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NUMBERS REPRESENTED BY RESTRICTED SUMS OF FOUR SQUARES

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ABSTRACT. In this paper, we prove some results of restricted sums of four squares using arithmetic of quaternions in the ring of Lipschitz integers. In particular, we show that every nonnegative integer n can be written as $x^2+y^2+z^2+t^2$, where x,y,z,t are integers and $x+3y+3z$ is a power of 2.

1. Introduction

In 1770 Lagrange established his well-known four-square theorem, which states that every positive integer n can be written as the sum of four integral squares.

Let a,b,c,d be integers and $\mathscr{S}$ be a subset of $\mathbb{Z}$, the ring of integers. For any positive integer m, Z.-W. Sun considered the system

$$E_m^{\mathscr{S}}(a\text{-}b\text{-}c\text{-}d):\begin{cases} m=x^2+y^2+z^2+t^2, \\ ax+by+cz+dt\in\mathscr{S}. \end{cases}$$

In [4], Z.-W. Sun proposed the following interesting 1-3-5 conjecture:

Let $\mathscr{S}$ denote the set of all the squares. The system $E_m^{\mathscr{S}}(1\text{-}3\text{-}5\text{-}0)$ has natural solutions for each positive integer m.

Y.-C. Sun and Z.-W. Sun [3] proved that any $n\in\mathbb{N}=\{0,1,2,3,\ldots\}$ can be written as $x^2+y^2+z^2+t^2$ with $x,y,5z,5t\in\mathbb{Z}$ and $x+3y+5z$ a square using Euler's four-square identity. Later, H.-L. Wu and Z.-W. Sun [6] investigated the integer version of 1-3-5 conjecture. Finally, with the help of the Lipschitz integers, the 1-3-5 conjecture was completely proved by A. Machiavelo and N. Tsopanidis [1] recently.

Let $\mathscr{P}=\{2^k:k\in\mathbb{N}\}$. Z.-W. Sun [5] proved some results concerning the solvability of $E_m^{\mathscr{P}}(a\text{-}b\text{-}c\text{-}d)$. For example, he [5, Theorem 1.1 (iv)] proved that the system $E_m^{\mathscr{P}}(1\text{-}1\text{-}1\text{-}1)$ has a solution for every positive integer m. Furthermore, Z.-W. Sun [5, Conjecture 4.4(i) and Conjecture 4.4(ii)] conjecture that, the system $E_m^{\mathscr{P}}(a\text{-}b\text{-}c\text{-}d)$ has a natural solution for each positive integer m if (a,b,c,d) is among

$$(1,3,-3,0),(4,-2,-1,0),(1,4,-2,0).$$

As an application of Lipschitz integers, in the present paper, we have the following result, which covers the integer version of the conjecture.

Theorem 1.1. *For any positive integer m, $E_m^{\mathscr{P}}(a\text{-}b\text{-}c\text{-}d)$ has an integral solution if (a,b,c,d) is among the four quadruples*

$$(1,3,3,0),(1,2,4,0),(1,1,2,5),(1,2,3,5).$$

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For each nonnegative integer m, define $r(m)$ to be the cardinality of the set

$$R(m) := \{(x_1,x_2,x_3,x_4) \in \mathbb{N}^4 : 0 \le x_1 \le x_2 \le x_3 \le x_4 \\ \text{and } x_1^2+x_2^2+x_3^2+x_4^2 = m\}.$$

Remark 1.1. In fact, the integral solution in the theorem can not be replaced by a natural solution. For example,

$$\begin{aligned} 46\cdot 4^r &= (6\cdot 2^r)^2+(3\cdot 2^r)^2+(1\cdot 2^r)^2+(0\cdot 2^r)^2 \\ &= (5\cdot 2^r)^2+(4\cdot 2^r)^2+(2\cdot 2^r)^2+(1\cdot 2^r)^2. \end{aligned}$$

In light of [2, Theorem 2.6], it is easy to see that $r(46\cdot 4^r)=2$. Let $A=\{0\cdot 2^r,1\cdot 2^r,3\cdot 2^r,6\cdot 2^r\}$ and $B=\{1\cdot 2^r,2\cdot 2^r,4\cdot 2^r,5\cdot 2^r\}$. One can easily verify that

$$\begin{aligned} &\{x+3y+3z : \{x,y,z\} \subset A\} \\ =&\{6\cdot 2^r, 9\cdot 2^r, 10\cdot 2^r, 12\cdot 2^r, 15\cdot 2^r, 18\cdot 2^r, 19\cdot 2^r, 21\cdot 2^r, \\ &24\cdot 2^r, 27\cdot 2^r, 28\cdot 2^r\}, \end{aligned}$$

and

$$\begin{aligned} &\{x+3y+3z : \{x,y,z\} \subset B\} \\ =&\{13\cdot 2^r, 14\cdot 2^r, 17\cdot 2^r, 19\cdot 2^r, 20\cdot 2^r, 22\cdot 2^r, 23\cdot 2^r, 25\cdot 2^r, \\ &28\cdot 2^r, 29\cdot 2^r\}. \end{aligned}$$

Neither of the above two sets has an intersection with $\mathscr{P}$. This implies that the system $E_m^{\mathscr{P}}$(1-3-3-0) has no natural solutions for $m=46\cdot 4^r$. For systems $E_m^{\mathscr{P}}$(1-2-4-0), $E_m^{\mathscr{P}}$(1-1-2-5) and $E_m^{\mathscr{P}}$(1-2-3-5), consider $m=12\cdot 4^r$, $m=18\cdot 4^r$ and $m=36\cdot 4^r$ respectively.

Our proof is based on properties of the Lipschitz integers defined by

$$\mathscr{L} = \{a+bi+cj+dk : a,b,c,d \in \mathbb{Z} \textit{ and } i^2=j^2=k^2=ijk=-1\}.$$

For a quaternion $\alpha = a_1+a_2i+a_3j+a_4k$, let $N(\alpha)=a_1^2+a_2^2+a_3^2+a_4^2$ be the norm of α and $\Re(\alpha)=a_1$ be the real part of α. For simplicity, we define $E_m^n(a\text{-}b\text{-}c\text{-}d) := E_m^{\{n\}}(a\text{-}b\text{-}c\text{-}d)$. It is clear that the system $E_m^n(a\text{-}b\text{-}c\text{-}d)$ is equivalent to

$$\begin{cases} m = N(\gamma), \\ n = \Re(\beta\gamma), \end{cases} \tag{1.1}$$

where $\gamma = x+yi+zj+tk$ and $\beta = a-bi-cj-dk$ are Lipschitz integers.

The paper is organized as follows. In section 2, we will show that, given a positive integer m, the solvability of $E_m^n(a\text{-}a\text{-}a\text{-}b)$ implies the solvability of $E_m^n(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ and prove that $E_m^n(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ has an integer solution under certain conditions and when all these four numbers are integers. We will prove Theorem 1.1 in section 3.

2. Solvability of $E_m^n(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$

Let a,b be distinct integers such that $a \equiv b \not\equiv 0 \pmod 5$, and suppose that $\ell = 3a^2 + b^2$ satisfies that $r(\ell) = 2$. One can easily verify that

$$\left(\frac{a+4b}{5}\right)^2 + \left(\frac{7a-2b}{5}\right)^2 + \left(\frac{3a+2b}{5}\right)^2 + \left(\frac{4a+b}{5}\right)^2 = \ell,$$

and that $\frac{a+4b}{5}$, $\frac{7a-2b}{5}$, $\frac{3a+2b}{5}$, $\frac{4a+b}{5}$ are pairwise distinct since $a \neq b$.

To show the solvability of $E_m^n(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$, we need the following result due to A. Machiavelo and N. Tsopanidis [1].

Lemma 2.1. ([1, Theorem 1]) *Let $m,n,\ell \in \mathbb{N}$ be such that $n^2 \leq \ell m$ and $\ell m - n^2$ is not of the form $4^r(8s+7)$ for any $r,s \in \mathbb{N}$. Then, for some $a,b,c,d \in \mathbb{N}$ satisfying that $a^2+b^2+c^2+d^2 = \ell$, $E_m^n(a\text{-}b\text{-}c\text{-}d)$ has an integral solution for m.*

Assume that $\eta = a + ai + aj + bk$ and $\gamma = x_0 - y_0 i - z_0 j - t_0 k$ be Lipschitz integers with

$$\begin{cases} m = N(\gamma), \\ n = \Re(\eta\gamma), \end{cases}$$

Note that for any $\alpha, \beta \in \mathscr{L}$ with $N(\alpha) = N(\beta) = 5$,we have

$$\begin{cases} N(\beta^{-1}\gamma\alpha) = m, \\ \Re(\alpha^{-1}\eta\beta \cdot \beta^{-1}\gamma\alpha) = \Re(\alpha^{-1}\eta\gamma\alpha) = \Re(\eta\gamma) = n, \end{cases}$$

where β^{-1} is the quarternion such that $\beta^{-1}\beta = 1$.

Now, we choose $\alpha = 1+2j, \beta = 1+2i$, One can easily verify that

$$\alpha^{-1}\eta\beta = \frac{a+4b}{5} + \frac{7a-2b}{5}i + \frac{3a+2b}{5}j + \frac{4a+b}{5}k,$$

and

$$\begin{aligned} \beta^{-1}\gamma\alpha = & \frac{x_0 - 2y_0 + 2z_0 + 4t_0}{5} + \frac{-2x_0 - y_0 - 4z_0 + 2t_0}{5}i \\ & + \frac{2x_0 - 4y_0 - z_0 - 2t_0}{5}j + \frac{-4x_0 - 2y_0 + 2z_0 - t_0}{5}k. \end{aligned}$$

By the equivalence of the system $E_m^n(a\text{-}a\text{-}a\text{-}b)$ and the system 1.1, if (x_0,y_0,z_0,t_0) is an integral solution of $E_m^n(a\text{-}a\text{-}a\text{-}b)$ for m and $x_0 - 2y_0 + 2z_0 - t_0 \equiv 0 \pmod 5$, then $E_m^n(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ has an integral solution for m.

Similarly, we have

$$\begin{aligned} (1-2j)^{-1}\eta(1-2i) &= \frac{a+4b}{5} + \frac{3a+2b}{5}i + \frac{7a-2b}{5}j + \frac{4a+b}{5}k, \\ (1+2k)^{-1}\eta(1+2j) &= \frac{3a+2b}{5} + \frac{7a-2b}{5}i + \frac{a+4b}{5}j + \frac{4a+b}{5}k, \\ (1-2k)^{-1}\eta(1-2j) &= \frac{7a-2b}{5} + \frac{3a+2b}{5}i + \frac{a+4b}{5}j + \frac{4a+b}{5}k, \\ (1+2i)^{-1}\eta(1+2k) &= \frac{7a-2b}{5} + \frac{a+4b}{5}i + \frac{3a+2b}{5}j + \frac{4a+b}{5}k, \end{aligned}$$

$$(1-2i)^{-1}\eta(1-2k) = \frac{3a+2b}{5} + \frac{a+4b}{5}i + \frac{7a-2b}{5}j + \frac{4a+b}{5}k.$$

By computation, we have

$$\begin{aligned}
(1-2i)^{-1}\gamma(1-2j) =& \frac{x_0+2y_0-2z_0+4t_0}{5} + \frac{2x_0-y_0-4z_0-2t_0}{5}i \\
&+ \frac{-2x_0-4y_0-z_0+2t_0}{5}j + \frac{-4x_0+2y_0-2z_0-t_0}{5}k, \\
(1+2j)^{-1}\gamma(1+2k) =& \frac{x_0+4y_0-2z_0+2t_0}{5} + \frac{-4x_0-y_0-2z_0+2t_0}{5}i \\
&+ \frac{-2x_0+2y_0-z_0-4t_0}{5}j + \frac{2x_0-2y_0-4z_0-t_0}{5}k, \\
(1-2j)^{-1}\gamma(1-2k) =& \frac{x_0+4y_0+2z_0-2t_0}{5} + \frac{-4x_0-y_0+2z_0-2t_0}{5}i \\
&+ \frac{2x_0-2y_0-z_0-4t_0}{5}j + \frac{-2x_0+2y_0-4z_0-t_0}{5}k, \\
(1+2k)^{-1}\gamma(1+2i) =& \frac{x_0+2y_0+4z_0-2t_0}{5} + \frac{2x_0-y_0-2z_0-4t_0}{5}i \\
&+ \frac{-4x_0+2y_0-z_0-2t_0}{5}j + \frac{-2x_0-4y_0+2z_0-t_0}{5}k, \\
(1-2k)^{-1}\gamma(1-2i) =& \frac{x_0-2y_0+4z_0+2t_0}{5} + \frac{-2x_0-y_0+2z_0-4t_0}{5}i \\
&+ \frac{-4x_0-2y_0-z_0+2t_0}{5}j + \frac{2x_0-4y_0-2z_0-t_0}{5}k.
\end{aligned}$$

Hence we can get the following lemma.

Lemma 2.2. *Given an integral solution (x_0, y_0, z_0, t_0) of $E^n_m(a\text{-}a\text{-}a\text{-}b)$ for m, if either of the following conditions holds:*

- $x_0 - 2y_0 + 2z_0 - t_0 \equiv 0 \pmod 5$,
- $x_0 + 2y_0 - 2z_0 - t_0 \equiv 0 \pmod 5$,
- $x_0 - y_0 - 2z_0 + 2t_0 \equiv 0 \pmod 5$,
- $x_0 - y_0 + 2z_0 - 2t_0 \equiv 0 \pmod 5$,
- $x_0 + 2y_0 - z_0 - 2t_0 \equiv 0 \pmod 5$,
- $x_0 - 2y_0 - z_0 + 2t_0 \equiv 0 \pmod 5$,

$E^n_m(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ *has an integral solution.*

In view of Lemma 2.1 and Lemma 2.2, we can deduce the following lemma.

Proposition 1. *Let a, b be two distinct integers such that $a \equiv b \not\equiv 0 \pmod 5$. Suppose that $\ell = 3a^2 + b^2$ is such that $r(\ell) = 2$. Assume that $m, n \in \mathbb{N}$ are such that $\ell m - n^2$ is nonnegative and not of the form of $4^r(8s+7)$ for any $r, s \in \mathbb{N}$. When $\ell m - n^2 \equiv 0, 1$ or $4 \pmod 5$, $E^n_m(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ has an integral solution.*

Proof. According to the assumptions, we have $\ell = 3a^2 + b^2$ or $\ell = \left(\frac{a+4b}{5}\right)^2 + \left(\frac{7a-2b}{5}\right)^2 + \left(\frac{3a+2b}{5}\right)^2 + \left(\frac{4a+b}{5}\right)^2$. By Lemma 2.1, either $E^n_m(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ or $E^n_m(a\text{-}a\text{-}a\text{-}b)$ has an integral solution. If

$E^n_m(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ has an integral solution, we are done. If not, we have an integral solution of $E^n_m(a\text{-}a\text{-}a\text{-}b)$. Using the notations above, we have

$$\gamma\eta = n + Ai + Bj + Ck,$$

where

$$\begin{cases} n = ax_0 + ay_0 + az_0 + bt_0, \\ A = ax_0 - ay_0 - bz_0 + at_0, \\ B = ax_0 + by_0 - az_0 - at_0, \\ C = bx_0 - ay_0 + az_0 - at_0. \end{cases}$$

By solving the equations and taking the coefficients modulo 5, we have

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix} \equiv a^{-1} \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} n \\ A \\ B \\ C \end{pmatrix} \pmod 5.$$

Hence the congruence condition of Lemma 2.2 is

$$\begin{pmatrix} 1 & -2 & 2 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & -1 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & 2 & -1 & -2 \\ 1 & -2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix} \equiv a^{-1} \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} n \\ A \\ B \\ C \end{pmatrix} \pmod 5,$$

which means that if either of the following conditions holds:

- $A \equiv \pm 2B \pmod 5$,
- $A \equiv \pm 2C \pmod 5$,
- $B \equiv \pm 2C \pmod 5$,

$E^n_m(\frac{a+4b}{5}\text{-}\frac{7a-2b}{5}\text{-}\frac{3a+2b}{5}\text{-}\frac{4a+b}{5})$ has an integral solution.

On the other hand, $A^2 + B^2 + C^2 = \ell m - n^2 \equiv 0, 1$ or $4 \pmod 5$. Since

$$\begin{cases} 0 \equiv 0 + 0 + 0 \equiv 0 + 1 + 4 \pmod 5, \\ 1 \equiv 0 + 0 + 1 \equiv 1 + 1 + 4 \pmod 5, \\ 4 \equiv 0 + 0 + 4 \equiv 1 + 4 + 4 \pmod 5. \end{cases}$$

are the only decompositions of 0, 1, 4 as a sum of three squares modulo 5, one can easily verify that one of $A^2 + B^2$, $A^2 + C^2$ and $B^2 + C^2$ must be divisible by 5, hence one of the conditions must hold. This completes the proof. □

Now we take (a,b) with $(1,-4)$, $(1,6)$, $(2,-3)$ and $(3,-2)$, then we have the following lemma.

Corollary 2.1. *When either $19m-n^2$, $39m-n^2$, $21m-n^2$, $31m-n^2$ is not of the form $4^r(8s+7)$, for any $r,s\in\mathbb{N}$ and is congruent to $0,1,4$ modulo 5, then E_m^n(1-3-3-0), E_m^n(1-2-3-5), E_m^n(1-2-4-0), E_m^n(1-1-2-5), respectively, has an integral solution.*

3. Proof of Theorem 1.1

Proof of Theorem 1.1. For $u\in\mathbb{N}$, define P_u as the set of $\{k\in\mathbb{N}:\ u-4^k\geq 0$ and $u-4^k$ is not of the form $4^r(8s+7)$ for any $r,s\in\mathbb{N}\}$. We describe some of the elements of P_u:

- If $u\equiv 1,2 \pmod 4$, any integer k satisfying $1\leq k\leq \log_4 u$ is in P_u.
- If $u\equiv 3 \pmod 8$, any integer k satisfying $0\leq k\leq \log_4 u$ and $k\neq 1$ is in P_u.
- If $u\equiv 7 \pmod 8$, $\{0,1\}\subseteq P_u$.

When m is divisible by 4, assume $m=4^\alpha m'$ with $\alpha\geq 1$ a positive integer and $4\nmid m'$. If

$$\begin{cases} m'=x^2+y^2+z^2+t^2,\\ ax+by+cz+dt=2^k\in\mathscr{P},\end{cases}$$

then

$$\begin{cases} m=(2^\alpha x)^2+(2^\alpha y)^2+(2^\alpha z)^2+(2^\alpha t)^2,\\ a(2^\alpha x)+b(2^\alpha y)+c(2^\alpha z)+d(2^\alpha t)=2^{k+\alpha}\in\mathscr{P}.\end{cases}$$

Therefore, it suffices to assume that m is not divisible by 4. When $\ell\in\{19,39,21,31\}$ and $\ell m\geq 64$, $\{0,1\}\subseteq P_{\ell m}$ or $\{2,3\}\subseteq P_{\ell m}$ according as $\ell m\equiv 7 \pmod 8$ or not. When $\ell m\not\equiv 1,2 \pmod 5$, $\ell m-4\equiv \ell m-4^3\equiv 0,1,4 \pmod 5$ and when $\ell m\equiv 1,2 \pmod 5$, $\ell m-4^0\equiv \ell m-4^2\equiv 0,1 \pmod 5$, therefore there exists $k\in P_{\ell m}$ such that $\ell m-4^k$ is congruent to $0,1,4$ modulo 5. Hence in this case, we may define

$$n_{\ell,m}:=\min\{2^k : k\in P_{\ell m}\cap\{0,1,2,3\} \text{ and } \ell m-4^k\equiv 0,1,4 \pmod 5\}.$$

(i) Solvability of $E_m^{\mathscr{P}}$(1-3-3-0).

When $m=1$,

$$\begin{cases} 1=1^2+0^2+0^2+0^2,\\ 1\cdot 1+3\cdot 0+3\cdot 0+0\cdot 0=1\in\mathscr{P}.\end{cases}$$

When $m=2$,

$$\begin{cases} 2=1^2+0^2+0^2+1^2,\\ 1\cdot 1+3\cdot 0+3\cdot 0+0\cdot 1=1\in\mathscr{P}.\end{cases}$$

When $m=3$,

$$\begin{cases} 3=1^2+(-1)^2+1^2+0^2,\\ 1\cdot 1+3\cdot(-1)+3\cdot 1+0\cdot 0=1\in\mathscr{P}.\end{cases}$$

When $m\geq 5$, Corollary 2.1 guarantees that $E_m^{n_{19,m}}$(1-3-3-0) has an integral solution, and so does $E_m^{\mathscr{P}}$(1-3-3-0) since $n_{19,m}\in\mathscr{P}$.

(ii) Solvability of $E_m^{\mathscr{P}}$(1-2-3-5).

When $m=1$,

$$\begin{cases} 1=1^2+0^2+0^2+0^2,\\ 1\cdot 1+2\cdot 0+3\cdot 0+5\cdot 0=1\in\mathscr{P}.\end{cases}$$

When $m \geq 2$, Corollary 2.1 guarantees that $E_m^{n_{39,m}}(1\text{-}2\text{-}3\text{-}5)$ has an integral solution, and so does $E_m^{\mathscr{P}}(1\text{-}2\text{-}3\text{-}5)$ since $n_{39,m} \in \mathscr{P}$.

(iii) Solvability of $E_m^{\mathscr{P}}(1\text{-}2\text{-}4\text{-}0)$.

When $m = 1$,

$$\begin{cases} 1 = 1^2 + 0^2 + 0^2 + 0^2, \\ 1 \cdot 1 + 2 \cdot 0 + 4 \cdot 0 + 0 \cdot 0 = 1 \in \mathscr{P}. \end{cases}$$

When $m = 2$,

$$\begin{cases} 2 = (-1)^2 + 1^2 + 0^2 + 0^2, \\ 1 \cdot (-1) + 2 \cdot 1 + 4 \cdot 0 + 0 \cdot 0 = 1 \in \mathscr{P}. \end{cases}$$

When $m = 3$,

$$\begin{cases} 3 = (-1)^2 + 1^2 + 0^2 + 1^2, \\ 1 \cdot (-1) + 2 \cdot 1 + 4 \cdot 0 + 0 \cdot 1 = 1 \in \mathscr{P}. \end{cases}$$

When $m \geq 5$, Corollary 2.1 guarantees that $E_m^{n_{21,m}}(1\text{-}2\text{-}4\text{-}0)$ has an integral solution, and so does $E_m^{\mathscr{P}}(1\text{-}2\text{-}4\text{-}0)$ since $n_{21,m} \in \mathscr{P}$.

(iv) Solvability of $E_m^{\mathscr{P}}(1\text{-}1\text{-}2\text{-}5)$.

When $m = 1$,

$$\begin{cases} 1 = 1^2 + 0^2 + 0^2 + 0^2, \\ 1 \cdot 1 + 1 \cdot 0 + 2 \cdot 0 + 5 \cdot 0 = 1 \in \mathscr{P}. \end{cases}$$

When $m = 2$,

$$\begin{cases} 2 = (-1)^2 + 0^2 + 1^2 + 0^2, \\ 1 \cdot (-1) + 1 \cdot 0 + 2 \cdot 1 + 5 \cdot 0 = 1 \in \mathscr{P}. \end{cases}$$

When $m \geq 3$, Corollary 2.1 guarantees that $E_m^{n_{31,m}}(1\text{-}1\text{-}2\text{-}5)$ has an integral solution, and so does $E_m^{\mathscr{P}}(1\text{-}1\text{-}2\text{-}5)$ since $n_{31,m} \in \mathscr{P}$. This completes our proof. $\square$

Remark 3.1. One can see from the proof that if m is not divisible by 4, $\mathscr{P}$ can be replaced by $\{1, 2, 4, 8\}$. However, for m divisible by 4, $\mathscr{P}$ can not be reduced to a finite set. In fact, by [2, Theorem 2.6], it is easy to see that for each $r \in \mathbb{N}$,

$$R(2 \cdot 4^r) = \{(0, 0, 2^r, 2^r)\}.$$

Setting $T_r(a\text{-}b\text{-}c\text{-}d) := \{ax + by + cz + dt : x, y, z, t \in \mathbb{Z},\ x^2 + y^2 + z^2 + t^2 = 2 \cdot 4^r\}$, one may verify that

$$T_r(1\text{-}2\text{-}3\text{-}5) = \{\pm 2^r, \pm 2 \cdot 2^r, \pm 3 \cdot 2^r, \pm 4 \cdot 2^r, \pm 5 \cdot 2^r, \pm 6 \cdot 2^r, \pm 7 \cdot 2^r, \pm 8 \cdot 2^r\}.$$

Hence if $\mathscr{S}'$ is a subset of integers, and $E_m^{\mathscr{S}'}(1\text{-}2\text{-}3\text{-}5)$ has an integral solution for every positive integer m, then $\mathscr{S}'$ must have an intersection with $T_r(1\text{-}2\text{-}3\text{-}5)$ for each $r \in \mathbb{N}$, which forces $\mathscr{S}'$ to be infinite since $T_r(1\text{-}2\text{-}3\text{-}5) \cap T_{r+4}(1\text{-}2\text{-}3\text{-}5) = \emptyset$. Similar argument works for the remaining three quadruples.

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