mathscr{S}$. The length of $\mathscr{S}'$ is at least $2b - \frac{b}{d}$, so by the above argument, the first term of $\mathscr{S}'$ is a multiple of b, which is impossible since $d < b$. Therefore, $\mathscr{S}$ is of length exactly $2b - \frac{b}{d}$. □

Proof of Theorem 3.5. Let $\mathscr{S}$ be a b-Niven d-AP of length at least $4b - \frac{2b}{d}$.

Suppose that $\mathscr{S}$ does not contain a multiple of d. Then, since $d \mid b$, there exists $a \geq 0$ such that $\mathscr{S}$ contains at least $2b - \frac{b}{d}$ terms strictly between ab^2 and $(a+1)b^2$. The smallest term in $\mathscr{S}$ that is greater than ab^2 is of the form $ab^2 + ib + j$ with $0 \leq i \leq b - 2d + 1$ and $1 \leq j \leq b - 1$. Then $\mathscr{S}$ contains the terms $ab^2 + ib + j$, $ab^2 + (i+1)b + j$, $\ldots$, $ab^2 + (i + d - 1)b + j$, whose digit sums are $s_b(a) + i + j$, $s_b(a) + i + 1 + j$, $\ldots$, $s_b(a) + i + d - 1 + j$, respectively. This sequence of digit sums forms a complete set of representatives of the congruence classes modulo d. Hence, there exists $i \leq k \leq i + d - 1$ such that $d \mid (s_b(a) + k + j)$. However, $ab^2 + kb + j$ is not a multiple of d, contradicting that $ab^2 + kb + j$ is in $\mathscr{S}$. Hence, $\mathscr{S}$ contains a multiple of d.

Now, since the length of $\mathscr{S}$ is more than $2b - \frac{b}{d}$, by Lemma 3.6, there exists $m \in \mathbb{Z}^+$ such that mb^2 is in $\mathscr{S}$. By Lemma 3.6 again, $\mathscr{S}$ has at most $2b - \frac{b}{d}$ terms less than mb^2 and at most $2b - \frac{b}{d}$ terms greater than mb^2. Suppose for a contradiction that $\mathscr{S}$ contains exactly $2b - \frac{b}{d}$ terms greater than mb^2. Then $mb^2 + d$ is the first term of a b-Niven d-AP of length $2b - \frac{b}{d}$ and is not a multiple of b, contradicting Lemma 3.6. Hence, $\mathscr{S}$ has at most $4b - \frac{2b}{d}$ terms, and the theorem follows. □

4. Arithmetic Progressions with $d = 2$

In this section, we consider b-Niven 2-APs for $b \geq 2$. We begin with Theorem 4.1, giving results for small values of b, which is proved at the end of the section. We then state and prove theorems that apply more generally. Theorems 4.2 and 4.4 provide bounds for odd bases, while Theorem 4.6 and Corollary 4.8 give bounds for even bases.

Theorem 4.1. *The maximum length of a*

(*a*) 2*-Niven* 2*-AP is* 6*. The only* 2*-Niven* 2*-AP of this length consists of numbers* 2, 4, 6, 8, 10, 12.
(*b*) 3*-Niven* 2*-AP is* 6*. The only* 3*-Niven* 2*-AP of this length consists of the numbers* 2, 4, 6, 8, 10, 12.
(*c*) 4*-Niven* 2*-AP is* 9*. Any* 4*-Niven* 2*-AP of this length is of the form* $16m-8, 16m-6, \ldots, 16m+8$ *for some* $m \in \mathbb{Z}^+$.
(*d*) 5*-Niven* 2*-AP is* 6 *or* 7*. If a* 5*-Niven* 2*-AP of length* 7 *exists, it is of the form* $5^{t+2}-5, 5^{t+2}-3, \ldots, 5^{t+2}+7$ *for some* $t \in \mathbb{Z}^+$.
(*e*) 6*-Niven* 2*-AP is* 7*. Any* 6*-Niven* 2*-AP of this length is of the form* $36m-6, 36m-4, \ldots, 36m+6$ *for some* $m \in \mathbb{Z}^+$.
(*f*) 7*-Niven* 2*-AP is* 8*. Any* 7*-Niven* 2*-AP of this length consists of numbers strictly between* $49m-8$ *and* $48m+9$ *for some* $m \in \mathbb{Z}^+$.

The next two theorems consider the case when b is odd.

Theorem 4.2. *Let $b \geq 7$ be odd. Then a b-Niven 2-AP is of length at most $\lfloor \frac{5b}{4} \rfloor$. Any b-Niven 2-AP of length greater than b consists of numbers strictly between mb^2-b-1 and $mb^2+b+\frac{b-3}{2}$ for some $m \in \mathbb{Z}^+$.*

Lemma 4.3. *Let $b \geq 7$ be odd. If there exist $a \geq 1$ and $1 \leq i \leq b-3$ such that $(a-1)b+i$, $(a-1)b+(i+2)$, $ab+(i-1)$, and $ab+(i+1)$ are all b-Niven numbers and $s_b(a)+i+1 > \frac{b-1}{2}$, then a is a multiple of b.*

Proof. Assume by way of contradiction that a is not a multiple of b. Then $s_b(a) = s_b(a-1)+1$. Applying Lemma 2.1 to $(a-1)b+(i+2)$ and $ab+(i+1)$ yields that $(s_b(a)+i+1) \mid (b-1)$. Since $s_b(a)+i+1 > \frac{b-1}{2}$, we have $s_b(a)+i+1 = b-1$. Similarly, applying Lemma 2.1 to $(a-1)b+i$ and $ab+(i-1)$ yields that $(s_b(a)+i-1) \mid (b-1)$. Noting that $s_b(a)+i-1 = b-3$, we have $(b-3) \mid (b-1)$, which is impossible for $b \geq 7$. □

Proof of Theorem 4.2. Let $\mathscr{S}$ be a b-Niven 2-AP of length greater than b. Since b is odd, each $0 \leq i \leq b-1$ occurs as the last digit of at least one term in $\mathscr{S}$. Let $(a-1)b+i$ be the smallest term of $\mathscr{S}$ satisfying $a \geq 1$ and $i = \frac{b-3}{2}$ or $\frac{b-1}{2}$. It follows that the first term of $\mathscr{S}$ is at least $(a-2)b+(i+1)$, which implies that $ab+(i+1)$ is also in $\mathscr{S}$. Since $1 \leq i \leq b-3$ and $s_b(a)+i+1 > \frac{b-1}{2}$, by Lemma 4.3, a is a multiple of b.

Now, suppose that $(a+1)b+i$ is in $\mathscr{S}$. Then $ab+(i-1)$, $ab+(i+1)$, $(a+1)b+(i-2)$, and $(a+1)b+i$ are all in $\mathscr{S}$. Moreover, $1 \leq i-1 \leq b-3$ and $s_b(a+1)+(i-1)+1 = s_b(a)+i+1 > \frac{b-1}{2}$. By Lemma 4.3 again, $a+1$ is also a multiple of b, contradicting that a is a multiple of b. Hence, $(a+1)b+i$ is not in $\mathscr{S}$.

Next, suppose that $\mathscr{S}$ contains $(a-2)b+(b-j)$ for some $j \in \{1,2\}$. Applying Lemma 2.1 to $(a-2)b+(b-j)$ and $(a-1)b+(b-j-1)$, we have $(s_b(a-1)+b-j-1) \mid (b-1)$, so $s_b(a-1) =$

$j \le 2$. Since a is a multiple of b, we have $s_b(a-1) \ge b-1 > 2$, a contradiction. Hence, $\mathscr{S}$ does not contain $(a-2)b+(b-j)$.

Therefore, letting $a = mb$ with $m \in \mathbb{Z}^+$, $\mathscr{S}$ consists of numbers strictly between mb^2-b-1 and $mb^2+b+\frac{b-3}{2}$, and the rest of the theorem follows. □

Theorem 4.4. *Let $b \ge 9$ and $n \ge 3$ both be odd such that $b \ge 2n$ and $n \nmid (b-1)$. Then a b-Niven 2-AP is of length at most $b+n-1$. Any b-Niven 2-AP of this length consists of numbers strictly between mb^2-b-1 and $mb^2+b+2n-2$ for some $m \in \mathbb{Z}^+$.*

Proof. Let $\mathscr{S}$ be a b-Niven 2-AP of length at least $b+n-1$. Since $b \ge 2n$ and b is odd, $\frac{b-1}{2} \ge n$, implying that $b+n-1 \ge \frac{b+1}{2}+2n-1$. By Theorem 3.2, $\mathscr{S}$ consists of numbers strictly between $mb^2-b-2n+1$ and $mb^2+b+2n-2$ for some $m \in \mathbb{Z}^+$.

Suppose that $\mathscr{S}$ contains a term less than mb^2-b. Then for some $j \in \{1,2\}$, $\mathscr{S}$ contains both $mb^2-b-j = (m-1)b^2+(b-2)b+(b-j)$ and $mb^2-j-1 = (m-1)b^2+(b-1)b+(b-j-1)$. By Lemma 2.1, $(s_b(m-1)+2b-2-j) \mid (b-1)$, a contradiction. Therefore, $\mathscr{S}$ consists of numbers strictly between mb^2-b-1 and $mb^2+b+2n-2$, and the rest of the theorem follows. □

The following corollary establishes that Theorem 4.4 provides better bounds than Theorem 4.2 for $b \ge 21$ and odd.

Corollary 4.5. *Let $b \ge 21$ be odd. Then a b-Niven 2-AP is of length at most $b+2\lfloor \frac{b+1}{8} \rfloor - 2$.*

Proof. We apply Theorem 4.4 with $n = 2\lfloor \frac{b+1}{8} \rfloor - 1$. Note that $n \ge 3$ and $b \ge 2n$. It is easy to verify that $n \nmid (b-1)$ when $21 \le b \le 25$ is odd. It remains to consider $b \ge 27$.

Note that $4n \le 8\left(\frac{b+1}{8}\right) - 4 < b-1$. On the other hand, since $b \ge 27$ is odd,

$$b-1 < 12\left(\frac{b+1-6}{8}\right) - 6 \le 12\left\lfloor \frac{b+1}{8} \right\rfloor - 6 = 6n.$$

Thus, $4n < b-1 < 6n$. Since n is odd and $b-1$ is even, $b-1 \ne 5n$. Hence $n \nmid (b-1)$, and Theorem 4.4 applies. □

We now consider even bases. The following theorem provides bounds for all even bases with at least one odd prime factor.

Theorem 4.6. *Let $b \ge 6$ be even and let p be an odd prime such that $p \mid b$. A b-Niven 2-AP is of length at most $b+p-2$. Any b-Niven 2-AP of length greater than b consists of terms strictly between $mb^2-b-p+2$ and $mb^2+b+p-2$ for some $m \in \mathbb{Z}^+$. In particular, any b-Niven 2-AP of length $b+p-2$ is of the form $mb^2-b-p+3, mb^2-b-p+5, \ldots, mb^2+b+p-3$ for some $m \in \mathbb{Z}^+$.*

Lemma 4.7. *Let $b \ge 6$ and let $p \ge 3$ be odd such that $b \ge 2p$ and $p \mid b$. If a b-Niven 2-AP has at least p terms strictly between $ab-1$ and $(a+1)b$ for some $a \in \mathbb{Z}^+$, then $p \mid s_b(a)$.*

Proof. Fix $i \ge 0$ such that $ab+i$ is the first of p terms between $ab-1$ and $(a+1)b$ of a b-Niven 2-AP. The digit sums of $ab+i$, $ab+i+2$, $\ldots$, $ab+i+2(p-1)$ are $s_b(a)+i$, $s_b(a)+i+2$, $\ldots$, $s_b(a)+i+2(p-1)$, respectively, which form a complete set of representatives of the congruence classes modulo p. Hence, there exists $0 \le j \le p-1$ such that $p \mid (s_b(a)+i+2j)$. Since $ab+i+2j$ is a b-Niven number, we have $p \mid (ab+i+2j)$, which further implies that $p \mid (i+2j)$. Thus $p \mid s_b(a)$, as desired. □

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Proof of Theorem 4.6. Let $\mathscr{S}$ be a b-Niven 2-AP of length greater than b. By Theorem 2.4, every term of $\mathscr{S}$ is even.

Suppose for a contradiction that no term of $\mathscr{S}$ is a multiple of b^2. Since the length of $\mathscr{S}$ is greater than b, there is some $a \in \mathbb{Z}^+$ such that both ab and $(a+1)b$ are in $\mathscr{S}$. Note that $b \geq 2p$ since b is an even multiple of the odd prime p. This implies that there are at least p terms of $\mathscr{S}$ strictly between $ab-1$ and $(a+1)b$. Hence, $p \mid s_b(a)$ by Lemma 4.7.

Since $\mathscr{S}$ contains no multiple of b^2, $s_b(a\pm 1) = s_b(a) \pm 1$. So, $s_b((a\pm 1)b + (p \mp 1)) = s_b(a) + p$, which is a multiple of p. Since the even numbers $(a\pm 1)b + (p\mp 1)$ are not multiples of p, they are not in $\mathscr{S}$. This implies that the length of $\mathscr{S}$ is less than $((a+1)b + (p-1) - (a-1)b + (p+1))/2 = b-1 < b$, a contradiction.

Thus, $\mathscr{S}$ contains mb^2 for some $m \in \mathbb{Z}^+$. If $mb^2 + b + p - 1$ is in $\mathscr{S}$, then by Lemma 4.7, $p \mid s_b(m)$. Thus $s_b(mb^2 + b + p - 1) = s_b(m) + p$ is a multiple of p. This is impossible since $p \nmid (mb^2 + b + p - 1)$. Similarly, if $mb^2 - b - p + 1 = (m-1)b^2 + (b-2)b + (b-p+1)$ is in $\mathscr{S}$, then by Lemma 4.7 again, $p \mid s_b((m-1)b + (b-1))$ and so $p \mid (s_b(m-1) + (b-1))$. It follows that $s_b((m-1)b^2 + (b-2)b + (b-p+1)) = s_b(m-1) + 2b - p - 1$ is a multiple of p. This is also impossible since $p \nmid ((m-1)b^2 + (b-2)b + (b-p+1))$. The theorem follows. □

The following corollary to Theorem 3.2 addresses the remaining case of b equal to a power of 2.

Corollary 4.8. *Let $b = 2^t$ for some integer $t \geq 3$. If $t = 4$, then a b-Niven 2-AP is of length at most 28; if t is odd, then a b-Niven 2-AP is of length at most $2^t + 4$; and if $t > 4$ and even, then a b-Niven 2-AP is of length at most $2^t + 2t(\log t + \log\log t) - 2$.*

Proof. The proofs for $t = 4$ and t odd follow from Theorem 3.2 with $n = 7$ and 3, respectively.

For $t \geq 6$, observe that $d(b-1) = 2(64 \cdot 2^{t-6} - 1) < 2 \cdot 81 \cdot 3^{t-6} = 2 \cdot 3^{t-2}$, and so $d(b-1)$ has fewer than $t-1$ distinct prime factors. Hence, there exists a prime p such that $p \nmid d(b-1)$ that is less than the t^{th} smallest prime. By Rosser and Schoenfeld [3], for $t \geq 6$, the t^{th} smallest prime is less than $t(\log t + \log\log t)$. Letting $n = p < t(\log t + \log\log t)$, the result follows from Theorem 3.2. □

We now prove Theorem 4.1. The examples provided in parts (*a*), (*b*), (*c*), and (*e*) are each the b-Niven 2-AP of maximal length with terms as small as possible. The numbers in parts (*c*) and (*e*) were found by a direct computer search. See Remark 4.9 for information on the example given in part (*f*).

Proof of Theorem 4.1. We consider each part of the theorem separately.

(*a*) Note that 2, 4, 6, 8, 10, 12 is a 2-Niven 2-AP of length 6. Suppose for a contradiction that $\mathscr{S}$ is a different 2-Niven 2-AP of length at least 6. By Theorem 2.4, every term of $\mathscr{S}$ is even. Since 14 is not a 2-Niven numbers, the first term of $\mathscr{S}$ is at least 16.

For any $a \geq 1$, if $16a + 10$ and $16a + 12$ are both in $\mathscr{S}$, then by Lemma 2.1, $(s_2(a) + 2) \mid 2$, which is impossible. Hence, since $\mathscr{S}$ is of length at least 6, for some $a \geq 2$, $\mathscr{S}$ contains the terms $16a + 2$, $16a + 4$, and $16a + 6$.

Applying Lemma 2.1 to $16a + 2$ and $16a + 4$ yields that $(s_2(a) + 1) \mid 2$. Thus, $s_2(a) = 1$, implying that $a = 2^t$ for some $t \in \mathbb{Z}^+$. It follows that $s_2(16a + 6) = s_2(16 \cdot 2^t + 6) = 3$, which does not divide $16 \cdot 2^t + 6$. Hence, $16a + 6$ is not in $\mathscr{S}$, a contradiction.

(b) Note that 2, 4, 6, 8, 10, 12 is a 3-Niven 2-AP of length 6. Suppose for a contradiction that $\mathscr{S}$ is a different 3-Niven 2-AP of length at least 6. Since 7 and 14 are not 3-Niven numbers, the first term of $\mathscr{S}$ is greater than 9.

Since $\mathscr{S}$ is a 3-Niven 2-AP of length at least 6, $\mathscr{S}$ must contains both $9a+4$ and $9a+6$ or both $9a+5$ and $9a+7$ for some $a \in \mathbb{Z}^+$. By Lemma 2.1, we have $(s_3(a)+2) \mid 2$ or $(s_3(a)+3) \mid 2$, contradictions. Hence, no such $\mathscr{S}$ exists.

(c) The number 17528 is the first term of a 4-Niven 2-AP of length 9. Let $\mathscr{S}$ be any 4-Niven 2-AP of length at least 9. By Theorem 2.4, every term of $\mathscr{S}$ is even. Since $\mathscr{S}$ is of length at least 9, there exists $m \in \mathbb{Z}^+$ such that $16m$ is a term in $\mathscr{S}$.

Suppose that $16(m-1)+6$ is in $\mathscr{S}$. Applying Lemma 2.1 to $16(m-1)+6$ and $16(m-1)+12$ yields that $(s_4(m-1)+3) \mid 6$, and so $s_4(m-1) = 0$ or 3. If $s_4(m-1) = 0$, then $s_4(16(m-1)+10) = 4$, which does not divide $16(m-1)+10$, a contradiction. If $s_4(m-1) = 3$, then $s_4(16(m-1)+14) = 8$, which does not divide $16(m-1)+14$, again a contradiction. Hence, $16(m-1)+6$ is not in $\mathscr{S}$.

Next, suppose that $16m+10$ is in $\mathscr{S}$. Applying Lemma 2.1 to $16m+2$ and $16m+8$ yields that $(s_4(m)+2) \mid 6$, so $s_4(m) = 1$ or 4. If $s_4(m) = 1$, then $s_4(16m+6) = 4$, which does not divide $16m+6$, a contradiction. If $s_4(m) = 4$, then $s_4(16m+10) = 8$, which does not divide $16m+10$, again a contradiction. Hence, $16m+10$ is not in $\mathscr{S}$, and the result follows.

(d) Since 2, 4, 6, 8, 10, 12 is a 5-Niven 2-AP of length 6, the maximum length is at least 6. Suppose that $\mathscr{S}$ is a 5-Niven 2-AP of length at least 7. Since 11, 14, 22, 23, and 34 are not 5-Niven numbers, the first term of $\mathscr{S}$ is at least 25.

Suppose that for some $a \geq 1$ and $1 \leq j \leq 4$, $25a+4j$ and $25a+4j+4$ are both in $\mathscr{S}$. By Lemma 2.1, $(s_5(a)+4) \mid 4$, a contradiction. Similarly, suppose that for some $a \geq 1$ and $2 \leq j \leq 4$, $25a+4j+1$ and $25a+4j+5$ are both in $\mathscr{S}$. Again by Lemma 2.1, $(s_5(a)+5) \mid 4$, also a contradiction. Finally, suppose that for some $a \geq 1$, $\mathscr{S}$ contains both $25a+18$ and $25a+22$ or both $25a+19$ and $25a+23$. Then $(s_5(a)+6) \mid 4$ or $(s_5(a)+7) \mid 4$, again contradictions.

Therefore, for some $a \in \mathbb{Z}^+$, $\mathscr{S}$ is a subsequence of $25a+20, 25a+22, \ldots, 25(a+1)+11$. Since $\mathscr{S}$ is of length at least 7, $25(a+1)+3$ and $25(a+1)+7$ are both in $\mathscr{S}$. By Lemma 2.1, $(s_5(a+1)+3) \mid 4$, which implies that $s_5(a+1) = 1$, and so $a+1 = 5^t$ for some $t \in \mathbb{Z}^+$. It follows that $s_b(25(a+1)+9) = 6$, which does not divide $25(a+1)+9 = 5^{t+2}+9$. Hence, $25(a+1)+9$ is not in $\mathscr{S}$. The result follows.

(e) By Theorem 4.6, the length of a 6-Niven 2-AP is at most 7. A computer check verifies that 10285274079642 is the first term of a 6-Niven 2-AP of length 7.

(f) By Theorem 4.2, the length of a 7-Niven 2-AP is at most 8. A computer check verifies that $7^{40}+7^{36}+7^{28}+7^4-6$ is the first term of a 7-Niven 2-AP of length 8. □

Remark 4.9. Finally, we provide some notes on finding the example given in the proof of Theorem 4.1(f). By Theorem 4.2, a 7-Niven 2-AP of length 8 consists of numbers strictly between $49m-8$ and $49m+9$ for some $m \in \mathbb{Z}^+$. We restrict our search to 2-APs of the form $49m-6, 49m-4, \ldots, 49m+8$. Applying Lemma 2.1 to $49m+2$ and $49m+8$ yields $(s_7(m)+2) \mid 6$, which implies that $s_7(m) = 1$ or 4. We further restrict our search to the case in which $s_7(m) = 4$.

Since $s_7(m) = 4$, we have $49m = 7^{k_4} + 7^{k_3} + 7^{k_2} + 7^{k_1}$ for some $2 \le k_1 \le k_2 \le k_3 \le k_4$. Note that $s_7(49m+4) = 8$, so $8 \mid (49m+4)$, implying that $(-1)^{k_4} + (-1)^{k_3} + (-1)^{k_2} + (-1)^{k_1} \equiv 49m \equiv 4 \pmod 8$. Hence, k_1, k_2, k_3, and k_4 are either all odd or all even. Suppose for a contradiction that they are all odd. Then, reducing modulo 4, $s_7(49m-2) = s_7(7^{k_4} + 7^{k_3} + 7^{k_2} + (7^{k_1} - 2)) = 3 + s_7(7^{k_1} - 2) = 3 + 6k_1 - 1 \equiv 0 \pmod 4$. Since $8 \mid (49m+4)$, we have that $49m - 2 \equiv 2 \pmod 4$, a contradiction. Therefore, k_1, k_2, k_3, and k_4 are all even.

Finally, note that $s_7(49m+6) = 10$. By reducing modulo 5, we have $0 \equiv 49m + 6 \equiv 2^{k_4} + 2^{k_3} + 2^{k_2} + 2^{k_1} + 1 \equiv (-1)^{k_4/2} + (-1)^{k_3/2} + (-1)^{k_2/2} + (-1)^{k_1/2} + 1 \pmod 5$, which implies that each of k_1, k_2, k_3, and k_4 is divisible by 4.

This characterization of $49m$ leads to an efficient computer search for a 7-Niven 2-AP of length 8, yielding the example given above.

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