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SOLVING QUADRATIC AND CUBIC DIOPHANTINE EQUATIONS USING 2-ADIC VALUATION TREES

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ABSTRACT. For fixed $D \in \mathbb{N}$, we demonstrate how to use 2-adic valuation trees of sequences to analyze Diophantine equations of the form $x^s + D = 2^t y$ for y odd, $s \in \{2,3\}$, and $t \in \mathbb{N}$. Further, we show for what values of D the numbers $x^3 + D$ will generate infinite valuation trees, which lead to infinite solutions to the above corresponding Diophantine equations.

1. Introduction

Special kinds of Diophantine equations called generalized Lebesgue-Ramanujan-Nagell equations have been investigated using various methods including elementary techniques in classical number theory, Diophantine approximation methods, the Baker method, the Bilu-Hanrot-Voutier theorem, and the modular approach [8, 11, 3]. In this paper, we present a straightforward way of analyzing the solutions to generalized Lebesgue-Ramanujan-Nagell equations using a visual approach. In particular, for $D \in \mathbb{N}$ fixed, we study families of Diophantine equations of the form

$$x^s + D = 2^t y, \tag{1.1}$$

for $s \in \{2,3\}$ with $t \in \mathbb{N}$. Oftentimes, the techniques used to solve such families do not apply to the case when $D \equiv 7 \pmod 8$. Therefore, it is often omitted from consideration. By contrast, we employ the construction of 2-adic valuation trees to visualize and easily identify relationships among the solutions $(x,y) \in \mathbb{Z}^2$ of equation (1.1).

Theorem 1.1. *Let $D \in \mathbb{N}$ in the family of Diophantine equations*

$$x^2 + D = 2^t y, \tag{1.2}$$

with $2 \nmid y$.

1. *If $D = 4^j(8k+7)$ for some $j,k \in \mathbb{Z}_{\geq 0}$, then equation (1.2) has non-trivial solutions $x,y \in \mathbb{N}$, with y odd, for all but finitely many $t \in \mathbb{N}$. Further, for $j = 0$, when $D = 8k+7$ the set of values for t that has non-trivial solutions does not include $t = 1$ or 2.*
2. *If $D \neq 4^j(8k+7)$ for any $j,k \in \mathbb{Z}_{\geq 0}$, then equation (1.2) has no non-trivial integer solutions $x,y \in \mathbb{N}$, with y odd, for infinitely many $t \in \mathbb{N}$.*

The cubic form of equation (1.1) for $s = 3$ has a similar rich history. In 1951, Nagell [10, p. 246-248] proved that the family of Diophantine equations $x^3 + y^3 = az^3$, for integer $a > 2$ not divisible by the cube of any prime has either no solution or infinitely many solutions in relatively prime integers x, y, and z, with $z \neq 0$. Nagell's theorem can be used to prove that, for some specific values of t, D, and y, the related cubic

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Diophantine equation (1.1) has either no solution or infinitely many solutions. Similar to what we see in the quadratic case, for $y = 1$, Beukers [4] proved that the cubic form of the equation (1.1) has at most five solutions in $x \in \mathbb{Z}$. In 2020, Alvarado et al [2, Theorem 8.2] analyzed the cubic Ramanujan-Nagell equation $x^3 + 3^k = q^n$ for prime $q > 3$ and integers $n, k > 0$. Letting q be a prime such that $3 < q \leq 500$, they list all integer solutions to the above equation, and further claim that their method can also be used to find the integer solutions to the equation $x^3 + p^k = q^n$ where p, q are distinct odd primes. By using the 2-adic valuation tree approach we prove the following theorem.

Theorem 1.2. *Let $D \in \mathbb{N}$. in the family of Diophantine equations*

$$x^3 + D = 2^t y, \tag{1.3}$$

with $2 \nmid y$.

(1) *If $D = 8^j(2k+1)$ for some $j, k \in \mathbb{Z}_{\geq 0}$, then equation (1.3), with y odd, has non-trivial solutions for all but finitely many integer values of $t \geq 0$.*

(2) *If $D \neq 8^j(2k+1)$ for any $j, k \in \mathbb{Z}_{\geq 0}$, then there are finitely many $t \in \mathbb{N}$ for which the equation equation (1.3), with y odd, has non-trivial solutions. Specifically, the following holds.*

   (a) *If $D = 2(2k+1)$ for some $k \in \mathbb{Z}_{\geq 0}$, then x is even with $t = 0$ or x is odd with $t = 1$.*

   (b) *If $D = 4(2k+1)$ for some $k \in \mathbb{Z}_{\geq 0}$, then x is odd with $t = 0$ or x is even with $t = 2$.*

   (c) *If $D = 2^{3k+i}$ for some $k \in \mathbb{N}$ and $i \in \{1, 2\}$, then $3 | t$ with $t = 3\ell$ for some $\ell < k + i$ and $x \equiv 2^\ell \pmod{2^{\ell+1}}$. Further, if, in this case, $t = 3\ell$ with $\ell = k + i$, then $x \equiv 0 \pmod{2^{k+i}}$.*

In Section 2, we demonstrate how the construction of 2-adic valuation trees can prove that a special case of the quadratic equation (1.1) with $s = 2$, $D = 7$ with $t = 0$ or $t \geq 3$, and y odd

$$x^2 + 7 = 2^t y \tag{1.4}$$

has an infinite number of positive integer solutions, and, further, we determine the form of the solutions. In Sections 3 and 4, this method is applied to prove Theorem 1.1 and Theorem 1.2, respectively. In Section 5, we provide a few examples of finding non-trivial integer solutions to families of equations (1.1) for specific $D \in \mathbb{N}$. We discuss further directions of this approach in Section 6 and highlight solutions for a few additional examples given in the Appendix.

We fix the following notations throughout the remainder of the article:

D	the constant term in the Diophantine equation (1.1)
ℓ	the level of the valuation tree, where $\ell \geq 0$
$\nu_p(n) = t$	for prime p, the p-adic valuation of integer n where $n = p^t b$ with $p \nmid b$
(x, y, t)	an integer solution to the Diophantine equation $x^s + D = 2^t y$ for $s \in \{2, 3\}$ for fixed D

2. 2-adic Valuation Tree Construction for $x^2 + 7$

We explicitly construct the 2-adic valuation tree described in [7, Section 4] where we incorporate more of the information encoded in the tree than the authors make use of.

In general for a prime p, a p-adic tree consists of vertices and edges which we will call *nodes* and *branches*, respectively. The initial node, called the *root* of the tree, is labeled with n_0. Proceeding to generate the tree in a downward direction, we will label each node and branch with a positive integer n_i and $b_{i,j}$, respectively for some $i, j \in \mathbb{N}$.

Note that leaves marked as square nodes occur when we know the exact valuation for a given classification of x, and circles when the valuation is still not known exactly, but we do know that it's greater than or equal to the value in the circle. Again, the tree will always branch from the circles and terminate on the squares.

To see this explicitly, let $p = 2$ and let the root n_0 have the form $n_0 = x^2 + 7$ for some $x \in \mathbb{Z}^+$ in Figure 1. We take the root n_0 to be at level 0. Since $x \equiv 0$ or $1 \pmod 2$, we have two branches leading to two nodes

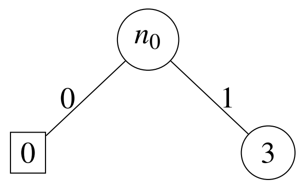

FIGURE 1. Level 1 of the 2-adic valuation tree for $x^2 + 7$.

at the next level, $\ell = 1$. First, the branches are labeled 0 and 1 as they are the two least residues of the equivalences classes modulo $2^{\ell+1} = 2$ when $\ell = 0$.

For the node at level 1 connecting to branch 0, we have $x \equiv 0 \pmod 2$ which we write $x = 2k_1$ for some $k_1 \in \mathbb{N}$. Then the value of that node is $\nu_2(x^2 + 7) = \nu_2((2k_1)^2 + 7) = 0$.

For the second node at level 1, connecting to branch 1, we take x odd. Then $x = 2k_1 + 1$, where $k_1 \in \mathbb{Z}_{\geq 0}$ gives the value of the node

$$
\begin{aligned}
\nu_2((2k_1+1)^2+7) &= \nu_2(4(k_1^2+k_1+2)) \\
&= \nu_2(4) + \nu_2(k_1^2+k_1+2) \\
&= 2 + \nu_2(k_1^2+k_1+2).
\end{aligned}
$$

Notice that the value of $\nu_2(k_1^2 + k_1 + 2)$ depends on the parity of k_1. Specifically, if k_1 is even, then $k_1^2 + k_1 + 2$ is also even, thus has at least one factor of 2. Now, if k_1 is odd, then $k_1 = 2k_2 + 1$ from some $k_2 \in \mathbb{Z}$ where $k_1^2 + k_1 + 2 = (4k_2^2 + 4k_2 + 1) + (2k_2 + 1) + 2 = 4k_2^2 + 6k_2 + 4 = 2(2k_2^2 + 3k_2 + 2)$, which contains at least one factor of 2. Therefore $\nu_2(k_1^2 + k_1 + 2) \geq 1$ in either case, yielding $\nu_2((2k_1 + 1)^2 + 7 \geq 3$.

Again, the 2-adic valuation of $x^2 + 7$ depends on whether k_1 is even or odd. Thus the tree grows further, meaning that we examine $x^2 + 7$ modulo 4.

Let us consider when k_1 is either even or odd, meaning that $k_1 = 2k_2$ or $k_1 = 2k_2 + 1$ for some $k_2 \in \mathbb{Z}_{\geq 0}$. Then either $x = 2^2k_2 + 1$ or $x = 2^2k_2 + 3$, that is, $x \equiv 1$ or $3 \pmod 4$, respectively. We see that

$$
(2^2k_2+1)^2+7 = 2^4k_2^2+2^3k_2+1+7
$$

and

$$
(2^2k_2+3)^2+7 = 2^4k_2^2+2^3\cdot 3k_2+9+7
$$

both depend on the parity of k_2 in order to determine the 2-adic valuation. (See Figure 2.)

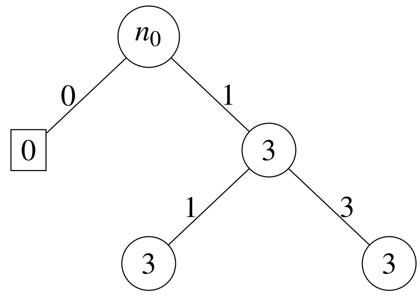

FIGURE 2. The first two levels of the 2-adic valuation tree for x^2+7.

Subsequent nodes are labeled with $\nu_2(x^2+7)$ when x is taken to be $2^\ell k_\ell + b_\ell$ where ℓ is the level of our tree, b_ℓ is the weight on the branch connecting level $\ell-1$ to current node and for some $k_\ell \in \mathbb{Z}_{\geq 0}$. Figure 3 showcases the continuation of this tree.

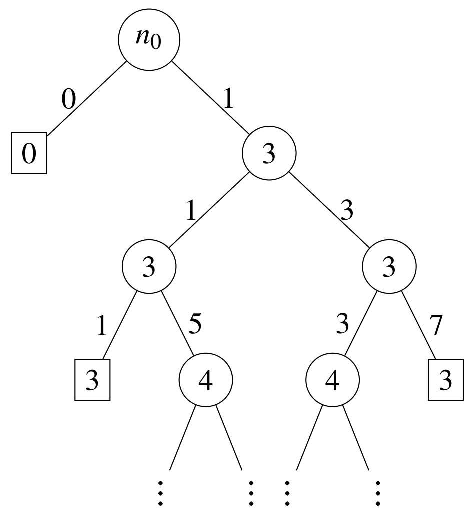

FIGURE 3. The first 3 levels of the 2-adic valuation tree for x^2+7.

It has been shown by Kozhushkina et al [7] that this tree is infinite, symmetrical, and has 2-adic valuation range $\{0,3,4,5,6,\ldots\}$. Consider

$$\nu_2(x^2+7) = t, \text{ for an arbitrary } t \geq 3. \tag{2.1}$$

We know by the 2-adic valuation tree for $t \geq 3$ there is always going to be a value for x that makes the equation true. This means that

$$x^2+7 = 2^t y$$

for some $y \in \mathbb{Z}$ where $2 \nmid y$. Thus, the 2-adic valuation range of t corresponds to a Diophantine equation, and it will have non-trivial solutions. We now turn our attention to finding those solutions (x,y,t) to the Diophantine equation $x^2+7=2^t y$.

If we know a solution (x,y,t) to the equation $x^2+7=2^t y$, with y odd, we can find the additional solutions recursively.

Proposition 2.1. *There exists a sequence of solution pairs $\{(x_t, y_t)\}_{t\geq 3}$ of the Diophantine equation $x^2 + 7 = 2^t y$ which corresponds to the minimum of the branch residues, when $t = \ell + 1$ (which we define here as the values x_ℓ, $x_\ell + 2^{\ell-1}$, $-(2^{\ell-1} + x_\ell)$, and $-x_\ell \bmod 2^\ell$) at level ℓ for x_t and the non-terminating node behavior at level $\ell - 1$ for y_t.*

Proof. We know by [7] that the tree for $x^2 + 7$ for every level $\ell \geq 3$ has four branches, each of which has a corresponding 2-adic valuation node, and those branches are symmetric. Through the properties of modular arithmetic and the fact that the tree is symmetric, we know, among these four nodes, two of the nodes are terminating with 2-adic valuation equal to ℓ while the other two nodes are non-terminating nodes with 2-adic valuation at least $\ell + 1$. Further, we can describe exactly what the values of these branches are.

In Figure 3, there are two infinite branches starting at level 2 of the tree. We'll refer to these left and right infinite branches as trunks. The two branches on the left trunk of Figure 3 represent integers of the form

$$2^\ell n + x_\ell, \quad 2^\ell n + x_\ell + 2^{\ell-1}$$

while the two branches on the right trunk of Figure 3 represent integers of the form

$$2^\ell n + 2^\ell - x_\ell, \quad 2^\ell n - (2^{\ell-1} + x_\ell),$$

where $n \in \mathbb{N}$. These are the branches of the tree as described by Kozhushkina et al [7] on page 100 with the notable difference that we are choosing not to take the values modulo 2^ℓ.

Observe that, for any given level ℓ, the four branch residues are determined by x_ℓ which depends on the 2-adic valuation node at the previous level $\ell - 1$.

The two branch residues on the left trunk, namely x_ℓ and $x_\ell + 2^{\ell-1}$, emanate from a non-terminating node in the $\ell - 1$ level, that is, the 2-adic valuation node with a value greater than or equal to ℓ. The branch residue x_ℓ is equal to the previous branch residue $x_{\ell-1}$.

Thus, in order to establish that $x_\ell = x_{t-1}$ is the minimum branch residue at level ℓ, we only need to check what the 2-adic valuation is at either the branch residue $x_\ell \pmod{2^\ell}$ or $-(2^{\ell-1} + x_\ell) \pmod{2^\ell}$. Moreover, due to the symmetry of the tree, if the 2-adic valuation at the branch residue x_ℓ leads to a non-terminating node, then it follows that the branch residue at $-(2^{\ell-1} + x_\ell) \pmod{2^\ell}$ leads to a terminating node. As a result, the other non-terminating node will come from the branch having residue $-x_\ell \pmod{2^\ell}$. Observe that $x_\ell < 2^\ell - x_\ell$ for all $\ell > 1$, because $x_\ell < 2^{\ell-1}$.

Hence, without loss of generality, let us evaluate the 2-adic valuation along the branch $2^\ell n + x_\ell$, for some $n \in \mathbb{N}$, at level ℓ, to get

$$\begin{aligned}
\nu_2((2^\ell n + x_\ell)^2 + 7) &= \nu_2\left(2^{2\ell} n^2 + 2^{\ell+1} n x_\ell + x_\ell^2 + 7\right) \\
&= \nu_2\left(2^\ell\right) + \nu_2\left(2^\ell n^2 + 2n x_\ell + \frac{x_\ell^2 + 7}{2^\ell}\right) \\
&= \ell + \nu_2\left(2^\ell n^2 + 2n x_\ell + \frac{x_\ell^2 + 7}{2^\ell}\right) \\
&= \ell + \nu_2\left(2^\ell n^2 + 2n x_\ell + y_\ell\right),
\end{aligned}$$

where $y_\ell = \dfrac{x_\ell^2 + 7}{2^\ell}$. There are two cases to consider, either $y_\ell = y_{t-1}$ is even or odd.

If $y_\ell = y_{t-1}$ is even, then

$$\nu_2(2^\ell n^2 + 2n x_\ell + y_\ell) \geq 1$$

so that $v_2((2^\ell n + x_\ell)^2 + 7) \geq \ell + 1$; this means that the branch $2^\ell n + x_\ell$ leads to a non-terminating node and hence, by symmetry, the branch $2^\ell n + 2^\ell - x_\ell$ leads to the other non-terminating node. Since $x_\ell < 2^\ell - x_\ell$, we have established that the minimum value of the branch residues occurs at $x_\ell = x_{t-1}$ in this case.

If $y_\ell = y_{t-1}$ is odd, then

$$v_2(2^\ell n^2 + 2nx_\ell + y_\ell) = 0$$

so that $v_2((2^\ell n + x_\ell)^2 + 7) = \ell$; this means that the branch residue x_ℓ leads to a terminating node and hence the branch residue $2^{\ell-1} - x_\ell$ leads to a non-terminating node. The other non-terminating node comes from the branch having residue $x_\ell + 2^{\ell-1}$. Since $2^{\ell-1} - x_\ell < x_\ell + 2^{\ell-1}$ for all $\ell > 1$, it follows that the minimum value of the branch residues occur at $2^{\ell-1} - x_\ell = 2^{\ell-2} - x_{t-1}$ in this case. □

A direct consequence of Proposition 2.1 to Diophantine equations are the following results.

Corollary 2.2. *For $\ell \in \mathbb{N}$ with $\ell \geq 3$, if $(x_{\ell-1}, y_{\ell-1}, \ell - 1)$ is an integer solution of the form (x, y, t) to the equation $x^2 + 7 = 2^t y$, then (x_ℓ, y_ℓ, ℓ) is also a solution given by the recursion*

$$x_\ell = \begin{cases} x_{\ell-1}, & y_{\ell-1} \text{ is even} \\ 2^{\ell-2} - x_{\ell-1}, & y_{\ell-1} \text{ is odd} \end{cases}$$

and

$$y_\ell = \frac{x_\ell^2 + 7}{2^\ell}.$$

Note that y_ℓ as defined above is always an integer.

Theorem 2.3. *The Diophantine equation $x^2 + 7 = 2^t y$, where y is odd, has positive integer solutions (x, y, t) for all $t \geq 3$ and $t = 0$. Further, for $\ell \in \mathbb{N}$ with $\ell \geq 4$, if $(x_{\ell-1}, y_{\ell-1}, \ell - 1)$ is a solution to the equation $x^2 + 7 = 2^t y$, then (x_ℓ, y_ℓ, ℓ) is another solution given by the recursion*

$$x_\ell = 2^{\ell-2} - x_{\ell-1}$$

and

$$y_\ell = \frac{x_\ell^2 + 7}{2^\ell}.$$

Proof. We know by [5] that there are solutions to the equations $x^2 + 7 = 2^t y$ if $t = 1, 2, 3$, and those solutions are $\{(1,4,1), (1,2,2), (1,1,3)\}$, respectively. If we further restrict y to be odd, then we find that the $t = 1$ and $t = 2$ cases are actually a form of the solution for $t = 3$; $(x, y, t) = (1, 1, 3)$.

By [7], we know the valuation tree for $x^2 + 7$ has range $\mathbb{Z}_{\geq 3} \cup \{0\}$, thus we have a solution for x to $v_2(x^2 + 7) = t$ for all $t \geq 3$ and $t = 0$. Further, by Proposition 2.2, we can recursively determine exactly the form that the solution for x should take for each t-value.

Notice that if $t = 0$ then all solutions of $x^2 + 7 = y$, in which y is odd, are of the form $(2k, (2k)^2 + 7, 0)$ for some $k \in \mathbb{N}$. (See Table 1 for example solutions.)

□

Notice that all of these solutions are in agreement with Bennett, Filaseta, and Trifonov [3, Theorem 1.1] because if x, t and y are positive integers satisfying equation (1.1), then either x is in the set of $\{1, 3, 5, 11, 181\}$ or $y > \sqrt{x}$.

3. Proof of Theorem 1.1

Medina, Moll, and Rowland [9, Theorem 2.1] have proven that a polynomial with roots in $\mathbb{Z}_2$, the 2-adic integers, will form infinite valuation trees. They focus on the sequence x^2+D in [1, Lemma 3.8 and Theorem 4.5], where they are able to describe the forms D must take in order for the tree to be infinite. This is further expanded upon in [6, Theorem 1 part 3] to general quadratic polynomials of the form ax^2+bx+c. The authors specify that our D should be of the form $4^j(8k+7)$ if we wish to have infinite trees for some $j,k \in \mathbb{Z}_{\geq 0}$. Here we use these results to describe which Diophantine equations of the form $x^2+D=2^t y$, y odd, will have infinitely many non-trivial solutions for t.

Proof of Theorem 1.1. If $D \neq 4^j(8k+7)$, then we know the tree is bounded by [1] and [6]. That means that the range of values t can take is finite, and so there exists a j such that for $t > j$, $x^2+D=2^t y$, with y odd has no non-trivial solutions. This is because no valuation branch exists with the value t, hence $\nu_2(x^2+D)=t$ does not exist for $t > j$.

We know from Theorem 1 part 3b in [6] that if $D=4^j(8k+7)$, then our tree will be infinite, that is, the range of t will be infinite as well. Starting at $t > j$, for some finite valuation of the tree j, there will be a valuation for every level of the tree. This means that our Diophantine equation $x^2+D=2^t y$, with y odd, will have solutions for all t in the range of valuations on the tree.

Consider $D=8k+7$. We know that substituting an even number into x^2+D will get us $\nu_2(x^2+D)=0$. Now, suppose $x=2n+1$, we then get

$$\begin{aligned}\nu_2\left(x^2+D\right) &= \nu_2\left(x^2+8k+7\right)\\ &= \nu_2\left((2n+1)^2+8k+7\right)\\ &= \nu_2\left(2^2n^2+2^2n+8k+8\right)\\ &= \nu_2\left(2^3\left(\frac{n^2+n}{2}+k+1\right)\right)\\ &= \nu_2\left(2^3\right)+\nu_2\left(\frac{n^2+n}{2}+k+1\right) \geq 3.\end{aligned}$$

Further, it is shown in [1, Theorem 4.4] that if there is valuation t on a terminating node of tree, then there will be valuation $t+1$ on the next level as a terminating node. The range of t-values in the solution of $x^2+D=2^t y$, y odd and $D \equiv 7 \pmod 8$ is at most $\{0,3,4,5,\ldots\}$. Using a similar proof for $D=4^j(8k+7)$, $k \geq 1$ we can show that $t \neq 1$. $\square$

4. A Cubic Diophantine Equation and its 2-adic Valuation Trees

In order to understand the Diophantine results for $x^3+D=2^t y$, y odd, we prove when the 2-adic valuation trees are finite and infinite.

Proposition 4.1. *If $D=8^j(2k+1)$, for $j,k \geq 0$, then the valuation tree of x^3+D is infinite.*

Proof. When D is odd, by Hensel's lemma we can see that the cube root of D is a root of the polynomial x^3+D in $\mathbb{Z}_2$ and therefore we have an infinite branch [9, Theorem 2.1]. When D is even, we have $j>0$

and we see that the cube root of D is given by $2^j(2k+1)^{1/3}$. By Hensel's lemma, we have that the cube root of D is a root of the polynomial x^3+D in $\mathbb{Z}_2$ and therefore we have an infinite branch. □

Here we work out the cases of D that would result in finite branches.

Proposition 4.2. *Let D be a positive integer. If $D \notin \{8^j(2k+1) : j,k \geq 0\}$ then*

(1) If $D = 2(2k+1)$ then

$$\nu_2(x^3+D) = \begin{cases} 1, & x \equiv 0 \pmod 2, \\ 0, & x \equiv 1 \pmod 2. \end{cases}$$

(2) If $D = 2^2(2k+1)$ then

$$\nu_2(x^3+D) = \begin{cases} 2, & x \equiv 0 \pmod 2, \\ 0, & x \equiv 1 \pmod 2. \end{cases}$$

(3) If $D = 2^{3k+i}$ for $k > 0$ and $i = 1$ or 2 then

$$\nu_2(x^3+D) = \begin{cases} 0, & \text{if } x \equiv 1 \pmod 2, \\ 3, & \text{if } x \equiv 2 \pmod{2^2}, \\ 3\ell, & \text{if } x \equiv 2^\ell \pmod{2^{\ell+1}}, \\ \vdots & \\ 3k, & \text{if } x \equiv 2^k \pmod{2^{k+1}}, \\ 3k+i, & \text{if } x \equiv 0 \pmod{2^{k+1}}, \end{cases}$$

where $\ell = 2, \ldots, k$.

Hence, the valuation tree is finite.

Proof. (1). Suppose $D = 2(2k+1)$ for some $k \geq 0$. If $x = 2n$ then

$$\nu_2((2n)^3+4k+2) = \nu_2(2)+\nu_2(4n^3+2k+1) = 1.$$

If $x = 2n+1$ then

$$\nu_2((2n+1)^3+4k+2) = 0,$$

since $(2n+1)^3+4k+2$ is odd for all k.

(2). Suppose $D = 2^2(2k+1)$ for some $k \geq 0$. If $x = 2n$ then

$$\nu_2((2n)^3+8k+4) = \nu_2(2^2)+\nu_2(2n^3+2k+1) = 2.$$

If $x = 2n+1$ then

$$\nu_2((2n+1)^3+8k+4) = 0,$$

since $(2n+1)^3+8k+4$ is odd for all t.

(3) Suppose $D = 2^{3k+i}$ for $k > 0$ and $i \in \{1,2\}$ then if $x = 2n+1$ we have

$$\nu_2((2n+1)^3+2^{3k+i}) = 0,$$

while if $x = 2n$ then

$$\nu_2((2n)^3+2^{3k+i}) = \nu_2(2^3)+\nu_2(n^3+2^{(3k+i)-3}) \geq 3.$$

Since this valuation is not constant, we have to look at two cases, $i = 1$ and $i = 2$.

Suppose $i = 1$. We will see that this case results in finite valuation trees that have different structures.

Diagramming the valuation tree, the first level of the tree in this case has two branches yielding a terminating valuation node value 0 (with residue 1) and a non-terminating node value 3 (with residue 0). Moreover, the exponent $(3k+1)-3 = 3k-2$ may only take values from the set $\{1,4,7,\dots\}$ since k is a natural number.

To create the next level of the tree, consider $x = 2^2n$ and $x = 2^2n+2$, respectively, to get

$$\nu_2((2^2n)^3 + 2^{3k+1}) = \nu_2(2^3) + \nu_2(2^3n^3 + 2^{3k-2}) > 3, \tag{4.1}$$

$$\nu_2((2^2n+2)^3 + 2^{3k+1}) = \nu_2(2^3) + \nu_2(2^3n^3 + 3\cdot 2^2n^2 + 3\cdot 2n + 1 + 2^{3k-2}) = 3, \tag{4.2}$$

since $3k-2 > 0$.

Observe that equation (4.2) gives a constant node value. In equation (4.1), if $3k-2 = 1$ then

$$\nu_2((2^2n)^3 + 2^{3k+1}) = \nu_2(2^3) + \nu_2(2^3n^3 + 2^1) = 4,$$

and in this case, the valuation tree terminates as shown in Figure 4.

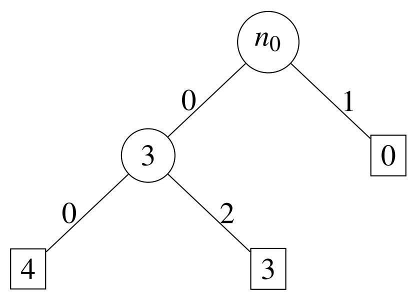

FIGURE 4. The finite valuation tree for $x^3 + 2^4$.

We see that when $3k-2 = 1$ then the 2-adic valuation tree terminates at the second level (with second-level branch residues $2^2n, 2^2n+2$) and having exact valuation nodes $0,3,4$.

If $3k-2 = 4$ then the first equation of (4.1) becomes

$$\nu_2((2^2n)^3 + 2^{3k+1}) = \nu_2(2^3) + \nu_2(2^3n^3 + 2^4) = \nu_2(2^3) + \nu_2(2^3) + \nu_2(n^3+2) \geq 6,$$

which then requires that we move on to the next level of the tree.

Consider $x = 2^3n$ and $x = 2^3n+4$, then the 2-adic valuations are given by

$$\nu_2((2^3n)^3 + 2^{3k+1}) = \nu_2(2^6) + \nu_2(2^3n^3 + 2^{3k-5}) > 6, \tag{4.3}$$

$$\nu_2((2^3n+4)^3 + 2^{3k+1}) = \nu_2(2^6) + \nu_2(2^3n^3 + 3\cdot 2^2n^2 + 3\cdot 2n + 1 + 2^{3k-5}) = 6,$$

since $3k-5 > 0$. Observe that the second equation in (4.3) gives a constant node value. Now we look at the first equation in (4.3).

If $3k-5 = 1$ then the first equation of (4.3) becomes

$$\nu_2((2^3n)^3 + 2^{3k+1}) = \nu_2(2^6) + \nu_2(2^3n^3 + 2^1) = 7,$$

and in this case, the valuation tree terminates as shown in Figure 5.

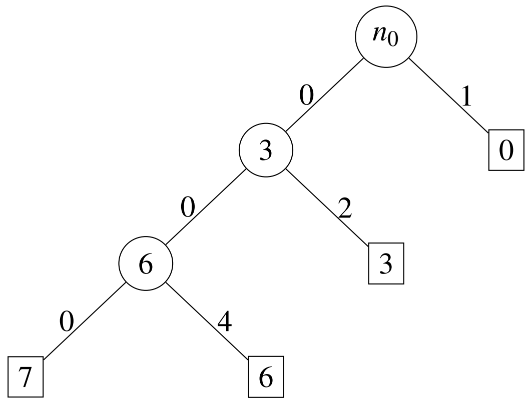

FIGURE 5. The finite valuation tree for $x^3 + 2^7$.

We see here that when $3k - 5 = 1$ then the 2-adic valuation tree terminates at the third level (with third-level branch residues $2^3 n, 2^3 n + 2^2$) and has exact valuation nodes $0, 3, 6, 7$.

The conclusion follows by induction on k. This ends the proof for the case $i = 1$. The proof for $i = 2$ is analogous. An example is shown in Figure 6.

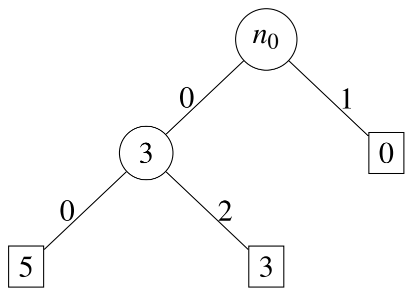

FIGURE 6. The finite valuation tree for $x^3 + 2^5$.

□

Now, we apply Propositions 4.1 and 4.2 to prove Theorem 1.2.

Proof of Theorem 1.2. Using the same techniques as our proof of Theorem 1.1 where we relate the valuations on the tree to our Diophantine equation we can show that for $D = 8^j(2k+1)$, $x^3 + D = 2^t y$, for y odd, has a solution for all $n \geq 0$ except for finitely many values. This is because Proposition 4.1 says the valuation trees are infinite.

If $D \neq 8^j(2k+1)$ we can see from Proposition 4.2 that the range of values of t depends on $D = 2(2k+1)$, $D = 2^2(2k+1)$ or $D = 2^{3k+i}$, $i \in \{1, 2\}$ and $k \in \mathbb{N}$. We know that there are finitely many t's where $x^3 + D = 2^t y$, y odd, will have solutions.

Note that we can do similar calculations as we did in the proof of Theorem 1.1 to show that for $D = 8^j(2k+1)$, $j \geq 1$ that $\nu_2(x^3 + D) \geq 3$. Our potential t valuations in this case are $\{0, 3, 4, \ldots\}$. And if $j = 0$ then we could have solutions for all $t \in \mathbb{Z}_{\geq 0}$. □

5. Examples of Using Valuation Trees to Solve $x^2 + D = 2^t y$ and $x^3 + D = 2^t y$ for Specific D

In [7], it was shown the exact forms of the valuation trees for $D = 1,\ 2,\ 3$ and 4. Using these trees we can find all the non-trivial solutions to the Diophantine equation $x^2 + D = 2^t y$ for $D = 1,\ 2,\ 3$ and 4. Here we show what solutions for the quadratic Diophantine equation would be for $D = 1,\ 3$, and 4.

Theorem 5.1. *The equation $x^2 + 1 = 2^t y$ has solutions only when $t = 0, 1$. If $t = 0$, then y is odd whenever x is even. If $t = 1$, then y is odd whenever x is odd.*

Proof. In order to see that the solutions to $x^2 + 1 = 2^t y$, with $2 \nmid y$, are of the form $(x \equiv 0 \pmod 2, x^2+1, 0)$ and
$(x \equiv 1 \pmod 2, \frac{x^2+1}{2}, 1)$, start with the corresponding tree in Figure 7.

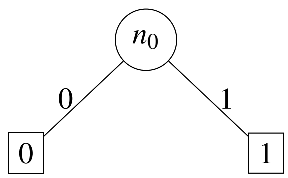

FIGURE 7. The finite valuation tree for $x^2 + 1$

From the finite tree in Figure 7, we get that the 2-adic valuations of our sequence $x^2 + 1$ can only ever be

$$\nu_2(x^2+1) = \begin{cases} 0, & x \equiv 0 \pmod 2, \\ 1, & x \equiv 1 \pmod 2. \end{cases}$$

Then we translate our valuation equations to get $x^2 + 1 = 2^0 y$ and $x^2 + 1 = 2^1 y$, respectively. These must have solutions since our tree implies that there are x-values that we can substitute in to get exact valuation 0 or 1.

Now we need to solve for y, which gives us

$$y = x^2 + 1,$$

where x is even and

$$y = \frac{x^2+1}{2},$$

where x is odd.

Since there are no 2-adic valuations greater than 1 in our valuation tree, there are no other non-trivial t-values that will be solutions to this Diophantine equation.

The tree is finite and so we know that any other classifications of our x-value modulo 2^t will only result in one of the two stated valuations. $\square$

Theorem 5.2. *The Diophantine equation $x^2 + 3 = 2^t y$ has solutions only when $t = 0, 2$. If $t = 0$, then y is odd, whenever x is even. If $t = 2$, then y is odd, whenever x is odd.*

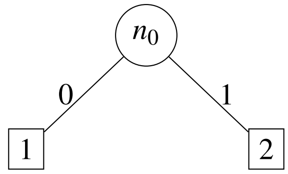

FIGURE 8. The finite valuation tree for $x^2 + 3$.

Proof. We summarize the finite tree in Figure 8 by

$$v_2(x^2+3) = \begin{cases} 0, & x \equiv 0 \pmod 2, \\ 2, & x \equiv 1 \pmod 2. \end{cases}$$

The 2-adic valuation translates to the equation

$$x^2 + 3 = 2^t y.$$

The corresponding finite valuation tree shows that the only possible values of t (where y is odd) will be 0 and 2. Therefore we have equations

$$x^2 + 3 = y, \text{ and} \tag{5.1}$$

$$x^2 + 3 = 2^2 y. \tag{5.2}$$

Further, the valuation tree indicates that we only have solutions for $t = 0$, if x is even. For equation (5.1), one solution is $(x, y, t) = (2, 7, 0)$. Similarly, we can find all solutions for the Diophantine equation to be of the form $(2n, (2n)^2 + 3, 0)$, for $n \in \mathbb{Z}$.

Now for equation (5.2), the tree says this valuation only occurs when x is odd. Hence our solutions are $(2n+1, \frac{(2n+1)^2+3}{4}, 2)$, for $n \in \mathbb{Z}$. $\square$

Theorem 5.3. *The Diophantine equation $x^2 + 4 = 2^t y$, y odd, has solutions only when $t = 0, 2, 3$.*

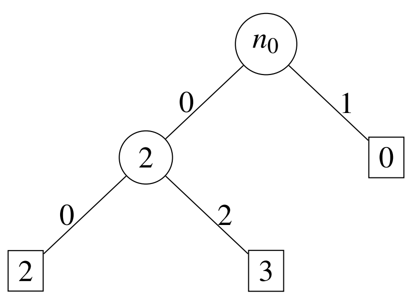

FIGURE 9. The finite valuation tree for $x^2 + 4$.

Proof. Summarizing the valuation tree in Figure 9 gives

$$v_2(x^2+4) = \begin{cases} 0, & x \equiv 1 \pmod 2, \\ 2, & x \equiv 0 \pmod 4, \\ 3, & x \equiv 2 \pmod 4. \end{cases}$$

Again, the 2-adic valuation translates to an equation

$$x^2 + 4 = 2^t y.$$

The finite valuation tree establishes that the only possible values of t (where y is odd) will be $0,\ 2$, and 3. Therefore we have equations

$$x^2 + 4 = y, \tag{5.3}$$

$$x^2 + 4 = 2^2 y, \text{ and} \tag{5.4}$$

$$x^2 + 4 = 2^3 y. \tag{5.5}$$

From the valuation tree we see that there are solutions only for $t = 0$, if x is odd. If we let $x = 1$, we find y in equation (5.3) for the solution $(1, 5, 0)$ to our Diophantine equation. Then all solutions for the Diophantine equation can be found to have the form $(2n+1, (2n+1)^2 + 4, 0)$, for $n \in \mathbb{Z}$.

Similarly, for equation (5.4), the tree implies that the 2-adic valuation equal to 2 only occurs when x is divisible by 4. Our solutions are then of the form $(2^2 n, \frac{(2^2 n)^2+4}{4}, 2)$, for $n \in \mathbb{Z}$.

Finally, for equation (5.5), the tree says the needed valuation only occurs when x is $2 \pmod 4$. Our solutions are $(2^2 n + 2, \frac{(2^2 n+2)^2+4}{8}, 3)$, for $n \in \mathbb{Z}$. □

Next, we will work out an example for cubic Diophantine equation $x^3 + D = 2^t y$ where $D = 8^j(2k+1) = 1$ when $j = k = 0$.

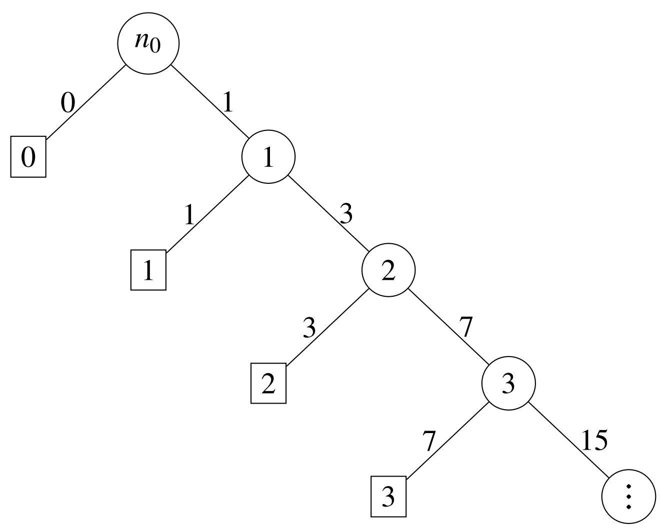

FIGURE 10. The infinite valuation tree for $x^3 + 1$ where the tree continues to split on the rightmost node indefinitely.

We'll see from the theorem below and the tree in Figure 10 that the 2-adic valuation tree for $\nu_2(x^3+1)$ has range of $\mathbb{Z}_{\geq 0}$. Also at each level ℓ, there are two branches yielding one terminating node with value $\ell - 1$ and one non-terminating node with minimum value ℓ.

Theorem 5.4. *The non-negative integer solutions of the cubic Diophantine equation $x^3 + 1 = 2^t y$, for y odd, follow from*

$$v_2(x^3+1) = \begin{cases} 0, & x \equiv 0 \pmod{2^1}, \\ 1, & x \equiv 1 \pmod{2^2}, \\ 2, & x \equiv 3 \pmod{2^3}, \\ 3, & x \equiv 7 \pmod{2^4}, \\ 4, & x \equiv 15 \pmod{2^5}, \\ \vdots & \\ n, & x \equiv 2^n - 1 \pmod{2^{n+1}}. \end{cases} \tag{5.6}$$

We can conclude there are solutions for all $n \in \mathbb{Z}_{\geq 0}$ with corresponding $x \equiv 2^n - 1 \pmod{2^{n+1}}$.

Proof. First, we show that the tree is infinite and will have the valuations as stated. Fix an integer $n \geq 0$. Suppose $x \equiv 2^n - 1 \pmod{2^{n+1}}$. Then

$$(2^n - 1)^3 + 1 = 2^{3n} - 3 \cdot 2^{2n} + 3 \cdot 2^n - 1 + 1 = 2^n(2^{2n} - 3 \cdot 2^n + 3).$$

Observe that $2^{2n} - 3 \cdot 2^n + 3$ is odd for any $n \geq 0$.

Because the rightmost branch in the tree is the one branch that will always continue, for each non-negative integer t, we have a corresponding x-value that we know is of the form $x \equiv 2^t - 1 \pmod{2^{t+1}}$. □

6. Conclusion

What is interesting about our approach is that it gives us a handle on Diophantine equations of the form $x^s + D = p^t y$, for any fixed prime p and positive integer s. Through studying classifications of x and creating our valuation trees we are able to determine, for which t, $x^s + D = p^t y$ has non-trivial integer solutions. For example, if we were to have studied $x^2 + 7 = 2^t y$ with a traditional tool such as the one in Bilu, Hanrot, and Voutier in [5] we would have discovered that there are finitely many solutions for $t = 1, 2$ and 3, but would have been unable to determine solutions for $t > 3$.

One direction for further study is to compute the p-adic valuation trees of more general polynomials like $x^n + D$ where $n, D \in \mathbb{Z}_{\geq 0}$ and determine the solutions of the related Diophantine equations.

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Appendix A. Tables of example solutions

t	x_t	y_t
3	1	1
4	3	1
5	5	1
6	11	2
7	11	1
8	53	11
9	75	11
10	181	32
11	181	16
12	181	8
13	181	4
14	181	2
15	181	1
16	16203	4006
17	16203	2003
18	49333	9284
19	49333	4642
20	49333	2321
21	474955	107566
22	474955	53783
23	1622197	313702
24	1622197	156851
25	6766411	1364479
26	10010805	1493338

TABLE 1. Some example solutions to our Diophantine equation $x^2 + 7 = 2^t y$, y odd, using our recursion.

We showcase some of the solutions (x, y, t) for $x^2 + 7 = 2^t y$, $x^2 + D = 2^t y$, and $x^3 + D = 2^t y$ which we found using the valuation tree method in Tables 1, 2 and 3.

D	t	(x,y,t)
8	0, 2, 3	$(1,9,0)$, $(2,3,2)$, $(4,3,3)$
9	0, 1	$(2,13,0)$, $(1,5,1)$
10	0, 1	$(1,11,0)$, $(2,7,1)$
11	0, 2	$(2,15,0)$, $(1,3,2)$
12	0, 2, 4	$(1,13,0)$, $(4,7,2)$, $(2,1,4)$
13	0, 1	$(2,17,0)$, $(1,7,1)$
14	0, 1	$(1,15,0)$, $(2,9,1)$
16	0, 2, 4, 5	$(1,17,0)$, $(2,5,2)$, $(8,5,4)$, $(4,1,5)$
17	0, 1	$(2,21,0)$, $(1,9,1)$
18	0, 1	$(1,19,0)$, $(2,11,1)$
19	0, 2	$(2,23,0)$, $(1,5,2)$
20	0, 2, 3	$(1,21,0)$, $(4,9,2)$, $(2,3,3)$
21	0, 1	$(2,25,0)$, $(1,11,1)$
22	0, 1	$(1,23,0)$, $(2,13,1)$
24	0, 2, 3	$(1,25,0)$, $(2,7,2)$, $(4,5,3)$
25	0, 1	$(2,29,0)$, $(1,13,1)$
26	0, 1	$(1,27,0)$, $(2,15,1)$
27	0, 2	$(2,31,0)$, $(1,7,2)$
29	0, 1	$(2,33,0)$, $(1,15,1)$
30	0, 1	$(1,31,0)$, $(2,17,1)$
32	0, 2, 4, 5	$(1,33,0)$, $(2,9,2)$, $(4,3,4)$, $(8,3,5)$
33	0, 1	$(2,37,0)$, $(1,17,1)$
34	0, 1	$(1,35,0)$, $(2,19,1)$
35	0, 2	$(2,39,0)$, $(1,9,2)$
36	0, 2, 3	$(1,37,0)$, $(4,13,2)$, $(2,5,3)$
37	0, 1	$(2,41,0)$, $(1,19,1)$
38	0, 1	$(1,39,0)$, $(2,21,1)$
40	0, 2, 3	$(1,41,0)$, $(2,11,2)$, $(4,7,3)$

TABLE 2. Examples of some solutions for $x^2 + D = 2^t y$ for specific D values.

D	t	(x,y,t)
2	0, 1	$(1,3,0)$, $(2,5,1)$
4	0, 2	$(1,5,0)$, $(2,3,2)$
6	0, 1	$(1,7,0)$, $(2,7,1)$
10	0, 1	$(1,11,0)$, $(2,9,1)$
12	0, 2	$(1,13,0)$, $(2,5,2)$
14	0, 1	$(1,15,0)$, $(2,11,1)$
16	0, 3, 4	$(1,17,0)$, $(2,3,3)$, $(4,5,4)$
18	0, 1	$(1,19,0)$, $(2,13,1)$
20	0, 2	$(1,21,0)$, $(2,7,2)$
22	0, 1	$(1,23,0)$, $(2,15,1)$
26	0, 1	$(1,27,0)$, $(2,17,1)$
28	0, 2	$(1,29,0)$, $(2,9,2)$
30	0, 1	$(1,31,0)$, $(2,19,1)$
32	0, 3, 5	$(1,33,0)$, $(2,5,3)$, $(4,3,5)$
34	0, 1	$(1,35,0)$, $(2,21,1)$
36	0, 2	$(1,37,0)$, $(2,11,2)$
38	0, 1	$(1,39,0)$, $(2,23,1)$
42	0, 1	$(1,43,0)$, $(2,25,1)$
44	0, 2	$(1,45,0)$, $(2,13,2)$
46	0, 1	$(1,47,0)$, $(2,27,1)$

TABLE 3. Examples of some solutions for $x^3 + D = 2^t y$ for specific D values.

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