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PROXIMAL NORMAL STRUCTURE ON BANACH SPACES

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ABSTRACT. Eldred, Kirk and Veeramani introduced the notion of proximal normal structure to prove the best proximity point theorems for relatively nonexpansive mappings and showed that uniformly convex Banach spaces has the proximal normal structure. In this paper, a characterization for the weak proximal normal structure is given. Using this characterization, it is proved that every weakly compact convex pair in a Banach space X has proximal normal structure whenever X satisfies: X is ε_0-inquadrate in every direction for some $\varepsilon_0 \in (0,1)$ or X has the modulus of $k-UC$ $\delta_X^k(1) > 0$, for $k \in \mathbb{N}$ or X has the modulus of k dimensional $U-$convexity $U_X^k(1) > 0$, for $k \in \mathbb{N}$ or X has the coefficient of noncompact convexity $\varepsilon_1(X) < 1$. Moreover, we generalize the notion ε_0-inquadrate in every direction to ε_0-inquadrate with respect to every $k-$dimensional subspace and showed that X has the weak proximal normal structure and the weak normal structure if $\varepsilon_0 \in (0,1)$. In case of $\varepsilon_0 \in [1,2)$, the Banach space X has the weak proximal normal structure with an additional assumption X has the WORTH property.

1. Introduction

Let A and B be nonempty bounded subsets of a Banach space X. The closed unit ball and the unit sphere of X are denoted by B_X and S_X respectively. Define for $a \in A$, $\delta(a,B) = \sup\{\|a-y\| : y \in B\}$, $\delta(A,B) = \sup_{a\in A}\{\delta(a,B)\}$, $dist(A,B) = \inf\{\|a-b\| : a \in A, b \in B\}$. The numbers $\delta(A,B)$ and $dist(A,B)$ are called respectively the diameter and distance between the sets A and B.

A pair of sets (A,B) is said to be a proximal pair [3] if for every $a \in A$, there exists an $b \in B$ such that $\|a-b\| = dist(A,B)$ and vice versa.

We say that a pair of sets (A,B) is nonempty bounded closed convex if both A and B are nonempty bounded closed convex sets in X. Similarly, the pair (A,B) is said to be weakly compact convex if both A and B are weakly compact convex sets.

Let $T : A\cup B \to A\cup B$ be a map. Then T is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. If $T(A) \subseteq A$ and $T(B) \subseteq B$, then T is called a non-cyclic map.

Eldred, Kirk and Veeramani [3] introduced the notion of proximal normal structure and proved:

Theorem 1.1. [3] *Let (A,B) be a nonempty weakly compact convex proximal pair having the proximal normal structure in a Banach space X. Suppose that $T : A\cup B \to A\cup B$ is a relatively nonexpansive map (i.e., for $x \in A, y \in B$ $\|Tx-Ty\| \le \|x-y\|$.).*

a) *If T is a non-cyclic map, then there exist $x \in A$ and $y \in B$ such that $Tx = x$, $Ty = y$ and $\|x-y\| = dist(A,B)$.*
b) *If T is a cyclic map, then there exists an $x \in A\cup B$ such that $\|x-Tx\| = dist(A,B)$.*

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Definition 1.2. [3] A nonempty bounded convex proximal pair (A,B) of a Banach space X is said to have the proximal normal structure if every convex proximal pair $(C,D) \subseteq (A,B)$ (i.e., $C \subseteq A$ and $D \subseteq B$) with $\delta(C,D) > dist(C,D) = dist(A,B)$ there exist points $c \in C$ and $d \in D$ satisfying

$$\delta(c,D) < \delta(C,D) \text{ and } \delta(d,C) < \delta(C,D).$$

We say that a Banach space has the proximal normal structure if every nonempty bounded convex proximal pair (A,B) has the proximal normal structure. A Banach space is said to have the weak proximal normal structure if every nonempty weakly compact convex pair (A,B) has the proximal normal structure. If we take $A = B$ in the definition 1.2, then the notion of proximal normal structure coincides with the normal structure concept [2].

Definition 1.3. [2] A nonempty bounded convex subset K of a Banach space X is said to have the normal structure if every nonempty convex subset F of K with $\delta(F) > 0$ there is a point $x \in F$ satisfying $\delta(x,F) < \delta(F)$.

We say that a Banach space X has the normal structure if every nonempty bounded closed convex set has the normal structure and X has the weak normal structure if every nonempty weakly compact convex set has the normal structure.

Kirk [10] proved that every nonexpansive self-map on a nonempty weakly compact convex subset K of a Banach space X has a fixed point whenever K has the normal structure. Motivated by this result, many interesting geometric conditions, implying the normal structure, on norms of Banach spaces were introduced. One may refer to [1, 5, 6, 7, 11, 13, 14] for various geometric conditions on norms implying the normal structure and the references therein.

It is known that the Banach space X has the weak normal structure if [13] X is ε_0-inquadrate in every direction for some $\varepsilon_0 \in (0,1)$ or [7] the characteristic of convexity $\varepsilon_0(X) < 1$ or [1] the modulus of $k-$uniform convexity $\delta_X^k(1) > 0$ or [5] the modulus of $k-$dimensional $U-$convexity $U_X^k(1) > 0$ or [7] the coefficient of noncompact convexity $\varepsilon_1(X) < 1$ or [14] X is uniformly rotund with respect to every $k-$dimensional subspace for some fixed $k \in \mathbb{N}$. Also, it is known from [9] that, UCED Banach spaces and Nearly uniformly convex Banach spaces have the proximal normal structure. Since $k-$uniformly convex Banach spaces are nearly uniformly convex, $k-$uniformly convex Banach space has the proximal normal structure.

Further, it is easy to see that the Banach space X has the (weak) normal structure whenever X has the (weak) proximal normal structure. Nevertheless, it is unclear that a Banach space X has the (weak) proximal normal structure whenever X has the (weak) normal structure. This motivates us to ask the following question:

Question: Does a Banach space X has the (weak) proximal normal structure whenever X has the (weak) normal structure?

In [9], Gopal and Veeramani gave the following characterization for the proximal normal structure.

Lemma 1.4. [9] *In a Banach space X, every weakly compact convex pair has proximal normal structure if there do not exist sequences $\{x_n\}$, $\{y_n\}$ such that*

$$x_n \xrightarrow{w} 0,$$
$$\|y_n\| \to 1,$$

$$\delta(\{x_n\},\{y_n\}) = 1,$$
$$\|x_n - y_n\| = dist(co\{x_m : m \in \mathbb{N}\}, co\{y_m : m \in \mathbb{N}\}) = d < 1, \text{ for all } n \in \mathbb{N},$$
$$dist(y_{n+1}, co\{x_1, x_2, \cdots, x_n\}) \to 1.$$

Remark 1.5. It is easy to see that for the sequence $\{x_n\}$ in Lemma 1.4 that the diameter $\delta(\{x_n\})$ satisfies $1-d \leq \delta(\{x_n\}) \leq 1+d$, where d is the distance $dist(co\{x_m : m \in \mathbb{N}\}, co\{y_m : m \in \mathbb{N}\})$ and $\delta(\{x_n\},\{y_n\}) = 1$. □

Using the lemma 1.4, the authors in [9] showed that a Banach space X has the weak proximal normal structure whenever X is UCED or X is nearly uniformly convex.

In this paper, it is proved that a Banach space X has the weak proximal normal structure whenever X is ε_0-inquadrate in every direction for some $\varepsilon_0 \in (0,1)$ or X has the modulus of k dimensional $U-$convexity $U_X^k(1) > 0$, for some $k \in \mathbb{N}$ or X has the modulus of $k-UC$ $\delta_X^k(1) > 0$, for some $k \in \mathbb{N}$ or X has the coefficient of noncompact convexity $\varepsilon_1(X) < 1$. Moreover, we generalize the notion ε_0-inquadrate in every direction to ε_0-inquadrate with respect to every $k-$dimensional subspace denoted by $\varepsilon_0 - URE_k$ and showed that such spaces have the weak proximal normal structure and the weak normal structure if $\varepsilon_0 \in (0,1)$. In case of $\varepsilon_0 \in [1,2)$, the $\varepsilon_0 - URE_k$ Banach space X has the weak proximal normal structure whenever it has the WORTH property.

2. Preliminaries

In this section, we give the definitions of some of the geometric coefficients used in this paper. Let X be a Banach space.

Definition 2.1. [7] The modulus of convexity in the direction of $z \in S_X$, $\delta_X(\varepsilon, z) : [0,2] \to [0,1]$, is defined as

$$\delta_X(\varepsilon, z) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in B_X, x - y = \lambda z \text{ and } |\lambda| \geq \varepsilon\}.$$

We say that X is uniformly convex in the direction of z if $\delta_X(\varepsilon, z) > 0$, for $\varepsilon \in (0,2]$ and X is uniformly convex in every direction (*UCED* in short) if $\delta_X(\varepsilon, z) > 0$ for all $z \in S_X$ and $\varepsilon \in (0,2]$.

Definition 2.2. [7] The modulus of convexity function, $\delta_X : [0,2] \to [0,1]$, is defined as

$$\delta_X(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in B_X \text{ and } \|x-y\| \geq \varepsilon\}.$$

We say that X is uniformly convex if $\delta_X(\varepsilon) > 0$ for $\varepsilon \in (0,2]$.

The coefficient of convexity of X is the number

$$\varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta_X(\varepsilon) = 0\}.$$

Definition 2.3. [1] The modulus of $k-$uniform rotundity of the Banach space X, $\delta_X^k(\varepsilon)$, is defined as

$$\delta_X^k(\varepsilon) = \inf\{1 - \frac{\|\sum_{i=1}^{k+1} x_i\|}{k+1} : \|x_i\| = 1 \; \forall \, i = 1, 2, \cdots, k+1 \text{ and } V(x_1, x_2, \cdots, x_{k+1}) \geq \varepsilon\},$$

where $V(x_1,x_2,\cdots,x_{k+1})$ is the $k-$dimensional volume of the convex hull $co\{x_1,x_2,\cdots,x_{k+1}\}$, that is,

$$V(x_1,x_2,\cdots,x_{k+1}) = \sup\{det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_{k+1}) \\ f_3(x_1) & f_3(x_2) & \cdots & f_3(x_{k+1}) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f_{k+1}(x_1) & f_{k+1}(x_2) & \cdots & f_{k+1}(x_{k+1}) \end{pmatrix} : f_i \in X^* \text{ and } \|f_i\| \leq 1 \; \forall \, i = 2,3,\cdots,k+1\}$$

We say that X is $k-$uniformly rotund, $k-UR$ in short, if $\delta_X^k(\varepsilon) > 0$, for $\varepsilon > 0$. The coefficient of $k-$convexity $\varepsilon_0^k(X)$ of a Banach space X is defined as follows

$$\varepsilon_0^k(X) = \sup\{\varepsilon \geq 0 : \delta_X^k(\varepsilon) = 0\}.$$

Definition 2.4. [7] The modulus of noncompact convexity of a Banach space X is the function $\triangle_X$: $[0,2] \to [0,1]$ defined by

$$\triangle_X(\varepsilon) = \inf\{1 - dist(0,A)\} = \inf\{1 - \inf_{x\in A}\{\|x\|\}\},$$

where the first infimum is taken over all convex subsets A of the unit ball with $\alpha(A) \geq \varepsilon$ and $\alpha(A) = \inf\{\varepsilon > 0 : A$ is contained in the union of a finite number of bounded sets each having diameter less than $\varepsilon\}$.

We say that the Banach space X is $\triangle-$uniformly convex if $\triangle_X(\varepsilon) > 0$ for $\varepsilon \in (0,2]$. The coefficient of noncompact convexity of X is the number

$$\varepsilon_1(X) = \sup\{\varepsilon \geq 0 : \triangle_X(\varepsilon) = 0\}.$$

3. Proximal Normal Structure

The following lemma gives a characterization for the weak proximal normal structure.

Lemma 3.1. *In a Banach space X, every weakly compact convex pair has proximal normal structure if there do not exist sequences $\{x_n\}$, $\{y_n\}$ such that*

$$\begin{gathered} x_n \xrightarrow{w} 0, \\ \delta(\{x_n\}) \geq 1, \\ \|y_n\| \to 1+d, \\ \|x_n - y_n\| = d, \text{ for all } n \in \mathbb{N}, \\ dist(y_{n+1}, co\{x_1,x_2,\cdots,x_n\}) \to 1+d, \end{gathered}$$

where $d = dist(co\{x_m : m \in \mathbb{N}\}, co\{y_m : m \in \mathbb{N}\})$ and $\delta(\{x_n\},\{y_n\}) = 1+d$.

Proof Suppose that the Banach space X does not have the weak proximal normal structure. Then by the lemma 1.4 there exist sequences $\{x_n\}$, $\{y_n\}$ such that

$$\begin{gathered} x_n \xrightarrow{w} 0, \\ \|y_n\| \to 1, \end{gathered}$$

$$\delta(\{x_n\},\{y_n\}) = 1,$$
$$\|x_n - y_n\| = dist(co\{x_m : m \in \mathbb{N}\}, co\{y_m : m \in \mathbb{N}\}) = d' < 1, \text{ for all } n \in \mathbb{N},$$
$$dist(y_{n+1}, co\{x_1, x_2, \cdots, x_n\}) \to 1.$$

Let $u_n = \dfrac{x_n}{1-d'}$ and $v_n = \dfrac{y_n}{1-d'}$, for all $n \in \mathbb{N}$. Then the sequences $\{u_n\}$ and $\{v_n\}$ satisfy:

$$u_n \xrightarrow{w} 0,$$
$$\delta(\{u_n\}) \geq 1,$$
$$\|v_n\| \to \frac{1}{1-d'},$$
$$\|u_n - v_n\| = \frac{d'}{1-d'} = dist(co\{u_m : m \in \mathbb{N}\}, co\{v_m : m \in \mathbb{N}\}), \text{ for all } n \in \mathbb{N},$$
$$dist(v_{n+1}, co\{u_1, u_2, \cdots, u_n\}) \to \frac{1}{1-d'}.$$

Note that $\delta(\{u_n\},\{v_n\}) = \dfrac{1}{1-d'}\delta(\{x_n\},\{y_n\}) = 1 + \dfrac{d'}{1-d'} = 1 + d,$

where $d = dist(co\{u_m : m \in \mathbb{N}\}, co\{v_m : m \in \mathbb{N}\})$. Hence proved. □

Remark 3.2. If (A,B) is a nonempty weakly compact convex proximal pair with $\delta(a,B) = \delta(A,B)$ for all $a \in A$ and $\delta(A,B) = \delta(A) + dist(A,B)$, then the sequence $\{x_n\}$ in the lemma 3.1 can be chosen such that $\delta(\{x_n\}) = 1$. □

The next result is used in the sequel.

Proposition 3.3. *Let $\{x_n\}$ and $\{y_n\}$ be sequences as in the lemma 3.1. Then*

a) $\lim\limits_{n\to\infty} \|x - y_n\| = D$, *for $x \in \overline{co}\{x_n\}$, where $D = \delta(\{x_n\},\{y_n\}) = 1 + d$ and $d = \|x_n - y_n\|$ for $n \in \mathbb{N}$,*

b) *for every $\varepsilon > 0$ and for $l \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $\|x_m - x_l\| > 1 - \varepsilon$ for $m > N$.*

c) *for every $\varepsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_l} - x_{n_k}\| > 1 - \varepsilon$ for $l \neq k$.*

Proof a) Fix $k \in \mathbb{N}$ and let $x \in co\{x_1, x_2, \cdots, x_k\}$.

Since $dist(y_{n+1}, co\{x_1, x_2, \cdots, x_n\}) \leq \|y_{n+1} - x\|$ for $k \leq n$, we have $\lim\limits_{n\to\infty} \|x - y_n\| = D$ for every $x \in co(\{x_n\})$. Now from the continuity of the function $f(x) = \lim\limits_{n\to\infty} \|x - y_n\|$, we get $\lim\limits_{n\to\infty} \|x - y_n\| = D$ for every $x \in \overline{co}(\{x_n\})$.

b) Since $\lim\limits_{n\to\infty} \|y_n - x_l\| = D$, for $\varepsilon \in (0, D-d)$ we can find an $N \in \mathbb{N}$ such that $\|y_n - x_l\| \geq D - \varepsilon$ for $n \geq N$. Then for $m > N$, we get $\|x_m - x_l\| \geq \|y_m - x_l\| - \|x_m - y_m\| > D - \varepsilon - d \geq 1 - \varepsilon$, as $D = 1 + d$.

c) Let $\varepsilon > 0$ and $n_1 = 1$. Now by b), there exists $n_2 \in \mathbb{N}$ such that $\|x_1 - x_n\| \geq 1 - \varepsilon$ for all $n \geq n_1$. For x_{n_2}, there exists $n_3 \in \mathbb{N}$ with $n_3 > n_2$ such that $\|x_{n_2} - x_n\| \geq 1 - \varepsilon$ for all $n \geq n_3$. Then $\|x_{n_i} - x_{n_j}\| \geq 1 - \varepsilon$ for $i \neq j$ and $i, j \in \{1,2,3\}$. By continuing this way, we can construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\|x_{n_l} - x_{n_k}\| \geq 1 - \varepsilon$ for $l \neq k$. □

Theorem 3.4. *If a Banach space X is ε_0-inquadrate in every direction for some fixed ε_0 in $(0,1)$, then every weakly compact convex pair (A,B) in X has the proximal normal structure.*

Proof Assume that X does not the proximal normal structure. Then there exists a pair of sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying the lemma 3.1.

Since $\delta(\overline{co}(\{x_n\})) \geq 1$, we can choose points u and v in $\overline{co}(\{x_n\})$ such that $\|u-v\| \geq \varepsilon_0$. Since $\lim_{n\to\infty} \|x-y_n\| = D$ for every $x \in \overline{co}(\{x_n\})$, we get $\lim_{n\to\infty} \|\frac{u+v}{2} - y_n\| \leq D(1-\delta_X(\varepsilon_0)) < D$. Hence X has the weak proximal normal structure. □

As a Banach space X has the coefficient of convexity $\varepsilon_0(X) < 1$ implies that X is ε_0-inquadrate in every direction, we get:

Corollary 3.5. *If X is a Banach space with the characeristic of convexity $\varepsilon_0(X) < 1$, then X has the proximal normal structure.*

Theorem 3.6. *If a Banach space X has the modulus of $k-$dimensional UC $\delta_X^k(1) > 0$, then X has the proximal normal structure.*

Proof Assume the contrary. Then by lemma 3.1 there exist sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$x_n \xrightarrow{w} 0 \text{ and } \delta(\{x_n\}) \geq 1$$
$$\|y_n\| \to \delta(\{x_n\},\{y_n\}),$$
$$\|x_n - y_n\| = dist(co\{x_m : m \in \mathbb{N}\}, co\{y_m : m \in \mathbb{N}\}), \text{ for all } n \in \mathbb{N},$$
$$dist(y_{n+1}, co\{x_1, x_2, \cdots, x_n\}) \to D := \delta(\{x_n\},\{y_n\}).$$

Since $\delta_X^k(1) > 0$ and $\delta_X^k(\cdot)$ is continuous on $[0,2)$, we can find $\varepsilon > 0$ such that $\delta_X^k(1-\varepsilon) > 0$. Now, choose $\varepsilon_1 \in (0,1)$ such that $1 > \varepsilon_1 + \sqrt[k]{1-\varepsilon}$. Since $x_n \xrightarrow{w} 0$ and $\delta(\{x_n\}) \geq 1$, we can find a linearly independent subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\|x_{n_l} - x_1\| \geq 1 - \varepsilon_1$ for $n_l \geq N(\varepsilon_1)$.

Without loss of generality assume that $\{x_n\}$ is a linearly independent set and $\|x_n - x_1\| \geq 1 - \varepsilon_l$ for all $n \neq 1$. Consider the finite set $\{x_1, x_2, \cdots, x_{k+1}\}$. By the Hahn-Banach theorem there exists linear functionals $f_{j+1} \in S_{X^*}$ such that $f_{j+1}(x_{j+1} - x_1) = \|x_{j+1} - x_1\|$ and $f_{j+1}(x_l) = 0$ for $j+1 \neq l$. Then the $k-$dimensional volume $V(x_1, x_2, x_3, \cdots, x_{k+1}) \geq (1-\varepsilon_1)^k > 1 - \varepsilon$.

Thus $\lim_{m\to\infty} \|y_m - \frac{x_1 + x_2 + x_3 + \cdots + x_{k+1}}{k+1}\| \leq (1 - \delta_X^k(1-\varepsilon))D < D$, for all $m \in \mathbb{N}$, which is a contradiction to $\lim_{m\to\infty} \|y_m - x\| = D$ for all $x \in \overline{co}(\{x_n\})$. Hence X has the proximal normal structure. □

Theorem 3.7. *If a Banach space X has the coefficient of noncompact convexity $\varepsilon_1(X) < 1$, then every weakly compact convex pair (A,B) has the proximal normal structure.*

Proof Suppose that there are sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying the lemma 3.1. Now, from the proposition 3.3, for every $\varepsilon > 0$ we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_l} - x_{n_k}\| \geq 1 - \varepsilon$ for $l \neq k$. Without loss of generality, assume that $\|x_n - x_m\| \geq 1 - \varepsilon > \varepsilon_1(X)$, for $n \neq m$. Then the measure of noncompactness of $\{x_n\}$ is $\alpha(\{x_n\}) > \varepsilon_1(X)$.

Since the sequence $\{x_n - y_k\}$ converges weakly to y_k and $\alpha(\{x_n - y_k\}) = \alpha(\{x_n\})$, we get $\|y_k\| \leq D(1 - \triangle_X(1-\varepsilon)) < D$ for all $k \in \mathbb{N}$. This is a contradiction to $\lim_{k\to\infty} \|y_k\| = D$. Hence the Banach space X has the proximal normal structure if $\varepsilon_1(X) < 1$. □

Since $\varepsilon_1(X) = 0$ for a nearly uniformly convex Banach space X, we get:

Corollary 3.8. [9] *In a nearly uniformly convex Banach space X, every nonempty weakly compact convex pair has the proximal normal structure.*

3.1. $\varepsilon_0 - URE_k$ ***Banach spaces.*** Sims [13] proved that if a Banach space X is ε_0-inquadrate in every direction for some $\varepsilon_0 \in [1,2)$ and X has the WORTH property, then X has the weak normal structure.

A Banach space X is said to have the WORTH property [12] if for every weakly null sequence (x_n) (i.e., $x_n \xrightarrow{w} 0$) in X we have

$$\|x_n + x\| - \|x_n - x\| \to 0 \; for \; all \; x \in X.$$

The authors in [14] introduced the notion of uniformly rotund with respect to every $k-$dimensional subspace (in short, URE_k) and proved that all such spaces have the weak normal structure.

Definition 3.9. [14] For a $k-$dimensional subspace Y of a normed linear space X, we define for each $\varepsilon > 0$

$\delta_X(\varepsilon, Y) = \inf\left\{1 - \dfrac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| : x_1, \cdots, x_{k+1} \in B_X, V(x_1, \cdots, x_{k+1}) \geq \varepsilon \;\; and \; span\{x_1 - x_{k+1}, x_2 - x_{k+1}, \cdots, x_k - x_{k+1}\} = Y\right\}$, where $V(x_1, \cdots, x_{k+1})$ denotes the $k-$dimensional volume of the $k+1$ points $x_1, x_2, \cdots, x_{k+1}$ defined by $V(x_1, \cdots, x_{k+1}) = \sup\left\{det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_{k+1}) \\ f_3(x_1) & f_3(x_2) & \cdots & f_3(x_{k+1}) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f_{k+1}(x_1) & f_{k+1}(x_2) & \cdots & f_{k+1}(x_{k+1}) \end{pmatrix}\right\}$, and the functionals $f_i \in S_{X^*}$ for $i = 2, 3, \cdots, k+1$.

The space X is said to be uniformly rotund with respect to Y if $\delta_X(\varepsilon, Y) > 0$ for all $\varepsilon > 0$. The space X is said to be uniformly rotund with respect to $k-$dimensional subspaces if $\delta_X(\varepsilon, Y) > 0$ for all $k-$dimensional subspaces Y and for all $\varepsilon > 0$. Such spaces are denoted by URE_k.

Motivated by the notion of ε_0-inquadrate in every direction by Sims [13] and the URE_k spaces, we define:

Definition 3.10. A Banach space X is said to be ε_0-inquadrate with respect to $k-$ dimensional subspaces (in short, $\varepsilon_0 - URE_k$) if $\delta_X(\varepsilon_0, Y) > 0$ for some $\varepsilon_0 > 0$ and for every $k-$dimensional subspace Y of X.

Remark 3.11. **Examples of ε_0-inquadrate Banach spaces:**

It is known from [8] that $\|(x,y)\|_a = \max\{\|(x,y)\|_\infty, \lambda_0\|(x,y)\|_1\}$ is a norm on $\mathbb{R}^2$ where $\lambda_0 = \dfrac{1+\sqrt{2}}{2+\sqrt{2}} < 1$ and the unit sphere $S[0,1]$ is the regular hexagon with side-length λ_0.

Hence $\mathbb{R}^2$ is not strictly convex w.r.to the norm $\|\cdot\|_a$. However, if $(x_1,y_1),(x_2,y_2) \in B[0,1]$ with $\|(x_1,y_1)-(x_2,y_2)\|_a > \lambda_0$, then $\|(x_1+x_2, y_1+y_2)\|_a < \|(x_1,y_1)\|_a + \|(x_2,y_2)\|_a$. Since the unit ball $B[0,1]$ is compact, we have (for $\varepsilon > \lambda_0$)

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2}\|(x_1+x_2, y_1+y_2)\|_a : (x_1,y_1),(x_2,y_2) \in B[0,1] \; with \; \|(x_1,y_1)-(x_2-y_2)\|_a \geq \varepsilon\} > 0.$$

Therefore the Banach space $(\mathbb{R}^2, \|\cdot\|_a)$ is uniformly $\varepsilon-$inquadrate for $\varepsilon > \lambda_0 = \dfrac{1+\sqrt{2}}{2+\sqrt{2}}$.

It is known [15] that $c_0(\mathbb{N})$ is $UCED$ with respect to the norm $\|(x(n))\|_b = \sqrt{\|(x_n)\|_\infty^2 + \sum_{n=1}^{\infty} \frac{|x(n)|^2}{4^n}}$.

Consider the Banach space $X = \{(x,y) : x \in \mathbb{R}^2, y \in c_0(\mathbb{N})\}$ with the norm $\|(x,y)\| = \sqrt{\|x\|_a^2 + \|y\|_b^2}$.

Since $(\mathbb{R}^2, \|\cdot\|_a)$ is isometrically isomorphic to the subspace $Y = \{(x,y) \in X : y$ is the zero vector in $c_0(\mathbb{N})\}$ of X, we have X is not strictly convex. This implies that X is not uniformly convex in every direction (UCED). Moreover, X is not reflexive, as $c_0(\mathbb{N})$ is isometrically isomorphic to the subspace $X_1 = \{(x,y) \in X : x = (0,0)\}$.

We claim that X is $\varepsilon-$inquadrate for $\varepsilon > \lambda_0$. Let $(x_0,y_0) \in X$ such that $\|(x_0,y_0)\| = 1$. That is

$$\delta_{(x_0,y_0)}(X,\varepsilon) = \inf\{1 - \frac{1}{2}\|(x+u, y+v)\| : \|(x,y)\| \le 1, \|(u,v)\| \le 1 \; with \; (x-u, y-v) = \varepsilon\,(x_0,y_0)\} > 0.$$

Suppose that $\delta_{(x,y)}(X,\varepsilon) = 0$. Then there exists sequences (x_n,y_n) and (u_n,v_n) in the unit ball $B[0,1]$ of X such that $(x_n - u_n, y_n - v_n) = \varepsilon\,(x_0,y_0)$ for all n and $\lim_{n\to\infty} \|(x_n+u_n, y_n+v_n\| = 2$.

Case 1: Let $\varepsilon\|x_0\|_a > \lambda_0$.

Since $(\mathbb{R}^2, \|\cdot\|_a)$ is $\varepsilon-$inquadrate for $\varepsilon > \lambda_0$ and $\|x_n - u_n\|_a = \varepsilon\|x_0\|_a > \lambda_0$, we have $\frac{1}{2}\|x_n+u_n\|_a \le \frac{\|x_n\|_a + \|u_n\|_a}{2} - \delta$ for some $\delta > 0$ and for all n. Then

$$\begin{aligned} 2 &= \limsup_{n\to\infty} \|(x_n+u_n, y_n+v_n)\| \\ &= \limsup_{n\to\infty} \sqrt{\|x_n+u_n\|_a^2 + \|y_n+v_n\|_b^2} \\ &\le \limsup_{n\to\infty} \sqrt{(\|x_n\|_a + \|u_n\|_a - 2\delta)^2 + (\|y_n+v_n\|_b)^2} \\ &\le \limsup_{n\to\infty} \|(x_n,y_n)\| + \|(u_n,v_n)\| \le 2. \end{aligned}$$

Hence $\limsup_{n\to\infty} [\|x_n\|_a + \|u_n\|_a - 2\delta] = \limsup_{n\to\infty}[\|x_n\|_a + \|u_n\|_a]$, when $\|x_n - u_n\|_a = \varepsilon\|x_0\|_a > \lambda_0$. This is a contradiction to $\mathbb{R}^2$ is $\varepsilon-$inquadrate for $\varepsilon > \lambda_0$.

Hence $\delta_{(x_0,y_0)}(X,\varepsilon) > 0$ if $\varepsilon > \lambda_0$ and $\varepsilon\|x_0\|_a > \lambda_0$.

Case 2: Let $\varepsilon\|x_0\|_a \le \lambda_0$.

Since $\varepsilon\|(x_0,y_0)\| > \lambda_0$, we have $\varepsilon\|y_0\|_b > 0$. Since $c_0(\mathbb{N})$ is UCED and $\|y_n - v_n\|_b = \varepsilon\|y_0\|_b > 0$, there exists a $\delta > 0$ such that $\frac{1}{2}\|y_n+v_n\|_b \le \frac{\|y_n\|_b + \|v_n\|_b}{2} - \delta$ for all n. Then, by using the arguments of the Case 1, it is easy to see that $\limsup_{n\to\infty}[\|y_n\|_b + \|v_n\|_b - 2\delta] = \limsup_{n\to\infty}[\|y_n\|_b + \|v_n\|_b]$, when $\|y_n - v_n\|_b = \varepsilon\|y_0\|_b > 0$. This is a contradiction to $c_0(\mathbb{N})$ is UCED. Hence $\delta_{(x_0,y_0)}(X,\varepsilon) > 0$ if $\varepsilon > \lambda_0$ and $\varepsilon\|x_0\|_a \le \lambda_0$.

Therefore the Banach space $X = \{(x,y) : x \in \mathbb{R}^2, y \in c_0(\mathbb{N})\}$ with the norm $\|(x,y)\| = \sqrt{\|x\|_a^2 + \|y\|_b^2}$ is $\varepsilon-$inquadrate for in every direction $\varepsilon > \lambda_0 = \frac{1+\sqrt{2}}{2+\sqrt{2}}$. □

The next theorem can be proved in a similar way as theorem 3.6.

Theorem 3.12. *If a Banach space X is $\varepsilon_0 - URE_k$ for some fixed $\varepsilon_0 \in (0,1)$, then every weakly compact convex pair (A,B) has the proximal normal structure. In particular, X has the weak normal structure.*

Corollary 3.13. *If a Banach space X is URE_k, then every weakly compact convex pair (A,B) has the proximal normal structure.*

Theorem 3.14. *If a Banach space X is $\varepsilon_0 - URE_k$ for some $\varepsilon_0 \in [1,2)$ and X has the WORTH property, then X has the weak normal structure.*

Proof Suppose that X does not have the weak normal structure. Then there exists a weakly null sequence (x_n) in X (i.e., x_n converges weakly to 0) and $dist(x_{n+1}, co(\{x_1, x_2, \cdots, x_n\})) \to 1$. This implies that $\lim_{n\to\infty} \|x_n - x\| = 1$, for all $x \in \overline{co}(\{x_n\})$. Since (x_n) is a weakly null sequence, we assume that $\{x_n\}$ is linearly independent.

Since X has the WORTH property, we get $\lim_{n\to\infty} \|x_n + x\| = 1$ for all $x \in \overline{co}\{x_n\}$. Choose $\varepsilon \in (0,1)$ such that $1 > \varepsilon + \sqrt[k]{\frac{\varepsilon_0}{2}}$. Since $\lim_{n\to\infty} \|x_n\| = 1$, for $\varepsilon \in (0,1)$ there exists $N \in \mathbb{N}$ such that $\|x_j\| \geq 1 - \varepsilon$ for all $n \geq N$.

Let $y_i = x_{N+i}$ for $i = 2, 3, \cdots, k+1$ and $y_1 = -y_2$. Since $\{y_2, y_3, \cdots, y_{k+1}\}$ is linearly independent, there are linear functionals $f_i \in S_{X^*}$ such that $f_i(y_i) = \|y_i\|$ and $f_i(y_j) = 0$ for $i \neq j$. Then

$$\begin{aligned} V(y_1, y_2, \cdots, y_{k+1}) &\geq det\{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ f_2(y_1) & f_2(y_2) & \cdots & f_2(y_{k+1}) \\ f_3(y_1) & f_3(y_2) & \cdots & f_3(y_{k+1}) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f_{k+1}(y_1) & f_{k+1}(y_2) & \cdots & f_{k+1}(y_{k+1}) \end{pmatrix}\} \\ &\geq 2\prod_{i=2}^{k} \|y_i\| = 2\prod_{i=2}^{k} \|x_{N+i}\| \geq 2(1-\varepsilon)^k > \varepsilon_0 \end{aligned}$$

Hence $\lim_{n\to\infty} \frac{1}{k+1} \| \sum_{i=1}^{k+1} x_n - y_i \| \leq (1 - \delta_X(V(y_1, y_2 \cdots, y_{k+1}), Y)) < 1$, where $Y = Span(\{y_1, y_2 \cdots, y_{k+1}\})$. Since $y_1 + y_2 = 0$ and $0 \in \overline{co}(\{x_n\})$, we have $z = \frac{1}{k+1} \sum_{i=3}^{k+1} y_i \in co\{x_n\}$ and $\frac{1}{k+1}(\sum_{i=1}^{k+1} x_n - y_i) = x_n - \frac{1}{k+1}(\sum_{i=3}^{k+1} y_i)$.

Therefore $\lim_{n\to\infty} \|x_n - z\| < 1$, where $z = \frac{1}{k+1} \sum_{i=3}^{k+1} y_i \in co\{x_n\}$, which is a contradiction. Thus the Banach space X has the weak normal structure if X is $\varepsilon_0 - URE_k$ for $\varepsilon_0 \in [1,2)$ and X has the WORTH property. □

The next theorem can be proved in a similar way.

Theorem 3.15. *If a Banach space X is $\varepsilon_0 - URE_k$ for $\varepsilon \in [1,2)$ and X has the WORTH property, then X has the weak proximal normal structure.*

Since $k-UC$ implies $k-UCED$, we have:

Corollary 3.16. *If a Banach space X has the modulus of $k-UC$ $\delta_X^k(\varepsilon_0) > 0$ for some $\varepsilon_0 \in [1,2)$ and the WORTH property, then X has the weak proximal normal structure. In particular, X has the weak normal structure.*

3.2. ***$U-$convex Banach spaces.*** Gao and Satit [5] introduced the notion of $k-$dimensional $U-$convex Banach space and proved that such spaces have the normal structure if the modulus of $k-$dimensional $U-$convexity $U_X^k(1) > 0$.

The modulus of $k-$dimensional $U-$convexity is defined as follows:

Definition 3.17. [5] Let X be a Banach space. The modulus of $k-$dimensional $U-$convexity function of X is defined by

$U_X^k(\varepsilon) = \inf\left\{1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| : x_1, \cdots, x_{k+1} \in S_X, v(x_1, \cdots, x_{k+1}) \geq \varepsilon\right\},$

where

$$v(x_1, \cdots, x_{k+1}) = \sup\{det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_{k+1}) \\ f_3(x_1) & f_3(x_2) & \cdots & f_3(x_{k+1}) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f_{k+1}(x_1) & f_{k+1}(x_2) & \cdots & f_{k+1}(x_{k+1}) \end{pmatrix} :$$
$$f_i \in S_{X^*} \; satisfies \; f_i(x_i) = 1 \; for \; i = 2, 3, \cdots, k+1\}$$

We say that a Banach space X is $k-$dimensional $U-$convex if $U_X^k(\varepsilon) > 0$ for all $\varepsilon \in (0, v_X^k)$, where $v_X^k = \sup\{v(x_1, x_2, \cdots, x_{k+1}) : x_1, \cdots, x_{k+1} \in S_X\}$.

Theorem 3.18. *If a Banach space X has the modulus of $k-$ dimensional $U-$convexity $U_X^k(1) > 0$, for some $k \in \mathbb{N}$, then X has the proximal normal structure.*

Proof Assume the contrary. Then by lemma 3.1 there exist sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$x_n \xrightarrow{w} 0 \text{ and } \delta(\{x_n\}) \geq 1$$
$$\|y_n\| \to \delta(\{x_n\}, \{y_n\}),$$
$$\|x_n - y_n\| = dist(co\{x_m : m \in \mathbb{N}\}, co\{y_m : m \in \mathbb{N}\}), \text{ for all } n \in \mathbb{N},$$
$$dist(y_{n+1}, co\{x_1, x_2, \cdots, x_n\}) \to D := \delta(\{x_n\}, \{y_n\}).$$

Since $x_n \xrightarrow{w} 0$ and $\delta(\{x_n\}) \geq 1$, we can find a linearly independent subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\|x_{n_l} - x_1\| \geq 1 - \varepsilon_1$ for $n_l \geq N(\varepsilon_1)$. Without loss of generality assume that $\{x_n\}$ is linearly independent.

Consider the finite linearly independent set $\{x_1, x_2, \cdots, x_{k+1}\}$. Since $\lim_{n\to\infty}\|y_n - x\| = D$ for all $x \in \overline{co}(\{x_j\})$, choose $N \in \mathbb{N}$ such that

$$D - \varepsilon < \|x_i - y_n\| \leq D,$$

for all $n \geq N$ and $i = 1, 2, \cdots, k+1$. Now, for $n \geq N$, define $z_i = \frac{x_i - y_n}{\|x_i - y_n\|}$ for $i = 1, 2, \cdots, k+1$. Then by the Hahn-Banach theorem there exists linear functionals $f_i \in S_{X^*}$ such that $f_i(z_i) = 1$ and $f_i(z_j) = 0$

for $i \neq j$. Then $v(z_1, z_2, \cdots, z_{k+1}) \geq 1$ for all $n \geq N$. This implies that

$$\|\frac{1}{k+1}\sum_{i=1}^{k+1} z_i\| \leq (1 - U_X^k(1)) < 1.$$

Now by the triangle inequality, we get

$$\frac{1}{k+1}\|\sum_{i=1}^{k+1}\frac{x_i - y_n}{D}\| \leq (1 - U_X^k(1)) + \frac{1}{k+1}\|\sum_{i=1}^{k+1}(x_i - y_n)\left(\frac{1}{D} - \frac{1}{\|x_i - y_n\|}\right)\|.$$

Since $\lim_{n\to\infty}\|x_i - y_n\| = D > 0$ for all $i \in \mathbb{N}$, we get

$$\lim_{n\to\infty}\|y_n - \frac{1}{k+1}\sum_{i=1}^{k+1} x_i\| \leq D(1 - U_X^k(1)) + 0 < D,$$

which is a contradiction to $\lim_{n\to\infty}\|y_n - x\| = D$ for all $x \in \overline{co}(\{x_j\})$.

Hence the Banach space X has the proximal normal structure whenever the modulus of $k-$dimensional $U-$convexity $U_X^k(1) > 0$. □

Using the proof technique of the theorem 3.14, we can prove:

Theorem 3.19. *If a Banach space X has the modulus of $k-$dimensional $U-$convexity $U_X^k(\varepsilon) > 0$ for some $\varepsilon \in [1,2)$ and X satisfies the WORTH property, then X has the weak proximal normal structure. In particular, X has the weak normal structure.*

In general, it is unknown that a Banach space X has the proximal normal structure whenever X has the normal structure. However, we observe:

Remark 3.20. Let (A,B) be a nonempty weakly compact convex proximal pair in a Banach space X with $\delta(A,B) = \delta(A) + dist(A,B) > \delta(A)$. If $\delta(x,B) = \delta(A,B)$ for all $x \in A$, then A does not have the normal structure. Moreover, if X is strictly convex Banach space, then both A and B do not have the normal structure.

Proof Suppose that (A,B) is a nonempty proximal pair satisfying

a) $\delta(A,B) = \delta(A) + dist(A,B)$ and
b) $\delta(x,B) = \delta(A,B)$ for all $x \in A$.

Then by the lemma 3.1 there are sequences $\{x_n\} \in A$ and $\{y_n\} \in B$ such that

$$x_n \xrightarrow{w} 0 \text{ and } \delta(\{x_n\}) = \delta(A),$$
$$\|y_n\| \to \delta(\{x_n\},\{y_n\}) = \delta(A,B),$$
$$\|x_n - y_n\| = dist(co\{x_m : m \in \mathbb{N}\}, co\{y_m : m \in \mathbb{N}\}) = d, \text{ for all } n \in \mathbb{N},$$
$$dist(y_{n+1}, co\{x_1, x_2, \cdots, x_n\}) \to \delta(\{x_n\},\{y_n\}).$$

Note that $\|y_n - x\| - d = \|y_n - x\| - \|x_n - y_n\| \leq \|x_n - x\|$, for all n and $x \in \overline{co}(\{x_n\})$. Since $\lim_{n\to\infty}\|y_n - x\| = \delta(A,B)$, we have $\limsup_{n\to\infty}\|x_n - x\| = \delta(A)$. Hence A does not have the normal structure.

Now, assume that X is strictly convex. Then it is known from [4] that there is a unique $h \in X$ such that $B = A + h$ and $\|h\| = dist(A,B)$. Since A does not have the normal structure and $B = A + h$, the set B does not have the normal structure. □

4. Relatively Nonexpansive Mappings

As the theorems 3.4, 3.6, 3.7, 3.12, 3.15, 3.18, and 3.19 give sufficient conditions for the proximal normal structure, we claim the following theorem for relatively nonexpansive mappings.

Theorem 4.1. *Let X be a Banach space as mentioned in any one of the theorems 3.4, 3.6, 3.7, 3.12, 3.15, 3.18, 3.19. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map (i.e., for $x \in A, y \in B$ $\|Tx - Ty\| \leq \|x - y\|$.), where (A,B) is a nonempty weakly compact convex proximal pair in X.*

a) *If T is a non-cyclic map, then there exist $x \in A$ and $y \in B$ such that $Tx = x$, $Ty = y$ and $\|x - y\| = dist(A,B)$.*
b) *If T is a cyclic map, then there exists an $x \in A \cup B$ satisfying $\|x - Tx\| = dist(A,B)$.*

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