Tensor products of Aluthge transforms and A-adjoints of m-isometric operators

B.P. Duggal, I.H. Kim

Abstract

Given an m-isometric Hilbert space operator $A \in \mathcal{B}(\mathcal{H})$, $\triangle^m_{A^*,A}(I) = \sum_{j=0}^m (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j}A^j = 0$, with polar decomposition $A = U|A|$, the Aluthge transform $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ preserves almost all the spectral properties of A. However, the m-isometric property of an operator neither implies nor is implied by the m-isometric property of its Aluthge transform. The operator A has an $|A|$-adjoint $\mathcal{A}$, $\mathcal{A}^* = [A]^* = U^*|A|$ [4, Definition 1.1]. If A_i, $i = 1, 2$, doubly commute and $\tilde{A}_i$ (resp., $\mathcal{A}_i$) is strict m_i-isometric, then $\widetilde{A_1A_2}$ (resp., $\mathcal{A}_1\mathcal{A}_2$) is strict $(m_1 + m_2 - 1)$-isometric. The converse fails for products A_1A_2, $\tilde{A}_1\tilde{A}_2$ and $\mathcal{A}_1\mathcal{A}_2$, but has a positive answer for tensor products $A_1 \otimes A_2$, $\tilde{A}_1 \otimes \tilde{A}_2$, $\mathcal{A}_1 \otimes \mathcal{A}_2$ (and their Hilbert-Schmidt class identifications with the elementary operators $L_{A_1}R_{A_2^*}$, $L_{\tilde{A}_1}R_{\tilde{A}_2^*}$ and $L_{\mathcal{A}_1}R_{\mathcal{A}_2^*}$); if $S \otimes T$, where $S \otimes T$ stands for either of the three tensor products above, is strict m-isometric, then there exist scalars c and d, $|cd| = 1$, and positive integers m_1 and m_2, $m = m_1 + m_1 - 1$, such that cS is strict m_1-isometric and dT is strict m_2-isometric.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of operators, i.e. bounded linear transformations, on an infinite dimensional complex Hilbert space $\mathcal{H}$ into itself. A generalisation of isometric operators $A \in \mathcal{B}(\mathcal{H})$ is obtained by calling A m-isometric, $A \in m$-isometric, if

$$\triangle^m_{A^*,A}(I) = (I - L_{A^*}R_A)^m(I) = \sum_{j=0}^m (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j}A^j = 0,$$

where L_{A^*}, $R_A \in B(\mathcal{B}(\mathcal{H}))$ are, respectively, the operator $L_{A^*}(X) = A^*X$ of left multiplication by A^* and the operator $R_A(X) = XA$ of right multiplication by A. Motivated by the work of W. Helton, the concept of m-isometric operators was introduced bt J. Agler [1], and a study of the structure of the class of m-isometric operators was initiated by Agler and Stankus in [2]. This class of operators has since been studied by a large number of authors, amongst them [3, 5, 8, 9, 11, 13, 14, 16, 18, 21]. If an $A \in \mathcal{B}(\mathcal{H})$ has the polar decomposition $A = U|A|$, then the Aluthge transform $\tilde{A}$

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of A is the operator $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$ [17]. The Aluthge transforms preserve, often improve upon, many a spectral property of the operator. However, as Botelho and Jamison [10] point out in their study of elementary operators and Aluthge transforms, the m-isometric property of an operator neither implies nor is implied by the m-isometric property of its Aluthge transform. For example, if $A_1 \in \mathcal{B}(\ell^2)$ is the operator $A_1x = A_1(x_1, x_2, x_3, x_4, \cdots) = \left(0, \frac{1}{2}x_1, 2x_2, \frac{1}{2}x_3, 2x_4, \cdots\right)$, then A_1 is not 1-isometric (i.e., isometric) and $\tilde{A}_1$, $\tilde{A}_1x = (0, x_1, x_2, x_3, x_4, \cdots)$, is 1-isometric. Again, the operator $A_2 \in \mathcal{B}(\ell^2)$, $A_2x = (0, a_1x_1, a_2x_2, a_3x_3, \cdots)$ and $a_j = e^{i\theta_j}\sqrt{\frac{1+j}{j}}$, is 2-isometric but $\tilde{A}x = \left(0, e^{i\theta_1}|a_1a_2|^{\frac{1}{2}}, e^{i\theta_2}|a_2a_3|^{\frac{1}{2}}, e^{i\theta_3}|a_3a_4|^{\frac{1}{2}}, \cdots\right)$ is not 2-isometric. Not all is, however, lost. In both the considered examples, the operator $|A_i|$, $i = 1, 2$, is invertible, hence defines an equivalent norm $\|\cdot\|_{|A_i|}$, $\|B\|_{|A_i|} = \left\||A_i|^{\frac{1}{2}}B\right\|$ and an $|A_i|$-adjoint operator (in the terminology of [4, Definition 1.1]) $[A_i]^* = U_i^*|A_i|$. It is seen that $A_1 \in (1, |A_1|)$-isometric (i.e., $\left\||A_1|^{\frac{1}{2}}A_1x\right\| = \left\||A_1|^{\frac{1}{2}}x\right\|$ for all $x \in \ell^2$), $[A_1] \in (1, |A_1|^{-1})$-isometric, $\tilde{A}_2 \in (2, |A_2|)$-isometric and $[A_2] \in$ 2-isometric. We prove in the following that such phenomena are typical of m-isometric operators.

Recall that $A \in \mathcal{B}(\mathcal{H})$ is said to be strict m-isometric if $\triangle^m_{A^*,A}(I) = 0$ and $\triangle^{m-1}_{A^*,A}(I) \neq 0$. If $A_i \in \mathcal{B}(\mathcal{H})$, $i = 1, 2$, commute, $[A_1, A_2] = A_1A_2 - A_2A_1 = 0$, and $A_i \in m_i$-isometric, then $A_1A_2 \in (m_1 + m_2 - 1)$-isometric [7]. A_1 and A_2 doubly commute if $[A_1, A_2] = 0 = [A_1^*, A_2]$. For doubly commuting strict m_i-isometric operators A_i, "the product property" extends to their Aluthge transforms $\tilde{A}_i$ and $|A_i|$-adjoints $\mathcal{A}_i = |A_i|U_i$ (see definition below). The converse, namely "does A_1A_2 (or, $\tilde{A}_1\tilde{A}_2$, or, $\mathcal{A}_1\mathcal{A}_2$) strict m-isometric imply the existence of positive integers m_i such that $m - m_1 + m_2 - 1$ and A_i (resp., $\tilde{A}_i$,or, $\mathcal{A}_i$), or some multiple thereof, is strict m_i-isometric fails, even for doubly commuting A_1 and A_2. An exception here is the tensor product $A_1 \otimes A_2$ (and its Hilbert-Schmidt class identification with the elementary opetrator $L_{A_1}R_{A_2^*}$). It is seen that if (S, T) is either of the pairs (A_1, A_2) or $(\tilde{A}_1, \tilde{A}_2)$ or $(\mathcal{A}_1, \mathcal{A}_2)$ and $S \otimes T$ is strict m-isometric, then there exist scalars c and d, $|cd| = 1$, and positive integers m_i, $m = m_1 + m_2 - 1$, such that cS is strict m_1-isometric and dT is strict m_2-isometric.

2. Results.

Let $P \geq 0$ be a positive operator in $\mathcal{B}(\mathcal{H})$. Given an operator $A \in \mathcal{B}(\mathcal{H})$ with adjoint A^*, an operator $\mathcal{A}^*$ is a P-adjoint of A if

$$\langle Ax, y\rangle_P = \langle PAx, y\rangle = \langle Px, \mathcal{A}^*y\rangle = \langle x, \mathcal{A}^*y\rangle_P$$

for all $x, y \in \mathcal{H}$ [4]. Equivalently, $\mathcal{A}^*$ is a P-adjoint of A if and only if

$$P\mathcal{A}^* = A^*P \;(\Longleftrightarrow \mathcal{A}P = PA).$$

Not every $A \in \mathcal{B}(\mathcal{H})$ has a P-adjoint: *A has a P-adjoint if and only if the operator equation $XP = PA$ has a solution and a necessary and sufficient condition for this to happen, guaranteed by the following theorem, is that $A^*P(\mathcal{H}) \subseteq P(\mathcal{H})$.*

Theorem 2.1 *[12] Given operators $A, B \in \mathcal{B}(\mathcal{H})$, the following conditions are equivalent:*

(i) $B(\mathcal{H}) \subseteq A(\mathcal{H})$.
(ii) There exists a positive scalar c such that $BB^* \leq cAA^*$.
(iii) There exists an operator $C \in \mathcal{B}(\mathcal{H})$ *such that* $AC = B$.

Furthermore, if one of these conditions is satisfied, then there exists a unique operator $D \in \mathcal{B}(\mathcal{H})$ *such that* $AD = B$, $D(\mathcal{H}) \subseteq \overline{A^*(\mathcal{H})}$, $D^{-1}(0) = B^{-1}(0)$ *and* $||D||^2 = \inf\{c > 0 : BB^* \leq cAA^*\}$.

If an operator $A \in \mathcal{B}(\mathcal{H})$ is m-isometric, then it is necessarily left invertible, hence has a polar decomposition $A = UP$, where U is isometric and $P \geq 0$ is invertible. Thus an m-isometric operator A always has a unique $P(= |A|)$-adjoint, namely the operator $\mathcal{A}$ defined by

$$\mathcal{A}^* = [A]^* = P^{-1}A^*P = U^*P \ (\Longleftrightarrow \mathcal{A} = PU).$$

The Aluthge transform $\tilde{A}$ (i.e., the operator $\tilde{A} = P^{\frac{1}{2}}UP^{\frac{1}{2}}$) of the m-isometric operator $A = UP$ is related to $\mathcal{A}$ via $P^{\frac{1}{2}}\tilde{A}P^{-\frac{1}{2}} = \mathcal{A}$. Henceforth, *given an operator* T *with polar decomposition* $T = U_T|T|, \mathcal{T}^* = [T]^* = U_T^*|T|$ *shall denote the* $|T|$*-adjoint of* T.

Remark 2.2 *For a given operator* $A = UP$, *the operator* $B = PU$ *has been called the Duggal transform of A [17]. It is a well known fact [6] that* $\sigma_x(\tilde{A}) = \sigma_x(\mathcal{A}) = \sigma_x(A)$ *for most of the distinguished parts of the spectrum* σ*: in particular, the equality holds for* $\sigma_x = \sigma$, *the spectrum, and* σ_a, *the approximate point spectrum.*

The following technical lemma is important to our deliberations below.

Lemma 2.3 *If* $A \in \mathcal{B}(\mathcal{H})$ *has the polar decomposition* $A = UP$, *Aluthge transform* $\tilde{A}$ *and P-adjoint* $\mathcal{A}^*$, *then:*

$$\begin{aligned} \{A \in m - isometric\} &\Longrightarrow \{\mathcal{A} \in m - isometric\} \\ &\Longrightarrow \{A \in (m, P^2) - isometric, 0 \notin \sigma_a(A)\} \wedge \\ &\quad\wedge \quad \{\tilde{A} \in (m, P) - isometric, 0 \notin \sigma_a(\tilde{A})\}, \end{aligned}$$

$$\{\mathcal{A} \in m - isometric\} \Longleftrightarrow \{\tilde{A} \in (m, P) - isometric,\ 0 \notin \sigma_a(\tilde{A})$$

and

$$\{A \in (m, P) - isometric\} \Longleftrightarrow \{\tilde{A} \in m - isometric, 0 \notin \sigma_a(A)\}.$$

Proof. Since $\mathcal{A} = PU$,

$$\begin{aligned} \{A \in m - \text{isometric}\} &\Longleftrightarrow \triangle^m_{A^*,A}(I) = 0 \\ &\Longleftrightarrow \sum_{j=0}^m (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j}A^j = 0 \\ &\Longleftrightarrow U^*\left(\sum_{j=1}^m (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} P\mathcal{A}^{*j-1}\mathcal{A}^{j-1}P + I\right)U = 0 \\ &\Longrightarrow \sum_{j=0}^m (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \mathcal{A}^{*j}\mathcal{A}^j = 0,\ 0 \notin \sigma_a(\mathcal{A}) \\ &\Longleftrightarrow \{\mathcal{A} \in m - \text{isometric}\} \end{aligned}$$

$$\begin{aligned}
&\iff P\left(\sum_{j=1}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}U^*A^{*j-1}P^2A^{j-1}U+I\right)P=0,\\
&\quad\; 0\notin\sigma_a(A)\\
&\iff \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}A^{*j}P^2A^j=0,\ 0\notin\sigma_a(A)\\
&\iff \triangle^m_{A^*,A}(P^2)=0,\ 0\notin\sigma_a(A)\\
&\iff P^{-\frac{1}{2}}\triangle^m_{A^*,A}(P^2)P^{-\frac{1}{2}}=0,\ 0\notin\sigma_a(A)\\
&\iff \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}\left(P^{\frac{1}{2}}U^*P^{\frac{1}{2}}\right)^jP\left(P^{\frac{1}{2}}UP^{\frac{1}{2}}\right)^j=0,\\
&\quad\; 0\notin\sigma_a(P^{\frac{1}{2}}UP^{\frac{1}{2}})\\
&\iff \triangle^m_{\tilde{A}^*,\tilde{A}}(P)=0.
\end{aligned}$$

For the two way implication, we have:

$$\begin{aligned}
\{A\in m-\text{isometric}\} &\iff \triangle^m_{A^*,A}(I)=0\iff\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}A^{*j}A^j=0\\
&\iff \sum_{j=1}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}U^*P^{\frac{1}{2}}\tilde{A}^{*j-1}P\tilde{A}^{j-1}P^{\frac{1}{2}}U+I=0\\
&\iff P^{\frac{1}{2}}\left(\sum_{j=1}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}U^*P^{\frac{1}{2}}\tilde{A}^{*j-1}P\tilde{A}^{j-1}P^{\frac{1}{2}}U+I\right)P^{\frac{1}{2}}=0,\\
&\quad\; 0\notin\sigma_a(\tilde{A})\\
&\iff \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}\tilde{A}^{*j}P\tilde{A}^j=0,\ 0\notin\sigma_a(\tilde{A})\\
&\iff \triangle^m_{\tilde{A}^*,\tilde{A}}(P)=0,\ 0\notin\sigma_a(\tilde{A}).
\end{aligned}$$

If $\tilde{A}\in m$-isometric, then

$$\begin{aligned}
&\triangle^m_{\tilde{A}^*,\tilde{A}}(I)=0\\
\iff\ & P^{\frac{1}{2}}[\triangle^m_{A^*,A}(I)]P^{\frac{1}{2}}=0,\ 0\notin\sigma_a(\tilde{A})=\sigma_a(A)\\
\iff\ & \triangle^m_{A^*,A}(P)=0,\ 0\notin\sigma_a(A).
\end{aligned}$$

This completes the proof. □

Observe that if the operator A in the first part of Lemma 2.3 is invertible, then U in the polar decomposition $A=UP$ is unitary, all the implications in the proof of the first set of implications are two way implications, hence the items in the statement of the lemma are equivalences. In the general case, $\mathcal{A}\in m$-isometric implies $\left(P\triangle^m_{\mathcal{A}^*,\mathcal{A}}(I)P=0\Longrightarrow\right)$ $A^*\triangle^m_{A^*,A}(I)A=0$. Such operators A have been called 1-quasi m-isometric and have been considered, amongst other papers, in [15]. $A\in m$-isometric is said to be strictly m-isometric, denoted $A\in$ strict m-isometric if

$$\triangle^m_{A^*,A}(I)=0 \text{ and } \triangle^{m-1}_{A^*,A}(I)\neq 0.$$

It is well known, see [15, Lemma 4.1], that if $A, B \in \mathcal{B}(\mathcal{H})$ commute, $[A, B] = AB - BA = 0, A \in m_1$-isometric and $B \in m_2$-isometric, then $AB \in (m_1 + m_2 - 1)$-isometric. Indeed:

Proposition 2.4 *(**a.**) Given $[A, B] = 0$, any two of the following three implications implies the other.*

(i) $AB \in (m_1 + m_2 - 1)$-isometric.

(ii) $A \in m_1$-isometric.

(iii) $B \in m_2$-isometric.

*(**b.**) Again, , given $[A, B] = 0$,*

(i) $AB \in$ strict $(m_1 + m_2 - 1)$-isometric if and only if $\triangle^{m_2-1}_{B^*,B}(\triangle^{m_1-1}_{A^*,A}(I) \neq 0$;

(ii) if $A \in m_1$-isometric and $B \in m_2$-isometric, then $AB \in$ strict (m_1+m_2-1)-isometric implies $A \in$ strict m_1-isometric and $B \in$ strict m_2-isometric. Furthermore, $AB \in$ strict $(m_1 + m_2 - 1)$-isometric and $A \in$ strict m_1 isometric (similarly, $AB \in$ strict $(m_1 + m_2 - 1)$-isometric and $B \in$ strict m_2 isometric) implies $B \in$ strict m_2-isometric (resp., $A \in$ strict m_1-isometric).

Does a similar result hold for products of Aluthge transforms and P-adjoints? More precisely, if $[A, B] = 0$, then does $\tilde{A}$ is m_1-isometric and $\tilde{B}$ is m_2-isometric (resp., $\mathcal{A}$ is m_1-isometric and $\mathcal{B}$ is m_2-isometric) imply $\widetilde{AB}$ is $m_1 + m_2 - 1$-isometric (resp., $\mathcal{AB}$ is $m_1 + m_2 - 1$-isometric)? The problem here is that of ensuring a reasonable relationship between the polar forms $U|A|$, $V|B|$ and $W|AB|$ of A, B and AB, respectively. Assuming merely that $[A, B] = 0$ is not enough to conclude $|AB| = |A||B|$, or $[U, |B|] = [V, |A|] = 0$; additional hypotheses are required.

The following terminology will come in handy in the statement of our result. Given a non-negative operator $P \in \mathcal{B}(\mathcal{H})$, $P \geq 0$, P generates a new semi-inner product $\langle ., . \rangle_P$ on $\mathcal{H}$ deined by

$$\langle x, y \rangle_P = \langle Px, y \rangle \text{ for every } x, y \in \mathcal{H}.$$

The semi-norm induced by this semi-inner product is a norm whenever P is injective and an equivalent norm whenever P is invertible. Choose $P \geq 0$ to be such that

$$||x||_P^2 = \langle Px, x \rangle = ||P^{\frac{1}{2}}x||^2, \ x \in \mathcal{H},$$

defines an equivalent norm on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$ such that $T \in (1, P)$-isometric,

$$\sum_{j=0}^{1}(-1)^j \begin{pmatrix} 1 \\ j \end{pmatrix} T^{*j}PT^j = 0 \Longleftrightarrow ||Tx||_P^2 = ||x||_P^2,$$

i.e., P defines an equivalent norm such that T is isometric in this equivalent norm. Generalising this concept, we say in the following that "an operator $T \in \mathcal{B}(\mathcal{H})$ is m-isometric in an equivalent norm on $\mathcal{H}$ if there exists a positive operator $P \in \mathcal{B}(\mathcal{H})$ defining an equivalent norm such that $T \in (m, P)$-isometric". Translated to this terminology, Lemma 2.3 says that " $A \in m$-isometric implies $\tilde{A}$ is m-isometric in the equivalent norm $||.||_{|A|}$" .

Theorem 2.5 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be doubly commuting operators (thus: $[A, B] = [A, B^*] = 0$) with polar decompositions $A = UP$ and $B = VQ$, let $\widetilde{AB}$ denote the Aluthge transform of AB and let $[AB]^*$ denote the $|AB|$-adjoint of AB.*

(i) If $\tilde{A} \in m_1$-isometric and $\tilde{B} \in m_2$-isometric (resp., $\mathcal{A} \in m_1$-isometric and $\mathcal{B} \in m_2$-isometric), then $\widetilde{AB} \in (m_1+m_2-1)$-isometric (resp., $\mathcal{AB} \in (m_1+m_2-1)$-isometric).

(ii) If $A \in m_1$-isometric and $B \in m_2$-isometric, then $\mathcal{AB} \in (m_1+m_2-1)$-isometric, and $\widetilde{AB} \in (m_1+m_2-1)$-isometric in an equivalent norm.

Proof. The hypotheses $\tilde{A} \in (m_1, P)$-isometric and $\tilde{B} \in (m_2, Q)$-isometric (similarly, $\mathcal{A} \in m_1$-isometric and $\mathcal{B} \in m_2$-isometric) imply $\sigma_a(A)$ and $\sigma_a(B)$ are contained in $\partial\mathbb{D}$; in particular, A and B are left invertible, hence U, V are isometries and P, Q are positive invertible in the polar decomposition for A and B. This, combined with the doubly commuting property of A and B, implies:

$$[A,Q]=[B,P]=[P,Q]=0=[U,Q]=[V,P]=[U,V]=[U,V^*]$$

and

$$AB = UPVQ = UVPQ = VUPQ = BA,$$

so that

$$\begin{aligned}&AB = W|AB| = UV|A||B|, [\tilde{A},\tilde{B}] = [\tilde{A},\tilde{B}^*] = 0 = [\mathcal{A},\mathcal{B}] = [\mathcal{A},\mathcal{B}^*],\\ &\widetilde{AB} = \tilde{A}\tilde{B}, [AB] = \mathcal{AB}.\end{aligned}$$

where $\tilde{A}, \tilde{B}$ are the Aluthge transforms of A, B, respectively, $\mathcal{A}^*$ and $\mathcal{B}^*$ are the $|A|$-adjoint of A and the $|B|$-adjoint of B, respectively. Let $m = m_1+m_2-1$.
(i) Since $\widetilde{AB} = \tilde{A}\tilde{B} = \tilde{B}\tilde{A}$ and $[AB] = [A][B] = [B][A]$, the proof of (i) is a straightforward consequence of Proposition 2.4(**a**).

(ii) Since, for an operator $T \in \mathcal{B}(\mathcal{H})$, $T \in m$-isometric implies $\mathcal{T} \in m$-isometric (by Lemma 2.3), the hypothesis that $A \in m_1$-isometric and $B \in m_2$-isometric implies $\mathcal{A} \in m_1$-isometric and $\mathcal{B} \in m_2$-isometric, and hence ,by Proposition 2.4, $\mathcal{AB} \in (m_1+m_2-1)$-isometric. To complete the proof of (ii), let $m = m_1+m_2-1$. Keeping in view the commutativity properties of P, Q, V, U etc., a straightforward argument proves that

$$\begin{aligned} A \in m_1-\text{isometric} &\iff \sum_{j=0}^{m_1}(-1)^j\begin{pmatrix} m_1\\ j\end{pmatrix}A^{*j}A^j = 0\\ &\iff P^{\frac{1}{2}}\left(\sum_{j=0}^{m_1}(-1)^j\begin{pmatrix} m_1\\ j\end{pmatrix}\tilde{A}^{*j}P^{-1}\tilde{A}^j\right)P^{\frac{1}{2}} = 0\\ &\iff \tilde{A} \in (m, P^{-1})-\text{isometric,}\end{aligned}$$

similarly

$$B \in m_2 - \text{isometric} \iff \tilde{B} \in (m, Q^{-1} - \text{isometric}$$

and

$$\begin{aligned} &\triangle^m_{(\widetilde{AB})^*,\widetilde{AB}}(P^{-1}Q^{-1}) = (I - L_{\widetilde{AB}^*}R_{\widetilde{AB}})^m(P^{-1}Q^{-1})\\ = \;&(I - L_{\tilde{A}^*}L_{\tilde{B}^*}R_{\tilde{A}}R_{\tilde{B}})^m(P^{-1}Q^{-1})\\ = \;&\left(L_{\tilde{A}^*}R_{\tilde{A}}\triangle_{\tilde{B}^*,\tilde{B}} + \triangle_{\tilde{A}^*,\tilde{A}}\right)^m(P^{-1}Q^{-1})\\ = \;&\sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}\left(L_{\tilde{A}^*}R_{\tilde{A}}\right)^{m-j}\triangle^{m-j}_{\tilde{B}^*,\tilde{B}}\left(\triangle^j_{\tilde{A}^*,\tilde{A}}(P^{-1}Q^{-1})\right).\end{aligned}$$

Since $\triangle^j_{\tilde{A}^*,\tilde{A}}(P^{-1}Q^{-1}) = Q^{-1}\left(\triangle^j_{\tilde{A}^*,\tilde{A}}(P^{-1})\right)$, and since $\triangle^j_{\tilde{A}^*,\tilde{A}}(P^{-1}) = 0$ for all $j \geq m_1$,

$$\begin{aligned} &\triangle^m_{(\widetilde{AB})^*,\widetilde{AB}}(P^{-1}Q^{-1}) \\ = &\sum_{j=0}^{m_1-1}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \left(L_{\tilde{A}^*}R_{\tilde{A}}\right)^{m-j} \triangle^{m-j}_{\tilde{B}^*,\tilde{B}}\left(Q^{-1}\triangle^j_{\tilde{A}^*,\tilde{A}}(P^{-1})\right) \\ = &\sum_{j=0}^{m_1-1}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \left(L_{\tilde{A}^*}R_{\tilde{A}}\right)^{m-j} \triangle^{j}_{\tilde{A}^*,\tilde{A}}\left(P^{-1}\triangle^{m-j}_{\tilde{B}^*,\tilde{B}}(Q^{-1}\right). \end{aligned}$$

But then $m-j \geq m_1+m_2-1-(m_1-1) = m_2$, and hence, since $\triangle^t_{\tilde{B}^*,\tilde{B}}(Q^{-1}) = 0$ for all $t \geq m_2$,

$$\triangle^m_{(\widetilde{AB})^*,\widetilde{AB}}(P^{-1}Q^{-1}) = 0,$$

equivalently, $\widetilde{AB}$ is m-isometric in the equivalent norm $||.||_{P^{-1}Q^{-1}}$. $\square$

Remark 2.6 If A, B are the doubly commuting operators of Theorem 2.5 such that $\tilde{A}$ is m_1-isometric and $\tilde{B}$ is m_2-isometric, then Proposition 2.4 implies that $\widetilde{AB}$ is strict (m_1+m_2-1)-isometric if and only if

$$\triangle^{m_1-1}_{\tilde{A}^*,\tilde{A}}\left(\triangle^{m_2-1}_{\tilde{B}^*,\tilde{B}}(I)\right) = \triangle^{m_2-1}_{\tilde{B}^*,\tilde{B}}\left(\triangle^{m_1-1}_{\tilde{A}^*,\tilde{A}}(I)\right) \neq 0;$$

$\widetilde{AB} \in$ strict (m_1+m_2-1)-isometric implies $\tilde{A} \in$ strict m_1-isometric and $\tilde{B} \in$ strict (m_2-1)-isometric. A similar statement holds for operator $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{AB}$.

Let $\mathcal{H}\bar{\otimes}\mathcal{H}$denote the completion, endowed with a reasonable uniform cross norm, of the algebraic tensor product of $\mathcal{H}$ with itself. Given operators $S, T \in \mathcal{B}(\mathcal{H})$, let $S \otimes T$ denote the tensor product of S and T. Define $A, B \in B(\mathcal{H}\bar{\otimes}\mathcal{H})$ by

$$A = S \otimes I \text{ and } B = I \otimes T.$$

Then A, B doubly commute and $\sigma_a(AB) = \sigma_a(S \otimes T) = \sigma_a(S)\sigma_a(T) = \sigma_a(A)\sigma_a(B)$; if S and T have the polar decompositions $S = U_S|S|$ and $T = U_T|T|$, then AB has the polar decomposition $AB = (U_S \otimes U_T)(|S| \otimes |T|)$. It is straightforward to see that

$$\begin{aligned} A \in \text{strict } m_1 - \text{isometric} &\iff S \otimes I \in \text{strict } m_1 - \text{isometric} \\ &\iff S \in \text{strict } m_1 - \text{isometric} \end{aligned}$$

and

$$\begin{aligned} B \in \text{strict } m_2 - \text{isometric} &\iff I \otimes T \in \text{strict } m_2 - \text{isometric} \\ &\iff T \in \text{strict } m_2 - \text{isometric} \end{aligned}$$

If we let $S \otimes I = A$ and $I \otimes T = B$, then (as already observed) $A \in m_1$-isometric, $B \in m_2$-isometric and $AB \in$ strict (m_1+m_2-1)-isometric implies A (hence S) in strict m_1-isometric and B (hence T) in strict m_2-isometric . The reverse implications fail; thus (i) A strict m_1-isometric and B strict m_2-isometric does not imply AB strict (m_1+m_2-1)-isometric (even for commuting A and B), and (ii), given commuting A and B, AB strict m-isometric does not imply the existence of positive integers $m_1, m_2 \leq m$, $m = m_1+m_2-1$, such that A, or some multiple of A, is strict m_1-isometric and B, or some multiple thereof, is strict m_2-isometric.

Example 2.7 (i). Let $A_1, B_1 \in \mathcal{B}(\mathcal{H})$ be such that $A_1, B_1 \in$ strict m-isometric. Then the operators $A = A_1 \oplus I$, $B = I \oplus B_1$ and AB are strict m-isometric operators (in $B(\mathcal{H} \oplus \mathcal{H})$). Evidently, $m = m + m - 1$ if and only $m = 1$.
(ii). Let $A_1, B_1 \in \mathcal{B}(\mathcal{H})$ be two commuting (Hilbert space) isometries and let $B_2 \in \mathcal{B}(\mathcal{H})$ be the operator $B_2 = I + V$, where V is the Volterra integral operator (and I is the identity of $\mathcal{B}(\mathcal{H})$). Define A_2, A and B by $A_2 = B_2^{-1}$, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. Then $[A, B] = 0$ and the operator $AB = A_1B_1 \oplus I$ is 1-isometric. However, neither of the operators A and B, or a multiple thereof, is 1-isometric. Even double commutativity fails to be sufficient. Choose, for example, A_1, B_1 to be commuting unitaries and choose B_2 to be a normal invertible operator such that no multiple of B_2 is 1-isometric, i.e,, no multiple of B_2 is unitary. Let $A_2 = B_2^{-1}$. Then $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ doubly commute, AB is 1-isometric, and no multiple of A or B is 1-isometric.

For tensor products, "the converse problem (ii)" has a positive answer. The following theorem, indeed a more general version, is proved in [20] using techniques from algebraic geometry. We give here a more transparent proof which uses little more than some basic operator theory.

Theorem 2.8 *If $S \otimes T \in$ strict m-isometric, then there exist non-zero scalars c and d, $|cd| = 1$, and positive integers $m_i \leq m, i = 1, 2$, satisfying $m = m_1 + m_2 - 1$ such that $dS \in$ strict m_1-isometric and $cT \in$ strict m_2-isometric.*

Proof. The operator $S \otimes T$ being m-isometric, $\sigma_a(S \otimes T) = \sigma_a(S)\sigma_a(T)$ is a subset of the boundary of the unit disc (i.e., a subset of the unit circle) in the comples plane $\mathtt{C}$. There exist non-zero scalar $c \in \sigma_a(S)$ and $d \in \sigma_a(T)$ such that $|cd| = 1$. Let $\{e_n\}$ be a sequence of unit vectors in $\mathcal{H}$ such that $lim_{n\to\infty}||(S - cI)e_n|| = 0$; let $x \in \mathcal{H}$. The $S \otimes T$ is m-isometric implies

$$0 = \triangle^m_{S^*\otimes T^*, S\otimes T}(I \otimes I) = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} (S^* \otimes T^*)^{m-j}(S \otimes T)^{m-j}$$

$$\Longrightarrow \quad 0 = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} (S^* \otimes T^*)^{m-j}(S \otimes T)^{m-j}\langle e_n \otimes x, e_n \otimes x\rangle$$

$$\Longleftrightarrow \quad 0 = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \langle S^{*(m-j)}S^{m-j}e_n, e_n\rangle\langle T^{*(m-j)}T^{m-j}x, x\rangle.$$

Taking limits as $n \to \infty$,

$$\begin{aligned} 0 &= \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} ||T^{m-j}x||^2|| \lim_{n\to\infty} ||S^{m-j}e_n||^2 \\ &= \sum_{j=0}^{m} (-1)^j \begin{pmatrix} n \\ j \end{pmatrix} ||(cT)^{m-j}x||^2 \end{aligned}$$

for all $x \in \mathcal{H}$. Hence, upon letting $cT = T_1$,

$$0 = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T_1^{*(m-j)}T^{m-j} = \triangle^m_{T_1^*, T_1}(I),$$

i.e., T_1 is m-isometric. A similar argument shows that $S_1 = dS$ is m-isometric.

Let m_1 and m_2 be the smallest positive integers such that S_1 is m_1-isometric and T_1 is m_2-isometric. Necessarily $m_1, m_2 \leq m$ and $m \geq m_1 + m_2 - 1$. (Observe that if $m_1 + m_2 - 1 > 1$, S_1 is strict m_1-isometric and T_1 is strict m_2-isometric, then $S_1 \otimes T_1$ is strictly $(m_1 + m_2 - 1)$-isometric, hence $S_1 \otimes T_1$ is not (strictly) $m_1 + m_2 - 1 > m$ isometric.) We need to prove $m = m_1 + m_2 - 1$ and that

$$\triangle^{m_1-1}_{S_1^*\otimes I, S_1\otimes I}(I\otimes I) \neq 0 \neq \triangle^{m_2-1}_{I\otimes T_1^*, I\otimes T_1}(I\otimes I).$$

Suppose that $m_1 + m_2 - 1 < m$. Since

$$\begin{aligned} & \triangle_{S^*\otimes T^*, S\otimes T}(I\otimes I) = \triangle_{S_1^*\otimes T_1^*, S_1\otimes T_1}(I\otimes I) \\ = \ & \{L_{I\otimes T_1^*}R_{I\otimes T_1}\triangle_{S_1^*\otimes I, S_1\otimes I} + \triangle_{I\otimes T_1^*, I\otimes T_1}\}(I\otimes I), \end{aligned}$$

$S \otimes T$ is strict m-isometric implies

$$\begin{aligned} 0 \ \neq \ & \triangle^{m-1}_{S^*\otimes T^*, S\otimes T}(I\otimes I) \\ = \ & \sum_{j=0}^{m-1}(-1)^j \binom{m-1}{j} (L_{I\otimes T_1^*}R_{I\otimes T_1})^{m-1-j}\triangle^{m-1-j}_{S_1^*\otimes I, S_1\otimes I}\triangle^{j}_{I\otimes T_1^*, I\otimes T_1}(I\otimes I) \\ = \ & \sum_{j=0}^{m_2-1}(-1)^j \binom{m-1}{j} (L_{I\otimes T_1^*}R_{I\otimes T_1})^{m-1-j}\triangle^{j}_{I\otimes T_1^*, I\otimes T_1}\triangle^{m-1-j}_{S_1^*\otimes I, S_1\otimes I}(I\otimes I) \\ & (\text{since } \triangle^{j}_{I\otimes T_1^*, I\otimes T_1}(I\otimes I) = 0 \text{ for all } j \geq m_2) \\ = \ & 0 \end{aligned}$$

for the reason that $0 \leq j \leq m_2 - 1$ implies $m - j - 1 \geq m - m_2 > m_1 - 1$ and $\triangle^{t}_{S_1^*\otimes I, S_1\otimes I}(I\otimes I) = 0$ for all $t \geq m_1$. □

If either of the hypotheses $S \in m_1$-isometric and $T \in m_2$-isometric or $\widetilde{S\otimes T} \in m$-isometric is satisfied, then the polar(decompositions $A = S\otimes I = UP$ and $B = I \otimes T = VQ$ satisfy all the properties listed in the proof of Theorem 2.5, hence the polar) decomposition $S\otimes T = (U_S \otimes U_T)(|S| \otimes |T|)$ satisfies

$$\widetilde{S\otimes T} = (\tilde{S}\otimes I)(I\otimes\tilde{T}), [S\otimes T]^* = \mathcal{S}^*\otimes\mathcal{T}^*$$

(where $[S\otimes T]^*$ is the $|S\otimes T|$-adjoint of $S\otimes T$, $\mathcal{S}^*$ is the $|S|$-adjoint of S and $\mathcal{T}^*$ is the $|T|$-adjoint of T). We note here that if $[S\otimes T]$ is left invertible (in particular, if $[S\otimes T]$ is m-isometric), then S and T are left invertible. This follows from the fact that $S\otimes T = (U_S\otimes U_T)(|S|\otimes|T|)$ implies $[S\otimes T] = (|S|\otimes|T|)(U_S\otimes U_T)$, hence $\sigma_a(S\otimes T) = \sigma_a(S)\sigma_a(T)$.

Theorem 2.5 translates to:

Theorem 2.9 *(i) If $\widetilde{S\otimes T} \in$ strict m-isometric, then there exist non-zero scalars c and d, $|cd| = 1$, and positive integers $m_i \leq m, i = 1, 2$, such that $m = m_1 + m_2 - 1, c\tilde{S} \in$ strict m_1-isometric and $d\tilde{T} \in$ strict m_2-isometric.*

(ii) If $[S\otimes T] \in$ strict m-isometric, then there exist non-zero scalars c and d, $|cd| = 1$, and positive integers $m_i \leq m$, $i = 1, 2$, such that $m = m_1 + m_2 - 1$, $c[S] = c\mathcal{S}$ is strict m_1-isometric and $d[T] = d\mathcal{T}$ is strict m_2-isometric.

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Proof. Since $\widetilde{S\otimes T}$, respectively $[S\otimes T]$, is strict m-isometric if and only if $\widetilde{S\otimes T}=\tilde{S}\otimes\tilde{T}$, respectively $[S\otimes T]=[S]\otimes[T]$, is strict m-isometric, $S\otimes I$ and $I\otimes T$ satisfy the doubly commutative hypothesis and $\sigma_a(S\tilde{\otimes}T)=\sigma_a(S\otimes T)=\sigma_a(A)\sigma_a(B)=\sigma_a(S)\sigma_a(T)$, the argument of Theorem 2.8 applies. □

We note here that if $S\otimes T$ has the polar decomposition $S\otimes T=W(|S|\otimes|T|)$, then $[S\otimes T]=(|S|\otimes|T|)W$ by our standing hypothesis. Hence $[S\otimes T]$ is left invertible implies S and T are left invertible, $S\otimes T=(U_S\otimes U_T)(|S|\otimes|T|)$ and $[S\otimes T]=|S|U_S\otimes|T|U_T=(|S|\otimes|T|)(U_S\otimes U_T)$.

Extension to multiplication operator $\mathcal{E}_{A,B}=L_AR_B$. The extension of tensor products results of Theorems 2.8 and 2.9 to multiplication operators $\mathcal{E}_{A,B}$ on the bimodule $\mathcal{C}_2(\mathcal{H})$, the Hilbert-Schmidt class, is almost automatic. We observe that "the prime condition" [16]

$$\mathcal{E}_{A,B}=0\in\mathcal{B}(\mathcal{C}_2(\mathcal{H}))\Longrightarrow 0\in\{A,B\}\subseteq\mathcal{B}(\mathcal{H})\cup\mathcal{B}(\mathcal{H})$$

says that the operators $\mathcal{E}_{A,B}$ induced by $A,B\in\mathcal{B}(\mathcal{H})$ on $\mathcal{C}_2(\mathcal{H})$ are just the tensor products $A\otimes B^*$ and "the ultra prime condition"

$$\|\mathcal{E}_{A,B}\|=\|A\|_2\|B\|_2,\ \|\cdot\|_2 \text{ the Hilbert}-\text{Schmidt norm},$$

ensures that the operator norm of $\mathcal{B}(\mathcal{C}_2(\mathcal{H}))$ induces a uniform cross norm on the tensor product $\mathcal{H}\bar{\otimes}\mathcal{H}$. We have:

Theorem 2.10 *Let $A,B\in\mathcal{B}(\mathcal{H})$. If $\mathcal{E}_{A,B}\in\mathcal{B}(\mathcal{C}_2(\mathcal{H}))$ is strictly m-isometric, then there exist scalars c,d, $|cd|=1$, and positive integers $m_i\leq m, i=1,2$, such that $m=m_1+m_2-1, cA\in$ strict m_1-isometric and $dB^*\in$ strict m_2-isometric.*

Theorem 2.10 generalises [10, Theorem 1.1] (see also [19, Theorem 7]).

Theorem 2.11 *Let $A,B\in\mathcal{B}(\mathcal{H})$, and let (as before) $\tilde{A}$ and $\mathcal{A}^*$ (etc.) denote the Aluthge transform and the $|A|$-adjoint of A, respectively. Let A,B have the polar decompositions $A=UP, B=VQ$.*
(i) If $A\in(m_1,P)$-isometric and $B\in(m_2,Q)$-isometric, then $\mathcal{E}_{\tilde{A},\tilde{B}^*}\in(m_1+m_2-1)$-isometric.
(ii) If $A\in m_1$-isometric and $B\in m_2$-isometric, then $\mathcal{E}_{\mathcal{A},\mathcal{B}^*}\in(m_1+m_2-1)$-isometric.
(iii) If $\mathcal{E}_{\tilde{A},\tilde{B}^*}\in$ strict m-isometric, then there exist scalars c,d, $|cd|=1$, and positive integers $m_i\leq m, i=1,2$, such that $m=m_1+m_2-1, cA\in$ strict (m_1,P)-isometric. and $dB^*\in$ strict (m_2,Q)-isometric.
(iv) If $\mathcal{E}_{\mathcal{A},\mathcal{B}^*}\in$ strict m-isometric, then there exist scalars c,d, $|cd|=1$, and positive integers $m_i\leq m, i=1,2$, such that $m=m_1+m_2-1, c\tilde{A}\in$ strict (m_1,P)-isometric. and $d\tilde{B}^*\in$ strict (m_2,Q)-isometric.

Proof. The proof of the theorem is immediate from Theorems 2.8 and 2.9. The only details needing attention here are the identifications $\widetilde{A\otimes B}$ and $\mathcal{E}_{\tilde{A},\tilde{B}^*}$, and $[A\otimes B]$ and $\mathcal{E}_{\mathcal{A},\mathcal{B}^*}$. If $A=UP$ and $B=VQ$, then $A\otimes B=(U\otimes V)(P\otimes Q)$, hence

$$\mathcal{E}_{A,B^*}=L_AR_{B^*}=L_UL_PR_{V^*}R_Q=L_UR_{V^*}L_PR_Q$$

and

$$\begin{aligned}\widetilde{\mathcal{E}}_{A,B^*} &= (L_P R_Q)^{\frac{1}{2}} L_U R_{V^*} (L_P R_Q)^{\frac{1}{2}} \\ &= L_P^{\frac{1}{2}} R_Q^{\frac{1}{2}} L_U R_{V^*} L_P^{\frac{1}{2}} R_Q^{\frac{1}{2}} \\ &= L_P^{\frac{1}{2}} L_U L_P^{\frac{1}{2}} R_Q^{\frac{1}{2}} R_{V^*} R_Q^{\frac{1}{2}} \\ &= L_{\tilde{A}} R_{\tilde{B}^*} \\ &= \mathcal{E}_{\tilde{A},\tilde{B}^*}.\end{aligned}$$

Again, since the adjoints $\mathcal{A}^*$ and $\mathcal{B}^*$ have the forms $\mathcal{A}^* = U^*P$ and $\mathcal{B}^* = V^*Q$,

$$\mathcal{E}_{\mathcal{A},\mathcal{B}^*} = L_{PU} R_{V^*Q} = L_P R_Q L_U R_{V^*}.$$

The hypothesis $\mathcal{E}_{\mathcal{A},\mathcal{B}^*} \in m$-isometric implies $\mathcal{E}_{\mathcal{A},\mathcal{B}^*}$ is left invertible; hence $L_P R_Q$, therefore $P \otimes Q$, is invertible. By definition, $[A \otimes B]|A \otimes B| = |A \otimes B|(A \otimes B)$, i.e., $[A \otimes B](P \otimes Q) = (P \otimes Q)(U \otimes V)(P \otimes Q)$, equivalently, $[A \otimes B] = (P \otimes Q)(U \otimes V)$. Conclusion $[\mathcal{E}_{A,B}]^* = \mathcal{E}_{\mathcal{A},\mathcal{B}^*}$. $\square$

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B.P. Duggal,University of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia.
e-mail: bpduggal@yahoo.co.uk

I. H. Kim, Department of Mathematics, Incheon National University, Incheon, 22012, Korea.
e-mail: ihkim@inu.ac.kr