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UNIFORM CONVERGENCE OF ITERATES ON THE UNIT BALL OF FUNCTIONS OF BICOMPLEX AND QUATERNIONIC VARIABLES

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ABSTRACT. For nontrivial self-maps of the complex unit disk, the Denjoy-Wolff theorem guarantees that sequences of iterates of such a map converge uniformly on compact subsets of the disk to a unique fixed point. Recently, Cowen, Ko, Thompson, and Tian developed conditions on these maps for when this convergence is uniform on the whole unit disk. We explore the analogous problem in two new contexts: the bicomplex numbers and the quaternions.

1. Introduction

In recent years, discrete dynamical systems have been intensely studied. The primary objects of interest are pairs (f,X), where f is a self-map of a set X. The general objective is the classification of the behavior of points in X under iteration by the function f. The situation when f is analytic and X is a subset of $\mathbb{C}$ is particularly well-studied [1, 2, 3] and has been popularized by the rich dynamics exhibited by the complex quadratic family of polynomials.

For (f,X) with $X \subset \mathbb{C}$ and f analytic, any iterate (repeated composition) of f, denoted

$$f^n = f \circ \cdots \circ f,$$

is also an analytic self-map of X. If we are concerned with the convergence behavior of sequences of iterates of f, then we need a topology on the set of analytic self-maps of X. The standard topology in this setting is the topology of uniform convergence on compact sets [3]. Indeed, a great deal of dynamical behavior for a map f is understood through the splitting of X into the two totally invariant sets, the Fatou set and the Julia set. The Fatou set is the set of points $x_0 \in X$ on which every infinite subsequence of f^n restricted to some open set U containing x_0 converges uniformly to some analytic function f on U; the Julia set is the complement of the Fatou set in X. Heuristically, the Fatou set is the set of points whose dynamics are locally stable. This set is always an open set, so the Julia set is always closed. For the purposes of this classification, the question of whether f^n converges (assuming, of course, that there is convergence) *uniformly* on the whole Fatou set as some open subset isn't relevant. Uniform convergence on compact subsets is sufficient to define the dynamical distinction between the Fatou and Julia sets.

The open complex unit disk, the set of all complex numbers with modulus less than 1, is denoted

$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}.$$

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Analytic self-maps of $\mathbb{D}$ play a central role in operator theory, the study of linear maps between function spaces. In a recent paper [4], it was shown that the image of the unit disk $\mathbb{D}$ can determine whether f^n converges uniformly on the entire *open* disk (not just compact subsets of the disk). To formalize the distinction between uniform convergence of iterates on compact sets and uniform convergence of iterates on the whole domain, we will use *UCI* (uniformly convergent iterates) to specifically indicate the latter case. That is,

Definition (Uniform Convergence of Iterates)**.** *Let X be a metric space and $D \subset X$ a domain. We say $f\colon D \to D$ has UCI if f^n converges uniformly on all of D to some function $g\colon D \to D$.*

A great deal of dynamical behavior is described by the Denjoy-Wolff Theorem for dynamical systems $(f, \mathbb{D})$.

Theorem (Denjoy-Wolff Theorem)**.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic and not an automorphism. Then there is a point $a \in \overline{\mathbb{D}}$ such that the iterates of φ converge to a uniformly on compact subsets of $\mathbb{D}$, and a is called the Denjoy-Wolff point.*

Two main results from [4] make use of the Denjoy-Wolff Theorem. The first deals with the case in which the Denjoy-Wolff point is in the interior of the disk:

Theorem 1 (UCI with Interior Fixed Point [4])**.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic and continuous on $\partial\mathbb{D}$. If the Denjoy-Wolff point a is in $\mathbb{D}$, then φ has UCI with φ^n converging uniformly to a, if and only if there is $N > 0$ such that $\varphi^N\left(\overline{\mathbb{D}}\right) \subseteq \mathbb{D}$.*

The second is for the case in which the Denjoy-Wolff point is on the boundary:

Theorem 2 (UCI with Boundary Fixed Point [4])**.** *Suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$ and continuous on $\partial\mathbb{D}$ and has Denjoy-Wolff point a with $|a| = 1$ and $\varphi'(a) < 1$. If $\varphi^N\left(\overline{\mathbb{D}}\right) \subseteq \mathbb{D} \cup \{a\}$ for some $N > 0$, then φ has UCI with φ^n converging uniformly to a.*

The criterion for uniform convergence of iterates to hold for φ when $|a| = 1$ and $\varphi'(a) < 1$ remains to be identified.

In this note, we wish to extend Theorems 1 and 2 into $\mathbb{BC}$, the set of bicomplex numbers, and Theorem 1 to $\mathbb{H}$, the quaternions. Along the way, we also prove bicomples versions of Schwarz Lemma, the Scwarz-Pick Theorem, and Julia's Lemma. The next two sections are devoted to these two settings, respectively. Both domains present distinct challenges. The quaternions are famously noncommutative, while the bicomplex numbers gain commutativity at the expense of some elements being non-invertible.

The Bicomplex Numbers

In this section, we focus on the bicomplex numbers and functions of a single bicomplex variable. First introduced in 1892 [5], the bicomplex numbers have seen a recent increase in popularity due to their applications in areas like quantum mechanics (see [6]) and the extension of standard results from $\mathbb{C}$ to $\mathbb{BC}$ (see, for example, [7],[8], and [9]), which we hope to further contribute to here.

Preliminaries. We define the four real dimensional set of bicomplex numbers

$$\mathbb{BC} = \{\zeta = z_1 + jz_2 \colon z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}\}$$

where $i^2 = j^2 = (ij)^2 = -1$ but i and j are independent. Bicomplex addition and multiplication are defined intuitively, with

$$\zeta + \omega = (z_1 + w_1) + j(z_2 + w_2)$$

and

$$\zeta \cdot \omega = (z_1 w_1 - z_2 w_2) + j(z_1 w_2 + w_1 z_2)$$

for bicomplex numbers $\zeta = z_1 + jz_2$ and $\omega = w_1 + jw_2$. Naturally, both of these operations are commutative. For any $\zeta \in \mathbb{BC}$ such that $z_1 = \pm iz_2$, we have that $1/\zeta$ is not defined, so ζ is non-invertible, also known as a zero-divisor [8]. Because of these non-invertible elements, $\mathbb{BC}$ is a commutative ring rather than a field.

The Euclidean norm of a bicomplex number ζ is $|\zeta| := z_1\overline{z_1} + z_2\overline{z_2}$. Properties like the Triangle Inequality still hold in $\mathbb{BC}$ [10]. However, there is an interesting inequality when considering the norm of a product:

$$|\zeta \cdot \omega| \leq \sqrt{2}|\zeta||\omega|.$$

This inequality can be verified easily through algebraic manipulations. Note that the familiar equality $|\zeta \cdot \omega| = |\zeta||\omega|$ holds if either ζ or ω are complex.

Alternatively, the set $\mathbb{BC}$ can written in terms of an idempotent representation

$$\zeta = \zeta_1 \boldsymbol{e}_1 + \zeta_2 \boldsymbol{e}_2,$$

where

$$\zeta_1 = z_1 - iz_2, \zeta_2 = z_1 + iz_2, \boldsymbol{e}_1 = \frac{1+ij}{2}, \text{ and } \boldsymbol{e}_2 = \frac{1-ij}{2}.$$

It is easy to check that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ form an orthogonal basis for $\mathbb{BC}$ as a vector space over $\mathbb{C}$. We define the set of hyperbolic numbers, $\mathbb{S} = \{a\boldsymbol{e}_1 + b\boldsymbol{e}_2 : a, b \in \mathbb{R}\}$, which is a strict subset of $\mathbb{BC}$. The subset of nonnegative hyperbolic numbers is $\mathbb{S}^+ = \{a\boldsymbol{e}_1 + b\boldsymbol{e}_2 : a, b \geq 0\}$. For all $\zeta \in \mathbb{BC}$ we can write $\zeta_1\boldsymbol{e}_1 + \zeta_2\boldsymbol{e}_2$ for some $\zeta_i \in \mathbb{C}$, and we define the hyperbolic norm as

$$|\cdot|_{\mathbb{S}} = |\zeta_1|\boldsymbol{e}_1 + |\zeta_2|\boldsymbol{e}_2$$

where $|\cdot|$ is the Euclidean norm defined above. Note that $|\zeta|_{\mathbb{S}} \in \mathbb{S}^+$ and that for all bicomplex ζ and ω, $|\zeta \cdot \omega|_{\mathbb{S}} = |\zeta|_{\mathbb{S}} \cdot |\omega|_{\mathbb{S}}$.

Though $|\cdot|_{\mathbb{S}}$ is not real valued, we use it to define a partial ordering on $\mathbb{S}^+$. For some $\zeta = \zeta_1\boldsymbol{e}_1 + \zeta_2\boldsymbol{e}_2$ and $\omega = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$, we say $\zeta \preceq \omega$ if and only if $|\zeta_1| \leq |\omega_1|$ and $|\zeta_2| \leq |\omega_2|$. This allows us to define a bicomplex open unit ball:

$$\mathscr{B} = \{\zeta \colon |\zeta|_{\mathbb{S}}^2 \prec 1\}.$$

One can algebraically verify that the above definition implies

$$\mathscr{B} = \{\zeta \colon |\zeta|^2 < 1\}.$$

Like complex numbers, each bicomplex number has a conjugate. Although there are numerous ways to define the conjugate of a bicomplex number (see source [11] for a list), we will use the following: For some

$$\zeta = \zeta_1 \boldsymbol{e}_1 + \zeta_2 \boldsymbol{e}_2,$$

we define

$$\zeta^* := \overline{\zeta_1} \boldsymbol{e}_1 + \overline{\zeta_2} \boldsymbol{e}_2,$$

where $\overline{\zeta_i}$ is the standard complex conjugate [12]. Using this particular conjugation, it follows that $\zeta\zeta^* = |\zeta|_{\mathbb{S}}$.

Finally, we will briefly discuss bicomplex functions (for an in-depth study of bicomplex functions, see [10]). Continuity is defined as one may expect [13]: For some $X \subseteq \mathbb{BC}$, a bicomplex function $\varphi\colon X \to \mathbb{BC}$ is continuous on X if and only if for every $\zeta_0 \in X$, we have that $\lim_{\zeta\to\zeta_0} \varphi(\zeta)$ exists and

$$\lim_{\zeta\to\zeta_0} \varphi(\zeta) = \varphi(\zeta_0).$$

Derivatives are defined with the familiar difference quotient. For a function $\varphi : \mathbb{BC} \to \mathbb{BC}$ at a point $\zeta_0 \in \mathbb{BC}$, the derivative of φ at the point ζ_0 is the limit

$$\varphi'(\zeta_0) := \lim_{h\to 0} \frac{\varphi(\zeta_0 + h) - \varphi(\zeta_0)}{h}$$

provided that the limit exists and that h is invertible [11]. Analyticity is also defined in the familiar way: a bicomplex function is analytic at a point if it can be written as a convergent power series.

The following two lemmas illustrate how we can appeal to the idempotent form when working with functions. These lemmas will prove useful in our results.

Lemma 3 (Bicomplex Holomorphic Functions [12])**.** *Define $\varphi\colon \mathbb{BC} \to \mathbb{BC}$ by*

$$\varphi(\zeta) = \varphi_1(\zeta_1)\boldsymbol{e}_1 + \varphi_2(\zeta_2)\boldsymbol{e}_2$$

where $\zeta = \zeta_1 \boldsymbol{e}_1 + \zeta_2 \boldsymbol{e}_2$ and $\varphi_i\colon \mathbb{C} \to \mathbb{C}$. We say that φ is holomorphic if and only if φ_1 and φ_2 are holormorphic.

The next lemma can be verified by appealing to the idempotent form and the fact that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal:

Lemma 4 (Iterates of Bicomplex Functions)**.** *Let $\varphi^n(\zeta)$ be the n^{th} iterate of φ. Then,*

$$\varphi^n(\zeta) = \varphi_1^n(\zeta_1)\boldsymbol{e}_1 + \varphi_2^n(\zeta_2)\boldsymbol{e}_2.$$

Note also that for some $\zeta = \zeta_1 \boldsymbol{e}_1 + \zeta_2 \boldsymbol{e}_2$ and some differentiable bicomplex function φ, we have from the definition of the derivative and the lemmas that $\varphi'(\zeta) = \varphi_1'(\zeta_1)\boldsymbol{e}_1 + \varphi_2'(\zeta_2)\boldsymbol{e}_2$.

Before we can prove results about uniform convergence of iterates of bicomplex-valued functions, it would be prudent to define uniform convergence with respect to the hyperbolic norm and UCI in this setting. For more on convergence of bicomplex sequences, see [11].

Definition (Uniform Convergence of Bicomplex-valued Functions)**.** *Let $X, Y \subseteq \mathbb{BC}$. We say that a sequence of functions $f^n\colon X \to Y$ converges uniformly if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $\zeta \in X$, $|f^n(\zeta) - f(\zeta)|_{\mathbb{S}} \prec \varepsilon$.*

Definition (Uniform Convergence of Iterates of Bicomplex-valued Functions)**.** *Let $D \subset \mathbb{BC}$ be a domain. We say $f\colon D \to D$ has UCI if f^n converges uniformly on all of D to some function $g\colon D \to D$.*

Now, we are ready to present our results.

Results. The following is the first of two main results for this section, and is analogous to Theorem 1:

Theorem 5 (Theorem 1 in $\mathbb{BC}$)**.** *Let $\mathscr{B} = \{\zeta : |\zeta|^2 < 1\}$. Suppose $\varphi : \mathscr{B} \to \mathscr{B}$ is analytic and continuous on $\partial\mathscr{B}$. If $a \in \mathscr{B}$ is the unique interior fixed point of φ, then $\varphi^n \to a$ uniformly on $\mathscr{B}$ if and only if there exists an $N > 0$ such that $\varphi^N(\overline{\mathscr{B}}) \subseteq \mathscr{B}$.*

Proof. First, suppose that there is an $N > 0$ such that $\varphi^N(\overline{\mathscr{B}}) \subseteq \mathscr{B}$. Note that since $\mathscr{B} = \{|\zeta|^2_{\mathbb{S}} \prec 1\} = \{\zeta = \zeta_1\boldsymbol{e}_1 + \zeta_2\boldsymbol{e}_2\colon\ |\zeta_1|, |\zeta_2| < 1\}$, we get $\overline{\mathscr{B}} = \boldsymbol{e}_1\overline{\mathbb{D}} + \boldsymbol{e}_2\overline{\mathbb{D}}$ where $\mathbb{D}$ is the standard complex open unit disk. Also note that since $a \in \mathscr{B}$, we have $a = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2$.

By Lemma 4, we have $\varphi^N(\overline{\mathscr{B}}) = \varphi_1^N(\overline{\mathbb{D}})\boldsymbol{e}_1 + \varphi_2^N(\overline{\mathbb{D}})\boldsymbol{e}_2 \subseteq \mathscr{B}$ for some $\varphi_i\colon \overline{\mathbb{D}} \to \mathbb{D}$. Since $\varphi_1^N(\overline{\mathbb{D}})\boldsymbol{e}_1 + \varphi_2^N(\overline{\mathbb{D}})\boldsymbol{e}_2 \subseteq \mathscr{B}$, we get

$$\begin{aligned}
\left|\varphi_1^N(\overline{\mathbb{D}})\boldsymbol{e}_1 + \varphi_2^N(\overline{\mathbb{D}})\boldsymbol{e}_2\right|^2_{\mathbb{S}} &\prec 1 \\
\left(\left|\varphi_1^N(\overline{\mathbb{D}})\right|\boldsymbol{e}_1 + \left|\varphi_2^N(\overline{\mathbb{D}})\right|\boldsymbol{e}_2\right)\left(\left|\varphi_1^N(\overline{\mathbb{D}})\right|\boldsymbol{e}_1 + \left|\varphi_2^N(\overline{\mathbb{D}})\right|\boldsymbol{e}_2\right) &\prec 1.
\end{aligned}$$

By the orthogonality of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$,

$$\left|\varphi_1^N(\overline{\mathbb{D}})\right|^2\boldsymbol{e}_1 + \left|\varphi_2^N(\overline{\mathbb{D}})\right|^2\boldsymbol{e}_2 \prec 1,$$

which implies that for each i, $\left|\varphi_i^N(\overline{\mathbb{D}})\right| < 1$. Thus, for all $N \geq 1$, $\varphi_i^N(\overline{\mathbb{D}})$ is a compact subset of $\mathbb{D}$. Because $\mathbb{D}$ is a bounded domain in a complex Banach space and all φ_i are holomorphic, the Earle-Hamilton Theorem [14] tells us that all φ_i^n converge uniformly on compact subsets of $\mathbb{D}$ to a unique point a. Since both $\varphi_1^n \to a_1$ and $\varphi_2^n \to a_2$ uniformly, we have $\varphi^n = \varphi_1^n\boldsymbol{e}_1 + \varphi^n\boldsymbol{e}_2 \to a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 = a$ uniformly. Thus, $\varphi^n \to a$ uniformly on $\varphi^N(\overline{\mathscr{B}})$

The rest of this proof is similar to the proof presented in [4]. Let M be the minimum distance between a and $\mathscr{B}$ (Note we are now using the real-valued Euclidean norm). Since $\varphi^n \to a$ uniformly on $\mathscr{B}$, for $\varepsilon = M/2$, there exists some $N > 0$ such that $\left|\varphi^N(\zeta) - a\right| < \varepsilon\ \forall\zeta \in \mathscr{B}$. Suppose that $\varphi^N(b_1) = b_2$, and $|b_1| = |b_2| = 1$. Then, for our given ε, since we have φ^N is continuous on $\partial\mathscr{B}$, for our ε there exists some $\delta > 0$ such that $|b_1 - \zeta| < \delta \implies \left|b_2 - \varphi^n(\zeta) - \varphi^N(\zeta)\right| < \varepsilon$. However, for each ζ such that $|b_1 - \zeta| < \delta$,

$$\begin{aligned}
M \leq |b_2 - a| &= \left|b_2 - \varphi^N(\zeta) + \varphi^N(\zeta) - a\right| \\
&\leq \left|b_2 - \varphi^N(\zeta)\right| + \left|\varphi^N(\zeta) - a\right| \\
&< 2\varepsilon = M
\end{aligned}$$

And so, $\varphi_N(\overline{\mathscr{B}}) \subseteq \mathscr{B}$. ∎

Before we can present the second of our two main results, we must first develop a version of Julia's Lemma in $\mathbb{BC}$. To do this, we need a Schwarz-Pick Theorem in $\mathbb{BC}$, which requires the following analogue of the classical Schwarz Lemma:

Theorem 6 (Schwarz Lemma in $\mathbb{BC}$)**.** *Let $\varphi\colon \mathscr{B} \to \mathscr{B}$ be holomorphic with $\varphi(0) = 0$ and $|\varphi|_{\mathbb{S}} \preceq 1$ for all $\zeta \in \mathscr{B}$. Then, $|\varphi(\zeta)|_{\mathbb{S}} \preceq |\zeta|_{\mathbb{S}}$ and $|\varphi'(0)|_{\mathbb{S}} \preceq 1$.*

Proof. Note that $|\varphi(\zeta)|_{\mathbb{S}} = |\varphi_1(\zeta_1)| + |\varphi_2(\zeta_2)|$ and that $\varphi(0) = \varphi_1(0)\boldsymbol{e}_1 + \varphi_2(0)\boldsymbol{e}_2$.

By Lemma 3, we have that φ_1 and φ_2 are holomorphic on $\mathbb{D}$. Applying the classical Schwarz Lemma, we have $|\varphi_1(\zeta_1)| \leq |\zeta_1|$ and $|\varphi_2(\zeta_2)| \leq |\zeta_2|$. Thus, $|\varphi(\zeta)|_{\mathbb{S}} \preceq |\zeta|_{\mathbb{S}}$.

We have that $|\varphi'(0)|_{\mathbb{S}} = |\varphi_1'(0)|\boldsymbol{e}_1 + |\varphi_2'(0)|\boldsymbol{e}_2$. Once again applying the classical Schwarz lemma, we have $|\varphi_i'(0)| \leq 1$. Thus, $|\varphi'(0)|_{\mathbb{S}} \preceq 1$. ■

Using our new Schwarz Lemma, we can now prove our Schwarz-Pick Theorem for $\mathbb{BC}$:

Lemma 7 (Schwarz-Pick Theorem in $\mathbb{B}i$)**.** *Let $\varphi\colon \mathscr{B} \to \mathscr{B}$ be analytic and $\omega, \zeta \in \mathscr{B}$. Then,*

$$\frac{|\varphi(\omega) - \varphi(\zeta)|_{\mathbb{S}}}{|1 - \varphi(\omega)^*\varphi(\zeta)|_{\mathbb{S}}} \preceq \frac{|\omega - \zeta|_{\mathbb{S}}}{|1 - \omega^*\zeta|_{\mathbb{S}}}.$$

Proof. In this proof, we appeal to the idempotent form and the classical Schwarz-Pick Theorem. Recall that the elements $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal:

$$\begin{aligned}
\frac{|\varphi(\omega) - \varphi(\zeta)|_{\mathbb{S}}}{|1 - \varphi(\omega)^*\varphi(\zeta)|_{\mathbb{S}}} &= \frac{|\varphi_1(\omega_1)\boldsymbol{e}_1 + \varphi_2(\omega_2)\boldsymbol{e}_2 - (\varphi_1(\zeta_1)\boldsymbol{e}_1 + \varphi_2(\zeta_2)\boldsymbol{e}_2|_{\mathbb{S}}}{|1 - (\varphi_1(\omega_1)\boldsymbol{e}_1 + \varphi_2(\omega_2)\boldsymbol{e}_2)^*(\varphi_1(\zeta_1) + \varphi_2(\zeta_2)\boldsymbol{e}_2)|_{\mathbb{S}}} \\
&= \frac{|(\varphi_1(\omega_1) + \varphi_1(\zeta_1))\boldsymbol{e}_1 + (\varphi_2(\omega_2) + \varphi_2(\zeta_2))\boldsymbol{e}_2|_{\mathbb{S}}}{\left|1 - \overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1)\boldsymbol{e}_1 + \overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2)\boldsymbol{e}_2)\right|_{\mathbb{S}}}
\end{aligned}$$

Note here that $|\zeta_1\boldsymbol{e}_1 + \zeta_2\boldsymbol{e}_2|_{\mathbb{S}} = |\zeta_1|\boldsymbol{e}_1 + |\zeta_2|\boldsymbol{e}_2$ [11]. Thus,

$$\begin{aligned}
\frac{|\varphi(\omega) - \varphi(\zeta)|_{\mathbb{S}}}{|1 - \varphi(\omega)^*\varphi(\zeta)|_{\mathbb{S}}} &= \frac{|\varphi_1(\omega_1) + \varphi_1(\zeta_1)|\boldsymbol{e}_1 + |\varphi_2(\omega_2) + \varphi_2(\zeta_2)|\boldsymbol{e}_2}{\left|\boldsymbol{e}_1 + \boldsymbol{e}_2 - \overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1)\boldsymbol{e}_1 + \overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2)\boldsymbol{e}_2)\right|_{\mathbb{S}}} \\
&= \frac{|\varphi_1(\omega_1) + \varphi_1(\zeta_1)|\boldsymbol{e}_1 + |\varphi_2(\omega_2) + \varphi_2(\zeta_2)|\boldsymbol{e}_2}{\left|(1 - \overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1))\boldsymbol{e}_1 + (1 - \overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2))\boldsymbol{e}_2)\right|_{\mathbb{S}}} \\
&= \frac{|\varphi_1(\omega_1) + \varphi_1(\zeta_1)|\boldsymbol{e}_1 + |\varphi_2(\omega_2) + \varphi_2(\zeta_2)|\boldsymbol{e}_2}{\left|1 - \overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1)\right|\boldsymbol{e}_1 + \left|1 - \overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2)\right|\boldsymbol{e}_2}
\end{aligned}$$

Now, we use the fact that for $a \in \mathbb{R}$, $(\zeta_1 \boldsymbol{e}_1 + \zeta_2 \boldsymbol{e}_2)^a = \zeta_1^a \boldsymbol{e}_1 + \zeta_2^a \boldsymbol{e}_2$ [12]:

$$
\begin{aligned}
\frac{|\varphi(\omega)-\varphi(\zeta)|_{\mathbb{S}}}{|1-\varphi(\omega)^*\varphi(\zeta)|_{\mathbb{S}}} &= [|\varphi_1(\omega_1)+\varphi_1(\zeta_1)|\boldsymbol{e}_1 + |\varphi_2(\omega_2)+\varphi_2(\zeta_2)|\boldsymbol{e}_2] \\
&\qquad \times \left[\left|1-\overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1)\right|\boldsymbol{e}_1 + \left|1-\overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2)\right|\boldsymbol{e}_2\right]^{-1} \\
&= [|\varphi_1(\omega_1)+\varphi_1(\zeta_1)|\boldsymbol{e}_1 + |\varphi_2(\omega_2)+\varphi_2(\zeta_2)|\boldsymbol{e}_2] \\
&\qquad \times \left[\left|1-\overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1)\right|^{-1}\boldsymbol{e}_1 + \left|1-\overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2)\right|^{-1}\boldsymbol{e}_2\right] \\
&= |\varphi_1(\omega_1)+\varphi_1(\zeta_1)|\left|1-\overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1)\right|^{-1}\boldsymbol{e}_1 \\
&\qquad + |\varphi_2(\omega_2)+\varphi_2(\zeta_2)|\left|1-\overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2)\right|^{-1}\boldsymbol{e}_2 \\
&= \frac{|\varphi_1(\omega_1)+\varphi_1(\zeta_1)|}{\left|1-\overline{\varphi_1(\omega_1)}\varphi_1(\zeta_1)\right|}\boldsymbol{e}_1 + \frac{|\varphi_2(\omega_2)+\varphi_2(\zeta_2)|}{\left|1-\overline{\varphi_2(\omega_2)}\varphi_2(\zeta_2)\right|^{-1}}\boldsymbol{e}_2
\end{aligned}
$$

We have now split the original expression into its idempotent components. Similar work gives us

$$
\frac{|\omega-\zeta|_{\mathbb{S}}}{|1-\omega^*\zeta|_{\mathbb{S}}} = \frac{|\omega_1+\zeta_1|}{|1-\overline{\omega_1}\zeta_1|}\boldsymbol{e}_1 + \frac{|\omega_2+\zeta_2|}{|1-\overline{\omega_2}\zeta_2|}\boldsymbol{e}_2
$$

Now, simply apply the classical Schwarz-Pick Theorem to each idempotent component to get the desired result. ■

We are almost ready to extend Julia's Lemma to $\mathbb{BC}$. But first, let us present a lemma that will be useful in our proof:

Lemma 8 (Bicomplex Mobius Transformations [12]). *Let $a \in \mathscr{B}$ and define the bicomplex Mobius transformation $T : \mathscr{B} \to \mathscr{B}$ as*

$$
T(\zeta) = \lambda \frac{a-\zeta}{1-a^*\zeta}
$$

where λ is a bicomplex scalar with $|\lambda|_{\mathbb{S}} = 1$ and the denominator is not a zero-divisor. Then,

$$
(1-|\zeta|_{\mathbb{S}}^2)\left|T'(\zeta)\right|_{\mathbb{S}} = 1-|T(\zeta)|_{\mathbb{S}}^2
$$

which is equivalent to

$$
(1-|\zeta|_{\mathbb{S}}^2)\frac{\left||a|_{\mathbb{S}}^2-1\right|_{\mathbb{S}}}{|(1-a^*\zeta)^2|_{\mathbb{S}}} = 1-\frac{|a-\zeta|_{\mathbb{S}}^2}{|1-a^*\zeta|_{\mathbb{S}}^2}
$$

For a proof of the above lemma, see [12].

Theorem 9 (Julia's Lemma in $\mathbb{BC}$). *Suppose $\varphi : \mathscr{B} \to \mathscr{B}$ is analytic. Let $\alpha \in \partial\mathscr{B}$ where $d(\alpha)$ is finite. Suppose the sequence $a_n \to \alpha$ satisfies*

$$
d(\alpha) = \lim_{n\to\infty} \frac{1-|\varphi(a_n)|_{\mathbb{S}}}{1-|a_n|_{\mathbb{S}}}
$$

and for $\omega \in \partial\mathscr{B}$, $\lim_{n\to\infty} \varphi(a_n) = \omega$. *Then for* $\zeta \in \mathscr{B}$

$$\frac{|\omega - \varphi(\zeta)|^2_{\mathbb{S}}}{1-|\varphi(\zeta)|^2_{\mathbb{S}}} \preceq d(\alpha)\frac{|\alpha-\zeta|^2_{\mathbb{S}}}{1-|\zeta|^2_{\mathbb{S}}}.$$

Proof. Note that because $\alpha, \omega \in \partial\mathscr{B}$, we have $|\alpha|_{\mathbb{S}} = |\omega|_{\mathbb{S}} = 1$. Then by Schwarz-Pick,

$$1 - \frac{|a_n - \zeta|^2_{\mathbb{S}}}{|1-a_n^*\zeta|^2_{\mathbb{S}}} \preceq 1 - \frac{|\varphi(a_n)-\varphi(\zeta)|^2_{\mathbb{S}}}{|1-\varphi(\zeta)\varphi(a_n)^*|^2_{\mathbb{S}}}.$$

Invoking the previous lemma gives us

$$\begin{aligned}
&\frac{(1-|\zeta|^2_{\mathbb{S}})\left||a_n|^2_{\mathbb{S}}-1\right|_{\mathbb{S}}}{|1-a_n^*\zeta|^2_{\mathbb{S}}} \preceq \frac{(1-|\varphi(\zeta)|^2_{\mathbb{S}})\left||\varphi(a_n)|^2_{\mathbb{S}}-1\right|_{\mathbb{S}}}{|1-\varphi(a_n)^*\varphi(\zeta)|^2_{\mathbb{S}}}\\
\implies &\frac{(1-|\zeta|^2_{\mathbb{S}})\left|(-1)(1-|a_n|^2_{\mathbb{S}})\right|_{\mathbb{S}}}{|1-a_n^*\zeta|^2_{\mathbb{S}}} \preceq \frac{(1-|\varphi(\zeta)|^2_{\mathbb{S}})\left|(-1)(1-|\varphi(a_n)|^2_{\mathbb{S}})\right|_{\mathbb{S}}}{|1-\varphi(a_n)^*\varphi(\zeta)|^2_{\mathbb{S}}}\\
\implies &\frac{(1-|\zeta|^2_{\mathbb{S}})\,|(-1)|_{\mathbb{S}}\left|(1-|a_n|^2_{\mathbb{S}})\right|_{\mathbb{S}}}{|1-a_n^*\zeta|^2_{\mathbb{S}}} \preceq \frac{(1-|\varphi(\zeta)|^2_{\mathbb{S}})\,|(-1)|_{\mathbb{S}}\left|(1-|\varphi(a_n)|^2_{\mathbb{S}})\right|_{\mathbb{S}}}{|1-\varphi(a_n)^*\varphi(\zeta)|^2_{\mathbb{S}}}\\
\implies &\frac{(1-|\zeta|^2_{\mathbb{S}})\left|1-|a_n|^2_{\mathbb{S}}\right|_{\mathbb{S}}}{|1-a_n^*\zeta|^2_{\mathbb{S}}} \preceq \frac{(1-|\varphi(\zeta)|^2_{\mathbb{S}})\left|1-|\varphi(a_n)|^2_{\mathbb{S}}\right|_{\mathbb{S}}}{|1-\varphi(a_n)^*\varphi(\zeta)|^2_{\mathbb{S}}}
\end{aligned}$$

Both a_n and $\varphi(a_n)$ are in $\mathscr{B}$ for all n. This implies that $a_n \preceq 1$ and $\varphi(a_n) \preceq 1$. So, $\left|1-|a_n|^2_{\mathbb{S}}\right|_{\mathbb{S}} = 1-|a_n|^2_{\mathbb{S}}$ and $\left|1-|\varphi(a_n)|^2_{\mathbb{S}}\right|_{\mathbb{S}} = 1-|\varphi(a_n)|^2_{\mathbb{S}}$. This gives us

$$\frac{(1-|\zeta|^2_{\mathbb{S}})(1-|a_n|^2_{\mathbb{S}})}{|1-a_n^*\zeta|^2_{\mathbb{S}}} \preceq \frac{(1-|\varphi(\zeta)|^2_{\mathbb{S}})(1-|\varphi(a_n)|^2_{\mathbb{S}})}{|1-\varphi(a_n)^*\varphi(\zeta)|^2_{\mathbb{S}}}$$

or equivalently,

$$\frac{|1-\varphi(a_n)^*\varphi(\zeta)|^2_{\mathbb{S}}}{1-|\varphi(\zeta)|^2_{\mathbb{S}}} \preceq \frac{(1-|\varphi(a_n)|^2_{\mathbb{S}})\,|1-a_n^*\zeta|^2_{\mathbb{S}}}{(1-|a_n|^2_{\mathbb{S}})(1-|\zeta|^2_{\mathbb{S}})}.$$

Taking the limit as n goes to ∞ gives

$$\frac{|1-\omega^*\varphi(\zeta)|^2_{\mathbb{S}}}{1-|\varphi(\zeta)|^2_{\mathbb{S}}} \preceq d(\alpha)\frac{|1-\alpha^*\zeta|^2_{\mathbb{S}}}{(1-|\zeta|^2_{\mathbb{S}})}.$$

Recall that for any bicomplex number ζ, $\zeta\zeta^* = |\zeta|_{\mathbb{S}}$. Also recall that $|\omega|_{\mathbb{S}} = |\alpha|_{\mathbb{S}} = 1$ This gives us

$$\frac{|1-\omega^*\varphi(\zeta)|^2_{\mathbb{S}}}{1-|\varphi(\zeta)|^2_{\mathbb{S}}} = \frac{|\omega|^2_{\mathbb{S}}}{|\omega|^2_{\mathbb{S}}}\cdot\frac{|1-\omega^*\varphi(\zeta)|^2_{\mathbb{S}}}{1-|\varphi(\zeta)|^2_{\mathbb{S}}} = \frac{|\omega-\omega\omega^*\varphi(\zeta)|^2_{\mathbb{S}}}{1\cdot(1-|\varphi(\zeta)|^2_{\mathbb{S}}} = \frac{|\omega-\varphi(\zeta)|^2_{\mathbb{S}}}{1-|\varphi(\zeta)^2|_{\mathbb{S}}}$$

and

$$d(\alpha)\frac{|1-\alpha^*\zeta|^2_{\mathbb{S}}}{1-|\zeta|^2_{\mathbb{S}}} = \frac{|1-\alpha^*\zeta|^2_{\mathbb{S}}}{1-|\zeta|^2_{\mathbb{S}}}\cdot\frac{|\alpha|^2_{\mathbb{S}}}{|\alpha|^2_{\mathbb{S}}} = \frac{|\alpha-\alpha\alpha^*\zeta|^2_{\mathbb{S}}}{(1)\cdot(1-|\zeta|^2_{\mathbb{S}})} = d(\alpha)\frac{|\alpha-\zeta|^2_{\mathbb{S}}}{1-|\zeta|^2_{\mathbb{S}}}.$$

Thus,

$$\frac{|\omega-\varphi(\zeta)|_{\mathbb{S}}^2}{1-|\varphi(\zeta)|_{\mathbb{S}}^2} \preceq d(\alpha)\frac{|\alpha-\zeta|_{\mathbb{S}}^2}{1-|\zeta|_{\mathbb{S}}^2}.$$

■

Geometrically, for some fixed, real-valued scalar $k>0$ and $\alpha\in\mathscr{B}$, we have closed balls of the form

$$E(\alpha,k)=\left\{\zeta\in\mathbb{BC}\colon\ |\alpha-\zeta|_{\mathbb{S}}^2\preceq k\left(1-|\zeta|_{\mathbb{S}}^2\right)\right\},$$

where $E(\alpha,k)$ is internally tangent to the unit ball at α with center $\alpha/(1+k)$ and radius $k/(1+k)$. Julia's lemma tells us that for each such $E(\alpha,k)$, we have $\varphi(E(\alpha,k))\subseteq E(\omega,kd(\alpha))$. This is similar to what we see in the complex setting (see [15]), but now we use the hyperbolic norm and the associated partial ordering. With this result, we are now ready to state and prove the second of our two main results for the section, which is analogous to Theorem 2:

Theorem 10 (Theorem 2 in $\mathbb{BC}$)**.** *Suppose $\varphi\colon\mathscr{B}\to\mathscr{B}$ is analytic in $\mathscr{B}$, continuous on $\partial\mathscr{B}$, and has a fixed point a with $|a|_{\mathbb{S}}=1$, $d(a)\prec 1$. If $\varphi^N(\overline{\mathscr{B}})\subseteq\mathscr{B}\cup\{a\}$ for some N, then $\varphi^n\to a$ uniformly in $\mathscr{B}$.*

Proof. This proof is similar to that of Theorem 4 in [4]. Without loss of generality, assume $\varphi(\overline{\mathscr{B}})\subseteq\mathscr{B}\cup\{a\}$. Because $\varphi(\overline{\mathscr{B}})\subseteq\mathscr{B}\cup\{a\}$ and $\varphi(\overline{\mathscr{B}})$ is connected, it fits within the ball $E(a,\lambda)$, which is a Euclidean subdisk of $\mathscr{B}$ centered at $a/(1+\lambda)$ with radius $\lambda/(1+\lambda)$ and tangent to $\mathscr{B}$ at a.

$$E(a,\lambda):=\left\{\zeta\in\mathbb{BC}\colon\ |a-\zeta|_{\mathbb{S}}^2\preceq\lambda\left(1-|\zeta|_{\mathbb{S}}^2\right)\right\}$$

for some constant $\lambda>0$. Applying Julia's Lemma for $\mathbb{BC}$, we know that $\varphi(E(a,\lambda))\subseteq E(a,d(a)\lambda)$. Applying φ iteratively, we see that for all $\zeta\in E(a,\lambda)$,

$$|a-\varphi^n(\zeta)|_{\mathbb{S}}^2\preceq d(a)^n\lambda\left(1-|\varphi^n(\zeta)|_{\mathbb{S}}^2\right)$$

and so

$$|a-\varphi^n(\zeta)|_{\mathbb{S}}\preceq\sqrt{\lambda}d(a)^{n/2}(1-|\varphi^n(\zeta)|_{\mathbb{S}})\preceq\sqrt{\lambda}d(a)^{n/2}.$$

Therefore, for any $\varepsilon\succ 0$, there exists some $N>0$ such that for $n>N$, $\sqrt{\lambda}d(a)^{n/2}\prec\varepsilon$. This implies that $|\varphi(\zeta)-a|_{\mathbb{S}}\prec\varepsilon$ for $n>N$, which completes the proof. ■

Quaternions

Let us now explore the second higher-dimensional domain of interest in this paper: the quaternions. The immediate difficulty is getting around their non-commutivity. This problem is side-stepped by defining a specific subset of quaternionic functions that obey a system of equations similar to the Cauchy-Riemann equations for $\mathbb{C}$. These functions, introduced by Gentili and Struppa in [16] and detailed in [17], are called *Slice-Regular*. There are left- and right-slice-regular functions, but for our purposes we will omit the qualifying left or right because the analysis can be accomplished with either (as long as the choice remains consistent). Furthermore, it will be shown that the left and right slice derivatives on the boundary of the unit ball are identical when defined with Julia's Lemma [18].

Preliminaries. The ring of quaternions is is similar to the field $\mathbb{C}$ in that it is an extension of $\mathbb{R}$ by imaginary units with specific algebraic properties. That is,

$$\mathbb{H} = \{q = q_1 + iq_2 + jq_3 + kq_4 \colon q_i \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = -1\}.$$

We'll use the notation $Re(q) = q_1$ and $Im(q) = iq_2 + jq_3 + kq_4$. The conjugate of each quaternion $q \in \mathbb{H}$ is defined to be

$$\bar{q} = q_1 - (iq_2 + jq_3 + kq_4) = Re(q) - Im(q) \in \mathbb{H}.$$

Note that $\mathbb{H}$ can be identified with $\mathbb{R}^4$, and

$$|q|^2 = q\bar{q} = Re(q)^2 + |Im(q)|^2,$$

where $Im(q)$ is thought of a vector in $\mathbb{R}^3$, and the norm is Euclidean. This allows the defining of the unit ball in $\mathbb{H}$ as

$$\mathscr{B} = \{q \colon |q|^2 < 1\}.$$

We adopt the superscript notation for iterates of a function used in the previous section. The definition of convergence is also similar. Let $X, Y \subseteq \mathbb{H}$. We say that a sequence of functions $f^n \colon X \to Y$ converges uniformly if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $q \in X$, $|f^n(q) - f(q)| < \varepsilon$.

In order to rigorously set up the definition of slice-regularity we must first define a geometric object in the set of quaternions that will replace the imaginary component in the analogous Cauchy-Riemann equations.

Definition (The Imaginary Sphere). *Let $q \in \mathbb{H}$ and $h_i \in \mathbb{R}$.*

$$\mathscr{S} = \{q = h_1 i + h_2 j + h_3 k \colon h_1^2 + h_2^2 + h_3^2 = 1\}.$$

Thus, for an element $I \in \mathscr{S}$, $I^2 = -1$ and

Definition. *For any fixed $I \in \mathbb{H}$, we define the complex slice*

$$\mathbb{C}_I = \{h_1 + Ih_2 \colon h_i \in \mathbb{R}\}.$$

With these definitions in mind it is possible to construct the hypotheses for a quaternionic function to be slice-regular.

Definition (Slice Regular Quaternionic Function [19]). *Let U be an open set in $\mathbb{H}$ and $f \colon U \to \mathbb{H}$ be real differentiable. The function f is said to be left slice regular if for every $I \in \mathscr{S}$, its restriction f_1 to $\mathbb{C}_I$ passing through the origin and containing I and 1 satisfies*

$$\bar{\partial}_1 f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_1(x+Iy) = 0 \text{ on } U \cup \mathbb{C}_I.$$

In order to achieve the analyticity needed for extending Theorem 1 we need the quaternionic version of an analytic function.

Definition 10.1 (σ-Analytic Quaternionic Functions [20]). *For every power series $f(q) = \sum_{n\in\mathbb{N}} q^n b_n$ there exists an $R \in [0, \infty)$, called the radius of convergence of $f(q)$, such that the series converges absolutely and uniformly on compact subsets in $\mathscr{B}_R = \{q \colon |q| < R\}$ and diverges everywhere else. f is thus considered σ-analytic.*

Slice-regular quaternionic functions and σ-analytic quaternionic functions are related.

Lemma 11 (Equality of Slice-Regular and σ-Analytic Functions [20])**.** *A quaternionic function is slice regular in a domain if, and only if, it is σ-analytic in the same domain.*

Now we are in need of Julia's Lemma for slice-regular quaternionic functions. In our analysis we will assume without loss of generality that $a = 1$. This is possible because there exists a conjugate map φ for which $a = 1$, and we will use φ.

Theorem 12 (Julia's Lemma in $\mathbb{H}$ [18])**.** *Let $f\colon \mathscr{B} \to \mathscr{B}$ be slice regular and let $\{a^n\}_{n\in\mathbb{N}} \subset \mathscr{B}$, converge to $a = 1$ as n tends towards ∞ such that*

$$\beta := \lim_{n\to\infty} \varphi(a^n) \text{ and } d(\beta) := \lim_{n\to\infty} \frac{1-|\varphi(a^n)|}{1-|a^n|}$$

exist and

$$\frac{|\beta - \varphi(a)|^2}{1-|\varphi(a)|^2} \le d(\beta)\frac{|1-a|^2}{1-|a|^2}.$$

As in the bicomplex case, Julia's Lemma tells us that for every open ball

$$E(a,\lambda) = \{q \in \mathbb{H}\colon |q-a|^2 \le \lambda\left(1-|q|^2\right)\},$$

internally tangent to the unit ball $\mathscr{B}$ at a, we have the useful containment

$$\varphi(E(a,\lambda)) \subseteq E(a, d(a)\lambda).$$

Results. We can now state and prove the quaternionic version of Theorem 1.

Theorem 13 (Theorem 2 in $\mathbb{H}$)**.** *Suppose $\varphi\colon \mathscr{B} \to \mathscr{B}$ is slice regular on $\mathscr{B}$, continuous on $\partial\mathscr{B}$, and has a fixed point, $a \in \partial\mathscr{B}$ with $|d(a)| < 1$. Also assume that $\varphi^n(q)$ converges to that boundary point, a, uniformly on compact subsets of $\mathscr{B}$. If $\varphi^N(\overline{\mathscr{B}}) \subseteq \mathscr{B} \cup \{a\}$ for some N, then $\varphi^n(q)$ converges uniformly to a on $\mathscr{B}$.*

Proof. We begin by noting that the structure of this proof follows the complex analog in [4] very closely; also, again we note that we may assume without loss of generality, that $a = 1$ because there exists a conjugate map whose conjugacy takes $a \in \partial\mathscr{B}$ to 1.

Assume $\varphi^N(\overline{\mathscr{B}}) \subseteq \mathscr{B} \cup \{a\}$. It follows that $\varphi(\overline{\mathscr{B}}) \subseteq E(a,\lambda)$, the open ball internally tangent to the unit ball with center $\frac{a}{1+\lambda}$ and radius $\frac{\lambda}{1+\lambda}$, with some $\lambda > 0$.

Using Julia's Lemma $\varphi(E(a,k)) \subseteq E(a,d(a)k)$. As a direct corollary of this set containment we achieve the inequality,

$$|\varphi^n(q) - a| \le \sqrt{\lambda}\, d(a)^{n/2}(1-|\varphi^n(q)|).$$

Since $1-|\varphi^n(q)| < 1$, we have

$$\sqrt{\lambda}\, d(a)^{n/2}(1-|\varphi^n(q)|) \le \sqrt{\lambda}\, d(a)^{n/2}.$$

These two inequalities imply uniform convergence. That is, for any $\varepsilon > 0$, $\exists\, N > 0$ such that $n > N$ implies $\sqrt{\lambda}\, d(a)^{n/2} < \varepsilon$ which implies that $|\varphi^n(q) - a| \le \sqrt{\lambda}\, d(a)^{n/2} < \varepsilon$.

■

An example is the Möbius transformation $\varphi(q) = \frac{q+1/2}{1+q/2}$. The fixed points are $q = \pm 1$, and defining the radial derivative via Julia's lemma gives $d(\pm 1) < 1$ at both. This example is corroborated by [21], which characterizes the convergence of Möbius transformations with certain hypotheses. In addition, all polynomials of the form $\varphi(q) = 1 - cq + cq^2$ with $0 < c < \frac{1}{2}$ exhibit UCI on $\mathscr{B}$. For this class of functions, the fixed point is $q = 1$ and the radial derivative is $d(1) < 1$. $\varphi(q) = 1 - cq + cq^2$ is slice regular because it is σ-analytic.

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