The three-dimensional divisor problems related to cusp form coefficients

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Abstract

Let f be a normalized primitive holomorphic cusp form of even integral weight for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Let $\lambda_{f\times f}(n)$ and $\lambda_{f\times f\times f}(n)$ be the normalized coefficients of the Dirichlet expansion of the Rankin-Selberg L-function and triple product L-function attached to f, respectively. In this paper, we establish the asymptotic formulae and the upper bounds for the three dimensional divisor problems related to these normalized coefficients, respectively.

Key words Fourier coefficients, Automorphic L-functions, Divisor problem

Mathematics Subject Classification 11F11, 11F30

1. Introduction

The Fourier coefficients of modular forms are important and interesting objects in number theory. Let k be an even integer and let H_k^* denote the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Then the $f(z) \in H_k^*$ has Fourier expansion at the cusp ∞ given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \quad \Im(z) > 0,$$

where $\lambda_f(n)$ are the normalized Fourier coefficients (Hecke eigenvalues) such that $\lambda_f(1) = 1$. Then $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where $m \geq 1$ and $n \geq 1$ are positive integer. In 1974, P. Deligne [3] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{1.1}$$

where $d(n)$ is the classical divisor function. By (1.1), Deligne's bound is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{1.2}$$

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21 Jun 2022 20:50:27 PDT
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More generally, for all positive integers $l \geq 1$ one has

$$\lambda_f(p^l) = \alpha_f(p)^l + \alpha_f(p)^{l-1}\beta_f(p) + \cdots + \alpha_f(p)\beta_f(p)^{l-1} + \beta_f(p)^l.$$

It is natural to define the Hecke L-function $L(f,s)$ associated to f by

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1-\lambda_f(p)p^{-s}+p^{-2s})^{-1} \\ &= \prod_p \left(1-\frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_f(p)}{p^s}\right)^{-1},\ \Re(s)>1, \end{aligned}$$

where $\alpha_f(p), \beta_f(p)$ are the local parameters satisfying (1.2). In a similar manner, we can also define the Rankin-Selberg L-function

$$L(f\times f,s) = \prod_p \big(1-p^{-s}\big)^{-2}\big(1-\alpha_f(p)^2p^{-s}\big)^{-1}\big(1-\beta_f(p)^2p^{-s}\big)^{-1} \tag{1.3}$$

for $\Re(s)>1$. We can also expand the Euler product (1.3) as a Dirichlet series

$$L(f\times f,s) = \zeta(2s)\sum_{n=1}^{\infty}\frac{\lambda_f^2(n)}{n^s} := \sum_{n=1}^{\infty}\frac{\lambda_{f\times f}(n)}{n^s}, \quad \Re(s)>1.$$

Let $\omega \geq 1$ is an integer, and define

$$\begin{aligned} \lambda_{\omega,f}(n) &:= \sum_{n=n_1n_2\cdots n_\omega} \lambda_f(n_1)\lambda_f(n_2)\cdots\lambda_f(n_\omega), \\ \lambda_{\omega,f\times f}(n) &:= \sum_{n=n_1n_2\cdots n_\omega} \lambda_{f\times f}(n_1)\lambda_{f\times f}(n_2)\cdots\lambda_{f\times f}(n_\omega). \end{aligned}$$

In the case $n=1$, we have $\lambda_{1,f}(n) = \lambda_f(n)$ and $\lambda_{1,f\times f}(n) = \lambda_{f\times f}(n)$. In 1927, Hecke [6] proved that

$$\sum_{n\leq x}\lambda_f(n) \ll x^{\frac{1}{2}}. \tag{1.4}$$

Later, the upper bound in (1.4) has been improved by several authors (See e.g. [3, 8, 23]). And the best record to date is given by Wu [26]:

$$\sum_{n\leq x}\lambda_f(n) \ll x^{\frac{1}{3}}\log^\rho x,$$

where

$$\rho = \frac{102+7\sqrt{21}}{210}\left(\frac{6-\sqrt{21}}{5}\right)^{\frac{1}{2}} + \frac{102-7\sqrt{21}}{210}\left(\frac{+\sqrt{21}}{5}\right)^{\frac{1}{2}} - \frac{33}{35} = -0.118\cdots.$$

Rankin [22] and Selberg [24] proved the following asymptotic formula by the powerful Rankin-Selberg method such that

$$\sum_{n\leq x}\lambda_{f\times f}(n) = c_f x + O(x^{3/5}), \tag{1.5}$$

where $c_f > 0$ is a positive constant depending on f and $\varepsilon > 0$ be arbitarily small positive number. Very recently, the exponent in (1.5) has been improved to $\frac{3}{5}-\delta$ in place of $\frac{3}{5}$ by Huang [9] , where $\delta \leq 1/560$. This remain the best possible result in this direction. Kanemitsu, Sankaranarayanan and Tanigawa [13] consider a general divisor problem and established

$$\sum_{n\leq x}\lambda_{\omega,f}(n) \ll x^{1-\frac{3}{2k+2}+\varepsilon}, \tag{1.6}$$

and

$$\sum_{n\leq x}\lambda_{\omega,f\times f}(n) = M_\omega(x) + O\big(x^{1-\frac{1}{2k}+\varepsilon}\big), \tag{1.7}$$

for any $\varepsilon > 0$, where $\omega \geq 2$ is an integer, and $M_\omega(x)$ has the form $xP_{\omega-1}(\log x)$ which is derived from the residue of the integral, and $P_{\omega-1}(t)$ denote a polynomial of t with degree $\omega - 1$. Later, the results (1.6) and (1.7) has been improved and generalized in various aspects by a number of authors (see e.g. [17–20]).

Let $1 < a < b$ be any fixed positive integers and define

$$\lambda_f^{a,b}(n) := \sum_{n=n_1^a n_2^b} \lambda_f(n_1)\lambda_f(n_2)$$

and

$$\lambda_{f\times f}^{a,b}(n) := \sum_{n=n_1^a n_2^b} \lambda_{f\times f}(n_1)\lambda_{f\times f}(n_2).$$

Very recently, Huang et al. [10] considered the average behaviour of two-dimensional divisor problems related to the Fourier coefficients $\lambda_f(n), \lambda_{f\times f}(n)$ in the following types

$$\begin{aligned} S_f(a,b;x) &:= \sum_{n\leq x} \lambda_f^{a,b}(n), \\ S_{f\times f}(a,b;x) &:= \sum_{n\leq x} \lambda_{f\times f}^{a,b}(n). \end{aligned}$$

More precisely, they proved

$$S_f(a,b;x) = \begin{cases} x^{1-\frac{3(2a-1)}{2(5a-b)}+\varepsilon}, & \text{if } b \leq 2a, \\ x^{\frac{1}{2a}+\varepsilon}, & \text{if } b > 2a \end{cases}$$

for any $\varepsilon > 0$. Furthermore, they also established the following results:

$$S_{f\times f}(a,b;x) = \begin{cases} c_{f,a,b}x^{\frac{1}{a}} + O\big(x^{1-\frac{84(2a-1)}{486a-131b}+\varepsilon}\big), & \text{if } 318a^2 - 131ab - 402a + 131b < 0, b \leq 2a, \\ O\big(x^{1-\frac{84(2a-1)}{486a-131b}+\varepsilon}\big), & \text{if } 318a^2 - 131ab - 402a + 131b > 0, b \leq 2a, \\ \tilde{c}_{f,a,b}x^{\frac{1}{2}} + O\big(x^{\frac{7}{16}+\varepsilon}\big), & \text{if } a = 2, b > 2a, \\ O\big(x^{1-\frac{3(2a-1)}{8a}+\varepsilon}\big), & \text{if } a \geq 3, b > 2a \end{cases}$$

for any $\varepsilon > 0$, where $c_{f,a,b}, \tilde{c}_{f,a,b}$ are given by

$$c_{f,a,b} = L(\mathrm{sym}^2 f, 1)L\left(f \times f, \frac{b}{a}\right), \qquad \tilde{c}_{f,a,b} = L(\mathrm{sym}^2 f, 1)L\left(f \times f, \frac{b}{2}\right).$$

Later, Huang et al. [11] generalized the above to three dimensional divisor problems related to Hecke eigenvalues. Let $1 < a < b < c$ be any fixed integers. Define

$$\lambda_f^{a,b,c}(n) := \sum_{n=n_1^a n_2^b n_3^c} \lambda_f(n_1)\lambda_f(n_2)\lambda_f(n_3)$$

and

$$\lambda_{f\times f}^{a,b,c}(n) := \sum_{n=n_1^a n_2^b n_3^c} \lambda_{f\times f}(n_1)\lambda_{f\times f}(n_2)\lambda_{f\times f}(n_3).$$

They proved by using the analytic properties of the associated L-functions that

$$S_f(a,b,c;x) := \sum_{n\leq x} \lambda_f^{a,b,c}(n) = \begin{cases} x^{\frac{1}{a}-\frac{3}{2(7a-b-c)}+\varepsilon}, & \text{if } c \leq 2a, \\ x^{\frac{1}{a}-\frac{3}{2(5a-b)}+\varepsilon}, & \text{if } b < 2a < c, \\ x^{\frac{1}{2a}+\varepsilon}, & \text{if } 2a \leq b \end{cases}$$

for any $\varepsilon > 0$. Define

$$S_{f\times f}(a,b,c;x) := \sum_{n\leq x} \lambda_f^{a,b,c}(n)$$

Furthermore, they also proved that

$$S_{f\times f}(a,b,c;x)=\begin{cases} M_1x^{\frac{1}{a}}+M_2x^{\frac{1}{b}}+M_3x^{\frac{1}{c}}++O\big(x^{\frac{1}{a}-\frac{84}{748a-131(b+c)}+\varepsilon}\big), & \text{if } c\leq 2a,\\ M_1x^{\frac{1}{a}}+M_2x^{\frac{1}{b}}+O\big(x^{\frac{1}{a}-\frac{84}{486a-131b}+\varepsilon}\big), & \text{if } b<2a<c,\\ M_1x^{\frac{1}{a}}+O\big(x^{\frac{5}{8a}+\varepsilon}\big), & \text{if } 2a\leq b\end{cases}$$

for any $\varepsilon>0$, where

$$M_1=L\Big(f\times f,\frac{b}{a}\Big)L\Big(f\times f,\frac{c}{a}\Big)L\big(\mathrm{sym}^2f,1\big),$$
$$M_2=L\Big(f\times f,\frac{a}{b}\Big)L\Big(f\times f,\frac{c}{b}\Big)L\big(\mathrm{sym}^2f,1\big),$$
$$M_3=L\Big(f\times f,\frac{a}{c}\Big)L\Big(f\times f,\frac{b}{c}\Big)L\big(\mathrm{sym}^2f,1\big).$$

Let $1<a<b<c$ be any fixed integers, we define

$$\lambda^{a,b,c}_{\omega_1,\omega_2,\omega_3,f\times f}(n)=\sum_{n=n_1^an_2^bn_3^c}\lambda_{\omega_1,f\times f}(n_1)\lambda_{\omega_2,f\times f}(n_2)\lambda_{\omega_3,f\times f}(n_3),$$

where $\omega_1,\omega_2,\omega_3\geq 1$ are any fixed positive integers. The first aim in this paper is to consider the divisor problem related to $\lambda^{a,b,c}_{\omega_1,\omega_2,\omega_3,f\times f}(n)$ in three dimensions. We are interested in studying the average behaviour of the following sum

$$S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x):=\sum_{n\leq x}\lambda^{a,b,c}_{\omega_1,\omega_2,\omega_3,f\times f}(n).$$

More precisely, we will be able to establish the following results.

Theorem 1.1. *Let $f\in H_k^*$ be a Hecke eigenform. And let $1<a<b<c$ be any fixed integers. Then we have*

$$\begin{aligned} &S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)\\ =&\begin{cases} x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)+x^{\frac{1}{c}}P_{\omega_3-1,3}(\log x) & \\ +O\big(x^{\frac{1}{a}-\frac{420}{697(\omega_1a+(2a-b)\omega_2+(2a-c)\omega_3)+423a}+\varepsilon}\big), & \textit{if } c<2a,\\ x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)+O\big(x^{\frac{1}{a}-\frac{420}{697(\omega_1a+(2a-b)\omega_2)+423a}+\varepsilon}\big), & \textit{if } b<2a\leq c,\\ x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+O\big(x^{\frac{1}{a}-\frac{420}{697\omega_1a+423a}+\varepsilon}\big), & \textit{if } 2a\leq b,\end{cases}\end{aligned}$$

where $P_{\omega_i,j}(t)(1\leq i,j\leq 3)$ denote the polynomials of t with degree ω_i.

Let $f\in H_k^*$ be a Hecke eigenform and denote by $\lambda_f(n)$ the nth normalized Fourier coefficients of f. The triple product L-function $L(f\times f\times f,s)$ is defined by

$$\begin{aligned} &L(f\times f\times f,s)\\ =&\prod_p\Big(1-\frac{\alpha_f(p)^3}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_f(p)}{p^s}\Big)^{-3}\Big(1-\frac{\alpha_f(p)^{-1}}{p^s}\Big)^{-3}\Big(1-\frac{\alpha_f(p)^{-3}}{p^s}\Big)^{-1}\\ :=&\sum_{n=1}^{\infty}\frac{\lambda_{f\times f\times f}(n)}{n^s},\quad \Re(s)>1.\end{aligned}$$

In 2017, Lü and Sankaranarayanan [18] investigated the average behaviour of $\lambda_{f\times f\times f}(n)$ in various aspects. Later, Liu [19] considered the average behaviour of $\lambda_{\omega,f\times f\times f}(n)$ and established the upper bound

$$\sum_{n\leq x}\lambda_{\omega,f\times f\times f}(n)\ll_{f,\omega}x^{1-\frac{3}{10\omega}+\varepsilon}$$

for any $\varepsilon > 0$, where $\omega \geq 1$ is any fixed integer. Let $1 < a < b < c$ be any fixed positive integers. Analogously, we define

$$\lambda^{a,b,c}_{\omega_1,\omega_2,\omega_3,f\times f\times f}(n) := \sum_{n=n_1^a n_2^b n_3^c} \lambda_{\omega_1,f\times f\times f}(n_1)\lambda_{\omega_2,f\times f\times f}(n_2)\lambda_{\omega_3,f\times f\times f}(n_3),$$

where $\omega_1, \omega_2, \omega_3 \geq 1$ are any fixed positive integers. And we also define the summatory function

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) := \sum_{n\leq x} \lambda^{a,b,c}_{\omega_1,\omega_2,\omega_3,f\times f\times f}(n).$$

We will prove the following theorem.

Theorem 1.2. *Let $f \in H_k^*$ be a Hecke eigenform. Set $1 < a < b < c$ be any fixed positive integers. Then we have*

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) \ll \begin{cases} x^{\frac{1}{a}-\frac{3}{(10\omega_1+1)a+10((2a-b)\omega_2+(2a-c)\omega_3)}+\varepsilon}, & \text{if } c < 2a, \\ x^{\frac{1}{a}-\frac{3}{(10\omega_1+1)a+10(2a-b)\omega_2}+\varepsilon}, & \text{if } b < 2a \leq c, \\ x^{\frac{1}{a}-\frac{3}{(10\omega_1+1)a}+\varepsilon}, & \text{if } 2a \leq b, \end{cases}$$

Throughout the paper, we always work on the vector space H_k^*. Denote by $\varepsilon > 0$ an arbitrarily small positive constant that may take different values in different occurrence.

2. Preliminaries

In this section, we introduce some lemmas which play an important role in the proof of the main results in this paper.

Let $f \in H_k^*$ be a normalized cuspidal Hecke eigenform. For $j \geq 1$, the j-th symmetric power L-function $L(\mathrm{sym}^j f, s)$ attached to $\mathrm{sym}^j f$ is defined by

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1}, \quad \Re(s) > 1. \tag{2.1}$$

We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right), \quad \Re(s) > 1. \end{aligned} \tag{2.2}$$

Apparently $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. From (1.2), (2.1), (2.2) and the theory of Hecke operators, we have

$$\lambda_f(p^j) = \sum_{m=0}^{j} \alpha_f(p)^{j-2m} = \lambda_{\mathrm{sym}^j f}(p), \quad j \geq 1.$$

Lemma 2.1. *For $\Re(s) > 1$, we have*

$$L(f\times f, s) = \zeta(s) L(sym^2 f, s) \tag{2.3}$$

and

$$L(f\times f\times f, s) = L(f, s)^2 L(sym^3 f, s). \tag{2.4}$$

21 Jun 2022 20:50:27 PDT
210913-GuodongHua Version 3 - Submitted to Rocky Mountain J. Math.

Proof Comparing the Euler product of both sides and invoking Deligne's bound (1.2), we can easily get the results. □

Lemma 2.2. *For any fixed $T \geq 1$ and $\varepsilon > 0$, we have*

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll T^{2+\varepsilon} \tag{2.5}$$

uniformly for $T \geq 1$, and

$$\zeta(\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon} \tag{2.6}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ and $|t| \geq 1$.

Proof The first result is due to Heath-Brown [7], and the second result is the new breakthrough of Bourgain [2]. □

Lemma 2.3. *For any $\varepsilon > 0$, we have*

$$\int_1^T \left|L\left(f,\frac{1}{2}+it\right)\right|^{2} dt \ll T^{1+\varepsilon} \tag{2.7}$$

uniformly for $T \geq 1$, and

$$L(f,\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{2}{3}(1-\sigma),0\}+\epsilon} \tag{2.8}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ and $|t| \geq 1$.

Proof These results follows from Good [5]. □

Lemma 2.4. *For $\frac{1}{2} \leq \sigma \leq 2$ and $|t| \geq 1$, we have*

$$L(sym^2 f,\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{27}{20}(1-\sigma),0\}+\epsilon}.$$

Proof The result follows from the Phragmén-Lindelöf convexity principle and the work of Aggarwal [1], respectively. □

As is well-known, to a primitive form $f(z)$ is associated an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$, and hence an automorphic L-function $L(\pi_f,s)$ which coincides with $L(f,s)$. It is predicted that π_f gives rise to a symmetric power lift–an automorphic representation whose L-function is the symmetric power L-function attached to f.

For $1 \leq j \leq 4$, the special Langlands functoriality conjecture which states that $\mathrm{sym}^j f$ is automorphic cuspidal has been showed by a series of important works of Gelbart and Jacquet [4], Kim [14], Kim and Shahidi [15, 16], Shahidi [25]. Then we know from Rankin-Selberg theory $L(\mathrm{sym}^j f,s), 1 \leq j \leq 4$ has analytic continuation to an entire function on the complex plane and satisfies a certain Riemann-type functional equation of degree $j+1$.

From above we observe that $L(\mathrm{sym}^j f,s), 1 \leq j \leq 4$ are general L-functions in the sense of Perelli [21]. For these general L-functions, we have the following averaged or individual convexity bounds.

Lemma 2.5. *Assume that $\mathcal{L}(s)$ is a general L-function of degree m. Then*

$$\int_T^{2T} \big|\mathcal{L}(\sigma+it)\big|^2 dt \ll T^{m(1-\sigma)+\varepsilon}, \tag{2.9}$$

uniformly for $\frac{1}{2} \le \sigma \le 1$ *and* $T \ge 1$, *and*

$$\mathcal{L}(\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{m}{2}(1-\sigma),0\}+\varepsilon} \tag{2.10}$$

uniformly for $\frac{1}{2} \le \sigma \le 1+\varepsilon$ *and* $|t| \ge 1$.

Proof This follows the results of Perelli's mean value theorem and convexity bounds for general L-functions in [21]. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by following the approach of [11] with some modifications.

Let $s=\sigma+it$ and $\eta=\frac{1}{a}+\varepsilon$, we have

$$L(f\times f,as)^{\omega_1}L(f\times f,bs)^{\omega_2}L(f\times f,cs)^{\omega_3}=\sum_{n=1}^{\infty}\frac{\lambda^{a,b,c}_{\omega_1\omega_2,\omega_3,f\times f}(n)}{n^s}, \quad \Re(s)>1. \tag{3.1}$$

Applying Perron's formula [12, Proposition 5.54] to the generating function (3.1) and using Deligne's bound (1.1), we can obtain

$$\begin{aligned} &S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)\\ &=\frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT}L(f\times f,as)^{\omega_1}L(f\times f,bs)^{\omega_2}L(f\times f,cs)^{\omega_3}\frac{x^s}{s}ds+O\bigg(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\bigg), \end{aligned} \tag{3.2}$$

where $1\le T\le x$ is a parameter to be chosen later. The points $s=\frac{1}{a}, s=\frac{1}{b}$ and $s=\frac{1}{c}$ are the only three possible simple poles of the integrand in the rectangle $\mathcal{R}_T:=\{s=\sigma+it:\frac{1}{2a}\le \sigma\le\eta, |t|\le T\}$.

We first consider the case $c<2a$. In this case, the points $s=\frac{1}{a}, s=\frac{1}{b}$ and $s=\frac{1}{c}$ are the simple poles of the integrand in the rectangle $\mathcal{R}_T$. By shifting the line of integral of (3.2) to the line $\Re(s)=\frac{1}{2a}+\varepsilon$ and invoking Cauchy's residue theorem, we can obtain

$$\begin{aligned} &S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)\\ =&\bigg(\mathrm{Res}_{s=\frac{1}{a}}+\mathrm{Res}_{s=\frac{1}{b}}+\mathrm{Res}_{s=\frac{1}{c}}\bigg)\bigg\{L(f\times f,as)^{\omega_1}L(f\times f,bs)^{\omega_2}L(f\times f,cs)^{\omega_3}\frac{x^s}{s}\bigg\}\\ &+I_1+I_2+I_3+O\bigg(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\bigg),\\ :=&\ x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)+x^{\frac{1}{c}}P_{\omega_3-1,3}(\log x)\\ &+I_1+I_2+I_3+O\bigg(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\bigg), \end{aligned} \tag{3.3}$$

where $P_{\omega_i,j}(t)(1\le i,j\le 3)$ is a polynomial of t with degree ω_i, and

$$\begin{aligned} I_1 &:= \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}L(f\times f,as)^{\omega_1}L(f\times f,bs)^{\omega_2}L(f\times f,cs)^{\omega_3}\frac{x^s}{s}ds,\\ I_2 &:= \frac{1}{2\pi i}\int_{\kappa+iT}^{1+\varepsilon+iT}L(f\times f,as)^{\omega_1}L(f\times f,bs)^{\omega_2}L(f\times f,cs)^{\omega_3}\frac{x^s}{s}ds,\\ I_3 &:= \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{\kappa-iT}L(f\times f,as)^{\omega_1}L(f\times f,bs)^{\omega_2}L(f\times f,cs)^{\omega_3}\frac{x^s}{s}ds, \end{aligned}$$

here $\kappa:=\frac{1}{2a}+\varepsilon$.

21 Jun 2022 20:50:27 PDT
210913-GuodongHua Version 3 - Submitted to Rocky Mountain J. Math.

Now we need to handle the three terms I_1, I_2, I_3, and the estimates of I_2, I_3 can be treated in a similar manner. We consider these estimates in three cases $c < 2a$, $b < 2a \leq c$ and $2a \leq b$.

To estimate the integrals on the horizontal segments I_2, I_3, we split the integral on the horizontal segments into four parts $I_{2,1}, I_{2,2}, I_{2,3}, I_{2,4}$ and apply Lemma 2.3. Let $\eta' := 1 + \varepsilon$.

Interval 1. We consider the interval

$$\begin{aligned}\mathfrak{l}_1 &:= \left\{s = \sigma + it : \frac{1}{2} + \varepsilon \leq a\sigma \leq \eta',\ \frac{b}{2a} + \varepsilon \leq b\sigma \leq \eta', \frac{c}{2a} + \varepsilon \leq c\sigma \leq \eta'\right\} \\ &= \left\{s = \sigma + it : \frac{1}{2a} + \varepsilon \leq \sigma \leq \frac{1}{c} + \varepsilon\right\}.\end{aligned}$$

By (2.3), (2.6) and Lemma 2.4, we have

$$\begin{aligned}I_{2,1} &:= \frac{1}{2\pi i}\int_{\mathfrak{l}_1} \zeta(as)^{\omega_1}\zeta(bs)^{\omega_2}\zeta(cs)^{\omega_3}L(\mathrm{sym}^2 f, as)^{\omega_1}L(\mathrm{sym}^2 f, bs)^{\omega_2}L(\mathrm{sym}^2 f, cs)^{\omega_3}\frac{x^s}{s}ds \\ &\ll \max_{\frac{1}{2a}\leq\sigma\leq\frac{1}{c}} x^\sigma T^{(\frac{13}{42}+\frac{27}{20})(1-a\sigma)\omega_1+(\frac{13}{42}+\frac{27}{20})(1-b\sigma)\omega_2+(\frac{13}{42}+\frac{27}{20})(1-c\sigma)\omega_3+\varepsilon}T^{-1} \\ &\ll x^{\frac{1}{2a}}T^{\frac{697}{420}(\frac{1}{2}\omega_1+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)-1+\varepsilon} + x^{\frac{1}{c}}T^{\frac{697}{420}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1+\varepsilon}. \end{aligned} \tag{3.4}$$

Interval 2. We consider the interval

$$\begin{aligned}\mathfrak{l}_2 &:= \left\{s = \sigma + it : \frac{1}{2} + \varepsilon \leq a\sigma \leq \eta',\ \frac{b}{2a} + \varepsilon \leq b\sigma \leq \eta',\ \eta' \leq c\sigma \leq c\eta',\right\} \\ &= \left\{s = \sigma + it : \frac{1}{c} + \varepsilon \leq \sigma \leq \frac{1}{b} + \varepsilon\right\}.\end{aligned}$$

In this interval, we have

$$\begin{aligned}I_{2,2} &:= \frac{1}{2\pi i}\int_{\mathfrak{l}_2} \zeta(as)^{\omega_1}\zeta(bs)^{\omega_2}\zeta(cs)^{\omega_3}L(\mathrm{sym}^2 f, as)^{\omega_1}L(\mathrm{sym}^2 f, bs)^{\omega_2}L(\mathrm{sym}^2 f, cs)^{\omega_3}\frac{x^s}{s}ds \\ &\ll \max_{\frac{1}{c}\leq\sigma\leq\frac{1}{b}} x^\sigma T^{\frac{697}{420}((1-a\sigma)\omega_1+(1-b\sigma)\omega_2)+\varepsilon}T^{-1} \\ &\ll x^{\frac{1}{c}}T^{\frac{697}{420}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1+\varepsilon} + x^{\frac{1}{b}}T^{\frac{697}{420}(1-\frac{a}{b})\omega_1-1+\varepsilon}. \end{aligned} \tag{3.5}$$

Interval 3. We consider the interval

$$\mathfrak{l}_3 := \left\{s = \sigma + it : \frac{1}{2} \leq a\sigma \leq \eta',\ \eta' \leq b\sigma \leq b\eta',\ \frac{c}{2a} \leq c\sigma \leq \eta'\right\} = \left\{s = \sigma + it : \frac{1}{b} \leq \sigma \leq \frac{1}{c}\right\}.$$

The interval is an empty set by noting that $\frac{1}{c} < \frac{1}{b}$.

Interval 4. We consider the interval

$$\begin{aligned}\mathfrak{l}_4 &:= \left\{s = \sigma + it : \frac{1}{2} + \varepsilon \leq a\sigma \leq \eta',\ \eta' \leq b\sigma \leq b\eta',\ \eta' \leq c\sigma \leq c\eta'\right\}, \\ &= \left\{s = \sigma + it : \frac{1}{b} + \varepsilon \leq \sigma \leq \frac{1}{a} + \varepsilon\right\}.\end{aligned}$$

In this interval, we have

$$\begin{aligned}I_{2,4} &:= \frac{1}{2\pi i}\int_{\mathfrak{l}_4} \zeta(as)^{\omega_1}\zeta(bs)^{\omega_2}\zeta(cs)^{\omega_3}L(\mathrm{sym}^2 f, as)^{\omega_1}L(\mathrm{sym}^2 f, bs)^{\omega_2}L(\mathrm{sym}^2 f, cs)^{\omega_3}\frac{x^s}{s}ds \\ &\ll \max_{\frac{1}{b}\leq\sigma\leq\frac{1}{a}} x^\sigma T^{\frac{697}{420}(1-a\sigma)\omega_1+\varepsilon}T^{-1} \\ &\ll x^{\frac{1}{a}+\varepsilon}T^{-1+\varepsilon} + x^{\frac{1}{b}+\varepsilon}T^{\frac{697}{420}(1-\frac{a}{b})\omega_1-1+\varepsilon}. \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5), (3.6), we have

$$I_2 + I_3 \ll I_{2,1} + I_{2,2} + I_{2,4}$$

21 Jun 2022 20:50:27 PDT
210913-GuodongHua Version 3 - Submitted to Rocky Mountain J. Math.

$$\ll x^{\frac{1}{2a}}T^{\frac{697}{420}(\frac{1}{2}\omega_1+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)-1+\varepsilon}+x^{\frac{1}{b}}T^{\frac{697}{420}(1-\frac{a}{b})\omega_1-1+\varepsilon}$$
$$+x^{\frac{1}{c}}T^{\frac{697}{420}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1+\varepsilon}+x^{\frac{1}{a}+\varepsilon}T^{-1+\varepsilon}. \tag{3.7}$$

Now we need to estimate I_1. For I_1, by applying Hölder's inequality and Lemma 2.2, Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned} I_1 \ll\ & x^{\frac{1}{2a}+\varepsilon}+x^{\frac{1}{2a}+\varepsilon}\log T\max_{1\le T_1\le T}T_1^{-1}\\ &\times\int_{T_1/2}^{T_1}\left|\zeta\left(\frac{b}{2a}+ibt\right)^{\omega_2}\zeta\left(\frac{c}{2a}+ict\right)^{\omega_3}L\left(\mathrm{sym}^2f,\frac{b}{2a}+ibt\right)^{\omega_2}L\left(\mathrm{sym}^2f,\frac{c}{2a}+ict\right)^{\omega_3}\right|\\ &\times\left|\zeta\left(\frac{1}{2}+iat\right)^{\omega_1}L\left(\mathrm{sym}^2f,\frac{1}{2}+iat\right)^{\omega_1}\right|dt\\ \ll\ & x^{\frac{1}{2a}+\varepsilon}\max_{1\le T_1\le T}T_1^{-1}T_1^{\frac{697}{420}(\frac{1}{2}(\omega_1-1)+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)+\varepsilon}\\ &\times\left(\int_{T_1/2}^{T_1}\left|\zeta\left(\frac{1}{2}+iat\right)\right|^{12}dt\right)^{\frac{1}{12}}\left(\int_{T_1/2}^{T_1}\left|L\left(\mathrm{sym}^2f,\frac{1}{2}+iat\right)\right|^{2}dt\right)^{\frac{1}{2}}T_1^{\frac{5}{12}}\\ \ll\ & x^{\frac{1}{2a}+\varepsilon}T^{-1}T^{\frac{697}{420}(\frac{1}{2}(\omega_1-1)+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)+\varepsilon}T^{2\times\frac{1}{12}+3\times\frac{1}{2}\times\frac{1}{2}+\frac{5}{12}}\\ \ll\ & x^{\frac{1}{2a}+\varepsilon}T^{\frac{697}{420}(\frac{1}{2}(\omega_1-1)+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)+\frac{1}{3}+\varepsilon}. \end{aligned}\tag{3.8}$$

From (3.3), (3.7), (3.8), then we get

$$\begin{aligned} &S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)\\ =\ & x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)+x^{\frac{1}{c}}P_{\omega_3-1,3}(\log x)\\ &+O\big(x^{\frac{1}{2a}+\varepsilon}T^{\frac{697}{420}(\frac{1}{2}(\omega_1-1)+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)+\frac{1}{3}+\varepsilon}\big)\\ &+O\big(x^{\frac{1}{b}}T^{\frac{697}{420}(1-\frac{a}{b})\omega_1-1+\varepsilon}+x^{\frac{1}{c}}T^{\frac{697}{420}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1+\varepsilon}+x^{\frac{1}{a}+\varepsilon}T^{-1+\varepsilon}\big) \end{aligned}\tag{3.9}$$

On taking $T=x^{\frac{420}{697(\omega_1a+(2a-b)\omega_2+(2a-c)\omega_3)+423a}}$ in (3.9), then we obtain

$$\begin{aligned} S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x) =\ & x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)+x^{\frac{1}{c}}P_{\omega_3-1,3}(\log x)\\ &+O\big(x^{\frac{1}{a}-\frac{420}{697(\omega_1a+(2a-b)\omega_2+(2a-c)\omega_3)+423a}+\varepsilon}\big). \end{aligned}\tag{3.10}$$

This prove the case when $c<2a$ of Theorem 1.1.

Now we treat the case $b<2a\le c$. In this case, the point $s=\frac{1}{a}$ and $s=\frac{1}{b}$ are the simple poles of the integrand in the rectangle $\mathcal{R}_T$. By the Cauchy residue theorem and shifting the line of integration to the parallel line with $\Re(s)=\frac{1}{2a}+\varepsilon$, we also have

$$\begin{aligned} S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x) =\ & x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)\\ &+I_1'+I_2'+I_3'+O\left(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\right), \end{aligned}\tag{3.11}$$

where I_1',I_2' and I_3' are defined in the same way corresponding to the case $c<2a$ as in (3.3).

In a similar argument, we divide the integral interval into four parts $\mathfrak{l}_1^*,\mathfrak{l}_2^*,\mathfrak{l}_3^*,\mathfrak{l}_4^*$. In this case, the intervals $\mathfrak{l}_1^*,\mathfrak{l}_3^*$ are empty sets. Following a similar argument of the previous corresponding parts of the case $c<2a$, we have

$$I_2'+I_3'\ll x^{\frac{1}{2a}}T^{\frac{697}{420}(\frac{1}{2}\omega_1+(1-\frac{b}{2a})\omega_2)-1+\varepsilon}+x^{\frac{1}{b}}T^{\frac{697}{420}(1-\frac{a}{b})\omega_1-1+\varepsilon}+x^{\frac{1}{a}+\varepsilon}T^{-1+\varepsilon}. \tag{3.12}$$

For I_1', by noting $\frac{c}{2a}\ge 1$, then we have

$$\begin{aligned} I_1' \ll\ & x^{\frac{1}{2a}+\varepsilon}+x^{\frac{1}{2a}+\varepsilon}\log T\max_{1\le T_1\le T}T_1^{-1}\\ &\times\int_{T_1/2}^{T_1}\left|\zeta\left(\frac{b}{2a}+ibt\right)^{\omega_2}L\left(\mathrm{sym}^2f,\frac{b}{2a}+ibt\right)^{\omega_2}\zeta\left(\frac{1}{2}+iat\right)^{\omega_1}L\left(\mathrm{sym}^2f,\frac{1}{2}+iat\right)^{\omega_1}\right|dt \end{aligned}$$

$$\ll \ x^{\frac{1}{2a}+\varepsilon}\max_{1\le T_1\le T}T_1^{-1}T_1^{\frac{697}{420}(\frac{1}{2}(\omega_1-1)+(1-\frac{b}{2a})\omega_2)+\varepsilon}$$
$$\times\Big(\int_{T_1/2}^{T_1}\Big|\zeta\Big(\frac{1}{2}+iat\Big)\Big|^{12}dt\Big)^{\frac{1}{12}}\Big(\int_{T_1/2}^{T_1}\Big|L\Big(\mathrm{sym}^2f,\frac{1}{2}+iat\Big)\Big|^{2}dt\Big)^{\frac{1}{2}}T_1^{\frac{5}{12}}$$
$$\ll \ x^{\frac{1}{2a}+\varepsilon}T^{\frac{697}{420}(\frac{1}{2}(\omega_1-1)+(1-\frac{b}{2a})\omega_2)+\frac{1}{3}+\varepsilon} \tag{3.13}$$

Then from (3.12) and (3.13) and recalling (3.11), we can obtain

$$\begin{aligned}&S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)\\ =\ &x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)+O\big(x^{\frac{1}{2a}+\varepsilon}T^{\frac{697}{420}(\frac{1}{2}(\omega_1-1)+(1-\frac{b}{2a})\omega_2)+\frac{1}{3}+\varepsilon}\\ &+x^{\frac{1}{b}}T^{\frac{697}{420}(1-\frac{a}{b})\omega_1-1+\varepsilon}+x^{\frac{1}{a}+\varepsilon}T^{-1+\varepsilon}\big).\end{aligned} \tag{3.14}$$

On taking $T=x^{\frac{420}{697(\omega_1a+(2a-b)\omega_2)+423a}}$ in (3.14), we get

$$S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)=x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+x^{\frac{1}{b}}P_{\omega_2-1,2}(\log x)+O\big(x^{\frac{1}{a}-\frac{420}{697(\omega_1a+(2a-b)\omega_2)+423a}+\varepsilon}\big).$$

This proves the case $b<2a\le c$ of Theorem 1.1.

Now we consider the case $2a\le b$. In this case, the point $s=\frac{1}{a}$ is the only simple pole of the integrand in the rectangle $\mathcal{R}_T$. By Cauchy residue theorem and shifting the line of integration to the parallel line with $\Re(s)=\frac{1}{2a}+\varepsilon$, we also have

$$S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)=x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+I_1''+I_2''+I_3''+O\Bigg(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\Bigg) \tag{3.15}$$

where I_1'',I_2'' and I_3'' are defined in the same way as (3.3). We can also divide the integral interval in the horizontal segment into four short intervals as in the case $c<2a$. In this instance, the corresponding short intervals $\mathfrak{l}_1^{**},\mathfrak{l}_2^{**},\mathfrak{l}_3^{**}$ are empty sets. In a similar manner, we can also estimate the integral on the horizontal segment as follows

$$I_2''+I_3''\ll x^{\frac{1}{2a}}T^{\frac{697}{840}\omega_1-1+\varepsilon}+x^{\frac{1}{a}+\varepsilon}T^{-1+\varepsilon}.$$

The estimate of the I_1 can be estimate as follows by noting $\frac{c}{2a}>\frac{b}{2a}>1$

$$\begin{aligned}I_1''\ &\ll\ x^{\frac{1}{2a}+\varepsilon}+x^{\frac{1}{2a}+\varepsilon}\log T\max_{1\le T_1\le T}T_1^{-1}\times\int_{T_1/2}^{T_1}\Big|\zeta\Big(\frac{1}{2}+iat\Big)^{\omega_1}L\Big(\mathrm{sym}^2f,\frac{1}{2}+iat\Big)^{\omega_1}\Big|dt\\ &\ll\ x^{\frac{1}{2a}+\varepsilon}\max_{1\le T_1\le T}T_1^{-1}T_1^{\frac{697}{420}(\frac{1}{2}(\omega_1-1))+\varepsilon}\\ &\quad\times\Big(\int_{T_1/2}^{T_1}\Big|\zeta\Big(\frac{1}{2}+iat\Big)\Big|^{12}dt\Big)^{\frac{1}{12}}\Big(\int_{T_1/2}^{T_1}\Big|L\Big(\mathrm{sym}^2f,\frac{1}{2}+iat\Big)\Big|^{2}dt\Big)^{\frac{1}{2}}T_1^{\frac{5}{12}}\\ &\ll\ x^{\frac{1}{2a}+\varepsilon}T^{\frac{697}{420}(\frac{1}{2}(\omega_1-1))+\frac{1}{3}+\varepsilon}\end{aligned} \tag{3.16}$$

Recalling (3.15), then we obtain

$$\begin{aligned}&S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)\\ =\ &x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+O\big(x^{\frac{1}{2a}+\varepsilon}T^{\frac{697}{420}(\frac{1}{2}(\omega_1-1))+\frac{1}{3}+\varepsilon}+x^{\frac{1}{a}+\varepsilon}T^{-1+\varepsilon}\big).\end{aligned} \tag{3.17}$$

On taking $T=x^{\frac{420}{697\omega_1a+423a}}$ in (3.17), then we get

$$S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x)=x^{\frac{1}{a}}P_{\omega_1-1,1}(\log x)+O\big(x^{\frac{1}{a}-\frac{420}{697\omega_1a+423a}+\varepsilon}\big).$$

Then we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

21 Jun 2022 20:50:27 PDT
210913-GuodongHua Version 3 - Submitted to Rocky Mountain J. Math.

In this section, we shall give the proof of Theorem 1.2, and the approach is similar as the proof of Theorem 1.1.

Set $s = \sigma + it$ and $\eta = \frac{1}{a} + \varepsilon$. Note that

$$L(f \times f \times f, as)^{\omega_1} L(f \times f \times f, bs)^{\omega_2} L(f \times f \times f, cs)^{\omega_3} = \sum_{n=1}^{\infty} \frac{\lambda^{a,b,c}_{\omega_1 \omega_2, \omega_3, f\times f\times f}(n)}{n^s} \tag{4.1}$$

for $\Re(s) > 1$. Applying the Perron's formula to the generating function (4.1) and using Deligne's bound (1.1), we can obtain

$$\begin{aligned} & S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) \\ = & \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} L(f \times f \times f, as)^{\omega_1} L(f \times f \times f, bs)^{\omega_2} L(f \times f \times f, cs)^{\omega_3} \frac{x^s}{s} ds \\ & + O\left(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\right), \end{aligned} \tag{4.2}$$

where $1 \le T \le x$ is a parameter to be chosen later.

We first consider the case $c < 2a$. Then by shifting the line of the integral of (4.2) to the line $\Re(s) = \frac{1}{2a} + \varepsilon$ and applying Cauchy's residue theorem, we can obtain

$$S_{\omega_1,\omega_2,\omega_3,f\times f}(a,b,c;x) = J_1 + J_2 + J_3 + O\left(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\right), \tag{4.3}$$

where

$$\begin{aligned} J_1 &:= \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} L(f \times f \times f, as)^{\omega_1} L(f \times f \times f, bs)^{\omega_2} L(f \times f \times f, cs)^{\omega_3} \frac{x^s}{s} ds, \\ J_2 &:= \frac{1}{2\pi i}\int_{\kappa+iT}^{\eta+iT} L(f \times f \times f, as)^{\omega_1} L(f \times f \times f, bs)^{\omega_2} L(f \times f \times f, cs)^{\omega_3} \frac{x^s}{s} ds, \\ J_3 &:= \frac{1}{2\pi i}\int_{\eta-iT}^{\kappa-iT} L(f \times f \times f, as)^{\omega_1} L(f \times f \times f, bs)^{\omega_2} L(f \times f \times f, cs)^{\omega_3} \frac{x^s}{s} ds, \end{aligned}$$

where $\kappa = \frac{1}{2a} + \varepsilon$.

Now we need to handle the three terms J_1, J_2, J_3, and the estimates of these integrals on the horizontal parts I_2, J_3 are analogous. In a similar approach as in the proof of Theorem 1.1, we split the integral interval into four parts $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4$ as defined in the proof of Theorem 1.1 and apply Lemmas 2.1-2.5.

Interval 1. From (2.4), (2.8) and (2.10), we have

$$\begin{aligned} J_{2,1} \quad := \quad & \frac{1}{2\pi i}\int_{\mathfrak{l}_1} L(f,as)^{2\omega_1} L(f,bs)^{2\omega_2} L(f,cs)^{2\omega_3} \\ & \times L(\mathrm{sym}^3 f, as)^{\omega_1} L(\mathrm{sym}^3 f, bs)^{\omega_2} L(\mathrm{sym}^3 f, cs)^{\omega_3} \frac{x^s}{s} ds \\ \ll \quad & \max_{\frac{1}{2a}\le\sigma\le\frac{1}{c}} x^{\sigma} T^{(\frac{4}{3}+2)((1-a\sigma)\omega_1+(1-b\sigma)\omega_2+(1-c\sigma)\omega_3)} T^{-1+\varepsilon} \\ \ll \quad & x^{\frac{1}{2a}+\varepsilon} T^{\frac{10}{3}(\frac{1}{2}\omega_1+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)-1} + x^{\frac{1}{c}+\varepsilon} T^{\frac{10}{3}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1}. \end{aligned} \tag{4.4}$$

Interval 2. In this interval, we have

$$\begin{aligned} J_{2,2} \quad := \quad & \frac{1}{2\pi i}\int_{\mathfrak{l}_2} L(f,as)^{2\omega_1} L(f,bs)^{2\omega_2} L(f,cs)^{2\omega_3} \\ & \times L(\mathrm{sym}^3 f, as)^{\omega_1} L(\mathrm{sym}^3 f, bs)^{\omega_2} L(\mathrm{sym}^3 f, cs)^{\omega_3} \frac{x^s}{s} ds \end{aligned}$$

$$\ll \max_{\frac{1}{c}\le\sigma\le\frac{1}{b}} x^{\sigma}T^{(\frac{4}{3}+2)((1-a\sigma)\omega_1+(1-b\sigma)\omega_2)}T^{-1+\varepsilon}$$
$$\ll x^{\frac{1}{c}+\varepsilon}T^{\frac{10}{3}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1+\varepsilon}+x^{\frac{1}{b}+\varepsilon}T^{\frac{10}{3}(1-\frac{a}{b})\omega_1-1+\varepsilon}. \tag{4.5}$$

Interval 3. This interval is empty by noting that $\frac{1}{c}<\frac{1}{b}$.

Interval 4. In this interval, we have

$$\begin{aligned} J_{2,4} &:= \frac{1}{2\pi i}\int_{\mathfrak{l}_4} L(f,as)^{2\omega_1}L(f,bs)^{2\omega_2}L(f,cs)^{2\omega_3}\\ &\quad\times L(\mathrm{sym}^3 f,as)^{\omega_1}L(\mathrm{sym}^3 f,bs)^{\omega_2}L(\mathrm{sym}^3 f,cs)^{\omega_3}\frac{x^s}{s}ds\\ &\ll \max_{\frac{1}{b}\le\sigma\le\frac{1}{a}} x^{\sigma}T^{(\frac{4}{3}+2)(1-a\sigma)\omega_1}T^{-1+\varepsilon}\\ &\ll x^{\frac{1}{a}}T^{-1+\varepsilon}+x^{\frac{1}{b}}T^{\frac{10}{3}(1-\frac{a}{b})\omega_1-1+\varepsilon}. \end{aligned} \tag{4.6}$$

Combining the three estimates (4.4), (4.5), (4.6), we have

$$\begin{aligned} J_2+J_3 &\ll J_{2,1}+J_{2,2}+J_{4,2}\\ &\ll x^{\frac{1}{2a}+\varepsilon}T^{\frac{10}{3}(\frac{1}{2}\omega_1+(1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)-1}+x^{\frac{1}{b}}T^{\frac{10}{3}(1-\frac{a}{b})\omega_1-1+\varepsilon}\\ &\quad+x^{\frac{1}{c}+\varepsilon}T^{\frac{10}{3}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1}+x^{\frac{1}{a}}T^{-1+\varepsilon}. \end{aligned} \tag{4.7}$$

For J_1, applying the Cauchy-Schwarz inequality, and Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned} J_1 &\ll x^{\frac{1}{2a}+\varepsilon}\log T\max_{1\le T_1\le T}T_1^{-1}\int_{T_1/2}^{T_1}\bigg|L\Big(f,\frac{b}{2a}+ibt\Big)^{2\omega_2}L\Big(f,\frac{c}{2a}+ict\Big)^{2\omega_3}\\ &\quad L\Big(\mathrm{sym}^3 f,\frac{b}{2a}+ibt\Big)^{\omega_2}L\Big(\mathrm{sym}^3 f,\frac{c}{2a}+ict\Big)^{\omega_3}L\Big(f,\frac{1}{2}+iat\Big)^{2\omega_1}\\ &\quad L\Big(\mathrm{sym}^3 f,\frac{1}{2}+iat\Big)^{2\omega_1}\bigg|dt+x^{\frac{1}{2a}+\varepsilon}\\ &\ll x^{\frac{1}{2a}+\varepsilon}\log T\max_{1\le T_1\le T}T_1^{-1}T_1^{\frac{10}{3}((1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)+\frac{5}{3}\omega_1-\frac{4}{3}}\\ &\quad\bigg(\int_{T_1/2}^{T_1}\Big|L\Big(f,\frac{1}{2}+iat\Big)\Big|^2dt\int_{T_1/2}^{T_1}\Big|L\Big(\mathrm{sym}^3 f,\frac{1}{2}+iat\Big)\Big|^2dt\bigg)^{\frac{1}{2}}\\ &\ll x^{\frac{1}{2a}+\varepsilon}T^{\frac{10}{3}((1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)+\frac{5}{3}\omega_1-\frac{5}{6}}. \end{aligned} \tag{4.8}$$

From (4.3), (4.7), (4.8), we know that

$$\begin{aligned} S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) &\ll x^{\frac{1}{2a}+\varepsilon}T^{\frac{10}{3}((1-\frac{b}{2a})\omega_2+(1-\frac{c}{2a})\omega_3)+\frac{5}{3}\omega_1-\frac{5}{6}}\\ &\quad+x^{\frac{1}{b}+\varepsilon}T^{\frac{10}{3}(1-\frac{a}{b})\omega_1-1+\varepsilon}+x^{\frac{1}{c}+\varepsilon}T^{\frac{10}{3}((1-\frac{a}{c})\omega_1+(1-\frac{b}{c})\omega_2)-1+\varepsilon}\\ &\quad+x^{\frac{1}{a}}T^{-1+\varepsilon}. \end{aligned} \tag{4.9}$$

On taking $T=x^{\frac{3}{(10\omega_1+1)a+10((2a-b)\omega_2+(2a-c)\omega_3)}}$ in (4.9), then

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x)\ll x^{\frac{1}{a}-\frac{3}{(10\omega_1+1)a+10((2a-b)\omega_2+(2a-c)\omega_3)}+\varepsilon}.$$

Then we complete the proof of Theorem 1.2 when $c<2a$.

Now we consider the case $b<2a\le c$. Then by Cauchy's residue theorem and shifting the line of integration the parallel line $\Re(s)=\frac{1}{2a}+\varepsilon$, we have

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x)=J_1'+J_2'+J_3'+O\bigg(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\bigg), \tag{4.10}$$

here J_1', J_2', J_3' are still defined in the same way corresponding to the case $c < 2a$ as in (4.3).

Now need to estimate the three terms J_1', J_2', J_3', and the estimates of J_2, J_3 can be handle in a similar manner. Following a similar approach as in the case $b < 2a \le c$ of the proof of Theorem 1.1, we divide the integral interval on the horizontal segment into the same four subintervals $\mathfrak{l}_1^*, \mathfrak{l}_2^*, \mathfrak{l}_3^*, \mathfrak{l}_4^*$. Then we have

$$J_2' + J_3' \ll x^{\frac{1}{2a}+\varepsilon} T^{\frac{10}{3}(\frac{1}{2}\omega_1 + (1-\frac{b}{2a})\omega_2) - 1 + \varepsilon} + x^{\frac{1}{b}+\varepsilon} T^{\frac{10}{3}(1-\frac{a}{b})\omega_1 - 1 + \varepsilon} + x^{\frac{1}{a}} T^{-1+\varepsilon}.$$

For J_1', by noting $\frac{c}{2a} \ge 1$ we have

$$\begin{aligned} J_1' \ll\ & x^{\frac{1}{2a}+\varepsilon} \log T \max_{1 \le T_1 \le T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\Big(f, \frac{b}{2a} + ibt\Big)^{2\omega_2} L\Big(\mathrm{sym}^3 f, \frac{b}{2a} + ibt\Big)^{\omega_2} \right. \\ & \left. L\Big(f, \frac{1}{2} + iat\Big)^{2\omega_1} L\Big(\mathrm{sym}^3 f, \frac{1}{2} + iat\Big)^{2\omega_1} \right| dt + x^{\frac{1}{2a}+\varepsilon} \\ \ll\ & x^{\frac{1}{2a}+\varepsilon} \log T \max_{1 \le T_1 \le T} T_1^{-1} T_1^{\frac{10}{3}(1-\frac{b}{2a})\omega_2 + \frac{5}{3}\omega_1 - \frac{4}{3}} \\ & \bigg(\int_{T_1/2}^{T_1} \Big| L\Big(f, \frac{1}{2} + iat\Big)\Big|^2 dt \int_{T_1/2}^{T_1} \Big| L\Big(\mathrm{sym}^3 f, \frac{1}{2} + iat\Big)\Big|^2 dt \bigg)^{\frac{1}{2}} \\ \ll\ & x^{\frac{1}{2a}+\varepsilon} T^{\frac{10}{3}(1-\frac{b}{2a})\omega_2 + \frac{5}{3}\omega_1 - \frac{5}{6}}. \end{aligned} \tag{4.11}$$

Recalling (4.10), we have

$$\begin{aligned} & S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) \\ \ll\ & x^{\frac{1}{2a}+\varepsilon} T^{\frac{10}{3}(1-\frac{b}{2a})\omega_2 + \frac{5}{3}\omega_1 - \frac{5}{6}} + x^{\frac{1}{b}+\varepsilon} T^{\frac{10}{3}(1-\frac{a}{b})\omega_1 - 1 + \varepsilon} + x^{\frac{1}{a}} T^{-1+\varepsilon} \end{aligned} \tag{4.12}$$

By taking $T = x^{\frac{3}{(10\omega_1+1)a + 10(2a-b)\omega_2}}$ in (4.12), then we get

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) \ll x^{\frac{1}{a} - \frac{3}{(10\omega_1+1)a+10(2a-b)\omega_2} + \varepsilon}.$$

This prove the case $b < 2a \le c$.

For the case $2a \le b$, we can prove by a similar argument as in the case $c < 2a$. By Cauchy's residue theorem and shifting the line of integration to the parallel line $\Re(s) = \frac{1}{2a} + \varepsilon$, we have

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) = J_1'' + J_2'' + J_3'' + O\bigg(\frac{x^{\frac{1}{a}+\varepsilon}}{T}\bigg), \tag{4.13}$$

here J_1'', J_2'', J_3'' are still defined in the same way corresponding to J_1, J_2, J_3 as (4.3).

We still divide the integral segment into four corresponding short intervals $\mathfrak{l}_1^{**}, \mathfrak{l}_2^{**}, \mathfrak{l}_3^{**}, \mathfrak{l}_4^{**}$ in the same manner as the case $2a \le b$ in the proof of Theorem 1.1, and in this instance $\mathfrak{l}_1^{**}, \mathfrak{l}_2^{**}, \mathfrak{l}_3^{**}$ are empty sets.

$$J_2'' + J_3'' \ll x^{\frac{1}{2a}} T^{\frac{5}{3}\omega_1 - 1 + \varepsilon} + x^{\frac{1}{a}} T^{-1+\varepsilon}.$$

The estimate can conduct as follows by noting $\frac{c}{2a} > \frac{b}{2a} \ge 1$

$$\begin{aligned} J_1'' \ll\ & x^{\frac{1}{2a}+\varepsilon} \log T \max_{1 \le T_1 \le T} T_1^{-1} \int_{T_1/2}^{T_1} \left| L\Big(f, \frac{1}{2} + iat\Big)^{2\omega_1} L\Big(\mathrm{sym}^3 f, \frac{1}{2} + iat\Big)^{2\omega_1} \right| dt + x^{\frac{1}{2a}+\varepsilon} \\ \ll\ & x^{\frac{1}{2a}+\varepsilon} \log T \max_{1 \le T_1 \le T} T_1^{-1} \bigg(\int_{T_1/2}^{T_1} \Big| L\Big(f, \frac{1}{2} + iat\Big)\Big|^2 dt \int_{T_1/2}^{T_1} \Big| L\Big(\mathrm{sym}^3 f, \frac{1}{2} + iat\Big)\Big|^2 dt \bigg)^{\frac{1}{2}} \\ \ll\ & x^{\frac{1}{2a}+\varepsilon} T^{\frac{5}{3}\omega_1 - \frac{5}{6}}. \end{aligned} \tag{4.14}$$

Hence recalling (4.13), we have

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) \ll x^{\frac{1}{2a}+\varepsilon} T^{\frac{5}{3}\omega_1 - \frac{5}{6}} + x^{\frac{1}{a}} T^{-1+\varepsilon}. \tag{4.15}$$

21 Jun 2022 20:50:27 PDT
210913-GuodongHua Version 3 - Submitted to Rocky Mountain J. Math.

On taking $T = x^{\frac{3}{(10\omega_1+1)a}}$ in (4.15), we obtain

$$S_{\omega_1,\omega_2,\omega_3,f\times f\times f}(a,b,c;x) \ll x^{\frac{1}{a}-\frac{3}{(10\omega_1+1)a}+\varepsilon}.$$

Then we complete the proof of Theorem 1.2.

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