REPRESENTATIONS BY ORDERED BELL AND DEGENERATE ORDERED BELL POLYNOMIALS

DAE SAN KIM AND TAEKYUN KIM

Abstract. In this paper, we consider the problem of representing any polynomial in terms of the ordered Bell and degenerate ordered Bell polynomials, and more generally of the higher-order ordered Bell and higher-order degenerate ordered Bell polynomials. We derive explicit formulas with the help of umbral calculus and illustrate our results with some examples.

1. Introduction and preliminaries

The ordered Bell numbers (also called Fubini numbers) b_n arise from number theory and various counting problems in enumerative combinatorics (see [5,14]). The ordered Bell numbers b_n appeared already in 1859 work of Cayley [3], who used them to count certain plane trees with $n+1$ totally ordered leaves. While the (unordered) Bell numbers Bel_n given by $e^{(e^t-1)} = \sum_{n=0}^{\infty} Bel_n \frac{t^n}{n!}$ count partitions of $[n] = \{1, 2, \dots, n\}$ into nonempty disjoint subsets, the ordered Bell numbers count ordered partitions of $[n]$. Equivalently, the ordered Bell numbers b_n count either the number of weak orderings on a set of n elements or the mappings from $[n]$ to itself whose image is $[\ell]$, $1 \le \ell \le n$. They also count formulas in Fubini's theorem when rearranging the order of summation in multiple sums. We let the reader refer to [19], for details on the numerous uses of the ordered Bell numbers in counting problems.

Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k B_k(x)$, where $B_n(x)$ are the Bernoulli polynomials (see (1.3)). Then it is known (see [12]) that

$$a_0 = \int_0^1 p(t)dt, \quad a_k = \frac{1}{k!}(p^{(k-1)}(1) - p^{(k-1)}(0)), \text{ for } k = 1, 2, \dots, n. \tag{1.1}$$

Applying the formulas in (1.1) to the polynomial $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x)$, we can obtain an identity (see [12,16]) which yields, afer slight modification, the following identity:

$$\begin{aligned}&\sum_{k=1}^{n-1} \frac{1}{2k\,(2n-2k)} B_{2k}\,(x)\, B_{2n-2k}\,(x) + \frac{2}{2n-1} B_1\,(x)\, B_{2n-1}\,(x) \\ =&\frac{1}{n} \sum_{k=1}^{n} \frac{1}{2k} \binom{2n}{2k} B_{2k} B_{2n-2k}\,(x) + \frac{1}{n} H_{2n-1} B_{2n}\,(x) + \frac{2}{2n-1} B_1\,(x)\, B_{2n-1},\end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 05A19; 05A40; 11B68; 11B83.

Key words and phrases. ordered Bell polynomial; degenerate ordered Bellpolynomial; higher-order ordered Bell polynomial; higher-order degenerate ordered Bell polynomial; umbral calculus.

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

where $n \geq 2$, and $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Letting $x = 0$ and $x = \frac{1}{2}$ in (1.2) respectively give a slight variant of the Miki's identity and the Faber-Pandharipande-Zagier (FPZ) identity. Here it should be emphasized that the other proofs of Miki's (see [8,17,21]) and FPZ identities (see [6,7]) are quite involved, while our proofs of Miki's and FPZ identities follow from the simple formulas in (1.1) involving only derivatives and integrals of the given polynomials.

Analogous formulas to Remark 3.2 can be obtained for the representations by Bernoulli, Euler and Genocchi polynomials. Many interesting identities have been derived by using these formulas (see [10,12,13]). The list in the References are far from being exhaustive. However, the interested reader can easily find more related papers in the literature. Also, we should mention here that there are other ways of obtaining the same result as the one in (1.2). One of them is to use Fourier series expansion of the function obtained by extending by periodicity 1 of the polynomial function restricted to the interval $[0, 1)$ (see [15,16]).

The aim of this paper is to derive formulas (see Theorem 3.1) expressing any polynomial in terms of the degenerate ordered Bell polynomials (see [11]) with the help of umbral calculus (see [4,20,22]) and to illustrate our results with some examples (see Chapter 5). This is generalized to the higher-order degenerate ordered Bell polynomials. Indeed, we deduce formulas of representing any polynomial in terms of the higher-order degenerate ordered Bell polynomials (see Theorems 4.1) again by using umbral calculus and illustrate them with some examples (see Chapter 6). Letting $\lambda \to 0$, we obtain formulas of expressing any polynomial in terms of ordered Bell and higher-order ordered Bell polynomials. These formulas are also illustrated in Chapters 5 and 6. The contribution of this paper is the derivation of such formulas which have many potential applications.

The outline of this paper is as follows. In Section 1, we recall some necessary facts that are needed throughout this paper. In Section 2, we go over umbral calculus briefly. In Section 3, we derive formulas expressing any polynomial in terms of the degenerate ordered Bell polynomials. In Section 4, we derive formulas representing any polynomial in terms of the higher-order degenerate ordered Bell polynomials. In Section 5, we illustrate our results for representations by ordered Bell and higher-order ordered Bell polynomials with some examples. In Section 6, we illustrate our results for representations by degenerate ordered Bell and higher-order degenerate ordered Bell polynomials with some examples. Finally, we conclude our paper in Section 7.

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. We observe that $B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} x^j$, $\frac{d}{dx} B_n(x) = n B_{n-1}(x)$, $B_n(x+1) - B_n(x) = n x^{n-1}$. The first few terms of B_n are given by:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},$$
$$B_{12} = -\frac{691}{2730}, \ldots;\ B_{2k+1} = 0,\ (k \geq 1).$$

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

The Euler polynomials $E_n(x)$ are defined by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1.4}$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers. We observe that $E_n(x)=\sum_{j=0}^{n}\binom{n}{j}E_{n-j}x^j$, $\frac{d}{dx}E_n(x)=nE_{n-1}(x)$, $E_n(x+1)+E_n(x)=2x^n$. The first few terms of E_n are given by:

$$E_0=1,\ E_1=-\frac{1}{2},\ E_3=\frac{1}{4},\ E_5=-\frac{1}{2},\ E_7=\frac{17}{8},\ E_9=-\frac{31}{2},\dots;$$
$$E_{2k}=0,\ \ (k\geq 1).$$

The Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}. \tag{1.5}$$

When $x=0$, $G_n=G_n(0)$ are called the Genocchi numbers. We observe that $G_n(x)=\sum_{j=0}^{n}\binom{n}{j}G_{n-j}x^j$, $\frac{d}{dx}G_n(x)=nG_{n-1}(x)$, $G_n(x+1)+G_n(x)=2nx^{n-1}$, and $\deg G_n(x)=n-1$, for $n\geq 1$. Note that $G_0(x)=0$. The first few terms of G_n are given by:

$$G_0=0,\ G_1=1,\ G_2=-1,\ G_4=1,\ G_6=-3,\ G_8=17,\ G_{10}=-155$$
$$G_{12}=2073,\dots;G_{2k+1}=0,\ \ (k\geq 1).$$

The ordered Bell polynomials $b_n(x)$ are defined by

$$\frac{1}{2-e^t}e^{xt}=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x=0$, $b_n=b_n(0)$ are called the ordered Bell numbers. We observe that $b_n(x)=\sum_{j=0}^{n}\binom{n}{j}b_{n-j}x^j$, $\frac{d}{dx}b_n(x)=nb_{n-1}(x)$, $2b_n(x)-b_n(x+1)=x^n$. The first few terms of b_n are given by:

$$b_0=1,\ b_1=1,\ b_2=3,\ b_3=13,\ b_4=75,\ b_5=541,\ b_6=4683,\ b_7=47293\dots.$$

More generally, for any nonnegative integer r, the ordered Bell polynomials $b_n^{(r)}(x)$ of order r are given by

$$\left(\frac{1}{2-e^t}\right)^r e^{xt}=\sum_{n=0}^{\infty}b_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.7}$$

For any nonzero real number λ, the degenerate exponentials are given by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{1.8}$$
$$e_\lambda(t)=e_\lambda^1(t)=(1+\lambda t)^{\frac{1}{\lambda}}=\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}.$$

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

In [11], in the spirit of [1] and as a degenerate version of them, the degenerate ordered Bell polynomials $b_{n,\lambda}(x)$ are introduced, which are given by

$$\frac{1}{2-e_{\lambda}(t)}e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}b_{n,\lambda}(x)\frac{t^{n}}{n!}. \tag{1.9}$$

For $x=0$, $b_{n,\lambda}=b_{n,\lambda}(0)$ are called the degenerate ordered Bell numbers.

More generally, for any nonnegative integer r, the degenerate ordered Bell polynomials $b_{n,\lambda}^{(r)}(x)$ of order r are given by

$$\left(\frac{1}{2-e_{\lambda}(t)}\right)^{r}e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!}. \tag{1.10}$$

We remark that $b_{n,\lambda}(x)\rightarrow b_{n}(x)$, and $b_{n,\lambda}^{(r)}(x)\rightarrow b_{n}^{(r)}(x)$, as λ tends to 0.

We recall some notations and facts about forward differences. Let f be any complex-valued function of the real variable x. Then, for any real number a, the forward difference Δ_{a} is given by

$$\Delta_{a}f(x)=f(x+a)-f(x). \tag{1.11}$$

If $a=1$, then we let

$$\Delta f(x)=\Delta_{1}f(x)=f(x+1)-f(x). \tag{1.12}$$

In general, the nth oder forward differences are given by

$$\Delta_{a}^{n}f(x)=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}f(x+ia). \tag{1.13}$$

For $a=1$, we have

$$\Delta^{n}f(x)=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}f(x+i). \tag{1.14}$$

Finally, we recall that the Stirling numbers of the second kind $S_{2}(n,k)$ can be given by means of

$$\frac{1}{k!}(e^{t}-1)^{k}=\sum_{n=k}^{\infty}S_{2}(n,k)\frac{t^{n}}{n!}. \tag{1.15}$$

2. Review of umbral calculus

Here we will briefly go over very basic facts about umbral calculus. For more details on this, we recommend the reader to refer to [4,20,22]. Let $\mathbb{C}$ be the field of complex numbers. Then $\mathcal{F}$ denotes the algebra of formal power series in t over $\mathbb{C}$, given by

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_{k}\frac{t^{k}}{k!}\;\middle|\;a_{k}\in\mathbb{C}\right\},$$

and $\mathbb{P}=\mathbb{C}[x]$ indicates the algebra of polynomials in x with coefficients in $\mathbb{C}$.

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. If $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, then the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \quad \langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex number.

For $f(t) \in \mathcal{F}$ with $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, we define the linear functional on $\mathbb{P}$ by

$$\langle f(t)|x^k\rangle = a_k. \tag{2.1}$$

From (2.1), we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \quad (n,k \geq 0),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Some remarkable linear functionals are as follows:

$$\begin{aligned} &\langle e^{yt}|p(x)\rangle = p(y), \\ &\langle e^{yt}-1|p(x)\rangle = p(y)-p(0), \\ &\left\langle \frac{e^{yt}-1}{t}\middle| p(x)\right\rangle = \int_0^y p(u)du. \end{aligned} \tag{2.2}$$

Let

$$f_L(t) = \sum_{k=0}^{\infty} \langle L|x^k\rangle \frac{t^k}{k!}. \tag{2.3}$$

Then, by (2.1) and (2.3), we get

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle.$$

That is, $f_L(t) = L$. Additionally, the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$.

Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra and the umbral calculus is the study of umbral algebra. For each nonnegative integer k, the differential operator t^k on $\mathbb{P}$ is defined by

$$t^k x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{2.4}$$

Extending (2.4) linearly, any power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$$

gives the differential operator on $\mathbb{P}$ defined by

$$f(t)x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}, \quad (n \geq 0). \tag{2.5}$$

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

It should be observed that, for any formal power series $f(t)$ and any polynomial $p(x)$, we have

$$\langle f(t)|p(x)\rangle = \langle 1|f(t)p(x)\rangle = f(t)p(x)|_{x=0}. \tag{2.6}$$

Here we note that an element $f(t)$ of $\mathcal{F}$ is a formal power series, a linear functional and a differential operator. Some notable differential operators are as follows:

$$\begin{aligned} &e^{yt}p(x) = p(x+y), \\ &(e^{yt}-1)p(x) = p(x+y) - p(x), \\ &\frac{e^{yt}-1}{t}p(x) = \int_x^{x+y} p(u)du. \end{aligned} \tag{2.7}$$

The order $o(f(t))$ of the power series $f(t)(\neq 0)$ is the smallest integer for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. If $o(f(t)) = 1$, then $f(t)$ is called a delta series.

For $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ $(\deg s_n(x) = n)$ of polynomials such that

$$\big\langle g(t)f(t)^k|s_n(x)\big\rangle = n!\delta_{n,k}, \quad (n, k \geq 0). \tag{2.8}$$

The sequence $s_n(x)$ is said to be the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. We observe from (2.8) that

$$s_n(x) = \frac{1}{g(t)}p_n(x), \tag{2.9}$$

where $p_n(x) = g(t)s_n(x) \sim (1, f(t))$.

In particular, if $s_n(x) \sim (g(t), t)$, then $p_n(x) = x^n$, and hence

$$s_n(x) = \frac{1}{g(t)}x^n. \tag{2.10}$$

It is well known that $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)}e^{x\overline{f}(t)} = \sum_{k=0}^{\infty}\frac{s_k(x)}{k!}t^k, \tag{2.11}$$

for all $x \in \mathbb{C}$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

The following equations (2.12), (2.13), and (2.14) are equivalent to the fact that $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$:

$$f(t)\, s_n(x) = n s_{n-1}(x), \quad (n \geq 0), \tag{2.12}$$

$$s_n(x+y) = \sum_{j=0}^{n}\binom{n}{j}s_j(x)\, p_{n-j}(y), \tag{2.13}$$

with $p_n(x) = g(t)\, s_n(x)$,

$$s_n(x) = \sum_{j=0}^{n}\frac{1}{j!}\big\langle g\left(\overline{f}(t)\right)^{-1}\overline{f}(t)^j\big|x^n\big\rangle x^j. \tag{2.14}$$

For $s_n(x) \sim (g(t), f(t))$, and $r_n(x) \sim (h(t), l(t))$, we have

$$s_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \quad (n \geq 0), \tag{2.15}$$

where

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h\left(\overline{f}(t)\right)}{g\left(\overline{f}(t)\right)} l\left(\overline{f}(t)\right)^k \middle| x^n \right\rangle. \tag{2.16}$$

3. Representation by degenerate ordered Bell polynomials

Our interest here is to derive formulas expressing any polynomial in terms of the degenerate ordered Bell polynomials.

From (1.9) and (2.11), we first observe that

$$b_{n,\lambda}(x) \sim \big(g(t) = 2 - e^t, f(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)\big), \tag{3.1}$$

$$(x)_{n,\lambda} \sim (1, f(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)). \tag{3.2}$$

From (1.11), (2.7), (2.12), (3.1) and (3.2), we note that

$$f(t) b_{n,\lambda}(x) = n b_{n-1,\lambda}(x) = \frac{1}{\lambda}(e^{\lambda t} - 1) b_n(x) = \frac{1}{\lambda} \Delta_\lambda b_n(x), \tag{3.3}$$

$$f(t)(x)_{n,\lambda} = n(x)_{n-1,\lambda}. \tag{3.4}$$

It is immediate to see from (1.9) and (1.12) that

$$2b_{n,\lambda}(x) - b_{n,\lambda}(x+1) = b_{n,\lambda}(x) - \Delta b_{n,\lambda}(x) = (I - \Delta) b_{n,\lambda}(x) = (x)_{n,\lambda}, \tag{3.5}$$

where I is the identity operator.

Now, we assume that $p(x) \in \mathbb{C}[x]$ has degree n, and write $p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$. Let $h(x) = 2p(x) - p(x+1)$. Then, from (3.5), we have

$$\begin{aligned} h(x) &= \sum_{k=0}^{n} a_k (2b_{k,\lambda}(x) - b_{k,\lambda}(x+1)) \\ &= \sum_{k=0}^{n} a_k (x)_{k,\lambda}. \end{aligned} \tag{3.6}$$

For $k \geq 0$, from (3.4) and (3.6) we obtain

$$\begin{aligned} (f(t))^k h(x) &= (f(t))^k \sum_{l=0}^{n} a_l (x)_{l,\lambda} \\ &= \sum_{l=k}^{n} l(l-1)\cdots(l-k+1) a_l (x)_{l-k,\lambda}. \end{aligned} \tag{3.7}$$

Letting $x = 0$ in (3.7), we finally get

$$a_k = \frac{1}{k!} (f(t))^k h(x)|_{x=0} = \frac{1}{k!} \langle (f(t))^k | h(x) \rangle, \ (k \geq 0). \tag{3.8}$$

An alternative expression of (3.8) is given by

$$\begin{aligned} a_k &= \frac{1}{k!\lambda^k}\Delta_\lambda^k h(x)|_{x=0} \\ &= \frac{1}{k!\lambda^k}(2\Delta_\lambda^k p(x) - \Delta_\lambda^k p(x+1))|_{x=0} \\ &= \frac{1}{k!\lambda^k}(2\Delta_\lambda^k p(0) - \Delta_\lambda^k p(1)), \end{aligned} \tag{3.9}$$

as $f(t)h(x) = \frac{1}{\lambda}(e^{\lambda t}-1)h(x) = \frac{1}{\lambda}\Delta_\lambda h(x)$.

From (1.13), we have another alternative expression of (3.8) which is given by

$$\begin{aligned} a_k &= \frac{1}{k!\lambda^k}\Delta_\lambda^k h(x)|_{x=0} \\ &= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(h(x+j\lambda)|_{x=0} \\ &= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\big(2p(x+j\lambda) - p(x+1+j\lambda)\big)|_{x=0} \\ &= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\big(2p(j\lambda) - p(1+k\lambda)\big). \end{aligned} \tag{3.10}$$

By using (1.15), we obtain yet another expression of (3.8) that is given by

$$\begin{aligned} a_k &= \frac{1}{k!}\langle (f(t))^k | h(x)\rangle \\ &= \frac{1}{\lambda^k}\left\langle \frac{1}{k!}(e^{\lambda t}-1)^k \middle| h(x)\right\rangle \\ &= \frac{1}{\lambda^k}\left\langle \sum_{l=k}^{\infty} S_2(l,k)\frac{\lambda^l t^l}{l!} \middle| h(x)\right\rangle \\ &= \sum_{l=k}^{n} S_2(l,k)\frac{\lambda^{l-k}}{l!}\big(2p^{(l)}(0) - p^{(l)}(1)\big), \end{aligned} \tag{3.11}$$

where $p^{(l)}(x) = (\frac{d}{dx})^l p(x)$.

Finally, from (3.8)–(3.11), we get the following theorem.

Theorem 3.1. *Let $p(x) \in \mathbb{C}[x], \deg p(x) = n$. Let $f(t) = \frac{1}{\lambda}(e^{\lambda t}-1)$. Then we have $p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$,* *where*

$$\begin{aligned}
a_k &= \frac{1}{k!}(f(t))^k(2p(x)-p(x+1))|_{x=0} \\
&= \frac{1}{k!}\langle (f(t))^k | 2p(x)-p(x+1)\rangle \\
&= \frac{1}{k!\lambda^k}\left\langle \left(e^{\lambda t}-1\right)^k \middle| 2p(x)-p(x+1)\right\rangle \\
&= \frac{1}{k!\lambda^k}(2\Delta_\lambda^k p(x)-\Delta_\lambda^k p(x+1))|_{x=0} \\
&= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\big(2p(j\lambda)-p(1+j\lambda)\big) \\
&= \sum_{l=k}^{n} S_2(l,k)\frac{\lambda^{l-k}}{l!}\big(2p^{(l)}(0)-p^{(l)}(1)\big), \text{ for } k=0,1,\dots,n.
\end{aligned}$$

Remark 3.2. *Let* $p(x)\in\mathbb{C}[x]$, *with* $\deg p(x)=n$. *Write* $p(x)=\sum_{k=0}^{n} a_k b_k(x)$. *As* λ *tends to* 0, $f(t)\to t$. *Thus we obtain the following result:*

$$a_k = \frac{1}{k!}(2p^{(k)}(0)-p^{(k)}(1)), \text{ for } k=0,1,\dots,n. \tag{3.12}$$

Remark 3.3. *Formulas similar to* (3.12) *for Bernoulli, Euler and Genocchi polynomials have been applied to many polynomials in oder to obtain interesting identities for certain special polynomials and numbers* (see [10,12,13]). *Some of the polynomials that have been considered are as follows:*

(a)

$$\sum B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)G_{k_1+1}(x)\cdots G_{k_t+1}(x)x^l,$$

where the sum is over all nonnegative integers $i_1,\cdots,i_r,j_1,\cdots,j_s,k_1,\cdots,k_t,l$ *such that* $i_1+\cdots+i_r+j_1+\cdots+j_s+k_1+\cdots+k_t+l=n$, *and* r,s,t,l *are nonnegative integers with* $r+s+t\geq 1$.

(b)

$$\sum \frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)G_{k_1+1}(x)\cdots G_{k_t+1}(x)x^l}{i_1!\cdots i_r!j_1!\cdots j_s!(k_1+1)!\cdots(k_t+1)!l!},$$

where the sum is over all nonnegative integers $i_1,\cdots,i_r,j_1,\cdots,j_s,k_1,\cdots,k_t,l$ *such that* $i_1+\cdots+i_r+j_1+\cdots+j_s+k_1+\cdots+k_t+l=n$, *and* r,s,t,l *are nonnegative integers with* $r+s+t\geq 1$.

(c)

$$\sum \frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)G_{k_1+1}(x)\cdots G_{k_t+1}(x)x^l}{i_1\cdots i_r j_1\cdots j_s(k_1+1)\cdots(k_t+1)l},$$

where the sum is over all positive integers $i_1,\cdots,i_r,j_1,\cdots,j_s,l$ *and nonnegative integers* $k_1,\cdots,k_t$ *such that* $i_1+\cdots+i_r+j_1+\cdots+j_s+k_1+\cdots+k_t+l=n$, *and* r,s,t,l *are nonnegative integers with* $r+s+t\geq 1$.

4. Representation by higher-order degenerate ordered Bell polynomials

Our interest here is to derive formulas expressing any polynomial in terms of the higher-order degenerate ordered Bell polynomials.

With $g(t)=2-e^t, f(t)=\frac{1}{\lambda}(e^{\lambda t}-1)$, and from (1.10) and (2.11), we note that

$$b_{n,\lambda}^{(r)}(x)\sim(g(t)^r,f(t)). \tag{4.1}$$

Also, from (4.1) and (2.12), we have

$$f(t)b_{n,\lambda}^{(r)}(x)=nb_{n-1,\lambda}^{(r)}(x), \tag{4.2}$$

and from (1.10), it is immediate to see that

$$g(t)b_{n,\lambda}^{(r)}(x)=2b_{n,\lambda}^{(r)}(x)-b_{n,\lambda}^{(r)}(x+1)=b_{n,\lambda}^{(r-1)}(x). \tag{4.3}$$

Thus, from (4.3) we have $g(t)^r b_{n,\lambda}^{(r)}(x)=b_{n,\lambda}^{(0)}(x)=(x)_{n,\lambda}$.

Now, we assume that $p(x)\in\mathbb{C}[x]$ has degree n, and write $p(x)=\sum_{k=0}^{n}a_k b_{k,\lambda}^{(r)}(x)$. Then we have

$$g(t)^r p(x)=\sum_{l=0}^{n}a_l\, g(t)^r b_{l,\lambda}^{(r)}(x)=\sum_{l=0}^{n}a_l(x)_{l,\lambda}. \tag{4.4}$$

By using (3.4) and (4.4), we observe that

$$\begin{aligned}f(t)^k g(t)^r p(x)&=\sum_{l=0}^{n}a_l f(t)^k(x)_{l,\lambda}\\&=\sum_{l=k}^{n}a_l\, l(l-1)\cdots(l-k+1)(x)_{l-k,\lambda}.\end{aligned} \tag{4.5}$$

By evaluating (4.5) at $x=0$, we obtain

$$a_k=\frac{1}{k!}f(t)^k g(t)^r p(x)|_{x=0}=\frac{1}{k!}\langle f(t)^k g(t)^r|p(x)\rangle. \tag{4.6}$$

This also follows from the observation $\langle g(t)^r f(t)^k|b_{l,\lambda}^{(r)}(x)\rangle=l!\,\delta_{l,k}$.

We would like to find more explicit expressions for (4.6).

For this purpose, we first observe that

$$g(t)^r p(x)=(I-\Delta)^r p(x)=2^r\sum_{j=0}^{r}\binom{r}{j}\big(-\frac{1}{2}\big)^j p(x+j). \tag{4.7}$$

Several alternative expressions of (4.6) follow from (4.7) and (1.13), which are given by

$$\begin{aligned} a_k &= \frac{1}{k!} g(t)^r f(t)^k p(x)|_{x=0} \\ &= \frac{1}{k!\lambda^k}(I-\Delta)^r \Delta_\lambda^k p(x)|_{x=0} \\ &= \frac{1}{k!\lambda^k}(I-\Delta)^{r-1}\Delta_\lambda^k (2p(x)-p(x+1))|_{x=0} \\ &= \frac{2^r}{k!\lambda^k}\sum_{l=0}^{k}\sum_{j=0}^{r}(-1)^{k+j-l}\binom{k}{l}\binom{r}{j}\frac{1}{2^j}p(j+l\lambda),\ (0 \le k \le n). \end{aligned} \tag{4.8}$$

Next, from (1.15) we observe that

$$f(t)^k p(x) = \frac{k!}{\lambda^k}\frac{1}{k!}(e^{\lambda t}-1)^k p(x) = \frac{k!}{\lambda^k}\sum_{l=k}^{n} S_2(l,k)\frac{\lambda^l}{l!}t^l p(x), \tag{4.9}$$

Combining (4.7) and (4.9), we see that

$$\begin{aligned} a_k &= \frac{1}{k!} f(t)^k g(t)^r p(x)|_{x=0} \\ &= \frac{1}{\lambda^k}\sum_{l=k}^{n} S_2(l,k)\frac{\lambda^l}{l!}t^l g(t)^r p(x)|_{x=0} \\ &= \frac{1}{\lambda^k}\sum_{l=k}^{n} S_2(l,k)\frac{\lambda^l}{l!}t^l 2^r \sum_{j=0}^{r}\binom{r}{j}\big(-\frac{1}{2}\big)^j p(x+j)|_{x=0} \\ &= \frac{2^r}{\lambda^k}\sum_{l=k}^{n}\sum_{j=0}^{r}\binom{r}{j}(-\frac{1}{2})^j\frac{\lambda^l}{l!}S_2(l,k)p^{(l)}(j). \end{aligned} \tag{4.10}$$

Now, from (4.8) and (4.10), we finally arrive at the following theorem.

Theorem 4.1. *Let $p(x) \in \mathbb{C}[x], \deg p(x) = n$. Let $g(t) = 2 - e^t, f(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)$. Then we have $p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}^{(r)}(x)$, where*

$$\begin{aligned} a_k &= \frac{1}{k!} f(t)^k g(t)^r p(x)|_{x=0} \\ &= \frac{1}{k!}\langle f(t)^k g(t)^r | p(x)\rangle \\ &= \frac{1}{k!\lambda^k}(I-\Delta)^r \Delta_\lambda^k p(x)|_{x=0} \\ &= \frac{1}{k!\lambda^k}(I-\Delta)^{r-1}\Delta_\lambda^k (2p(x)-p(x+1))|_{x=0} \\ &= \frac{2^r}{k!\lambda^k}\sum_{l=0}^{k}\sum_{j=0}^{r}(-1)^{k+j-l}\binom{k}{l}\binom{r}{j}\frac{1}{2^j}p(j+l\lambda) \\ &= 2^r\sum_{l=k}^{n}\sum_{j=0}^{r}\binom{r}{j}(-\frac{1}{2})^j\frac{\lambda^{l-k}}{l!}S_2(l,k)p^{(l)}(j),\ (0 \le k \le n). \end{aligned}$$

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

Remark 4.2. *Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k b_k^{(r)}(x)$. As λ tends to 0, $f(t) \to t$. Thus, from Theorem 4.1, we have the following result:*

$$\begin{aligned} a_k &= \frac{1}{k!} f(t)^k g(t)^r p(x)|_{x=0} \\ &= \frac{2^r}{k!} \sum_{j=0}^{r} \binom{r}{j} (-\frac{1}{2})^j p^{(k)}(j) \\ &= \frac{1}{k!} (I-\Delta)^r p^{(k)}(x)|_{x=0} \\ &= \frac{1}{k!} \sum_{j=0}^{r} \binom{r}{j} (-1)^j \Delta^j p^{(k)}(x)|_{x=0} \\ &= \frac{1}{k!} (I-\Delta)^{r-1} \big(2p^{(k)}(x) - p^{(k)}(x+1)\big)|_{x=0} \\ &= \frac{1}{k!} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \big(2\Delta^j p^{(k)}(x) - \Delta^j p^{(k)}(x+1)\big)|_{x=0}, \end{aligned} \tag{4.11}$$

where $g(t) = 2 - e^t$, $f(t) = t$.

5. Examples on representations by ordered Bell polynomials

Here we illustrate our formulas in Remarks 3.2 and 4.2 with some examples.

(a) Let $p(x) = B_n(x) = \sum_{k=0}^{n} a_k b_k(x)$. Then, noting that $B_n(1) - B_n = \delta_{n,1}$ from (1.3), we have

$$\begin{aligned} a_k &= \frac{1}{k!}(2p^{(k)}(0) - p^{(k)}(1)) = \binom{n}{k}(2B_{n-k} - B_{n-k}(1)) \\ &= \binom{n}{k}\big(B_{n-k} - (B_{n-k}(1) - B_{n-k})\big) \\ &= \binom{n}{k}(B_{n-k} - \delta_{n-k,1}). \end{aligned}$$

Thus $B_n(x) = \sum_{k=0}^{n} \binom{n}{k}(B_{n-k} - \delta_{n-k,1}) b_k(x)$.

Now, we let $B_n(x) = \sum_{k=0}^{n} c_k b_k^{(r)}(x)$. Then we obtain the following:

$$\begin{aligned} c_k &= \frac{2^r}{k!} \sum_{j=0}^{r} \binom{r}{j} \big(-\frac{1}{2}\big)^j p^{(k)}(j) \\ &= 2^r \binom{n}{k} \sum_{j=0}^{r} \binom{r}{j} (-\frac{1}{2})^j B_{n-k}(j). \end{aligned}$$

Hence $B_n(x) = 2^r \sum_{k=0}^{n} \sum_{j=0}^{r} \binom{n}{k} \binom{r}{j} \big(-\frac{1}{2}\big)^j B_{n-k}(j)\, b_k^{(r)}(x)$.

These results on representations of $B_n(x)$ by $b_k(x)$ and $b_k^{(r)}(x)$ also follow from (2.16).

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

(b) Let $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x)$, $(n \geq 2)$. For this, we first recall from [12] that

$$p(x) = \frac{2}{n} \sum_{l=0}^{n-2} \frac{1}{n-l} \binom{n}{l} B_{n-l} B_l(x) + \frac{2}{n} H_{n-1} B_n(x), \tag{5.1}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the harmonic number. A slight modification of (5.1) gives the Miki's and FPZ identities. Let $p(x) = \sum_{k=0}^{n} a_k b_k(x)$. Then, for $0 \leq k \leq n-2$, we have

$$\begin{aligned} k! a_k &= 2p^{(k)}(0) - p^{(k)}(1) \\ &= \frac{2}{n} \sum_{l=k}^{n-2} \frac{1}{n-l} \binom{n}{l} B_{n-l} (l)_k (B_{l-k} - \delta_{l-k,1}) + \frac{2}{n} H_{n-1} (n)_k B_{n-k}; \end{aligned}$$

For $k = n-1$ or $k = n$, we have

$$k! a_k = \frac{2}{n} H_{n-1} (n)_k (B_{n-k} - \delta_{n-k,1}).$$

Hence we get

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x) \\ &= \sum_{k=0}^{n-2} \left\{ \frac{2}{n} \sum_{l=k}^{n-2} \frac{1}{n-l} \binom{n}{l} \binom{l}{k} B_{n-l} (B_{l-k} - \delta_{l-k,1}) + \frac{2}{n} \binom{n}{k} H_{n-1} B_{n-k} \right\} b_k(x) \\ &\qquad - 3 H_{n-1} b_{n-1}(x) + \frac{2}{n} H_{n-1} b_n(x). \end{aligned}$$

(c) Let $\Omega_l = \sum_{a=1}^{s} \binom{s}{a} 2^a (-1)^{s-a} \sum_{i_1 + \cdots + i_a = l} \prod_{j=1}^{a} b_{i_j} - \sum_{i_1 + \cdots + i_s = l} \prod_{j=1}^{s} b_{i_j}$. Let $p(x) = \sum_{i_1 + \cdots + i_s = n} \prod_{j=1}^{s} b_{i_j}(x)$. Then, from Theorems 2 and 3 in [5], it is immediate to see that

$$p(x) = \frac{1}{n+s} \sum_{j=0}^{n} \binom{n+s}{j} \Omega_{n-j+1} B_j(x).$$

Let $p(x) = \sum_{k=0}^{n} a_k b_k(x)$. Then, for $0 \leq k \leq n$, we have

$$k! a_k = 2p^{(k)}(0) - p^{(k)}(1) = \frac{1}{n+s} \sum_{j=k}^{n} \binom{n+s}{j} \Omega_{n-j+1} (j)_k (B_{j-k} - \delta_{j-k,1}).$$

Thus we obtain

$$\sum_{i_1 + \cdots + i_s = n} \prod_{j=1}^{s} b_{i_j}(x) = \frac{1}{n+s} \sum_{k=0}^{n} \left\{ \sum_{j=k}^{n} \binom{n+s}{j} \binom{j}{k} \Omega_{n-j+1} (B_{j-k} - \delta_{j-k,1}) \right\} b_k(x).$$

(d) In [16], it is proved that the following identity is valid for $n \geq 2$:

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} G_k(x) G_{n-k}(x) = -\frac{4}{n} \sum_{k=0}^{n-2} \binom{n}{k} \frac{G_{n-k}}{n-k} B_k(x). \tag{5.2}$$

Write $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} G_k(x) G_{n-k}(x) = \sum_{k=0}^{n-2} a_k b_k^{(r)}(x)$.

$$\begin{aligned} a_k &= \frac{2^r}{k!} \sum_{j=0}^{r} \binom{r}{j} \big(-\frac{1}{2}\big)^j p^{(k)}(j) \\ &= -\frac{2^{r+2}}{n} \sum_{j=0}^{r} \sum_{l=k}^{n-2} \big(-\frac{1}{2}\big)^j \binom{r}{j} \binom{n}{l} \binom{l}{k} \frac{G_{n-l}}{n-l} B_{l-k}(j). \end{aligned}$$

Therefore we obtain the following result:

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{1}{k(n-k)} G_k(x) G_{n-k}(x) \\ &= -\frac{2^{r+2}}{n} \sum_{k=0}^{n-2} \Big\{ \sum_{j=0}^{r} \sum_{l=k}^{n-2} (-\frac{1}{2})^j \binom{r}{j} \binom{n}{l} \binom{l}{k} \frac{G_{n-l}}{n-l} B_{l-k}(j) \Big\} b_k^{(r)}(x). \end{aligned}$$

(e) Nielsen [18,2] represented products of two Euler polynomials in terms of Bernoulli polynomials as follows:

$$\begin{aligned} E_m(x)E_n(x) = &-2 \sum_{i=1}^{m} \binom{m}{i} E_i \frac{B_{m+n-i+1}(x)}{m+n-i+1} \\ &- 2 \sum_{j=1}^{n} \binom{n}{j} E_j \frac{B_{m+n-j+1}(x)}{m+n-j+1} \\ &+ 2(-1)^{n+1} \frac{m!\, n!}{(m+n+1)!} E_{m+n+1}. \end{aligned}$$

Write $p(x) = E_m(x)E_n(x) = \sum_{k=0}^{m+n} a_k b_k^{(r)}(x)$. We now observe that

$$\begin{aligned} a_k &= \frac{2^r}{k!} \sum_{j=0}^{r} \binom{r}{j} \big(-\frac{1}{2}\big)^j p^{(k)}(j) \\ &= \frac{2^r}{k!} \sum_{j=0}^{r} \sum_{a+b=k} \binom{r}{j} \binom{k}{a,b} \big(-\frac{1}{2}\big)^j (E_m(x))^{(a)} (E_n(x))^{(b)} |_{x=j}, \end{aligned}$$

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

where $(E_n(x))^{(a)} = (\frac{d}{dx})^a E_n(x)$, and $\binom{k}{a,b} = \frac{k!}{a!b!}$. Thus we obtain the following:

$$\begin{aligned}
&-2\sum_{i=1}^{m}\binom{m}{i}E_i\frac{B_{m+n-i+1}(x)}{m+n-i+1} - 2\sum_{j=1}^{n}\binom{n}{j}E_j\frac{B_{m+n-j+1}(x)}{m+n-j+1}\\
&\quad + 2(-1)^{n+1}\frac{m!\,n!}{(m+n+1)!}E_{m+n+1}\\
&\quad = \sum_{k=0}^{m+n}\frac{2^r}{k!}\bigg\{\sum_{j=0}^{r}\sum_{a+b=k}\binom{r}{j}\binom{k}{a,b}(-\frac{1}{2})^j(E_m(x))^{(a)}(E_n(x))^{(b)}|_{x=j}\bigg\}b_k^{(r)}(x).
\end{aligned}$$

6. Examples on representations by degenerate ordered Bell polynomials

Here we illustrate our formulas in Theorems 3.1 and 4.1 with some examples.

(a) Let $p(x) = E_n(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$. Then, noting $E_n(1) + E_n = 2\delta_{n,0}$, we have

$$\begin{aligned}
a_k &= \frac{1}{k!\lambda^k}(2\Delta_\lambda^k E_n(x) - \Delta_\lambda^k E_n(x+1))|_{x=0}\\
&= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\big(2E_n(j\lambda) - E_n(1+j\lambda)\big)\\
&= \sum_{l=k}^{n}\binom{n}{l}\lambda^{l-k}S_2(l,k)\big(3E_{n-l} - 2\delta_{n,l}\big).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
E_n(x) &= \sum_{k=0}^{n}\frac{1}{k!\lambda^k}\big((2\Delta_\lambda^k E_n(x) - \Delta_\lambda^k E_n(x+1))|_{x=0}\big)b_{k,\lambda}(x)\\
&= \sum_{k=0}^{n}\frac{1}{k!\lambda^k}\bigg\{\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\big(2E_n(j\lambda) - E_n(1+j\lambda)\big)\bigg\}b_{k,\lambda}(x)\\
&= \sum_{k=0}^{n}\bigg\{\sum_{l=k}^{n}\binom{n}{l}\lambda^{l-k}S_2(l,k)\big(3E_{n-l} - 2\delta_{n,l}\big)\bigg\}b_{k,\lambda}(x).
\end{aligned}$$

Now, we let $p(x) = E_n(x) = \sum_{k=0}^{n} c_k b_{k,\lambda}^{(r)}(x)$. Recalling that $E_n(x+1) + E_n(x) = 2x^n$, we get

$$\begin{aligned}
c_k &= \frac{1}{k!\lambda^k}(I-\Delta)^r\Delta_\lambda^k E_n(x)|_{x=0}\\
&= \frac{1}{k!\lambda^k}(I-\Delta)^{r-1}\Delta_\lambda^k(3E_n(x) - 2x^n)|_{x=0}
\end{aligned}$$

$$= \frac{2^r}{k!\lambda^k}\sum_{l=0}^{k}\sum_{j=0}^{r}(-1)^{k+j-l}\binom{k}{l}\binom{r}{j}\frac{1}{2^j}E_n(j+l\lambda)$$
$$= \frac{2^r}{\lambda^k}\sum_{l=k}^{n}\sum_{j=0}^{r}\binom{r}{j}\binom{n}{l}(-\frac{1}{2})^j\lambda^l S_2(l,k)E_{n-l}(j).$$

This shows the following:

$$\begin{aligned}E_n(x) &= \sum_{k=0}^{n}\frac{1}{k!\lambda^k}\big((I-\Delta)^r\Delta_\lambda^k E_n(x)|_{x=0}\big)b_{k,\lambda}^{(r)}(x)\\
&= \sum_{k=0}^{n}\frac{1}{k!\lambda^k}\big((I-\Delta)^{r-1}\Delta_\lambda^k(3E_n(x)-2x^n)|_{x=0}\big)b_{k,\lambda}^{(r)}(x)\\
&= \sum_{k=0}^{n}\left\{\frac{2^r}{k!\lambda^k}\sum_{l=0}^{k}\sum_{j=0}^{r}(-1)^{k+j-l}\binom{k}{l}\binom{r}{j}\frac{1}{2^j}E_n(j+l\lambda)\right\}b_{k,\lambda}^{(r)}(x)\\
&= \sum_{k=0}^{n}\left\{\frac{2^r}{\lambda^k}\sum_{l=k}^{n}\sum_{j=0}^{r}\binom{r}{j}\binom{n}{l}(-\frac{1}{2})^j\lambda^l S_2(l,k)E_{n-l}(j)\right\}b_{k,\lambda}^{(r)}(x).\end{aligned}$$

These results on representations of $E_n(x)$ by $b_{k,\lambda}(x)$ and $b_{k,\lambda}^{(r)}(x)$ also follow from (2.16).

(b) Working similarly to (a) and recalling that $2b_n(x) - b_n(x+1) = x^n$, we have

$$\begin{aligned}b_n(x) &= \sum_{k=0}^{n}\left\{\frac{1}{k!\lambda^k}(2\Delta_\lambda^k b_n(x) - \Delta_\lambda^k b_n(x+1))|_{x=0}\right\}b_{k,\lambda}(x)\\
&= \sum_{k=0}^{n}\left\{\frac{1}{k!\lambda^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\big(2b_n(j\lambda)-b_n(1+j\lambda)\big)\right\}b_{k,\lambda}(x)\\
&= \sum_{k=0}^{n}S_2(n,k)\lambda^{n-k}b_{k,\lambda}(x).\end{aligned}$$

More generally, we also have

$$\begin{aligned}b_n(x) &= \sum_{k=0}^{n}\left\{\frac{1}{k!\lambda^k}(I-\Delta)^r\Delta_\lambda^k b_n(x)|_{x=0}\right\}b_{k,\lambda}^{(r)}(x)\\
&= \sum_{k=0}^{n}\left\{\frac{1}{k!\lambda^k}(I-\Delta)^{r-1}\Delta_\lambda^k(2b_n(x)-b_n(x+1))|_{x=0}\right\}b_{k,\lambda}^{(r)}(x)\\
&= \sum_{k=0}^{n}\left\{\frac{2^r}{k!\lambda^k}\sum_{l=0}^{k}\sum_{j=0}^{r}(-1)^{k+j-l}\binom{k}{l}\binom{r}{j}\frac{1}{2^j}b_n(j+l\lambda)\right\}b_{k,\lambda}^{(r)}(x)\\
&= \sum_{k=0}^{n}\left\{\frac{2^r}{\lambda^k}\sum_{l=k}^{n}\sum_{j=0}^{r}\binom{r}{j}\binom{n}{l}(-\frac{1}{2})^j\lambda^l S_2(l,k)b_{n-l}(j)\right\}b_{k,\lambda}^{(r)}(x).\end{aligned}$$

(c) As we mentioned earlier in (4.11), the following identity holds:

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

$$\sum_{i_1+\cdots+i_s=n}\prod_{j=1}^{s} b_{i_j}(x) = \frac{1}{n+s}\sum_{i=0}^{n}\binom{n+s}{i}\Omega_{n-i+1}B_i(x),$$

where $\Omega_l = \sum_{a=1}^{s}\binom{s}{a}2^a(-1)^{s-a}\sum_{i_1+\cdots+i_a=l}\prod_{j=1}^{a} b_{i_j} - \sum_{i_1+\cdots+i_s=l}\prod_{j=1}^{s} b_{i_j}$.

Then, by proceeding similarly to (a), we have

$$\begin{aligned}
&\sum_{i_1+\cdots+i_s=n}\prod_{j=1}^{s} b_{i_j}(x)\\
&=\sum_{k=0}^{n}\left\{\frac{1}{k!\lambda^k}\frac{1}{n+s}\sum_{i=0}^{n}\binom{n+s}{i}\Omega_{n-i+1}(I-\Delta)^r\Delta_\lambda^k B_i(x)|_{x=0}\right\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{n}\left\{\frac{1}{n+s}\frac{1}{k!\lambda^k}\sum_{i=0}^{n}\binom{n+s}{i}\Omega_{n-i+1}(I-\Delta)^{r-1}\Delta_\lambda^k (B_i(x)-jx^{j-1})|_{x=0}\right\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{n}\left\{\frac{2^r}{k!\lambda^k}\frac{1}{n+s}\sum_{l=0}^{k}\sum_{j=0}^{r}\sum_{i=0}^{n}(-1)^{k+j-l}\binom{k}{l}\binom{r}{j}\binom{n+s}{i}\frac{1}{2^j}\Omega_{n-i+1}B_i(j+l\lambda)\right\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{n}\left\{\frac{2^r}{\lambda^k}\frac{1}{n+s}\sum_{l=k}^{n}\sum_{j=0}^{r}\sum_{i=l}^{n}\binom{r}{j}\binom{i}{l}\binom{n+s}{i}(-\frac{1}{2})^j\lambda^l S_2(l,k)\Omega_{n-i+1}B_{i-l}(j)\right\}b_{k,\lambda}^{(r)}(x)
\end{aligned}$$

(d) Nielsen [18,2] expressed products of two Bernoulli polynomials in terms of Bernoulli polynomials. Namely, for positive integers m and n, with $m+n\geq 2$,

$$B_m(x)B_n(x) = \sum_r\left\{\binom{m}{2r}n+\binom{n}{2r}m\right\}\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r}+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}.$$

Then, by proceeding analogously to (a), we get

$$\begin{aligned}
&\sum_r\left\{\binom{m}{2r}n+\binom{n}{2r}m\right\}\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r}+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}\\
&=\sum_{k=0}^{m+n}\left\{\frac{1}{k!\lambda^k}(I-\Delta)^r\Delta_\lambda^k\big(B_m(x)B_n(x)\big)|_{x=0}\right\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{m+n}\left\{\frac{1}{k!\lambda^k}(I-\Delta)^{r-1}\Delta_\lambda^k\big(2B_m(x)B_n(x)-B_m(x+1)B_n(x+1)\big)|_{x=0}\right\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{m+n}\left\{\frac{2^r}{k!\lambda^k}\sum_{l=0}^{k}\sum_{j=0}^{r}(-1)^{k+j-l}\binom{k}{l}\binom{r}{j}\frac{1}{2^j}B_m(j+l\lambda)B_n(j+l\lambda)\right\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{m+n}\left\{\frac{2^r}{\lambda^k}\sum_{l=k}^{n}\sum_{j=0}^{r}\sum_{a+b=l}\binom{r}{j}\binom{l}{a,b}(-\frac{1}{2})^j\frac{\lambda^l}{l!}S_2(l,k)\big(B_m(x)\big)^{(a)}\big(B_n(x)\big)^{(b)}|_{x=j}\right\}b_{k,\lambda}^{(r)}(x),
\end{aligned}$$

where $(B_n(x))^{(a)} = (\frac{d}{dx})^a B_n(x)$.

6 Oct 2021 20:59:30 PDT
211006-TaekyunKim Version 1 - Submitted to Rocky Mountain J. Math.

(e) In (4.20) of [9], it is shown that the following identity holds for $n \geq s$:

$$\sum_{\substack{i_1+\cdots+i_s=n\\ i_1,\dots,i_s\geq 1}} G_{i_1}(x)\dots G_{i_s}(x) = \frac{1}{n+s}\sum_{l=1}^{s}\binom{s}{l}(-2)^{l-1}\sum_{\substack{i_0+i_1+\cdots+i_{s-l}=n+1-l\\ i_0,i_1,\dots,i_{s-l}\geq 1}}\binom{n+s}{i_0}G_{i_1}\cdots G_{i_s}G_{i_0}(x).$$

Then, proceeding similarly to (b), we can show that

$$\begin{aligned}
&\sum_{\substack{i_1+\cdots+i_s=n\\ i_1,\dots,i_s\geq 1}} G_{i_1}(x)\dots G_{i_s}(x) = \sum_{k=0}^{n-s}\bigg\{\frac{1}{k!\lambda^k}\frac{1}{n+s}\sum_{l=1}^{s}\binom{s}{l}(-2)^{l-1}\\
&\times \sum_{\substack{i_0+i_1+\cdots+i_{s-l}=n+1-l\\ i_0,i_1,\dots,i_{s-l}\geq 1}}\binom{n+s}{i_0}G_{i_1}\cdots G_{i_s}(I-\Delta)^r\Delta_\lambda^k G_{i_0}(x)|_{x=0}\bigg\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{n-s}\bigg\{\frac{1}{k!\lambda^k}\frac{1}{n+s}\sum_{l=1}^{s}\binom{s}{l}(-2)^{l-1}\sum_{\substack{i_0+i_1+\cdots+i_{s-l}=n+1-l\\ i_0,i_1,\dots,i_{s-l}\geq 1}}\binom{n+s}{i_0}G_{i_1}\cdots G_{i_s}\\
&\times (I-\Delta)^{r-1}\Delta_\lambda^k\big(3G_{i_0}(x)-2i_0x^{i_0-1}\big)|_{x=0}\bigg\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{n-s}\bigg\{\frac{2^r}{k!\lambda^k}\frac{1}{n+s}\sum_{a=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{s}(-1)^{k+j+a+l-1}\binom{k}{a}\binom{r}{j}\binom{s}{l}2^{l-j-1}\\
&\times \sum_{\substack{i_0+i_1+\cdots+i_{s-l}=n+1-l\\ i_0,i_1,\dots,i_{s-l}\geq 1}}\binom{n+s}{i_0}G_{i_1}\cdots G_{i_s}G_{i_0}(j+a\lambda)\bigg\}b_{k,\lambda}^{(r)}(x)\\
&=\sum_{k=0}^{n-s}\bigg\{\frac{2^r}{\lambda^k}\frac{1}{n+s}\sum_{a=k}^{n}\sum_{j=0}^{r}\sum_{l=1}^{s}\binom{r}{j}\binom{s}{l}\binom{i_0}{a}(-2)^{l-j-1}\lambda^a S_2(a,k)\\
&\times \sum_{\substack{i_0+i_1+\cdots+i_{s-l}=n+1-l\\ i_1,\dots,i_{s-l}\geq 1,\, i_0\geq a+1}}\binom{n+s}{i_0}G_{i_1}\cdots G_{i_s}G_{i_0-a}(j)\bigg\}b_{k,\lambda}^{(r)}(x).
\end{aligned}$$

7. Conclusion

In this paper, we were interested in representing any polynomial in terms of the ordered Bell and degenerate ordered Bell polynomials, and more generally of the higher-order ordered Bell and higher-order degenerate ordered Bell polynomials. We were able to derive formulas for such representations with the help of umbral calculus. Further, we illustrated the formulas with some examples.

Even though the method adopted in this paper is elementary, they are very useful and powerful. Indeed, as we mentioned in the Section 1, both a variant of Miki's identity and Faber-Pandharipande-Zagier (FPZ) identity follow from the one identity (see (1.2)) that can be derived from a formula (see (1.1)) involving only derivatives and integrals of the given polynomial, while all the other proofs are quite involved. We recall here that the FPZ identity was a conjectural relations between Hodge integrals in Gromov-Witten theory.

It is one of our future research projects to continue to find formulas representing polynomials in terms of some specific special polynomials and to apply those in discovering some interesting identities.

References

[1] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51—88.
[2] L. Carlitz, *The product of two Eulerian polynomials,* Math. Mag. **36** (1963), no. 1, 37–41.
[3] A. Cayley, *On the analytical forms called trees, second part,* Philisophical Magazine, Series IV **18** (1859), no. 121, 374–378.
[4] R. Dere, Y. Simsek, *Applications of umbral algebra to some special polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 433—438.
[5] D. V. Dolgy, D. S. Kim, T. Kim, G.-W. Jang, T. Mansour, *Sums of finite products of ordered Bell functions,* Adv. Stud. Contemp. Math. (Kyungshang) **28** (2018), no. 3, 381–391.
[6] G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys. **7** (2013), no. 2, 225–249.
[7] C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139** (2000), no. 1, 173–199.
[8] I. M. Gessel, *On Miki's identities for Bernoulli numbers,* J. Number Theory **110** (2005), no. 1, 75–82.
[9] Y. He, T. Kim, *General convolution identities for Apostol-Bernoulli, Euler and Genocchi polynomials,* J. Nonlinear Sci. Appl. **9** (2016), 4780–4797.
[10] D. S. Kim, T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials,* Int. J. Math. Math. Sci. 2012, Article ID 463659, 12 pp.
[11] D. S. Kim, T. Kim, *Degenerate ordered Bell numbers and polynomials associated with umbral calculus,* J. Nonlinear Sci. Appl. **10** (2017), 5142— 5155.
[12] D. S. Kim, T, Kim, S.-H. Lee, Y.-H. Kim, *Some identities for the product of two Bernoulli and Euler polynomials,* Adv. Differ. Equ. 2012, 2012:95, 14 pp.
[13] D. S. Kim, T. Kim, T. Mansour, *Euler basis and the product of several Bernoulli and Euler polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **24** (2014), no. 4, 535–547.
[14] T. Kim, D. S. Kim, G.-W. Jang, *Some formulas of ordered Bell numbers and polynomials arising from umbral calculus,* Proc. Jangjeon Math. Soc. **20** (2017), no. 4, 659–670.
[15] T. Kim, D. S. Kim, G.-W. Jang, J. Kwon, *Series of sums of products of higher-order Bernoulli functions,* J. Inequal. Appl. 2017, 2017:221, 16 pp.
[16] T. Kim, D. S. Kim, G.-W. Jang, J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications,* J. Nonlinear Sci. Appl. **10** (2017), 1683—1694.
[17] H. Miki, *A relation between Bernoulli numbers,* J. Number Theory **10** (1978), no. 3, 297–302.
[18] N. Nielsen, Traité élémentaire des nombres de Bernoulli, Paris, 1923.
[19] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A000670.
[20] S. Roman, *The umbral calculus,* Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
[21] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36** (1982), no. 1, 73–83.
[22] T. Kim, D. S. Kim, *Some Identities on Truncated Polynomials Associated with Degenerate Bell Polynomials,* Russ. J. Math. Phys **28** (2021), no. 2, 342–355

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
Email address: dskim@sogang.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
Email address: tkkim@kw.ac.kr