On isolated points of the surjective spectrum of a closed linear relation

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Abstract

We characterize the isolated points of the surjective spectrum of a closed linear relation acting on a complex Banach space by means of the local spectral theory. The found results generalize some known characterizations in the frame of bounded operators to include the cases of closed operators and more generally closed linear relations.

Key words: Linear relation, local spectral theory, surjective spectrum.
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Introduction

Throughout this paper, $(X.\|.\|)$ will denote a complex Banach space. The first phase of our investigation consists in studying the local spectral theory of closed linear relations. More precisely, we develop the basic properties of the local and glocal spectral subspaces, the quasinilpotent part $H_0(T)$ and the analytic core $K(T)$ of a closed not necessary bounded linear relation T. This seem a generalization of the later developments of this theory in both cases of closed and bounded linear relations and bounded operators. As an application, we use the found results to give some properties of isolated points of the surjective spectrum of a closed linear relation. To describe these achievements, let's start by recalling some known results in the case of bounded operators. In 2008, González et al. [5] have shown that if T is a bounded operator then

$$\lambda \text{ is an isolated point of the surjective spectrum of } T \text{ if and only if } X = H_0(T-\lambda I) + K(T-\lambda I).$$

Recently, the characterization of isolated points of the spectrum has been extended in [6] to the case of linear relations. It was proved that for a closed and bounded linear relation T such that 0 is a point of its spectrum, we have the equivalence:

$$0 \text{ is an isolated in the spectrum of } T \text{ if and only if } H_0(T) \text{ and } K(T) \text{ are closed and } X = H_0(T) \oplus K(T).$$

25 Jul 2021 14:49:52 PDT
210725-Mnif Version 1 - Submitted to Rocky Mountain J. Math.

All of the above motivated us to establish a necessary and sufficient condition for which a point of the surjective spectrum of a closed linear relation be isolated. This will be considered as a continuation of the study made in the case of linear relations but also a generalization of the investigation carried out for the case of operators since it covers the case of closed operators which are not necessary bounded. More precisely, the purpose of this paper is to show that, under some conditions, the results mentioned above remain valid in the general setting of closed linear relations. The demonstrations provided are essentially based on the local spectral theory that we have developed in the framework of closed linear relations. We now describe this approach in greater detail.
Section 1: The first section is mainly dedicated to introduce the basic tools of linear relations. Then, we study different types of invariance of a subspace of X by a linear relation. Section 2: The first part of this section focuses on providing several tools for studying the local spectral theory. Additionally, we give a further look at Leiterer's result [7, Theorem 3.2.1] which is needed later. Subsequently, we deal with a localized version of the quasinilpotent part and the analytic core of a closed linear relation. Section 3: In this section, we are interested in closed linear relations that verify two further conditions. For these classes of linear relations we derive more properties of the local and glocal spectral subspaces and the quasinilpotent part $H_0(T)$. These results are then applied to characterize the isolated points of the surjective spectrum of a linear relation.

1. Preliminaries

In this first section, a brief introduction of the linear relation theory is given. We essentially aim to recall some basic definitions and properties which are needed in the rest of this work. A linear relation (or a multivalued linear operator) in a Banach space X, $T : X \to X$, is a mapping from a subspace $D(T) = \{x \in X : Tx \neq \emptyset\}$, called the domain of T into the set of nonempty subsets of X verifying $T(\alpha_1 x + \alpha_2 y) = \alpha_1 T(x) + \alpha_2 T(y)$ for all non zero scalars α_1, α_2 and vectors x and $y \in D(T)$. We denote by $\mathcal{LR}(X)$ the class of all linear relations in X. A linear relation $T \in \mathcal{LR}(X)$ is completely determined by its graph defined by $G(T) := \{(x, y) \in X \times X : x \in D(T), y \in Tx\}$. Let $T \in \mathcal{LR}(X)$. The inverse of T is the relation T^{-1} given by $G(T^{-1}) := \{(u, v) \in X \times X : (v, u) \in G(T)\}$. The closure $\overline{T}$ of a linear relation T is defined by $G(\overline{T}) := \overline{G(T)}$. We say that T is closed if its graph is a closed subspace of $X \times X$. The set of all closed linear relations is denoted by $\mathcal{CR}(X)$. We say that T is continuous if the operator $Q_T T$ is continuous when Q_T is the quotient map from X onto $\dfrac{X}{\overline{T(0)}}$. In such a case the norm of T is defined by $\|T\| := \|Q_T T\|$. We say that T is bounded if it is continuous and everywhere defined. The set of all bounded and closed linear relations acting between two Banach spaces X and Y is denoted by $\mathcal{BCR}(X, Y)$. If $X = Y$, we write $\mathcal{BCR}(X, X) := \mathcal{BCR}(X)$. The subspaces $ker(T) := T^{-1}(0)$

and $Im(T) := T(D(T))$ are called respectively the null space and the range space of T. We say that T is surjective if $T(D(T)) = X$ and T is injective if $ker(T) = \{0\}$. Note that T is an operator if and only if $T(0) = \{0\}$. For linear relations S,$T \in \mathcal{LR}(X)$, $S\hat{+}T$ is defined by

$$S\hat{+}T := \{(x+u, y+v) : (x,y) \in G(S) \text{ and } (u,v) \in G(T)\}.$$

This last sum is direct when $G(S) \cap G(T) = \{(0,0)\}$. In such case, we write $S \oplus T$. We denote by $\mathcal{B}(X,Y)$ the Banach algebra of all bounded operators on X and Y. If $X = Y$, we write $\mathcal{B}(X,X) := \mathcal{B}(X)$. For $r > 0$ we denote $D(0,r) := \{\lambda \in \mathbb{C}; 0 \leq |\lambda| < r\}$ and $D^*(0,r) := D(0,r)\backslash\{0\}$. Now, we aim to define and study some basic tools of the spectral theory. Given a closed linear relation T. For $\lambda \in \mathbb{C}$, we denote by $R_\lambda(T) = (\lambda I - T)^{-1}$ the resolvent of T at λ. The resolvent set of T is the set defined by:

$$\rho(T) = \{\lambda \in \mathbb{C}; (\lambda I - T)^{-1} \text{ is everywhere defined and single valued}\}.$$

We say that T is invertible if $0 \in \rho(T)$. The spectrum of T is the set $\sigma(T) = \mathbb{C}\backslash\rho(T)$. The extended spectrum of T is the subset $\tilde{\sigma}(T)$ of the extended complex plane $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that is equal to

$$\tilde{\sigma}(T) = \begin{cases} \sigma(T), & \text{if } T \in \mathcal{B}(X) \\ \sigma(T) \cup \{\infty\}, & \text{otherwise.} \end{cases}$$

The surjective spectrum of T is defined by

$$\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}.$$

We give in the sequel different types of invariance by a linear relation.

Definition 1.1. *Let $T \in \mathcal{LR}(X)$ and let Z be a subspace of X such that $D(T) \cap Z \neq \emptyset$. We say that Z is invariant by T if $T(Z) \subseteq Z$. The restriction T/Z is defined in terms of its graph by*

$$G(T/Z) := \{(x,y) \in G(T) \text{ such that } x \in Z\}.$$

Let Y be a closed subspace of X. We say that Y is weakly invariant by T if $Ty \cap Y \neq \emptyset$ for all $y \in Y \cap D(T)$. The restriction $T|Z$ of T in Z is defined by

$$G(T|Z) := G(T) \cap (Z \times Z).$$

Note that, by definition, $D(T|Z) \subseteq Z$ and $Im(T|Z) \subseteq Z$. Assume that Y and Z are two subspaces of X such that $X = Y \oplus Z$. We say that T is completely reduced by the pair of subspaces (Y,Z), denoted as $(Y,Z) \in Red(T)$, if it can be decomposed as $T = T|Y \oplus T|Z$.

25 Jul 2021 14:49:52 PDT
210725-Mnif Version 1 - Submitted to Rocky Mountain J. Math.

Definition 1.2. *[3, Definition 2.6] A closed linear subspace Z of X is said to be strongly invariant by a relation $T \in \mathcal{LR}(X)$ with non empty $\rho(T)$ if Z is invariant by all operators $R_\lambda(T)$, $\lambda \in \rho(T)$. By the restriction of the relation $T \in \mathcal{LR}(X)$ to the subspace Z we shall mean the relation $T_Z \in \mathcal{LR}(Z)$ whose resolvent is the restriction $R_0 : \rho(T) \to \mathcal{B}(Z)$, $R_0(\lambda) = R_\lambda(T)/Z$, $\lambda \in \rho(T)$, of the resolvent $R(.,T) : \rho(T) \to \mathcal{B}(X)$ to Z.*

Lemma 1.1. *Let $T \in \mathcal{CR}(X)$ with $\rho(T) \neq \emptyset$ and let Z be a closed subspace of X. If Z is weakly invariant by T and strongly invariant by T, then we have*

$$D(T_Z) = D(T) \cap Z \quad \text{and} \quad G(T_Z) = G(T) \cap (Z \times Z).$$

2. Local spectral theory for closed linear relations

In the following, we will introduce some elements of the local spectral theory for closed relations. We note that this theory was first developed for the case of bounded operators by [7, 1], then it was extended to the case of bounded and closed linear relations by [10]. What we are going to do next is an extension of the works cited above to include the cases of closed operators and closed relations. Let $T \in \mathcal{CR}(X)$. We consider the graph norm $\|.\|_T$ on $D(T)$ defined by

$$\|x\|_T := \|x\| + \|Tx\|.$$

In what follows X_T denotes $D(T)$ endowed with the graph norm. Observe that X_T is a Banach space (since $Q_T T$ is a closed operator). Consider the relation $\tilde{T}$ defined by

$$\tilde{T} : X_T \to X, x \mapsto Tx.$$

Evidently, $\widetilde{T}$ is closed and $D(\tilde{T}) = D(T)$. Then, by virtue of [4, II.5.1] we get that $\tilde{T} \in \mathcal{BCR}(X_T, X)$.

Lemma 2.1. *Let $T \in \mathcal{CR}(X)$ and $x \in X$. Then,*

$$\tilde{R}_.(T)x : \rho(T) \to X_T, \mu \mapsto \tilde{R}_\mu(T)x = R_\mu(T)x := (\mu I - T)^{-1}x \text{ is analytic.}$$

Proof : Let $\lambda \in \rho(T)$. Then, by virtue of [4, Corollary VI.1.9], we get that if $|\lambda - \mu| < \|R_\lambda(T)\|^{-1}$ then,

$$R_\mu(T) = \sum_{n=0}^{\infty} R_\lambda(T)^{n+1}(\mu - \lambda)^n.$$

Which implies that $\tilde{R}_\mu(T)x = \sum_{n=0}^{\infty} \tilde{R}_\lambda(T)^{n+1}x(\mu-\lambda)^n$. It was like proving that $\sum_{n=0}^{\infty} \tilde{R}_\lambda(T)^{n+1}x(\mu-\lambda)^n$ is convergent on X_T. Observe that $\|\tilde{R}_\lambda(T)^{n+1}x\|_T =$

25 Jul 2021 14:49:52 PDT
210725-Mnif Version 1 - Submitted to Rocky Mountain J. Math.

$\|\tilde{R}_\lambda(T)^{n+1}x\| + \|T\tilde{R}_\lambda(T)^{n+1}x\|$. Moreover, we have

$$\begin{aligned}
\|T\tilde{R}_\lambda(T)^{n+1}x\| &= \|Q_T T\tilde{R}_\lambda(T)^{n+1}x\| \\
&= \|Q_T(T-\lambda I+\lambda I)\tilde{R}_\lambda(T)^{n+1}x\| \\
&\leq \|Q_T R_\lambda(T)^n x\| + |\lambda|\|Q_T R_\lambda(T)^{n+1}x]\| \\
&\leq \|R_\lambda(T)^n x\| + |\lambda|\|R_\lambda(T)^{n+1}x]\|.
\end{aligned}$$

Then, $\|\tilde{R}_\lambda(T)^{n+1}x\|_T \leq (1+|\lambda|)\|R_\lambda(T)^{n+1}x\|+\|\tilde{R}_\lambda(T)^n x\|$. Since $\sum\limits_{n\geq 0} R_\lambda(T)^{n+1}x(\mu-\lambda)^n$ and $\sum\limits_{n\geq 0} R_\lambda(T)^n x(\mu-\lambda)^n$ are absolutely convergent in X, then $\sum\limits_{n\geq 0} \|\tilde{R}_\lambda(T)^n x\|_T|\mu-\lambda|^n$ is convergent. Therefore, $\sum\limits_{n\geq 0} \tilde{R}_\lambda(T)^{n+1}x(\mu-\lambda)^n$ is convergent on X_T, as required. ■

Definition 2.1. *The local resolvent set of $T \in \mathcal{CR}(X)$ at the point $x \in X$, denoted by $\rho_T(x)$, is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U_λ and an analytic function $f_{\lambda,x} : U_\lambda \to X_T$ such that $(\mu I - T)f_{\lambda,x}(\mu) = x + T(0)$ holds for all $\mu \in U_\lambda$.*
The local spectrum of T at the point x is the set $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$.

Remark 2.1. *It is easy to see that*

(i) $\rho_T(x) := \bigcup\limits_{\lambda\in\rho_T(x)} U_\lambda$ *is an open subset of* $\mathbb{C}$.

(ii) For all $x \in X$, $\rho(T) \subseteq \rho_T(x)$.

(iii) For all $x \in T(0)$, $\rho_T(x) = \mathbb{C}$.

Proposition 2.1. *Let $T \in \mathcal{CR}(X)$. Then, $\sigma_T(\alpha x+\beta y) \subseteq \sigma_T(x)\cup\sigma_T(y)$, for all $x, y \in X$ and $\alpha, \beta \in \mathbb{C}$.*

Proof : Show that for all $x, y \in X$ and $\alpha, \beta \in \mathbb{C}$, $\rho_T(x)\cap\rho_T(y) \subseteq \rho_T(\alpha x+\beta y)$. In the trivial case where $\alpha = \beta = 0$ there is nothing to prove. Otherwise, let $\lambda \in \rho_T(x) \cap \rho_T(y)$. Then, there exist an open neighborhood U_λ and an analytic function $f_{\lambda,x} : U_\lambda \to X_T$ such that $(\mu I - T)f_{\lambda,x}(\mu) = x+T(0)$ holds for all $\mu \in U_\lambda$ and there exist an open neighborhood V_λ and an analytic function $g_{\lambda,x} : V_\lambda \to X_T$ such that $(\mu I - T)g_{\lambda,x}(\mu) = y + T(0)$ holds for all $\mu \in V_\lambda$. Put $O_\lambda := U_\lambda \cap V_\lambda$ and $h := \alpha f_{\lambda,x} + \beta g_{\lambda,x}$. Evidently, $h : O_\lambda \to X_T$ is analytic and for all $\mu \in O_\lambda$, $(\mu I - T)h(\mu) = (\alpha x + \beta y) + T(0)$. Thus, $\lambda \in \rho_T(\alpha x + \beta y)$, as required. ■

Definition 2.2. *For every subset F of $\mathbb{C}$ the local spectral subspace of T associated to F is the set*

$$X_T(F) := \{x \in X, \sigma_T(x) \subseteq F\}.$$

25 Jul 2021 14:49:52 PDT
210725-Mnif Version 1 - Submitted to Rocky Mountain J. Math.

Definition 2.3. *Let $F \subseteq \mathbb{C}$ be a closed subset and let $T \in \mathcal{CR}(X)$. We define the glocal spectral subspace $\chi_T(F)$ as the set of all $x \in X$ such that there exists an analytic function $f : \mathbb{C}\backslash F \to X_T$ checking :*

$$(\lambda I - T)f(\lambda) = x + T(0) \quad \textit{for all} \quad \lambda \in \mathbb{C}\backslash F.$$

Remark 2.2. *We note that $X_T(F)$ and $\chi_T(F)$ are two subspaces of X and that $\chi_T(F) \subseteq X_T(F)$.*

The following result generalizes Leiterer's theorem [7, Theorem 3.2.1] to the setting of closed linear relations.

Proposition 2.2. *Let $T \in \mathcal{CR}(X)$ and $\lambda \in \mathbb{C}$. Let U_λ be an open neighborhood of λ such that $\mu I - T$ is surjective for all $\mu \in U_\lambda$. Then, for all analytic function $g : U_\lambda \to X$ there exists an analytic function $f : U_\lambda \to X_T$ such that*

$$(\mu I - T)f(\mu) = g(\mu) + T(0) \quad \textit{for all} \quad \mu \in U_\lambda.$$

Proof : As T is closed then $Q_T T$ is a closed operator from the Banach space X to the Banach space $\dfrac{X}{T(0)}$. Note that $Q_T\tilde{T} : X_T \to \dfrac{X}{T(0)}$ is bounded. We claim that for every $\mu \in U_\lambda$ the operator $Q_T(\mu i_T - \tilde{T})$ is bounded and surjective from X_T to $\dfrac{X}{T(0)}$, where $i_T : X_T \to X, x \mapsto x$. Indeed, as $(\mu i_T - \tilde{T})$ is surjective, then it is clear that $Q_T(\mu i_T - \tilde{T})$ is also surjective. On the other hand, we have $Q_T(\mu i_T - \tilde{T}) = \mu Q_T i_T - Q_T\tilde{T}$ with $Q_T\tilde{T}$ is a bounded operator from X_T to $\dfrac{X}{T(0)}$ and for all $x \in D(T)$, $\|Q_T i_T x\| = d(x, T(0)) \leq \|x\| \leq \|x\|_T$. Then, $Q_T i_T$ considered as an operator from X_T to $\dfrac{X}{T(0)}$ is bounded. Hence, $Q_T(\mu i_T - \tilde{T})$ is bounded from X_T to $\dfrac{X}{T(0)}$ for all $\mu \in U_\lambda$. Now, let us consider the function $Q_T g$ on U_λ. Since Q_T is a bounded operator, then $Q_T g$ is an analytic function from U_λ to $\dfrac{X}{T(0)}$. So, let's recap, consider the operator function $\hat{T}$ defined on U_λ by

$$\begin{aligned} \hat{T} :& U_\lambda \to \mathcal{B}(X_T, \frac{X}{T(0)}) \\ & \mu \to \hat{T}(\mu) := Q_T(\mu i_T - \tilde{T}). \end{aligned}$$

We have $\hat{T}$ is analytic on U_λ for which the mapping $\hat{T}(\mu)$ is surjective for all $\mu \in U_\lambda$. On the other hand, $Q_T g$ is an analytic function from U_λ to $\dfrac{X}{T(0)}$, then from Leiterer's theorem [7, Theorem 3.2.1] there exists an analytic function f from U_λ to X_T such that for all $\mu \in U_\lambda$,

$$Q_T(\mu i_T - \tilde{T})f(\mu) = Q_T g(\mu).$$

Hence, $Q_T[(\mu i_T - \tilde{T})f(\mu) - g(\mu)] = 0$, for all $\mu \in U_\lambda$. Thus, $(\mu i_T - \tilde{T})f(\mu) - g(\mu) \subseteq T(0)$. Which implies that $(\mu i_T - \tilde{T})f(\mu) = g(\mu) + T(0)$. ∎

Remark 2.3. *We note, by the proof of Proposition 2.2, that Q_T and $Q_T\tilde{T}$ considered from X_T to $\dfrac{X}{T(0)}$ are bounded operators.*

Quasinilpotent part and analytic core of a closed linear relation

Now, let's further extend the concept of quasinilpotent part and the analytic core developed in [8, 9] to the case of closed not necessary bounded linear relations.

Definition 2.4. *Let $T \in \mathcal{CR}(X)$.*

(i) The quasinilpotent part of T, denoted by $H_0(T)$, is the set of all $x \in D(T)$ for which there exists a sequence $(x_n)_n \subseteq D(T)$ satisfying

$$x_0 = x, \quad x_{n+1} \in Tx_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \|x_n\|_T^{\frac{1}{n}} \to 0.$$

(ii) The analytic core of T, denoted by $K(T)$, is defined as the set of all $x \in X$ for which there exist $c > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_0 = x$ and for all $n \geq 0$, $x_{n+1} \in D(T)$, $x_n \in Tx_{n+1}$ and

$$d(x_n, ker(T) \cap T(0)) \leq c^n d(x, ker(T) \cap T(0)).$$

It is easy to see that for each $j \geq 0$, $ker(T^j) \subseteq H_0(T)$.

In the next lemma, we collect some elementary properties of $K(T)$.

Lemma 2.2. *Let $T \in \mathcal{CR}(X)$. Then the following statements hold.*

(i) $T(D(T) \cap K(T)) = K(T)$;

(ii) If F is a closed subspace of X such that $T(D(T) \cap F) = F$ then $F \subseteq K(T)$.

(iii) If $x \in K(T)$, then there exist $d > 0$ and a sequence $(x_n)_n$ satisfying $x_0 = x$ and for all $n \geq 0$, $x_{n+1} \in D(T)$, $x_n \in Tx_{n+1}$ and for all $n \geq 1$,

$$\|x_n\|_T \leq d^n \|x\|.$$

Proof : (i) The proof is similar to the proof of [6, Lemma 2.1].
(ii) First, we claim that $F \cap D(T)$ is closed in X_T. Indeed, let $(x_n)_n \subseteq F \cap D(T)$ be such that $x_n \xrightarrow[n\to\infty]{X_T} x$. Trivially, $x \in D(T)$. On the other hand, we have $\|x_n - x\|_T \xrightarrow[n\to\infty]{} 0$. As F is closed in X, then $x \in F$. Hence, $F \cap D(T)$ is closed in X_T, as claimed. Recall that the relation $\tilde{T}$ is closed. Let us consider $T_0 : D(T) \cap F \to F$, the restriction of $\tilde{T}$. We have $G(T_0) = G(T) \cap ((D(T) \cap F) \times F)$ is closed in $X_T \times X$. Then, T_0 is closed. We have, by hypothesis, $ImT_0 = F$ then, by the open mapping theorem [4, Theorem III.4.2], we deduce that T_0 is open. Thus, there exists a constant $\gamma > 0$ such that for all $x \in D(T_0) = D(T) \cap F$,

$$d_T(x, ker(T_0)) \leq \gamma \|T_0 x\|,$$

where $d_T(x,G) := \inf_{\alpha\in G} \|x-\alpha\|_T$. As, for all $x \in D(T_0)$ and $\alpha \in kerT_0$, $\|x-\alpha\| \leq \|x-\alpha\|_T$ then, $d(x, kerT_0) \leq d_T(x, kerT_0)$. Hence,

$$d(x, kerT_0) \leq \gamma\|T_0 x\|. \tag{2.1}$$

Now, consider $\epsilon > 0$ and let $u \in F$. Then, there exists $x \in D(T) \cap F$ such that $u \in Tx$. By (2.1) there exists $y \in ker(T_0) \subseteq ker(T)$ such that $\|x-y\| \leq (\gamma+\epsilon)d(u, T(0))$. Take $u_1 = x - y \in D(T) \cap F$. We have $u \in T(u_1)$ and

$$d(u_1, T(0) \cap ker(T)) \leq (\gamma+\epsilon)d(u, T(0) \cap ker(T)).$$

Continuing in the same manner, we build a sequence $(u_n)_n$ such that $u_0 = u$, for all $n \geq 0$, $u_{n+1} \in D(T) \cap F$, $u_n \in Tu_{n+1}$ and

$$d(u_n, T(0) \cap ker(T)) \leq (\gamma+\epsilon)^n d(u, T(0) \cap ker(T)).$$

Hence, $u \in K(T)$. Thus, $F \subseteq K(T)$.
(iii) Let $x \in K(T)$. Then, there exist $c > 0$ and a sequence $(y_n)_n$ such that

$$\begin{cases} y_0 = x, \\ \text{for all } \ n \geq 0, \quad y_{n+1} \in D(T) \ \text{ and } \ y_n \in Ty_{n+1}, \\ d(y_n, ker(T) \cap T(0)) \leq c^n d(x, ker(T) \cap T(0)). \end{cases}$$

Let $d > c$. Then, for all $n \geq 1$ there exists $\alpha_n \in T(0) \cap ker(T) \subseteq D(T)$ such that $\|y_n - \alpha_n\| \leq d^n\|x\|$. Let $(x_n)_n$ be the sequence defined by $x_{n+1} = y_{n+1} - \alpha_{n+1}$ for all $n \geq 0$ and $x_0 = x$. Then, for all $n \geq 0$, $x_{n+1} \in D(T)$, $x_n \in Tx_{n+1}$ and $\|x_n\| \leq d^n\|x\|$. On the other hand, we have $\|x_n\|_T = \|x_n\| + \|Q_T T x_n\| = \|x_n\| + \|Q_T(x_{n-1})\|$. Then, $\|x_n\|_T = \|x_n\| + d(x_{n-1}, T(0))$. Which implies that

$$\|x_n\|_T \leq d^n\|x\| + \|x_{n-1}\| \leq (d^n + d^{n-1})\|x\|.$$

Consequently, there exists $\delta > 0$ such that $\|x_n\|_T \leq \delta^n\|x\|$. ■

The next lemma describes $K(T)$ and $H_0(T)$ in terms of the local and glocal spectral subspaces.

Lemma 2.3. *Let $T \in \mathcal{CR}(X)$. Then, the following assertions hold.*

(i) $H_0(T) + T(0) \subseteq \chi_T(\{0\})$.

(ii) $K(T) = X_T(\mathbb{C}\backslash\{0\})$.

Proof : (i) Let $x \in H_0(T)$. Then, there exists $(x_n)_n \subseteq D(T)$ such that $x_0 = x$, $x_{n+1} \in Tx_n$ and $\lim_{n\to\infty} \|x_n\|_T^{\frac{1}{n}} = 0$. Thus, the series $f(\lambda) := \sum_{n\geq 1} \lambda^{-n} x_{n-1}$ converges in X_T uniformly on $\mathbb{C}\backslash\{0\}$. Therefore, f is analytic throughout $\mathbb{C}\backslash\{0\}$ with values in X_T. Using Remark 2.3, we get for all $\lambda \neq 0$,

$$Q_T T(\sum_{n\geq 1} \lambda^{-n} x_{n-1}) = \sum_{n\geq 1} \lambda^{-n} Q_T T x_{n-1} = \sum_{n\geq 1} \lambda^{-n} Q_T x_n = Q_T(\sum_{n\geq 1} \lambda^{-n} x_n).$$

25 Jul 2021 14:49:52 PDT
210725-Mnif Version 1 - Submitted to Rocky Mountain J. Math.

Whence, $(\lambda I-T)f(\lambda)=\sum_{n\geq 0}\lambda^{-n}x_n-\sum_{n\geq 1}\lambda^{-n}x_n+T(0)=x+T(0)$. Therefore, $x\in \chi_T(\{0\})$. Furthermore, we have $\chi_T(\{0\})$ is a subspace of X and $T(0)\subseteq \chi_T(\{0\})$ which provides the required inclusion.
(ii) Let $x\in X_T(\mathbb{C}\backslash\{0\})$. Then, $0\in\rho_T(x)$ which implies that there exist an open disc $D(0,\epsilon)$ and an analytic function $f: D(0,\epsilon)\rightarrow X_T$ such that $(\mu I-T)f(\mu)=x+T(0)$ holds for all $\mu\in D(0,\epsilon)$. As f is an analytic function, then there exists a sequence $(u_n)_{n\geq 1}\subseteq D(T)$ such that

$$f(\lambda)=-\sum_{n\geq 1}\lambda^{n-1}u_n,\quad \text{for all}\quad \lambda\in D(0,\epsilon).$$

But, we have $f(0)=-u_1$ then $Tu_1=x+T(0)$. Therefore, $x\in Tu_1$. Take $u_0=x$. We can show by induction that $u_n\in Tu_{n+1}$ for all $n\in\mathbb{N}$. It remains to prove that there exists $b>0$ such that $d(u_n,T(0)\cap kerT)\leq b^n d(x,T(0)\cap kerT)$ for all $n\geq 0$. Trivially, if $x\in T(0)\cap ker(T)$, then there is nothing to prove. Otherwise, as $\sum_{n\geq 1}\lambda^{n-1}u_n$ converges in X_T, then $|\lambda|^{n-1}\|u_n\|_T\rightarrow 0$ as $n\rightarrow\infty$ for all $|\lambda|<\epsilon$. Particularly, $\frac{1}{\mu^{n-1}}\|u_n\|_T \xrightarrow[n\to\infty]{} 0$ for all $\mu>\frac{1}{\epsilon}$. Now, take $\mu_0>\frac{1}{\epsilon}$. Then, there exists $c>0$ such that for all $n\geq 1$, $\|u_n\|_T\leq c\mu_0^{n-1}$. Besides this, we have for all $n\geq 1$,

$$\left(\frac{\mu_0}{\mu_0+\frac{c}{d(x,T(0)\cap kerT)}}\right)^{n-1}\leq 1+\frac{d(x,T(0)\cap kerT)\mu_0}{c}.$$

Hence, for all $n\geq 1$, we get

$$\|u_n\|_T\leq \left(\mu_0+\frac{c}{d(x,T(0)\cap kerT)}\right)^n d(x,T(0)\cap kerT).$$

Consequently, for all $n\geq 1$, we obtain

$$d(u_n,T(0)\cap kerT)\leq \|u_n\|\leq \|u_n\|_T\leq b^n d(x,T(0)\cap kerT),$$

with $b=\mu_0+\frac{c}{d(x,T(0)\cap kerT)}$. Observe that the last inequality holds for $n=0$. Thus, $x\in K(T)$ and hence, $X_T(\mathbb{C}\backslash\{0\})\subseteq K(T)$. Conversely, assume that $x\in K(T)$. Then, there exist $\delta>0$ and a sequence $(x_n)_n$ satisfying $x_0=x$ and for all $n\geq 0$, $x_{n+1}\in D(T)$, $x_n\in Tx_{n+1}$ and $d(x_n,ker(T)\cap T(0))\leq\delta^n d(x,ker(T)\cap T(0))$. Therefore, by Lemma 2.2 (iii) we get that there exist $b>0$ and a sequence $(y_n)_n$ such that

$$\begin{cases} y_0=x, \\ \text{for all}\quad n\geq 0,\quad y_{n+1}\in D(T)\quad\text{and}\quad y_n\in Ty_{n+1}, \\ \text{for all}\quad n\geq 1,\quad \|y_n\|_T\leq b^n\|x\|. \end{cases}$$

Let f be the analytic function $f: B(0,\frac{1}{b})\rightarrow X_T$ defined by

$$f(\lambda)=-\sum_{n\geq 1}\lambda^{n-1}y_n.$$

Using Remark 2.3, we get $T(\sum_{n\geq 1}\lambda^{n-1}y_n)-\sum_{n\geq 1}\lambda^{n-1}y_{n-1}\subseteq T(0)$, for all $\lambda\in B(0,\frac{1}{b})$. Whence,

$$(\lambda I-T)f(\lambda)=\sum_{n\geq 1}\lambda^{n-1}y_{n-1}+T(0)-\lambda\sum_{n\geq 1}\lambda^{n-1}y_n=y_0+T(0)=x+T(0).$$

Thus, $0\in\rho_T(x)$ and so, $x\in X_T(\mathbb{C}\backslash\{0\})$, then we have the required inclusion. ■

Definition 2.5. *Let S be a subset of X. We say that S is nowhere dense if the interior of its closure is empty. A subset $E\subseteq X$ is called of first category if it is a countable union of nowhere dense subsets. If E fails to be of first category we say that E is of second category. In addition, the union of any countable family of first category is of first category.*

In the next lemma, we gather some properties of local and glocal spectral subspaces and the surjective spectrum of a closed linear relation.

Lemma 2.4. *Let $T\in\mathcal{CR}(X)$ and $F\subseteq\mathbb{C}$ be a closed subset. Then,*

(i) $\sigma_{su}(T)=\bigcup_{x\in X}\sigma_T(x)$ *and it is closed.*

(ii) The set $\{x\in X\ \text{ such that }\ \sigma_T(x)=\sigma_{su}(T)\}$ *is of the second category in* X.

(iii) $\chi_T(F)=X$ *if and only if* $\sigma_{su}(T)\subseteq F$.

(iv) $\chi_T(F\cap\sigma(T))=\chi_T(F)$ *and* $X_T(F\cap\sigma(T))=X_T(F)$.

Proof : (i) We shall show that $\rho_{su}(T)=\bigcap_{x\in X}\rho_T(x)$. To see this, let $\lambda\in\bigcap_{x\in X}\rho_T(x)$. Then, $\lambda\in\rho_T(x)$ for all $x\in X$. Let $x\in X$. Then there exists an open neighborhood U_λ of λ and an analytic function $f_x:U_\lambda\to X_T$ such that

$$(\mu I-T)f_x(\mu)=x+T(0)\ \text{ for all }\ \mu\in U_\lambda.$$

Which implies that $x\in(\lambda I-T)f_x(\lambda)\subseteq Im(\lambda I-T)$ for all $x\in X$. Then, $(\lambda I-T)$ is surjective. Whence, $\lambda\in\rho_{su}(T)$. Moving to the direct inclusion, let $\lambda\in\rho_{su}(T)$. Then, $(\lambda I-T)$ is surjective. According to Lemma 2.2 (ii), we get $K(\lambda I-T)=X$. In addition, it follows from Lemma 2.3 (ii) that $0\in\rho_{\lambda I-T}(x)$. Therefore, there exist an open neighborhood U_0 of 0 and an analytic function $f:U_0\to X_T$ such that $((\mu-\lambda)I+T)f(\mu)=x+T(0)$ for all $\mu\in U_0$. Consequently, $(\gamma I-T)g(\gamma)=x+T(0)$ for all $\gamma\in U_\lambda$, where U_λ be the open neighborhood of λ given by $U_\lambda=\lambda-U_0$ and g be the analytic function defined on U_λ by $g(\delta)=-f(\lambda-\delta)$. Hence, $\lambda\in\rho_T(x)$ for all $x\in X$.
(ii) Let E be a dense countable subset of $\sigma_{su}(T)$. Then, for all $\lambda\in E$ we have $\mathrm{Im}(\lambda I-T)\neq X$. We note that $\mathrm{Im}(\lambda I-T)$ is of the first category in X. In fact, assume that $\mathrm{Im}(\lambda I-T)$ is of the second category in X. We start by proving that

$\mathrm{Im}(\lambda I - T) = X$, which is absurd. To do that, it suffices to prove that $(\lambda I - T)$ is open. Let us suppose that U is the open ball in X_T with center 0 and radius $r > 0$. We show that $(\lambda i_T - \tilde{T})(U)$ contains a neighborhood of 0 in X. Let us define,

$$U_n := \{x \in D(T), \|x\|_T < 2^{-n} r\} \quad (n \in \mathbb{N}).$$

Observe that $U_1 \supseteq U_2 - U_2$. Then, $(\lambda i_T - \tilde{T})U_1 \supseteq (\lambda i_T - \tilde{T})U_2 - (\lambda i_T - \tilde{T})U_2$. Whence,

$$\overline{(\lambda i_T - \tilde{T})U_1} \supseteq \overline{(\lambda i_T - \tilde{T})U_2 - (\lambda i_T - \tilde{T})U_2} \supseteq \overline{(\lambda i_T - \tilde{T})U_2} - \overline{(\lambda i_T - \tilde{T})U_2}. \quad (2.2)$$

On the other hand, we have $Im(\lambda i_T - \tilde{T}) = \bigcup_{k=1}^{\infty} k(\lambda i_T - \tilde{T})(U_2)$. Since $\mathrm{Im}(\lambda I - T)$ is of second category, then at least one $k(\lambda i_T - \tilde{T})U_2$ is of second category of X and hence, $(\lambda i_T - \tilde{T})U_2$ is of the second category. Thus, $int(\overline{(\lambda i_T - \tilde{T})U_2}) \neq \emptyset$. Hence, by (2.2), there exists a neighborhood W of 0 in X such that

$$W \subseteq \overline{(\lambda i_T - \tilde{T})U_1}.$$

We claim that $\overline{(\lambda i_T - \tilde{T})(U_1)} \subseteq (\lambda i_T - \tilde{T})(U)$. Indeed, take $y_1 \in \overline{(\lambda i_T - \tilde{T})(U_1)}$. As, what just proved for U_1 holds true by proceeding with the same way for U_2, then $(\lambda i_T - \tilde{T})(U_2)$ contains a neighborhood of 0. Consequently,

$$(y_1 - \overline{(\lambda i_T - \tilde{T})U_2}) \cap (\lambda i_T - \tilde{T})(U_1) \neq \emptyset.$$

Hence, there is some $\alpha_1 \in (\lambda i_T - \tilde{T})x_1$ with $x_1 \in U_1$ such that $y_2 = y_1 - \alpha_1 \in \overline{(\lambda i_T - \tilde{T})U_2}$. We proceed by induction. Then, we may construct the sequences $(\alpha_n)_{n\geq 1}$, $(x_n)_{n\geq 1} \subseteq D(T)$ and $(y_n)_{n\geq 1}$, such that for all $n \geq 1$, $x_n \in U_n$, $\alpha_n \in (\lambda i_T - \tilde{T})(x_n)$ and $y_{n+1} = y_n - \alpha_n \in \overline{(\lambda i_T - \tilde{T})(U_{n+1})}$. As, $\|x_n\|_T \leq \frac{r}{2^n}$, then $\sum_{n\geq 1} x_n$ converges in X_T. Let $x = \sum_{n=1}^{\infty} x_n$. Then, $\|x\|_T < r$ and hence, $x \in U$. Moreover, we have by the construction of $(\alpha_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ that

$$\sum_{n=1}^{m} Q_T \alpha_n = \sum_{n=1}^{m} Q_T (y_n - y_{n+1}) = Q_T y_1 - Q_T y_{m+1}.$$

Then,

$$Q_T (\lambda i_T - \tilde{T})(\sum_{n=1}^{m} x_n) = Q_T y_1 - Q_T y_{m+1}.$$

Now, we note that $Q_T y_m \xrightarrow[m\to\infty]{} 0$. In fact, assume that $\alpha \in (\lambda i_T - \tilde{T})(U_n)$. Then, there is some $\beta \in U_n$ such that $\alpha \in (\lambda i_T - \tilde{T})(\beta)$. Thus, we get $d(\alpha, \tilde{T}(0)) = \|(\lambda i_T - \tilde{T})\beta\| \leq \|(\lambda i_T - \tilde{T})\|_{\mathcal{B}(X_T, X)} \|\beta\|_T$. Hence, we obtain

$$(\lambda i_T - \tilde{T})(U_n) \subseteq \{\alpha \in X, d(\alpha, T(0)) \leq \|(\lambda i_T - \tilde{T})\|_{\mathcal{B}(X_T, X)} \frac{r}{2^n}\}.$$

Observe that the map $\alpha \to d(\alpha, T(0))$ is continuous on X. Then,

$$\{\alpha \in X; d(\alpha, T(0)) \leq \|(\lambda i_T - \tilde{T})\|_{\mathcal{B}(X_T,X)} \frac{r}{2^n}\} \text{ is closed}$$

and we have

$$y_n \in \overline{(\lambda i_T - \tilde{T})(U_n)} \subseteq \{\alpha \in X; d(\alpha, T(0)) \leq \|(\lambda i_T - \tilde{T})\|_{\mathcal{B}(X_T,X)} \frac{r}{2^n}\}.$$

Whence, $d(y_n, T(0)) \leq \|(\lambda i_T - \tilde{T})\|_{\mathcal{B}(X_T,X)} \frac{r}{2^n}$. So, $\|Q_T y_n\| \xrightarrow[n\to\infty]{} 0$. Therefore, as $Q_T(\lambda i_T - \tilde{T}) : X_T \to \dfrac{X}{T(0)}$ is a bounded operator, then we infer that $Q_T(\lambda i_T - \tilde{T})(x) = Q_T(y_1)$ and so, $y_1 \in (\lambda i_T - \tilde{T})(x) = (\lambda I - T)(x)$. Which implies that

$$W \subseteq \overline{(\lambda I - T)U_1} \subseteq (\lambda I - T)(U).$$

Thus, $(\lambda I - T)$ is an open mapping. Now, since $(\lambda I - T)$ is open, then $\overline{Im(\lambda I - T)}$ is closed. On the other hand, $\mathrm{Im}(\lambda I - T)$ is of second category then, $\mathrm{int}(\overline{Im(\lambda I - T)}) \neq \emptyset$. Whence, $\mathrm{int}(Im(\lambda I - T)) \neq \emptyset$. So, $\mathrm{Im}(\lambda I - T) = X$ which is absurd. Therefore, $\mathrm{Im}(\lambda I - T)$ is of first category for all $\lambda \in E$. So, $F = \bigcup_{\lambda \in E} Im(\lambda I - T)$ is also of the first category. Thus, $X \backslash F$ is of second category. We note that for all $x \in X \backslash F$, $\sigma_{su}(T) \subseteq \sigma_T(x)$. In fact, let $x \in X \backslash F$. Then $E \subseteq \sigma_T(x)$. Whence,

$$\sigma_{su}(T) = \overline{E} \subseteq \overline{\sigma_T(x)} = \sigma_T(x).$$

Therefore, $\sigma_{su}(T) = \sigma_T(x)$. Hence, the set $\{x \in X \ \text{ such that } \ \sigma_{su}(T) = \sigma_T(x)\}$ is of second category.
(iii) Suppose that $\sigma_{su}(T) \subseteq F$. Show that $\chi_T(F) = X$. Note that $T - \lambda I$ is surjective for all $\lambda \in \mathbb{C} \backslash \sigma_{su}(T)$. Let $x \in X$ and let

$$\begin{aligned} g : \mathbb{C} \backslash \sigma_{su}(T) &\to X \\ \lambda &\mapsto x. \end{aligned}$$

Then, using Proposition 2.2, there exists an analytic function $f : \mathbb{C} \backslash \sigma_{su}(T) \to X_T$ such that

$$(\lambda I - T) f(\mu) = x + T(0) \quad \text{for all } \ \mu \in \mathbb{C} \backslash \sigma_{su}(T).$$

Thus, $x \in \chi_T(\sigma_{su}(T))$ and hence, $\chi_T(\sigma_{su}(T)) = X$. On the other hand, we have $\sigma_{su}(T) \subseteq F$. Then, $\chi_T(\sigma_{su}(T)) \subseteq \chi_T(F)$. It follows that $\chi_T(F) = X$. Conversely, we suppose that $\chi_T(F) = X$. By (i) we have $X_T(F) = X$. Which implies that for all $x \in X$, $\sigma_T(x) \subseteq F$. But, by (i), we have $\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x)$. Then, $\sigma_{su}(T) \subseteq F$.
(iv) Let $y \in \chi_T(F)$. Then, there exists an analytic function $f : \mathbb{C} \backslash F \to X_T$

such that for all $\lambda \in \mathbb{C}\backslash F$, $(\lambda I - T)f(\lambda) = y + T(0)$. Set $U := \mathbb{C}\backslash(F \cap \sigma(T)) = (\mathbb{C}\backslash F) \cup \rho(T)$. We have for all $\lambda \in \rho(T) \cap (\mathbb{C}\backslash F)$, $f(\lambda) = R_\lambda(T)y$. Define

$$h(\lambda) = \begin{cases} f(\lambda), & \text{if } \ \lambda \in \mathbb{C}\backslash F \\ R_\lambda(T)y, & \text{if } \ \lambda \in \rho(T). \end{cases}$$

According to Lemma 2.1 we get that h is an analytic function $U \to X_T$ such that

$$(\lambda I - T)h(\lambda) = y + T(0) \ \text{ for each } \ \lambda \in \mathbb{C}\backslash(F \cap \sigma(T)).$$

Hence, $y \in \chi_T(F \cap \sigma(T))$. Consequently, $\chi_T(F) \subseteq \chi_T(F \cap \sigma(T))$. The reversed inclusion is straightforward. Thus,

$$\chi_T(F) = \chi_T(F \cap \sigma(T)).$$

Now, let $x \in X_T(F)$. Then, $\sigma_T(x) \subseteq F$. On the other hand, as $\rho(T) \subseteq \rho_T(x)$, then $\sigma_T(x) \subseteq \sigma(T)$. Consequently, $x \in X_T(F \cap \sigma(T))$. Conversely, if $x \in X_T(F \cap \sigma(T))$ then, $\sigma_T(x) \subseteq F \cap \sigma(T) \subseteq F$, as required. ■

3. On the isolated points of the surjective spectrum

Before stating the main result of this section, we gathered some technical lemmas which are crucial for the proof. Let's consider now the following assumptions.

Assumption 3.1. *$T \in \mathcal{CR}(X)$ be a relation with extended spectrum $\tilde{\sigma}(T)$ of the form $\tilde{\sigma}(T) = \sigma(T) \cup \{\infty\}$, where $\sigma(T)$ is a compact subset of $\mathbb{C}$.*

Example 3.1. *Let $T \in \mathcal{B}(X)$ be generalized Drazin invertible. Then, 0 is isolated in $\sigma(T)$, where $\sigma(T)$ is a compact of $\mathbb{C}$. Therefore, $S := T^{-1}$ is a closed linear relation and $\tilde{\sigma}(S) = \{\infty\} \cup \sigma(S)$ with $\sigma(S)$ is a compact subset of $\mathbb{C}$. Hence, S satisfies Assumption 3.1.*

Assumption 3.2. *$T \in \mathcal{CR}(X)$ be such that $D(T)$ is closed and $D(T) \oplus T(0) = X$.*

Example 3.2. *Let P be a bounded projection operator and let $T := P^{-1}$. Then, T is a closed linear relation. Since $\sigma(P) = \{0, 1\}$, then $\tilde{\sigma}(T) = \{1, \infty\}$. Thus, T satisfies Assumption 3.1. Besides this, we have $D(T) = Im(P)$ and it is closed. As $D(T) \oplus T(0) = Im(P) \oplus Ker(P) = X$, then, Assumption 3.2 is fulfilled.*

As a particular case of Baskakov's Theorem 2.10 in [2], we infer to the following remark.

Remark 3.1. *Let $T \in \mathcal{CR}(X)$ satisfying Assumption 3.1 and let Γ be a closed Jordan curve around $\sigma(T)$ lying in $\rho(T)$. Consider the Riesz projection*

$$P_\sigma := \frac{1}{2\pi i} \int_\Gamma R_\lambda(T) d\lambda.$$

It is well known by [2] that P_σ is a bounded projection and using [2, Theorem 2.10], we get that $ker P_\sigma$ and $Im P_\sigma$ are strongly invariant by T and that $T = T_0 \oplus T_1$ where T_0 is a bounded operator defined by $T_0 = T_{ImP_\sigma}$ and T_1 is a closed linear relation given by $T_1 = T_{kerP_\sigma}$. Furthermore, we have

$$\tilde{\sigma}(T_0) = \sigma(T) \text{ and } \tilde{\sigma}(T_1) = \{\infty\}.$$

Lemma 3.1. *Let $T \in \mathcal{CR}(X)$, satisfying Assumption 3.1 and Assumption 3.2. Let K be a compact of $\mathbb{C}$, Γ be a contour in the complement $U := \mathbb{C}\backslash K$ that surrounds K and let $x \in X$. If there exists an analytic function $f : U \to X_T$ such that for each $\lambda \in U$, $(\lambda I - T)f(\lambda) = x + T(0)$, then*

$$P_\sigma x = \frac{1}{2\pi i}\int_\Gamma f(\lambda)d\lambda.$$

Proof : As f is analytic on U, it follows from Cauchy's theorem that

$$\int_\Gamma f(\lambda)d\lambda = \int_\Upsilon f(\lambda)d\lambda,$$

where Υ denotes a positively oriented boundary of a disc that is centered in the origin and large enough to include both the contour Γ and the spectrum of T in its interior. Now, for $\lambda \in \Upsilon$ we have $\lambda \in \rho(T)$ and $f(\lambda) = \tilde{R}_\lambda(T)x$. Using Remark 3.1, we get for all $x \in X$,

$$P_\sigma x = \frac{1}{2\pi i}\int_\Upsilon R_\lambda(T)x d\lambda.$$

As $R_\lambda(T)x$ is analytic by Lemma 2.1 and that $\|.\|$ and $\|.\|_T$ are comparable norms on $D(T)$ by Assumption 3.2, then we get

$$P_\sigma x = \frac{1}{2\pi i}\int_\Upsilon \tilde{R}_\lambda(T)x d\lambda = \frac{1}{2\pi i}\int_\Gamma f(\lambda)d\lambda.$$

∎

Lemma 3.2. *If $T \in \mathcal{CR}(X)$ satisfies Assumption 3.1 and Assumption 3.2 then, for all disjoint closed sets F_1, $F_2 \subseteq \mathbb{C}$, such that F_1 is compact and connected, we have*

$$\chi_T(F_1 \cup F_2) \subseteq \chi_T(F_1) \cap D(T) + X_T(F_2) \cap D(T) + ker P_\sigma,$$

where P_σ is the Riesz projection defined in Remark 3.1.

Proof : Assume that $x \in \chi_T(F_1 \cup F_2)$. Then there exists $f : \mathbb{C}\backslash(F_1 \cup F_2) \to X_T$ an analytic function such that

$$(\mu I - T)f(\mu) = x + T(0) \quad \text{for all } \mu \in \mathbb{C}\backslash(F_1 \cup F_2).$$

25 Jul 2021 14:49:52 PDT
210725-Mnif Version 1 - Submitted to Rocky Mountain J. Math.

We may assume that F_2 is compact since we have, by Lemma 2.4, for all sets $F \subseteq \mathbb{C}$, $\chi_T(F) = \chi_T(F \cap \sigma(T))$ and $X_T(F) = X_T(F \cap \sigma(T))$. Let G_1, G_2 be two disjoints compact sets such that for $i = 1, 2$ the set G_i is a neighborhood of F_i whose boundary Γ_i be a contour in the complement $\mathbb{C}\backslash F_i$ that surrounds F_i and that G_1 is connected. By Lemma 3.1, we have

$$P_\sigma x = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} f(\lambda) d\lambda.$$

On the other hand, we have

$$P_\sigma x = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} f(\lambda) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma_2} f(\lambda) d\lambda. \tag{3.1}$$

Put $x_i = \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_i} f(\lambda) d\lambda \in D(T)$ for $i = 1, 2$. Now, show that $x_i \in \chi_T(G_i)$. Let

$$g_i(\lambda) := \frac{1}{2\pi i} \int_{\Gamma_i} \frac{f(\mu)}{\lambda - \mu} d\mu \quad \text{for all} \quad \lambda \in \mathbb{C}\backslash G_i.$$

Note that g_i considered as a function from $\mathbb{C}\backslash G_i$ to X_T is analytic. We have for every $\lambda \in \mathbb{C}\backslash G_i$, the operator $Q_T(\lambda I - T)$ is bounded from X_T to $\dfrac{X}{T(0)}$. Whence, for all $\lambda \in \mathbb{C}\backslash G_i$,

$$\begin{aligned}
Q_T(\lambda I - T) g_i(\lambda) &= \frac{1}{2\pi i} \int_{\Gamma_i} Q_T(\mu I - \mu I + \lambda I - T) \frac{f(\mu)}{\lambda - \mu} d\mu \\
&= \frac{1}{2\pi i} \int_{\Gamma_i} Q_T(\mu I - T) \frac{f(\mu)}{\lambda - \mu} d\mu + Q_T(\frac{1}{2\pi i} \int_{\Gamma_i} f(\mu) d\mu) \\
&= Q_T(\frac{1}{2\pi i} \int_{\Gamma_i} \frac{x}{\mu - \lambda} d\mu + x_i).
\end{aligned}$$

Using Cauchy's theorem we get that

$$\int_{\Gamma_i} \frac{d\mu}{\lambda - \mu} = 0, \quad \text{for all} \quad \lambda \in \mathbb{C}\backslash G_i.$$

Thus, for each $\lambda \in \mathbb{C}\backslash G_i$, $(\lambda I - T) g_i(\lambda) - x_i = T(0)$. Whence, $x_i \in \chi_T(G_i)$ for any neighborhood G_i of F_i described as above. Hence, it remains to prove that $x_1 \in \chi_T(F_1) \cap D(T)$ and $x_2 \in X_T(F_2) \cap D(T)$. First, we claim that $x_1 \in \chi_T(F_1) \cap D(T)$. Indeed, as we have $x_1 \in D(T)$ then we need only to prove that there exists an analytic function $f : \mathbb{C}\backslash F_1 \to X_T$ such that $(\mu I - T) f(\mu) = x_1 + T(0)$ for all $\mu \in \mathbb{C}\backslash F_1$. We note that for every connected compact neighborhood G of F_1, whose boundary Γ in $\mathbb{C}\backslash F_1$ that surrounds F_1, there exists an analytic function $f_G : \mathbb{C}\backslash G \to X_T$ such that

$$(\mu I - T) f_G(\mu) = x_1 + T(0), \quad \text{for all} \quad \mu \in \mathbb{C}\backslash G.$$

Let $\lambda \in \mathbb{C}\backslash F_1$. Then, there exists an infinite choice of G, with the properties mentioned above, such that $\lambda \in \mathbb{C}\backslash G$. Let G_1 and G_2 be any two of these choices. We shall prove that $f_{G_1}(\lambda) = f_{G_2}(\lambda)$. Indeed, we have for all $\mu \in (\mathbb{C}\backslash G_1) \cap (\mathbb{C}\backslash G_2) = \mathbb{C}\backslash(G_1 \cup G_2)$,

$$\begin{cases} (\mu I - T) f_{G_1}(\mu) = x_1 + T(0) \\ (\mu I - T) f_{G_2}(\mu) = x_1 + T(0). \end{cases}$$

Observe that $\mathbb{C}\backslash(G_1 \cup G_2)$ is an open connected set which intersects the open set $\rho(T)$ and that on $\rho(T) \cap \mathbb{C}\backslash(G_1 \cup G_2)$ we have $f_{G_1}(\mu) = (\mu I - T)^{-1}$ and $f_{G_2}(\mu) = (\mu I - T)^{-1}$. So, $f_{G_1} = f_{G_2}$ on $\rho(T) \cap \mathbb{C}\backslash(G_1 \cup G_2)$. But, we have $\mathbb{C}\backslash(G_1 \cup G_2)$ is an open connected set and that $\rho(T) \cap \mathbb{C}\backslash(G_1 \cup G_2)$ has an accumulation point, then the identity theorem entails that $f_{G_1} = f_{G_2}$ on $\mathbb{C}\backslash(G_1 \cup G_2)$ and hence, $f_{G_1}(\lambda) = g_{G_2}(\lambda)$. This allows us to define a function f on $\mathbb{C}\backslash F_1$ as follows: For all $\lambda \in \mathbb{C}\backslash F_1$, $f(\lambda) = f_G(\lambda)$, where G is any connected compact neighborhood of F_1 such that $\lambda \in \mathbb{C}\backslash G$. Whence, the function f defined above is analytic on $\mathbb{C}\backslash F_1$ and for every $\mu \in \mathbb{C}\backslash F_1$, $(\mu I - T) f(\mu) = x_1 + T(0)$. Hence,

$$x_1 \in \chi_T(F_1) \cap D(T). \tag{3.2}$$

Second, let us prove that $x_2 \in X_T(F_2)$. We have $x_2 \in \chi_T(G) \subseteq X_T(G)$ for every compact neighborhood G of F_2 whose boundary Γ_2 is a contour surrounding F_2. Then, $\sigma_T(x_2) \subseteq G$ for all G a compact neighborhood of F_2 and therefore, $\sigma_T(x_2) \subseteq \overline{F_2}$. Consequently,

$$x_2 \in X_T(F_2) \cap D(T). \tag{3.3}$$

Thus, (3.3), (3.2) and (3.1) ensure that

$$x = x_1 + x_2 - (P_\sigma x - x) \in \chi_T(F_1) \cap D(T) + X_T(F_2) \cap D(T) + ker P_\sigma.$$

This achieves the proof. ■

Lemma 3.3. *Let $T \in \mathcal{CR}(X)$ with $\rho(T) \neq \emptyset$ and unbounded. Then, for all $\lambda \in \rho(T)$,*

(i) $(\lambda i_T - \tilde{T})^{-1} \in \mathcal{B}(X, X_T)$, *where $i_T : X_T \to X, x \mapsto x$.*

(ii) If T satisfies Assumption 3.2, then

$$\lim_{|\lambda| \to \infty} \|(\lambda i_T - \tilde{T})^{-1} i_T\|_{\mathcal{B}(X_T)} = 0.$$

(iii) If T satisfies Assumption 3.2 and $\sigma(T)$ is bounded, then

$$\chi_T(\{0\}) \cap D(T) \subseteq H_0(T) + T(0).$$

Proof : (i) We start by proving that $\lambda i_T - \tilde{T}$ is injective. Let $x \in ker(\lambda i_T - \tilde{T})$. Then, $0 \in (\lambda i_T - \tilde{T})x = \lambda x - Tx = (\lambda I - T)x$. So, $x \in ker(\lambda I - T) = \{0\}$. Hence, $\lambda i_T - \tilde{T}$ is injective. Show that $\lambda i_T - \tilde{T}$ is surjective. Let $y \in X$. Then, there exists $x \in D(T)$ such that $y \in (\lambda I - T)x = \lambda x - Tx = (\lambda i_T - \tilde{T})x$. Thus, $(\lambda i_T - \tilde{T})$ is surjective. Therefore, $(\lambda i_T - \tilde{T})$ is invertible and $(\lambda i_T - \tilde{T})^{-1} : X \to X_T$ is a bounded operator.
(ii) As a preliminary to the proof we begin by showing that for $\lambda, \mu \in \rho(T)$, we have

$$(\lambda i_T - \tilde{T})^{-1} - (\mu i_T - \tilde{T})^{-1} = (\mu - \lambda)(\lambda i_T - \tilde{T})^{-1} i_T (\mu i_T - \tilde{T})^{-1}. \tag{3.4}$$

We have, for all $\lambda, \mu \in \rho(T)$ and $x \in D(T)$, $(\mu - \lambda)(\lambda i_T - \tilde{T})^{-1} i_T (\mu i_T - \tilde{T})^{-1} x = (\lambda i_T - \tilde{T})^{-1}(\mu - \lambda) i_T (\mu i_T - \tilde{T})^{-1} x = (\lambda i_T - \tilde{T})^{-1}(\mu i_T - \tilde{T} + \tilde{T} - \lambda i_T)(\mu i_T - \tilde{T})^{-1} x$. Therefore, $(\mu - \lambda)(\lambda i_T - \tilde{T})^{-1} i_T (\mu i_T - \tilde{T})^{-1} x = (\lambda i_T - \tilde{T})^{-1}[(\mu i_T - \tilde{T})(\mu i_T - \tilde{T})^{-1} x - (\lambda i_T - \tilde{T})(\mu i_T - \tilde{T})^{-1} x]$. Using (i) we get the desired equality (3.4). Whence, for all $\lambda, \mu \in \rho(T)$, we get $[(\lambda i_T - \tilde{T})^{-1} - (\mu i_T - \tilde{T})^{-1}](\mu i_T - \tilde{T}) = (\mu - \lambda)(\lambda i_T - \tilde{T})^{-1} i_T (\mu i_T - \tilde{T})^{-1}(\mu i_T - \tilde{T})$. Using again (i) we obtain $(\lambda i_T - \tilde{T})^{-1}(\mu i_T - \tilde{T}) - I_{X_T} = (\mu - \lambda)(\lambda i_T - \tilde{T})^{-1} i_T I_{X_T}$, where I_{X_T} is the identity on X_T. Therefore,

$$(\lambda i_T - \tilde{T})^{-1}(\mu - \lambda) i_T = (\lambda i_T - \tilde{T})^{-1}(\mu i_T - \tilde{T}) - I_{X_T}. \tag{3.5}$$

On the other hand, by Assumption 3.2, there exists a projection $P \in \mathcal{B}(X)$ such that $ImP = D(T)$ and $kerP = T(0)$. We note that P considered as operator from X to X_T is bounded. Furthermore, by (3.5), we get

$$\begin{aligned}
&\|(\mu - \lambda)(\lambda i_T - \tilde{T})^{-1} i_T\|_{\mathcal{B}(X_T)} = \|(\lambda i_T - \tilde{T})^{-1}(P + I - P)(\mu i_T - \tilde{T}) - I_{X_T}\|_{\mathcal{B}(X_T)} \\
&= \|(\lambda i_T - \tilde{T})^{-1} i_T P(\mu i_T - \tilde{T}) + (\lambda i_T - \tilde{T})^{-1}(I - P)(\mu i_T - \tilde{T}) - I_{X_T}\|_{\mathcal{B}(X_T)} \\
&\leq \|P(\mu i_T - \tilde{T})\|_{\mathcal{B}(X_T)} \|\|(\lambda i_T - \tilde{T})^{-1} i_T\|_{\mathcal{B}(X_T)} + 1.
\end{aligned}$$

Then, $(|\mu - \lambda| - \|P(\mu i_T - \tilde{T})\|_{\mathcal{B}(X_T)})\|(\lambda i_T - \tilde{T})^{-1} i_T\|_{\mathcal{B}(X_T)} \leq 1$ and so,

$$\|(\lambda i_T - \tilde{T})^{-1} i_T\|_{\mathcal{B}(X_T)} \leq \frac{1}{|\mu - \lambda| - \|P(\mu i_T - \tilde{T})\|_{\mathcal{B}(X_T)}}.$$

Thus, letting $|\lambda| \to \infty$, we get the desired result.
(iii) Assume that $x \in \chi_T(\{0\}) \cap D(T)$. Then, there exists an analytic function $f : \mathbb{C}\backslash\{0\} \to X_T$ such that $(\lambda I - T)f(\lambda) = x + T(0)$ holds for every $\lambda \in \mathbb{C}\backslash\{0\}$. By hypothesis, we have $\sigma(T)$ is bounded then $V := \mathbb{C}\backslash\{0\} \cap \rho(T)$ is not empty and open and the function f is analytic throughout V. We claim that

$$\lim_{\substack{|\lambda| \to \infty \\ \lambda \in V}} \|f(\lambda)\|_T = 0. \tag{3.6}$$

Indeed, we have $(\lambda I - T)f(\lambda) = x + T(0)$, then, $\lambda i_T f(\lambda) - \tilde{T} f(\lambda) = x + T(0)$. So,

$$(\lambda i_T - \tilde{T})f(\lambda) = x + T(0). \tag{3.7}$$

Using (i) we get, $f(\lambda) = (\lambda i_T - \tilde{T})^{-1}x + (\lambda i_T - \tilde{T})^{-1}(\lambda i_T - \tilde{T})(0) = (\lambda i_T - \tilde{T})^{-1} i_T x$. Now, by virtue of (ii), the equality (3.6) holds. Which means that $\lim\limits_{\substack{|\lambda|\to\infty \\ \lambda\in V}} f(\lambda) = 0$ on X_T. Let us consider the analytic function g defined by

$$g(\mu) := \begin{cases} f(\frac{1}{\mu}) & \text{if } \ \mu \neq 0, \\ 0 & \text{if } \ \mu = 0. \end{cases}$$

As g is analytic on $\mathbb{C}$ and $g(0) = 0$, then there exists a sequence $(x_n)_n \subseteq D(T)$ such that $x_0 = 0$ and $g(\mu) = \sum\limits_{n\geq 0} \mu^n x_n$ holds for all $\mu \in \mathbb{C}$. Whence, $\limsup\limits_{n\to\infty} \|x_n\|_T^{\frac{1}{n}} = 0$. Which implies that $\lim\limits_{n\to\infty} \|x_n\|_T^{\frac{1}{n}} = 0$. Furthermore, we have $f(\lambda) = g(\frac{1}{\lambda}) = \sum\limits_{n\geq 0} \lambda^{-n} x_n$. It follows from Lemma 5.2 in [10], that for all $n \in \mathbb{N}$, there exists $\alpha_{n+1} \in \tilde{T}x_n$ such that for all $\lambda \neq 0$ we have

$$\begin{aligned}(\lambda i_T - \tilde{T})f(\lambda) &= \sum_{n\geq 0} \lambda^{-n+1} x_n - \sum_{n\geq 0} \lambda^{-n} \alpha_{n+1} + \tilde{T}(0) \\ &= \sum_{n\geq 0} \lambda^{-n}(x_{n+1} - \alpha_{n+1}) + T(0).\end{aligned}$$

This implies, by the use of (3.7), that $Q_T(\lambda i_T - \tilde{T})f(\lambda) = Q_T(x + T(0)) = \sum\limits_{n\geq 0} \lambda^{-n} Q_T(x_{n+1} - \alpha_{n+1})$. Hence,

$$\begin{cases} Q_T(x_1 - \alpha_1) = Q_T(x), \\ Q_T(x_{n+1} - \alpha_{n+1}) = 0, \quad \text{for every } \ n \geq 1. \end{cases}$$

Whence, $x_{n+1} \in \tilde{T}x_n = Tx_n$ for all $n \geq 1$ and there exists $\alpha \in T(0)$ such that $x = x_1 + \alpha$. Thus, it remains to prove that $x_1 \in H_0(T)$. Let $(y_n)_n$ be the sequence defined by $y_n := x_{n+1}$ for all $n \in \mathbb{N}$. Trivially, we have $y_0 = x_1$ and $y_{n+1} = x_{n+2} \in Tx_{n+1} = Ty_n$. Moreover, we have $\lim\limits_{n\to\infty} \|y_n\|_T^{\frac{1}{n}} = \lim\limits_{n\to\infty} \|x_{n+1}\|_T^{\frac{1}{n}} = 0$. Therefore, $x_1 \in H_0(T)$ and so, $x = x_1 + \alpha \in H_0(T) + T(0)$. Hence, $\chi_T(\{0\}) \cap D(T) \subseteq H_0(T) + T(0)$. ■

For bounded operators, González et al. [5] presented a characterization of the isolated points of the surjective spectrum. Recently, the authors of [10] have extended a part of this result for closed and bounded linear relations as follows:

Lemma 3.4. *[10, Theorem 6.2] Let $T \in \mathcal{BCR}(X)$. If $X = H_0(\lambda I - T) + K(\lambda I - T)$ then λ is isolated in $\sigma_{su}(T)$.*

In the following theorem, we intend to further generalize the result of González et al. in [5] by moving from the case of bounded operators to the more general setting of closed linear relations.

25 Jul 2021 14:49:52 PDT
210725-Mnif Version 1 - Submitted to Rocky Mountain J. Math.

Theorem 3.1. *Let $T \in \mathcal{CR}(X)$ satisfying Assumption 3.1 and Assumption 3.2. If $0 \in \sigma_{su}(T)$, then we have the equivalence:*

$$0 \text{ is isolated in } \sigma_{su}(T) \text{ if and only if } X = H_0(T) + K(T).$$

Proof : Suppose that $0 \in \sigma_{su}(T)$ and $X = H_0(T) + K(T)$. Let $x \in X$. Then, $x = x_1 + x_2$ with $x_1 \in H_0(T)$ and $x_2 \in K(T)$. It follows from Lemma 2.3 (i) that $\sigma_T(x_1) \subseteq \{0\}$. Therefore, by virtue of Proposition 2.1, we get

$$\sigma_T(x) \subseteq \sigma_T(x_1) \cup \sigma_T(x_2) \subseteq \{0\} \cup \sigma_T(x_2). \tag{3.8}$$

Now, by Lemma 2.3 (ii) and since $\sigma_T(x_2)$ is closed we conclude that 0 is isolated in $\sigma_T(x) \cup \{0\}$ for any $x \in X$. Using Lemma 2.4 (i) and (ii) we get that 0 is isolated in $\sigma_{su}(T)$ as desired. For the only if part, since 0 is an isolated point in $\sigma_{su}(T)$, then it follows from Lemma 3.2, that

$$\chi_T(\sigma_{su}(T)) \subseteq \chi_T(\{0\}) \cap D(T) + X_T(\sigma_{su}(T)\backslash\{0\}) \cap D(T) + kerP_\sigma.$$

Observe that, by the use of Lemma 2.4 together with Lemma 2.3, we have $\chi_T(\sigma_{su}(T)) = X$ and $X_T(\sigma_{su}(T)\backslash\{0\}) \cap D(T) \subseteq X_T(\mathbb{C}\backslash\{0\}) = K(T)$. Whence, by Lemma 3.3 (iii), we get

$$X \subseteq H_0(T) + T(0) + K(T) + kerP_\sigma. \tag{3.9}$$

Adhering the notations of Remark 3.1 and according to Lemma 1.1, we get that $D(T_1) = D(T) \cap kerP_\sigma$ with $T_1 = T_{kerP_\sigma}$. Thus, we obtain $T_1(kerP_\sigma \cap D(T_1)) = T_1(D(T_1))$. Furthermore, it follows from Remark 3.1 that $0 \notin \sigma(T_1)$. Then, T_1 is surjective. Consequently, we obtain $T_1(kerP_\sigma \cap D(T_1)) = kerP_\sigma$. By virtue of Lemma 2.2, we get that $kerP_\sigma \subseteq K(T_1)$. Now, we claim that

$$K(T_1) \subseteq K(T).$$

Indeed, let $x \in K(T_1)$. Then, it follows from the proof of Lemma 2.2 (iii) that there exist $d > 0$ and a sequence $(x_n)_n$ such that $x_0 = x$; for all $n \geq 0, x_n \in T_1 x_{n+1}$ and $x_{n+1} \in D(T) \cap X_1$ and $\|x_n\| \leq d^n \|x\|$. Let $\tilde{x}_n = 0 \oplus x_n$. We have $\tilde{x}_0 = 0 + x_0 = x$. Since $x_n \in T_1 x_{n+1}$, then $\tilde{x}_n = 0 + x_n \in T_0(0) \oplus T_1 x_{n+1} \in T(0 \oplus x_{n+1}) \in T(\tilde{x}_{n+1})$. Furthermore, we have $\|\tilde{x}_n\| = \|x_n\| \leq d^n \|x\|$. Now, if $x \in T(0) \cap kerT$ then $x \in K(T)$. Otherwise, a short calculation reveals the existence of some $d' > d(x, T(0) \cap kerT)$ such that

$$d(\tilde{x_n}, T(0) \cap kerT) \leq d'^n d(x, T(0) \cap kerT).$$

Whence, $x \in K(T)$. Thus, $kerP_\sigma \subseteq K(T)$. Consequently, it follows from (3.9) that $X \subseteq H_0(T) + K(T)$. Hence, we get the desired result. ■

Let us denote by $\mathrm{acc}(\sigma_T(x))$ the set of all accumulation points of the local spectrum of $T \in \mathcal{CR}(X)$ at the point $x \in X$.

Corollary 3.1. *Let $T \in \mathcal{CR}(X)$ satisfying Assumption 3.1 and Assumption 3.2. Then we have the equivalence:*

$$X = H_0(T) + K(T) \text{ if and only if } 0 \notin acc(\sigma_T(x)) \text{ for every } x \in X.$$

Proof : Suppose that $X = H_0(T) + K(T)$. Using (3.8), we get that 0 is isolated in $\{0\} \cup \sigma_T(x)$ for all $x \in X$. Then, $0 \notin acc(\sigma_T(x))$ for every $x \in X$. On the other hand, by Lemma 2.4 (ii), there exists $x_0 \in X$ such that $\sigma_T(x_0) = \sigma_{su}(T)$. Hence, $0 \notin acc(\sigma_{su}(T))$. Now, assume that $0 \notin \sigma_{su}(T)$ then $X = K(T)$, as desired. Otherwise, if $0 \in \sigma_{su}(T)$ then, from Theorem 3.1, we have $X = H_0(T) + K(T)$. Which ends the proof. ∎

Data availability. All data generated or analysed during this study are included in this published article.

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