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HARMONIC CURVATURE-LIKE TENSORS WITH SECOND BIANCHI IDENTITY ON KÄHLER MANIFOLDS

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ABSTRACT. In this paper we introduce the notion of curvature-like tensors on Kähler manifolds satisfying the second Bianchi equation. As examples, we could mention *concircular curvature tensor*, *projective curvature tensor*, *conformal curvature tensor* and *Bochner curvature tensor* which are all closed curvature-like tensors. In particular, we show that all of them, except the concircular curvature tensor, are harmonic curvature-like tensors and give some conditions for them to be harmonic.

1. Introduction

Let R and S be, respectively, the Riemannian curvature tensor and the Ricci tensor of a Riemannian manifold M. From the second Bianchi identity it is easily seen that $dR = 0$, where d is the exterior derivative on M, that is, R is closed. We will say that R is harmonic if the adjoint co-derivative $\delta R = 0$, which means that R is coclosed. Then the Ricci tensor S satisfies $(\nabla_X S)Y = (\nabla_Y S)X$ for any vector fields X and Y on M.

On a Kähler manifold due to Besse [2] we can consider the following curvature-like tensors: the concircular curvature tensor Z, the projective curvature tensor G, the conformal curvature tensor H and the Bochner curvature tensor B.

On such a manifold we can consider a unitary basis $\{U_A\} = \{U_j, \bar{U}_j\}$, $j = 1, \cdots, n$, which is given by

$$U_j = \frac{1}{\sqrt{2}}(E_j - iJE_j), \qquad \bar{U}_j = \frac{1}{\sqrt{2}}(E_j + iJE_j)$$

for an orthonormal frame $\{E_\alpha\} = \{E_j, E_{j^*}\}$, $E_{j^*} = JE_j$, so that the curvature tensor R on M can be expressed by $R_{\bar{i}jk\bar{l}}$ and it satisfies the 1st Bianchi identity $R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}}$ and the second Bianchi identity $R_{\bar{i}jk\bar{l}m} = R_{\bar{i}jm\bar{l}k}$. Then the components $S_{i\bar{j}l}$ of the covariant derivative of the Ricci tensor S are given by

$$\begin{aligned} S_{i\bar{j}l} &= \sum_k R_{\bar{j}ik\bar{k}l} = \sum_k R_{\bar{j}il\bar{k}k} \\ &= \sum_k R_{\bar{j}li\bar{k}k} = \sum_k R_{\bar{j}lk\bar{k}i} \\ &= S_{l\bar{j}i}. \end{aligned}$$

This means that the Ricci tensor S is of Codazzi type. So in Theorem 5.2 we will prove that the Ricci tensor S is closed. Moreover, it satisfies the equation of Codazzi if and only if $S_{\bar{i}jk} = 0$ (see Proposition 5.1). That is, the Ricci tensor S is parallel if and only if the coderivative $\delta^\nabla R$ of the curvature tensor R becomes $\delta^\nabla R = 0$, that is, the curvature tensor R is coclosed. Since the curvature tensor R always

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satisfies $d^{\nabla}R = 0$ by the second Bianchi identity, R satisfies $\triangle R = (d^{\nabla}\delta^{\nabla} + \delta^{\nabla}d^{\nabla})R = 0$ on a Kähler manifold M with parallel Ricci tensor S. In this case the curvature tensor R is said to be harmonic on M (see also Theorem 1.14, page 25, in Besse [2], and Matsushima [14]).

In this paper we will prove that the *concircular curvature tensor* Z, the *projective curvature tensor* G, the *conformal curvature tensor* H and the *Bochner curvature tensor* B satisfy both the 1st and the 2nd Bianchi identities, respectively.

The concircular curvature tensor Z introduced in section 6 is a formal complex version of a concircular curvature tensor on a Riemannian manifold. So the tensor Z defined on a Kähler manifold M vanishes if and only if M is a complex space form, that is, a Kähler manifold with constant holomorphic sectional curvature. The projective curvature tensor G(resp. the conformal curvature tensor H) is an invariant of the projectively related connections(resp. of the conformally related connections). These two tensors G and H will be defined in sections 7 and 8 respectively. Of course, they are closely related to the curvature tensor R when a Kähler manifold becomes a complex space form. That is, if both of them vanish identically on a Kähler manifold M, that is, $G = H = 0$, then M is of constant holomorphic sectional curvature (see Bochner [3], Yano and Bochner [30]). In section 9 we introduce the notion of Bochner curvature tensor due to Bochner [3].

Then on such a Kähler manifold we introduce the notion of closed curvature-like tensor T, and want to find out some equivalent conditions for such a closed curvature-like tensor T to be a harmonic curvature like tensor.

In order to do this, first, in section 4 we define an associated curvature-like 2 form $\Phi = \Phi(T)$ in such a way that

$$\Phi_{ij} = \sum_{k,l} T_{\bar{i}jk\bar{l}}\,\omega_k \wedge \overline{\omega}_l,$$

where $T_{\bar{i}jk\bar{l}}$ are the components of the tensor T.

Usually we call such an associated curvature-like 2 form Φ just a *curvature-like* form of the curvature-like tensor T. Next in sections 4 and 5 we associate two different forms which are said to be respectively the *curvature-like* 4 *form* $\Psi = \Psi(T)$ and the *Ricci-like* 2 *form* $\psi = \psi(T)$ for T in such a way that

$$\Psi = \sum \Phi_{ij}\,\omega_i \wedge \overline{\omega}_j,$$

and

$$\psi = -2i \sum T_{\bar{i}j}\,\overline{\omega}_i \wedge \omega_j.$$

When the associated *curvature-like 4-form* $\Psi = \Psi(T)$ for T satisfies $d\Psi = 0$, then the *curvature-like tensor* T is said to be *closed*. In sections 4 and 5 by using such a curvature-like 4-form or a Ricci-like 2-form respectively we give some necessary and sufficient conditions for the curvature-like tensor T to be *closed*. Accordingly, we are able to assert that its associated *Rici-like tensor* $\mathrm{Ric}(T)$ is *closed* if the curvature-like tensor T is *closed*. Here, if the curvature-like 2-form $\Phi = \Phi(T)$ satisfies

$$\triangle \Phi(T) = (d\delta + \delta d)\Phi(T) = 0,$$

we say that the curvature-like 2-form $\Phi = \Phi(T)$ is *harmonic*.

In sections 7,8 and 9, we can confirm that the curvature-like 2-forms $\Phi(G)$, $\Phi(H)$ and $\Phi(B)$ could satisfy the Laplacian equations $\triangle\Phi(G) = 0$, $\triangle\Phi(H) = 0$ and $\triangle\Phi(B) = 0$ by using the second Bianchi equations $d\Phi(G) = \Phi\wedge\omega - \omega\wedge\Phi$, $d\Phi(H) = \Phi\wedge\omega - \omega\wedge\Phi$, and $d\Phi(B) = \Phi\wedge\omega - \omega\wedge\Phi$, respectively. So in these cases the curvature-like 2-forms $\Phi(G)$, $\Phi(H)$ and $\Phi(B)$ are said to be *harmonic curvature-like* 2-*forms* and the curvature-like tensors G, H and B are said to be *harmonic* curvature tensors.

The notions of harmonic curvature like 2-forms and harmonic curvature tensors on Kähler manifolds are very important to study harmonic Weyl tensor, Einstein metrics, Yang-Mills connections, bounded Ricci curvature and Betti numbers, Ricci soliton and Yamabe problems in Mathematical Physics (see Besse [2], Li [13], and Morgan and Tian [17]).

Motivated by such notions of harmonic curvature-like 2-forms and the corresponding second Bianchi equation, and together with applications of Theorems 4.2 and 5.2, the purpose of this paper is to give some equivalent conditions to be harmonic curvature-like tensors as follows:

Theorem 1. *For the concircular curvature tensor Z on a complex $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The concircular curvature tensor Z is closed, that is, $d\Psi(Z)=0$,*
(b) *The Ricci-like tensor $Ric(Z)$ is closed ; $d\psi(Z)=0$,*
(c) *The concircular curvature-like form $\Phi=\Phi(Z)$ satisfies the second Bianchi equation $d\Phi=\Phi\wedge\omega-\omega\wedge\Phi$,*
(d) *The scalar curvature r on M is constant.*

Theorem 2. *For the projective curvature tensor G on a complex $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The projective curvature tensor G is closed, that is, $d\Psi(G)=0$,*
(b) *The curvature-like form $\Phi=\Phi(G)$ satisfies the second Bianchi equation $d\Phi=\Phi\wedge\omega-\omega\wedge\Phi$,*
(c) *$\nabla S\in C^\infty(\mathscr{I})$, that is, the curvature-like form $\Phi=\Phi(G)$ is coclosed, where $\nabla S\in C^\infty(\mathscr{I})$ means that $S_{j\bar{i}k}=\frac{1}{2(n+1)}\{r_j\delta_{ik}+r_k\delta_{ij}\}$.*
(d) *The Ricci tensor S on M is parallel.*

By virtue of Theorem 2 we know that the projective curvature tensor G becomes *harmonic*. Concerned with the closeness of curvature-like tensors, we also give some equivalent conditions for the conformal curvature tensor H and for the Bochner curvature tensor B respectively as follows:

Theorem 3. *For the conformal curvature tensor H on a complex $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The conformal curvature tensor H is closed, that is, $d\Psi(H)=0$,*
(b) *The conformal curvature-like form $\Phi=\Phi(H)$ satisfies the second Bianchi equation $d\Phi=\Phi\wedge\omega-\omega\wedge\Phi$,*
(c) *The Ricci tensor S on M is parallel,*
(d) *The conformal curvature-like form $\Phi=\Phi(H)$ is coclosed.*

Theorem 4. *For the Bochner curvature-like tensor B on a complex $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The Bochner curvature tensor B is closed, that is, $d\Psi(B)=0$,*
(b) *The Bochner curvature-like form $\Phi=\Phi(B)$ satisfies the second Bianchi equation $d\Phi=\Phi\wedge\omega-\omega\wedge\Phi$,*
(c) *$\nabla S\in C^\infty(\mathscr{I})$, that is, the Bochner curvature-like form $\Phi=\Phi(B)$ is coclosed, where $\nabla S\in C^\infty(\mathscr{I})$ means that $S_{j\bar{i}k}=\frac{1}{2(n+1)}\{r_j\delta_{ik}+r_k\delta_{ij}\}$.*

It can be easily verified that the conformal curvature tensor and the Bochner curvature tensor satisfy the second Bianchi equation. So, by virtue of Theorem 3(resp. Theorem 4), we know that the conformal curvature tensor H (resp., the closed Bochner curvature tensor B) is closed and also coclosed, which means that it is harmonic. In sections 7 and 8 respectively we will show that a Kähler manifold M could be locally symmetric if M has *parallel projective* curvature tensor G, or *parallel conformal* curvature tensor H.

In section 9 we will show that a Kähler manifold M is locally symmetric if M has *parallel Bochner* curvature tensor and constant scalar curvature. Moreover, we will prove that there exists an example of non-locally symmetric space with non-parallel Bochner curvature tensor.

2. Geometric Structures on Kähler manifolds

This section is concerned with recalling basic formulas on Kähler manifolds (cf. [2], [3], [4], [5], [6], [8], [9], [17] and [30]).

Let M be a complex $n(\geq 2)$-dimensional Kähler manifold equipped with the Kähler metric tensor g and almost complex structure J. We can choose a local field $\{E_j, E_{j^*}\} = \{E_1, \ldots, E_n, E_{1^*}, \ldots, E_{n^*}\}$ of orthonormal frames on a neighborhood of M, where $E_{j^*} = JE_j$ and $j^* = n + j$. Here and in the sequel, the indices $i, j, k, \ldots$ run from 1 to n. We set $U_j = \frac{1}{\sqrt{2}}(E_j - iE_{j^*})$ and $\overline{U}_j = \frac{1}{\sqrt{2}}(E_j + iE_{j^*})$, where i is the imaginary unit. Then $\{U_j\}$ constitutes a local field of unitary frames on the neighborhood of M. With respect to the Kähler metric we have $g(U_j, \overline{U}_k) = \delta_{jk}$.

Now, let $\{\omega_j\} = \{\omega_1, \ldots, \omega_n\}$ be the dual coframe field with respect to the local field $\{U_j\}$ of unitary frames on the neighborhood of M. Then $\{\omega_j\}$ consists of complex valued 1-forms of type $(1,0)$ on M such that $\omega_j(U_k) = \delta_{jk}$ and $\{\omega_j, \overline{\omega}_j\} = \{\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n\}$ are linearly independent. The Kähler metric g of M can be expressed as $g = 2\sum_j \omega_j \otimes \overline{\omega}_j$.

Associated with the frame field $\{U_j\}$, there exist complex valued 1-forms ω_{jk}, which are usually called *complex connection forms* on M such that they satisfy the structure equations of M :

$$
\begin{aligned}
&d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \quad \omega_{ij} + \overline{\omega}_{ji} = 0, \\
&d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \quad \Omega_{ij} = \sum_{k,l} R_{\bar{i}jk\bar{l}}\, \omega_k \wedge \overline{\omega}_l \ ,
\end{aligned} \tag{2.1}
$$

where Ω_{ij} (resp. $R_{\bar{i}jk\bar{l}}$) denotes the curvature form (resp. the components of the Riemannian curvature tensor R) of M. The equation (2.1) means that the skew-Hermitian symmetry of Ω_{ij}, which is equivalent to the symmetric condition

$$
R_{\bar{i}jk\bar{l}} = \overline{R}_{\bar{j}il\bar{k}} \ . \tag{2.2}
$$

Moreover, by the exterior differential of the first and the third equations of (2.1), the first Bianchi equation

$$
\sum_j \Omega_{ij} \wedge \omega_j = 0 \tag{2.3}
$$

is given. It implies the further symmetric relations

$$
R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} = R_{\bar{l}jk\bar{i}} = R_{\bar{l}kj\bar{i}} \ . \tag{2.4}
$$

Now, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows :

$$
S = \sum_{i,j} (S_{i\bar{j}}\, \omega_i \otimes \overline{\omega}_j + S_{\bar{i}j}\, \overline{\omega}_i \otimes \omega_j), \tag{2.5}
$$

where $S_{i\bar{j}} = \sum_k R_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j}$. The scalar curvature r of M is also given by

$$r = 2\sum_j S_{j\bar{j}} \,. \tag{2.6}$$

An n-dimensional Kähler manifold M is said to be *Einstein* if the Ricci tensor S satisfies the condition

$$S_{i\bar{j}} = \frac{r}{2n}\delta_{ij} \,. \tag{2.7}$$

The components $R_{\bar{i}jk\bar{l}r}$ and $R_{\bar{i}jk\bar{l}\bar{r}}$ (resp. $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are given by

$$\begin{aligned}\sum_r (R_{\bar{i}jk\bar{l}r}\omega_r + R_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r) &= dR_{\bar{i}jk\bar{l}} \\ &\quad - \sum_r (R_{\bar{r}jk\bar{l}}\overline{\omega}_{ri} + R_{\bar{i}rk\bar{l}}\omega_{rj} + R_{\bar{i}jr\bar{l}}\omega_{rk} + R_{\bar{i}jk\bar{r}}\overline{\omega}_{rl}),\end{aligned} \tag{2.8}$$

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\overline{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\overline{\omega}_{kj}). \tag{2.9}$$

The second Bianchi equation is given by the exterior derivative of the third equation of (2.1), that is, we have

$$d\Omega_{ij} = \sum_k (\Omega_{ik} \wedge \omega_{kj} - \omega_{ik} \wedge \Omega_{kj}) \tag{2.10}$$

with the help of the property $d^2 = 0$ and the property of the exterior derivative. We can regard $\omega_0 = (\omega_j)$ as an n-vector, and $\Omega = (\Omega_{ij})$ and $\omega = (\omega_{ij})$ can be complex matrices of order n. Then the equations (2.3) and (2.10) can be reformed as

$$\Omega \wedge \omega_0 = 0, \quad d\Omega = \Omega \wedge \omega - \omega \wedge \Omega, \tag{2.11}$$

which means that the Riemannian curvature form Ω satisfies the first and the second Bianchi equations. By the straightforward calculation we have

$$R_{\bar{i}jk\bar{l}h} = R_{\bar{i}jh\bar{l}k} \,, \tag{2.12}$$

and hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_l R_{\bar{j}il\bar{l}k} = \sum_l R_{\bar{j}kl\bar{l}i}. \tag{2.13}$$

On the other hand, the exterior differential dr of the scalar curvature r on M is given by

$$dr = \sum_j (r_j\omega_j + r_{\bar{j}}\overline{\omega}_j). \tag{2.14}$$

Now, a Kähler manifold of constant holomorphic sectional curvature is called a *complex space form*. An n-dimensional complex space form of constant holomorphic sectional curvature c is denoted by $M^n(c)$. The Riemannian curvature tensor $R_{\bar{i}jk\bar{l}}$ of $M^n(c)$ is given by

$$R_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.15}$$

3. Vector subbundles

In Kähler geometry, it is well known that the properties of the Ricci tensor S and the covariant derivative ∇S are very important. Moreover the second Bianchi identity of the Riemannian curvature tensor R plays an important role. It has been shown in Besse [2] and Gray [10] that there exist a few classes of Riemannian metrics which are some generalizations of Einstein metrics and characterized by tensorial conditions. In this section we discuss a certain generalization of Einstein metrics on Kähler manifolds and the second Bianchi identity for the curvature-like tensor.

Let $M = M^n$ be an n-dimensional Kähler manifold with Kähler metric g. Let ∇ be a Kähler connection on M. It is originally defined for real vector fields, but it can be extended by complex linearity to act on complex vector fields. Then the extended connection is denoted by the same symbol. Afterwards, such natural extensions of all kinds of geometric quantities are denoted by the same symbol without fear of confusion. Let TM (*resp.* T^*M) be the tangent bundle (*resp.* the cotangent bundle) of M and let T^cM (*resp.* $T^{*^c}M$) be the complexfication of the tangent bundle TM (*resp.* the cotangent bundle T^*M). Then it can be decomposed into

$$T^cM = T^{1,0}M \oplus T^{0,1}M, \quad T^{*^c}M = T_{1,0}M \oplus T_{0,1}M,$$

where $T^{1,0}M$ (*resp.* $T^{0,1}M$) denotes the subbundle consisting of complex vector fields of type (1,0) (*resp.* type (0,1)) in T^cM and $T_{1,0}M$ (*resp.* $T_{0,1}M$) denotes the subbundle consisting of differentiable 1-form of degree (1,0) (*resp.* degree of (0,1)) in $T^{*^c}M$. The space DM consisting of all complex differentiable forms on the Kähler manifold M may be bigraded as follows ; $DM = \sum_{r,s=0}^{n} D^{r,s}M$, where $D^{r,s}M$ is the subspace of DM spanned by $\alpha \wedge \beta$, $\alpha \in \wedge^r T_{1,0}M$ and $\beta \in \wedge^s T_{0,1}M$. An element in $D^{r,s}M$ is called a *complex form of degree* (r,s).

Let $\mathscr{H} = \mathscr{H}(M,g)$ be the vector subbundle of the fibre $\otimes^3 T^{*^c}M$ of which, at any point p in M, consists of all trilinear mapping ξ of $\otimes^3 T_p^cM$ into C such that $\xi(U,\bar{V},W) = 0$ for any vector fields U in $T^{1,0}M$ and any vector fields V in $T^{0,1}M$, and

$$\sum_j \xi(U_j,\bar{U}_j,W) = \sum_j \xi(W,\bar{U}_j,U_j) \tag{3.1}$$

for any complex vector W in $T_p^{1,0}M$ and any unitary basis $\{U_j\}$ for $T_p^{1,0}M$. We put $T_3M = T_{1,0}M \otimes T_{0,1}M \otimes T_{1,0}M$ which is a subbundle of $\otimes^3 T^{*^c}M$ and homomorphic to $D^{2,1}M$. There is a naturally defined scalar product on $\mathscr{H} = \mathscr{H}(M,g)$ given by

$$<\xi,\eta>= \sum_{i,j,k} \xi(U_i,\bar{U}_j,U_k)\bar{\eta}(U_i,\bar{U}_j,U_k). \tag{3.2}$$

We here give some examples of the cross section of $\mathscr{H}$.

Example 3.1.

(1) $\xi = 0$. This is the most trivial example of smooth sections on $\mathscr{H}(M,g)$.

(2) As a non-trivial examples of the smooth section of $\mathscr{H}(M,g)$, we can adduce that the covariant derivative ∇S of the Ricci tensor S is defined by

$$\nabla S(V,\bar{W},Z) = \nabla_Z S(V,\bar{W})$$

for any complex vector fields V,W and Z in T^cM. Then it satisfies

$$\nabla S(V,\bar{W},Z) = \nabla S(Z,\bar{W},V),$$

from which it yields that ∇S is the cross section on $\mathscr{H}(M,g)$.

(3) Let $M = M^n$ be an n-dimensional Kähler submanifold in an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ and of constant holomorphic sectional curvature c. Let α be the second fundamental form of M and let α_2 be the tensor of degree (1.1) with components $h_{i\bar{j}}{}^2$. Then, by virtue of the formula for complex submanifolds given in [2], [3], [4], [5], [6] and [30] we have $\nabla\alpha_2 \in C^\infty(\mathscr{H})$.

So, we can regard the smooth section ξ on $\mathscr{H}$ as the generalization of the covariant derivative of the Ricci tensor S. Given a Kähler manifold (M,g) one has the following natural bundle homomorphisms associated with T_3M. The partial alternation $a : T_3M \to T_3M$, the partial symmetrization $b : T_3M \to T_3M$. the contraction $c : T_3M \to T^{*^c}M$, and the mapping $i : \mathscr{H}(M,g) \to \mathscr{H}(M,g)$ such that

$$a(\xi)(U,\bar{V},W) = \frac{1}{2}\{\xi(U,\bar{V},W) - \xi(W,\bar{V},U)\},$$
$$b(\xi)(U,\bar{V},W) = \frac{1}{2}\{\xi(U,\bar{V},W) + \xi(W,\bar{V},U)\},$$
$$c(\xi)(U) = 2\sum_j \xi(U,\bar{U}_j,U_j),$$
$$i(\xi)(U,\bar{V},W) = \frac{1}{2(n+1)}\{c(\xi)(U)g(W,\bar{V}) + c(\xi)(W)g(U,\bar{V})\}$$

for any cross section ξ in $C^\infty(\mathscr{H})$, any vector fields U,V and W in $T^{1.0}M$ and any unitary frame $\{U_j\}$.

The subbundles $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{I}$ in $\mathscr{H}$ are defined by

(3.3) $$\mathscr{A} = \mathrm{Ker}\ a, \quad \mathscr{B} = \mathrm{Ker}\ b \cap \mathscr{H}, \quad \mathscr{C} = \mathrm{Ker}\ c \cap \mathscr{H}, \quad \mathscr{I} = \mathrm{Im}\ i.$$

In fact, each element ξ in $\mathscr{A}$ satisfies $\xi(U,\bar{V},W) = \xi(W,\bar{V},U)$, which implies that $\mathscr{A} \subset \mathscr{H}$ as subsets.

On the other hand, by the definition of any cross section η on $\mathscr{I}$ there exists an element η' in $C^\infty(\mathscr{H})$ such that

(3.4) $$\eta(U,\bar{V},W) = \frac{1}{2(n+1)}\{c(\eta')(U)g(W,\bar{V}) + c(\eta')(W)g(U,\bar{V})\}.$$

Accordingly, we have $\eta(U,\bar{V},W) = \eta(W,\bar{V},U)$. It implies that

$$\sum_j \eta(U_j,\bar{U}_j,W) = \sum_j \eta(W,\bar{U}_j,U_j),$$

from which it follows that $\mathscr{I}$ is contained in $\mathscr{H}$.

Let $M = M^n$ be an n-dimensional Kähler manifold. Let U be a complex symmetric bilinear form satisfying

$$U(\bar{V},\bar{W}) = \overline{U(V,W)} \quad \text{for any } V,W \in T^cM,$$
$$U(V,W) = 0 \quad \text{for any } V,W \in T^{1,0}M.$$

Then U is called a *tensor field of degree* (1,1) and we put $u = 2C_{12}U$, where C_{12} denotes the contraction. There exist many examples of such tensors. In fact, for the Kähler metric g and the Ricci tensor S are important examples of the tensors of degree (1,1). For the Ricci tensor $U = S$ the formula $u = 2C_{12}U$ gives that

$$r = 2C_{12}U = 2C_{12}S = 2\mathrm{Tr}S = 2\textstyle\sum_j S_{j\bar{j}},$$

where r denotes the scalar curvature of M in (2.6).

Remark 3.1 When $i(\xi) = \nabla S \in \mathscr{I}$ for the Ricci tensor S on a Kähler manifold M, it follows that

(3.5) $$c(\xi)(U_j) = 2\textstyle\sum_k (\nabla_{U_j}S)(U_k,\bar{U}_k) = r_j,$$

where the scalar curvature r is given by $r = 2\sum_k S(U_k, \bar{U}_k) = 2\sum_k S_{k\bar{k}}$. So by (3.4) it follows that

$$S_{j\bar{i}k} = \frac{1}{2(n+1)}\{r_j\delta_{ik} + r_k\delta_{ij}\},$$

where we have put $U = U_j$, $\bar{V} = \bar{U}_i$, and $W = U_k$ for the unitay frame field $\{U_A\} = \{U_j, \bar{U}_j\}$ on M.

4. Curvature-like tensors on Kähler manifolds

In this section, we introduce the concept of Bianchi identities for the curvature-like tensor on the Kähler manifold with almost complex structure J.

Let M be a complex n-dimensional Kähler manifold with Kähler connection ∇. We denote by TM the tangent bundle of M. Let T^CM be the complexification of TM. Let T be a quadrilinear mapping of $T^CM \times T^CM \times T^CM \times T^CM$ into $\mathbb{C}$ satisfying the curvature-like properties

(a) $$T(X,Y,Z,U) = -T(Y,X,Z,U) = -T(X,Y,U,Z),$$

(b) $$T(X,Y,Z,U) = T(Z,U,X,Y),$$

(c) $$T(JX,JY,Z,U) = T(X,Y,JZ,JU) = T(X,Y,Z,U).$$

Then T is called a *curvature-like tensor*. By the complex linearity the Kähler connection ∇ is extended to act on complex vector fields. Depended only on this extension, the operator T is naturally extended to the complexification of TM. For an unitary frame $\{U_j\}$, let $\omega_0 = \{\omega_j\}$, $\omega = \{\omega_{ij}\}$ and $\Omega = \{\Omega_{ij}\}$ be the complex canonical form, the complex connection form and the complex curvature form on M, respectively. Let T_{ABCD} be the components of T associated with the unitary frame $\{U_A\} = \{U_j, \overline{U}_j\}$. Accordingly the components T_{ABCD} are given by $T_{ABCD} = T(U_A, U_B, U_C, U_D)$. Then we have $T_{ABkl} = T_{AB\bar{k}\bar{l}} = 0$. Because we have

$$\begin{aligned} T_{ABkl} &= T(U_A,U_B,U_k,U_l) = T(U_A,U_B,JU_k,JU_l) \\ &= i^2 T(U_A,U_B,U_k,U_l) = -T(U_A,U_B,U_k,U_l) = -T_{ABkl}\ , \end{aligned}$$

and hence we have $T_{ABkl} = 0$, where the first and the last equalities are given by its definition, the second one can be derived from (c) and the third one is followed by the fact that vectors U_k are of type (1,0). Similarly we obtain $T_{AB\bar{k}\bar{l}} = 0$. By the condition (a)-(c), essential components T_{ABCD} of the curvature-like tensor T consist only of $\{T_{\bar{i}jk\bar{l}}\}$. Consequently, by the conditions (a), (b) and (c) the following properties of the components of T hold ;

(1) $$T_{\bar{i}jk\bar{l}} = -T_{j\bar{i}k\bar{l}} = -T_{\bar{i}j\bar{l}k}\ ,$$

(2) $$T_{\bar{i}jk\bar{l}} = T_{\bar{l}kj\bar{i}}\ .$$

If the components T_{ABCD} of a tensor T in $\bigotimes^4 T^{*C}M$ consist of only $\{T_{\bar{i}jk\bar{l}}\}$ and if they satisfy (1) and (2), then T becomes a curvature-like tensor.

In the same way as we associate a curvature form Ω to the Riemannian curvature tensor R, we associate a 2-form $\Phi = \{\Phi_{ij}\}$ to the curvature-like tensor T in such a way that

(4.1) $$\Phi_{ij} = \sum_{k,l} T_{\bar{i}jk\bar{l}}\omega_k \wedge \overline{\omega}_l\ .$$

In such a case we say a 2-form $\Phi = \{\Phi_{ij}\}$ a *curvature-like 2-form* for T. The canonical form $\omega_0 = \{\omega_j\}$ can be regarded as a vector in $\mathbb{C}^n$ and the connection form $\omega = \{\omega_{ij}\}$ and the curvature form $\Phi = \{\Phi_{ij}\}$

can be regarded as $n \times n$ complex matrices. Then the equation

$$\Phi \wedge \omega_0 = 0 \tag{4.2}$$

is called the *first Bianchi equation* for the curvature-like form Φ. By (4.1) and (4.2), the tensor T satisfies the first Bianchi equation if and only if its components satisfies

$$T_{\bar{i}jk\bar{l}} = T_{\bar{i}kj\bar{l}}. \tag{4.3}$$

So (4.3) is called the *first Bianchi identity* for T. The components $T_{\bar{i}jk\bar{l}r}$ and $T_{\bar{i}jk\bar{l}\bar{r}}$ of the covariant derivative ∇T of T are defined by

$$\begin{aligned}(4.4)\qquad &\sum_r (T_{\bar{i}jk\bar{l}r}\omega_r + T_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r)\\ &= dT_{\bar{i}jk\bar{l}} - \sum_r (T_{\bar{r}jk\bar{l}}\overline{\omega}_{ri} + T_{\bar{i}rk\bar{l}}\omega_{rj} + T_{\bar{i}jr\bar{l}}\omega_{rk} + T_{\bar{i}jk\bar{r}}\overline{\omega}_{rl}).\end{aligned}$$

Corresponding to (2.11), we call the equation

$$d\Phi = \Phi \wedge \omega - \omega \wedge \Phi \tag{4.5}$$

the *second Bianchi equation* for the curvature-like 2-form Φ.

Proposition 4.1. *On a Kähler manifold M, the curvature-like form Φ satisfies the second Bianchi equation if and only if*

$$T_{\bar{i}jk\bar{l}r} = T_{\bar{i}jr\bar{l}k}\ , \quad T_{\bar{i}jk\bar{l}\bar{r}} = T_{\bar{i}jk\bar{r}\bar{l}}\ . \tag{4.6}$$

Proof. Since Φ_{ij} is the 2-form, the left hand side of (4.5) is given by

$$\begin{aligned}
d\Phi_{ij} &= \sum_{k,l}(dT_{\bar{i}jk\bar{l}} \wedge \omega_k \wedge \overline{\omega}_l + T_{\bar{i}jk\bar{l}}d\omega_k \wedge \overline{\omega}_l - T_{\bar{i}jk\bar{l}}\omega_k \wedge d\overline{\omega}_l)\\
&= \sum_{k,l}\{\sum_r (T_{\bar{i}jk\bar{l}r}\omega_r + T_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r) \wedge \omega_k \wedge \overline{\omega}_l\\
&\quad + \sum_r (T_{\bar{r}jk\bar{l}}\overline{\omega}_{ri} + T_{\bar{i}rk\bar{l}}\omega_{rj} + T_{\bar{i}jr\bar{l}}\omega_{rk} + T_{\bar{i}jk\bar{r}}\overline{\omega}_{rl}) \wedge \omega_k \wedge \overline{\omega}_l\\
&\quad + T_{\bar{i}jk\bar{l}}(-\sum_r \omega_{kr} \wedge \omega_r) \wedge \overline{\omega}_l - T_{\bar{i}jk\bar{l}}\omega_k \wedge (-\sum_r \overline{\omega}_{lr} \wedge \overline{\omega}_r)\}\\
&= \sum_{k,l}\{\sum_r (T_{\bar{i}jk\bar{l}r}\omega_r + T_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r) \wedge \omega_k \wedge \overline{\omega}_l\\
&\qquad + \sum_r (T_{\bar{r}jk\bar{l}}\overline{\omega}_{ri} + T_{\bar{i}rk\bar{l}}\omega_{rj}) \wedge \omega_k \wedge \overline{\omega}_l\}\\
&= \sum_{k,l}\{\sum_r (T_{\bar{i}jk\bar{l}r}\omega_r + T_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r) \wedge \omega_k \wedge \overline{\omega}_l\}\\
&\quad + \sum_r\{(\sum_{k,l} T_{\bar{i}rk\bar{l}}\omega_k \wedge \overline{\omega}_l) \wedge \omega_{rj} - \omega_{ir} \wedge (\sum_{k,l} T_{\bar{r}jk\bar{l}}\omega_k \wedge \overline{\omega}_l)\}\\
&= \sum_{k,l}\sum_r (T_{\bar{i}jk\bar{l}r}\omega_r + T_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r) \wedge \omega_k \wedge \overline{\omega}_l\\
&\qquad + \sum_r (\Phi_{ir} \wedge \omega_{rj} - \omega_{ir} \wedge \Phi_{rj}),
\end{aligned}$$

where the first equality follows from the fact that the complex canonical form is a 1-form, the second equality is derived from (4.4) and the first equation of (2.1), the fourth term in the third equality follows from the second equation of (2.1) and the fourth equality is derived by (4.1). Hence we obtain

$$d\Phi_{ij} = \sum_{k,l}\sum_{r}(T_{\bar{i}jk\bar{l}r}\omega_r + T_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r)\wedge\omega_k\wedge\overline{\omega}_l + (\Phi\wedge\omega-\omega\wedge\Phi)_{ij} \tag{4.7}$$

From (4.7) it follows that (4.5) is equivalent to (4.6). This completes the proof. □

Remark 4.1. The first equation in (4.6) is essential and the second equation

$$T_{\bar{i}jk\bar{l}\bar{r}} = T_{\bar{i}jk\bar{r}\bar{l}}$$

in (4.6) naturally can be derived by the properties $d^2 = 0$. So the first of (4.6) is called the *second Bianchi identity* for the curvature-like tensor T.

Remark 4.2. The Riemannian curvature tensor R satisfies the first and the second Bianchi identities. In fact, the Bianchi equations (2.11) are given by the exterior differentiation of the first and the third equations of (2.1), respectively.

Remark 4.3. By Proposition 4.1, it can be easily seen that the parallel curvature-like tensor on M satisfies the second Bianchi identity. The converse does not necessarily hold. In section 6 we will give such an example.

Next, we define a curvature-like 4-form $\Psi = \Psi(T)$ for the curvature-like tensor T in such a way that

$$\Psi = \sum_{i,j}\Phi_{ij}\overline{\omega}_i\wedge\omega_j = \sum_{i,j,k,l}T_{\bar{i}jk\bar{l}}\omega_k\wedge\overline{\omega}_l\wedge\overline{\omega}_i\wedge\omega_j, \tag{4.8}$$

where $\Phi_{ij} = \sum_{k,l}T_{\bar{i}jk\bar{l}}\omega_k\wedge\overline{\omega}_l$. Then the 4-form $\Psi = \Psi(T)$ is said to be a *curvature-like 4-form* for the curvature-like tensor T. It can be regarded as a 2-form with values in the subbundle consisting of complex 2-forms of degree $(1,1)$ of $\bigotimes^2 T^{*C}M$. If $d\Psi = 0$, then the curvature-like tensor T is said to be *closed*.

Now, related to such a curvature-like 4-form $\Psi = \Psi(T)$, we give the following

Theorem 4.2. *For the curvature-like tensor T on a complex n-dimensional Kähler manifold we have the following equivalent assertions :*

(a) *T is closed i.e., for the curvature-like 4-form $\Psi = \Psi(T)$, $d\Psi = 0$.*
(b) *The curvature-like 2-form $\Phi = \Phi(T)$ satisfies the second Bianchi equation $d\Phi = \Phi\wedge\omega-\omega\wedge\Phi$.*
(c) $T_{\bar{i}jk\bar{l}h} = T_{\bar{i}jh\bar{l}k}$.

Proof. By Proposition 4.1, the statements (b) and (c) are equivalent. Next we show that the contents (a) and (b) are equivalent. Since the curvature-like 4-form $\Psi = \Psi(T)$ is given by (4.8) we have

$$\Psi = \sum_{i,j}\Phi_{ij}\wedge\overline{\omega}_i\wedge\omega_j\ .$$

Accordingly, by (2.1) we have

$$\begin{aligned} d\Psi &= \sum_{i,j}(d\Phi_{ij}\wedge\overline{\omega}_i\wedge\omega_j+\Phi_{ij}\wedge d\overline{\omega}_i\wedge\omega_j-\Phi_{ij}\wedge\overline{\omega}_i\wedge d\omega_j) \\ &= \sum_{i,j}(d\Phi_{ij}\wedge\overline{\omega}_i\wedge\omega_j-\sum_k\Phi_{ij}\wedge\overline{\omega}_{ik}\wedge\overline{\omega}_k\wedge\omega_j \\ &\quad+\sum_k\Phi_{ij}\wedge\overline{\omega}_i\wedge\omega_{jk}\wedge\omega_k) \\ &= \sum_{i,j}(d\Phi_{ij}+\sum_k\omega_{ik}\wedge\Phi_{kj}-\sum_k\Phi_{ik}\wedge\omega_{kj})\wedge\overline{\omega}_i\wedge\omega_j \\ &= \sum_{i,j}(d\Phi+\omega\wedge\Phi-\Phi\wedge\omega)_{ij}\wedge\overline{\omega}_i\wedge\omega_j\ , \end{aligned}$$

where the first equality follows from the fact that Φ is the 2-form and the second one can be derived from the structure equation. This equation means that $d\Psi=0$ if and only if

$$d\Phi+\omega\wedge\Phi-\Phi\wedge\omega=0,$$

which gives that the statements (a) and (b) are equivalent to each other. The statement (c) can be derived from (4.7) and Remark 4.1. This completes the proof. □

5. Ricci-like tensors

In this section we introduce the concept of the Ricci-like tensor associated with the curvature-like tensor defined on a Kähler manifold with almost complex structure J.

Let M be a complex n-dimensional Kähler manifold with curvature-like tensor T with the components $T_{\bar{i}jk\bar{l}}$. The Ricci-like tensor $Ric(T)$ associated with the curvature-like tensor T is defined by $Ric(T)(U,V)$ denotes the trace of the map $\{W\to T(U,V)W\}$, where $T(U,V)W$ is a complex vector field defined by $T(U,V,W,Z)=g(T(U,V)W,Z)$ for any complex vector fields U, V, W and Z. By the definition the Ricci-like tensor $Ric(T)$ of T is a a symmetric tensor of type (0,2)) and its components $T_{\bar{i}j}$ of $Ric(T)$ are given by

$$T_{\bar{i}j}=\sum_k T_{\bar{i}jk\bar{k}}.$$

From the definition of Ricci-like tensor $Ric(T)$ we know that

$$T_{\bar{i}j}=\sum\nolimits_k T_{\bar{i}jk\bar{k}}=\sum\nolimits_k T_{\bar{k}kj\bar{i}}=T_{j\bar{i}}=\overline{T}_{\bar{j}i}.$$

The scalar-like curvature t associated with T is defined by $t=\sum_{j,k}T_{\bar{j}jk\bar{k}}$. In the same way as we associate a curvature-like form $\Phi=\{\Phi_{ij}\}$ to the curvature-like tensor T, we associate a 1-form $\phi=\{\phi_j\}$ to the Ricci-like tensor $Ric(T)$ for T in the following (see Kobayashi and Nomizu [9])

(5.1) $$\phi_i=-2i\sum_j T_{\bar{i}j}\omega_j\ ,$$

which is called a *Ricci-like form* for the curvature-like tensor T.

The canonical form $\omega_0=\{\omega_j\}$ can be regarded as a vector in $\mathbb{C}^n$ and the connection form $\omega=\{\omega_{ij}\}$ and the Ricci-like form $\phi=\{\phi_{ij}\}$ can be regarded as $n\times n$ complex matrix. The components $T_{\bar{i}jk}$ and $T_{\bar{i}j\bar{k}}$ of the covariant derivative of the Ricci-like tensor are defined by

(5.2) $$\sum_k(T_{\bar{i}jk}\omega_k+T_{\bar{i}j\bar{k}}\overline{\omega}_k)=dT_{\bar{i}j}-\sum_k(T_{\bar{k}j}\overline{\omega}_{ki}+T_{\bar{i}k}\omega_{kj}).$$

We call the equation

$$d\phi + \omega \wedge \phi = 0 \tag{5.3}$$

a *Codazzi equation* for the Ricci-like form ϕ.

Proposition 5.1. *Let M be a Kähler manifold. Then the Ricci-like form $\phi = \phi(T)$ satisfies the equation of Codazzi if and only if the Ricci-like tensor $Ric(T)$ is parallel, i.e., its components satisfy*

$$T_{\bar{i}jk} = 0. \tag{5.4}$$

Proof. Since $\phi = \{\phi_i\}$ is the 1-form, the left hand side of (5.1) is given by

$$\begin{aligned} d\phi_i &= -2i \sum_j (dT_{\bar{i}j} \wedge \omega_j + T_{\bar{i}j} d\omega_j) \\ &= -2i \sum_{j,k} \{(T_{\bar{i}jk}\omega_k + T_{\bar{i}j\bar{k}}\overline{\omega}_k + T_{\bar{k}j}\overline{\omega}_{ki} + T_{\bar{i}k}\omega_{kj}) \wedge \omega_j \\ &\qquad + T_{\bar{i}j}(-\omega_{jk} \wedge \omega_k)\} \\ &= -2i \sum_{j,k} (T_{\bar{i}jk}\omega_k + T_{\bar{i}j\bar{k}}\overline{\omega}_k + T_{\bar{k}j}\overline{\omega}_{ki}) \wedge \omega_j \\ &= -2i \sum_{j,k} (T_{\bar{i}jk}\omega_k + T_{\bar{i}j\bar{k}}\overline{\omega}_k) \wedge \omega_j - (\omega \wedge \phi)_i \ , \end{aligned}$$

and hence we have

$$(d\phi + \omega \wedge \phi)_i = -2i \sum_{j,k} (T_{\bar{i}jk}\omega_k + T_{\bar{i}j\bar{k}}\overline{\omega}_k) \wedge \omega_j \ , \tag{5.5}$$

which implies that $d\phi + \omega \wedge \phi = 0$ holds on M if and only if $T_{\bar{i}jk} = T_{\bar{i}kj}$, $T_{\bar{i}j\bar{k}} = 0$. Since the components $T_{\bar{i}j}$ satisfy $T_{\bar{i}j} = T_{j\bar{i}}$, the last equalities are equivalent to $T_{\bar{i}jk} = 0$.

In fact, from $T_{\bar{i}j\bar{k}} = \bar{T}_{i\bar{j}k} = 0$ it follows that

$$0 = T_{i\bar{j}k} = T_{\bar{j}ik},$$

because of $T_{i\bar{j}} = T_{\bar{j}i}$. This completes the proof of our Proposition. □

Remark 5.1. The Ricci form $\phi = \phi(S)$ of the Ricci tensor $S = Ric(R)$ satisfies the Codazzi equation $d\phi + \omega \wedge \phi = 0$ if and only if S is parallel.

Now let us define a Ricci-like 2-form $\psi = \psi(T)$ for the curvature-like tensor T in $\bigotimes^2 T^{*C}M$ in such a way that

$$\psi = -2i \sum_{i,j} T_{\bar{i}j}\overline{\omega}_i \wedge \omega_j \ . \tag{5.6}$$

This is called a *Ricci-like 2-form* for the curvature-like tensor T. Then it can be easily checked that $\bar{\psi} = \psi$. Moreover, it can be regarded as a 1-form with values in the subbundle consisting of complex 2-forms of degree (1,0) of $T^{*C}M$. If $d\psi = 0$, then the Ricci-like tensor $Ric(T)$ is said to be *closed*.

Theorem 5.2. *Let $Ric(T)$ be the Ricci-like tensor on a Kähler manifold M associated to the curvature-like tensor T with components $T_{\bar{i}j}$. Then the following assertions are equivalent :*

(a) *The Ricci-like tensor $Ric(T)$ is closed, i.e., $d\psi = 0$.*

(b) $T_{\bar{i}jk} = T_{\bar{i}kj}$.

Proof. From (5.1) and (5.6), the Ricci-like 2-form ψ is given by

$$\psi = -2i\sum_{i,j} T_{\bar{i}j}\overline{\omega}_i \wedge \omega_j = -2i\sum_{j,k} T_{\bar{j}k}\overline{\omega}_j \wedge \omega_k = \sum_j \overline{\omega}_j \wedge \phi_j \,.$$

Accordingly we obtain

$$\begin{aligned} d\psi &= \sum_j (d\overline{\omega}_j \wedge \phi_j - \overline{\omega}_j \wedge d\phi_j) \\ &= -\sum_j (\sum_k \overline{\omega}_{jk} \wedge \overline{\omega}_k \wedge \phi_j + \overline{\omega}_j \wedge d\phi_j) \\ &= -\sum_j (\sum_k \overline{\omega}_j \wedge \omega_{jk} \wedge \phi_k + \overline{\omega}_j \wedge d\phi_j) \\ &= -\sum_i \overline{\omega}_i \wedge (d\phi + \omega \wedge \phi)_i \\ &= 2i\sum_i \overline{\omega}_i \wedge \sum_{j,k} \varepsilon_j\varepsilon_k (T_{\bar{i}jk}\omega_k + T_{\bar{i}j\bar{k}}\overline{\omega}_k) \wedge \omega_j \\ &= -2i\sum_{i,j,k} (T_{\bar{i}jk}\omega_k + T_{\bar{i}j\bar{k}}\overline{\omega}_k) \wedge \overline{\omega}_i \wedge \omega_j \,, \end{aligned}$$

where the fifth equality follows from (5.5). This equation means that $d\psi = 0$ if and only if the condition (b) holds on M. The latter part $T_{\bar{i}j\bar{k}} = T_{\bar{k}j\bar{i}}$ is equivalent to $T_{i\bar{j}k} = T_{k\bar{j}i}$ by the complex conjugation. Accordingly this gives our assertion. □

Combining Theorems 4.2 and 5.2, we have the following

Theorem 5.3. *Let M be a complex n-dimensional Kähler manifold. For any curvature-like tensor T let $Ric(T)$ be the Ricci-like tensor associated with T. If T is closed and T satisfies the first Bianchi identity* (4.3)*, then $Ric(T)$ is closed.*

Proof. By Theorem 4.2 and (4.3) we have

$$T_{\bar{i}jl} = \sum_k T_{\bar{i}jk\bar{k}l} = \sum_k T_{\bar{i}jl\bar{k}k} = \sum_k T_{\bar{i}lj\bar{k}k} = \sum_k T_{\bar{i}lk\bar{k}j} = T_{\bar{i}lj}.$$

Then by Theorem 5.2 we get our assertion. □

Remark 5.2. It can be easily verified that the Ricci tensor $S = Ric(R)$ is always closed. Because the curvature tensor R satisfies the 1st and the 2nd Bianchi identities.

In fact, by the first and the second Bianchi identities, the derivative of the Ricci tensor S is given by

$$\begin{aligned} S_{i\bar{j}l} &= \sum\nolimits_k R_{\bar{j}ik\bar{k}l} = \sum\nolimits_k R_{\bar{j}il\bar{k}k} \\ &= \sum\nolimits_k R_{\bar{j}li\bar{k}k} = \sum\nolimits_k R_{\bar{j}lk\bar{k}i} \\ &= S_{l\bar{j}i}. \end{aligned}$$

Then the Ricci tensor S is of Codazzi type. So by Theorem 5.2, the Ricci tensor S is closed.

Remark 5.3. If the Ricci tensor S is parallel on a Kähler manifold M, that is, $S_{i\bar{j}k} = 0$ on M, we can assert that the curvature 2-form $\Omega = (\Omega_{ij})$, $\Omega_{ij} = \sum_{k,l} R_{\bar{i}jk\bar{l}}\omega_k \wedge \bar{\omega}_l$ on M is harmonic. Since the curvature 2-form Ω satisfies the second Bianchi equation we know that $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$. Moreover, by the properties of the curvature 2-form in section 2, its coderivative $\delta\Omega$ of the curvature 2-form Ω in

(2.1) is coclosed as follows:

$$\begin{aligned}\delta\Omega &= -c_{15}\nabla R = -\sum_i R_{\bar{i}jk\bar{l}i} = -\sum_i R_{\bar{l}kj\bar{i}i}\\ &= -\sum_i R_{\bar{l}ki\bar{i}j}\\ &= -S_{\bar{l}kj} = 0.\end{aligned}$$

Accordingly, we can assert that any curvature 2-form Ω on M satisfies

$$\begin{aligned}\triangle\Omega =&(d\delta+\delta d)\Omega = \delta d\Omega\\ =&\delta(\Omega\wedge\omega-\omega\wedge\Omega)\\ =&\Omega\wedge\delta\omega-\delta\omega\wedge\Omega\\ =&0,\end{aligned}$$

where we have used $\delta\Omega = 0$ in the second and fourth equalities and in the third equality the second Bianchi equation in (2.10) is used, and finally in the fifth equality just used a differentiable function $\delta\omega$. So the curvature 2-form Ω with parallel Ricci tensor becomes a harmonic 2-form on M (see also Kobayashi and Nomizu [11], [12] and Matsushima [14]).

Remark 5.4. In the following sections 6, 7, 8 and 9 we introduce the notion of the concircular curvature tensor Z, the projective curvature tensor G, the conformal curvature tensor H and the Bochner curvature tensor B respectively and will make an application of Theorems 4.2 and 5.3 to such tensors.

Remark 5.5. Those four kind of tensors Z, G, H and B in Remark 5.4 satisfy the curvature-like properties (a), (b) and (c) in section 4. So they are curvature-like tensors and moreover, satisfy the first Bianchi identity (4.3). Then by (4.8) and the skew-symmetry of the wedge product $\omega_j\wedge\omega_k$ we know that the corresponding curvature-like 4-form Ψ identically vanishes on a Kähler manifold M. This gives $d\Psi = 0$ on M. Accordingly, such tensors Z, G, H and B are always closed. So naturally, by Theorem 5.3, its corresponding Ricci-like tensors are also closed.

6. The concircular curvature tensor Z

In this section we will investigate some equivalent conditions for a Kähler manifold with closed concircular curvature tensor Z.

Let M be a complex n-dimensional Kähler manifold with Kaehler connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M.

Now, let us denote by Z the *concircular curvature tensor* with components $Z_{\bar{i}jk\bar{l}}$ on M, which is defined by

$$Z_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{r}{2n(n+1)}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}). \tag{6.1}$$

Corresponding to a concircular curvature tensor on a Riemannian manifold (Yano and Bochner [30]), we consider the concircular curvature tensor Z, which is another kind of curvature-like tensor defined on a Kähler manifold M.

Though in Theorem 5.3 we have proved that if the curvature-like tensor T is closed and satisfies the first Bianchi identity, then the Ricci-like tensor $Ric(T)$ is closed.

Let $\Phi = \Phi(Z)$ (resp. $\Psi = \Psi(Z)$) be a curvature-like 2-form (resp. the curvature-like 4-form) for Z. The concircular curvature tensor Z is said to *satisfy the second Bianchi identity* (or is said to be *closed*) if it satisfies $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$ or it satisfies $d\Psi(Z) = 0$ respectively, where ω denotes the connection form on M. Let $Ric(Z)$ (resp. $\psi = \psi(Z)$) be the Ricci-like tensor (resp. Ricci-like 2-form) of Z. Then $Ric(Z)$ is said to be *closed* if $d\psi(Z) = 0$.

Let $Z_{\bar{i}j}$ be components of $Ric(Z)$ for Z. By (6.1) we have

$$Z_{\bar{i}j} = \sum_k Z_{\bar{i}jk\bar{k}} = S_{\bar{i}j} - \frac{r}{2n}\delta_{ij}\ . \tag{6.2}$$

This means that if $d(S - \frac{r}{2n}I) = 0$, then $Ric(Z)$ is closed.

Now summing up above contents and Remark 5.3 for the concircular curvature tensor Z, we assert the following

Theorem 6.1. *For the concircular curvature tensor Z on a complex $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The concircular curvature tensor Z is closed, that is, $d\Psi(Z) = 0$,*
(b) *The Ricci-like tensor $Ric(Z)$ is closed ; $d\psi(Z) = 0$,*
(c) *The curvature-like 2-form $\Phi = \Phi(Z)$ satisfies the second Bianchi equation $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$,*
(d) *The scalar curvature r on M is constant.*

Proof. By Theorem 4.2, we know that the assertions (a) and (c) are equivalent to each other. Now let us consider the statements (a) and (b).

Now we show that (a)⇒(b). Before giving this assertion, we note that the concircular curvature tensor Z in (6.1) satisfies the first Bianchi identity, that is, $Z_{\bar{i}jk\bar{l}} = Z_{\bar{i}kj\bar{l}}$. Moreover, from (c) the second Bianchi identity holds if Z is closed. So it follows that

$$Z_{\bar{i}jl} = \sum_k Z_{\bar{i}jk\bar{k}l} = \sum_k Z_{\bar{i}jl\bar{k}k} = \sum_k Z_{\bar{i}lj\bar{k}k} = \sum_k Z_{\bar{i}lk\bar{k}j} = Z_{\bar{i}lj}.$$

Then by Theorem 5.2, we know that the Ricci-like tensor $Ric(Z)$ is closed. So (b) holds.

Conversely, let us show (b)⇒(a), and (b)⇒(d). First we show (b)⇒(d) holds on M. In order to do this, let us suppose that $Ric(Z)$ is closed. Then we have by the definition $d\psi(Z) = 0$. By Theorem 5.2, we get $Z_{\bar{i}jk} = Z_{\bar{i}kj}$. By (6.2) we know that $Z_{\bar{i}jk} = S_{\bar{i}jk} - \frac{1}{2n} r_k \delta_{ij}$. So it follows $r_k \delta_{ij} = r_j \delta_{ik}$. Accordingly we have $(n-1)r_j = 0$ for any index j, which means that the scalar curvature r is constant provided that $n \geq 2$. Hence (b)⇒(d) holds.

Next let us show (b)⇒(a). Then by (6.1), together with constant scalar curvature r, we have $Z_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h}$ by (6.1) and hence $Z_{\bar{i}jk\bar{l}h} = Z_{\bar{i}jh\bar{l}k}$ by (2.12). Then, by Theorem 4.2, the concircular curvature tensor Z, which is a kind of curvature-like tensor, is closed. So (b)⇒(a) holds.

Now, let us show that (d)⇒(a). If the scalar curvature is constant, then it is easily seen that we have $Z_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h}$ by (6.1). That is, the second Bianchi identity holds. Then by virtue of (a)⇔(c), the concircular curvature tensor Z is closed, that is, $Z_{\bar{i}jk} = Z_{\bar{i}kj}$.

Then summing up all of the documents mentioned above, we completes the proof of our theorem. □

Now we here want to give an example and a remark concerned with the closed concircular curvature tensor defined on an indefinite complex hypersurface as follows:

Example 6.1.([5], [24], and [27]) For any integer p (≥ 2) an indefinite complex hypersurface $M = M(p,\lambda)$ of a $(2n+1)$-dimensional indefinite complex Euclidean space C_n^{2n+1} of index $2n$ is defined as follows:

Let $(z^A) = (z^j, z^{j^*}, z^{2n+1}) = (z^1, \ldots, z^{2n+1})$ be a complex coordinate of C_n^{2n+1} and let λ be a complex number such that $|\lambda| = 1$. Then $M(p,\lambda)$ is an indefinite complete complex hypersurface of index $2n$ defined by

$$z^{2n+1} = \sum_j f_j(z^j + \lambda z^{j^*}) , \quad j^* = n+j , \quad f_j(z) = z^p .$$

Then for the components h_{AB} of the second fundamental form we have

$$\begin{aligned} &h_{ij} = p(p-1)\delta_{ij}{\mu_i}^{p-2} , \quad h_{i^*j} = p(p-1)\lambda\delta_{ij}{\mu_i}^{p-2} , \\ &h_{i^*j^*} = p(p-1)\lambda^2\delta_{ij}{\mu_i}^{p-2} , \quad \mu_i = z^i + \lambda z^{i^*} , \end{aligned} \tag{6.3}$$

and then for the components h_{ABC} of the covariant derivatives of the second fundamental form we have

$$\begin{aligned} &h_{ijk} = p(p-1)(p-2)\delta_{ij}\delta_{ik}{\mu_i}^{p-3} , \\ &h_{i^*jk} = p(p-1)(p-2)\lambda\delta_{ij}\delta_{ik}{\mu_i}^{p-3} , \\ &h_{i^*j^*k} = p(p-1)(p-2)\lambda^2\delta_{ij}\delta_{ik}{\mu_i}^{p-3} , \\ &h_{i^*j^*k^*} = p(p-1)(p-2)\lambda^3\delta_{ij}\delta_{ik}{\mu_i}^{p-3} . \end{aligned} \tag{6.4}$$

Since the holomorphic sectional curvature of the ambient space C_n^{2n+1} is vanishing, by the equation of Gauss in [24], the components of the curvature tensor R of the space-like complex hypersurface M in C_n^{2n+1} is given by

$$R_{\bar{i}jk\bar{l}} = -\sum \varepsilon_x h^x_{jk}\bar{h}^x_{il} = h_{jk}\bar{h}_{il}.$$

Then differentiating by the exterior derivative d to this curvature tensor of M gives the following

$$R_{\bar{i}jk\bar{l}m} = h_{jkm}\bar{h}_{il} + h_{jk}\bar{h}_{ilm}.$$

But from the above expressions of (6.4), we know that

$$\begin{aligned} R_{\bar{i}^*j^*k^*\bar{l}^*m} =& h_{j^*k^*m}\bar{h}_{i^*l^*} + h_{j^*k^*}\bar{h}_{i^*l^*m} \\ =& p^2(p-1)^2(p-2)\delta_{jk}\delta_{jm}\delta_{il}\mu_j^{p-3}\bar{\mu}_i^{p-2} \\ &+ p^2(p-1)^2(p-2)\delta_{jk}\delta_{il}\delta_{im}\bar{\mu}_i^{p-3}\mu_j^{p-2} \\ =& p^2(p-1)^2(p-2)\delta_{jk}\delta_{il}\mu_j^{p-3}\bar{\mu}_i^{p-3}(\delta_{jm}\bar{\mu}_i + \delta_{im}\mu_j), \end{aligned} \tag{6.5}$$

where we have used the fact that $|\lambda|^2 = \lambda\bar{\lambda} = 1$. So without $\mu_i = 0$ the curvature component $R_{\bar{i}^*j^*k^*\bar{l}^*m}$ is never vanishing. This means that the curvature tensor $R_{\bar{i}jk\bar{l}}$ can not be parallel in general, that is, non-parallel.

Remark 6.1. In this Example 6.1, since components of the second fundamental form are given by (6.3), the following facts can be guaranteed : The Ricci tensor of M is flat provided that $|\lambda| = 1$, but the curvature tensor is not flat. Accordingly, the Ricci tensor is flat and the scalar curvature r is zero on M, which means that *the concircular curvature tensor Z coincides with the Riemannian curvature tensor R*. This means that the concircular curvature tensor Z of this indefinite complex hypersurface is not flat, but it satisfies the second Bianchi identity and closed.

Furthermore, taking account of (6.5), we can see that its derivative of the concircular curvature tensor $Z_{\bar{i}jk\bar{l}}$ is non-vanishing. That is, the *concircular curvature tensor Z is non-parallel* provided that $p \geq 3$. So there exist indefinite Kähler manifolds whose concircular curvature tensor Z is *closed,* *but not parallel*. So there are no meaning to consider the parallel concircular curvature tensor ∇Z on Kähler manifolds.

But hereafter in sections 7,8 and 9, it will be meaningful to consider the following notions of parallel projective curvature tensor, parallel conformal curvature tensor and parallel Bochner curvature tensor on Kähler manifolds respectively.

7. The projective curvature tensor G

In this section we want to find out some equivalent assertions for a Kähler manifold with projective curvature tensor G.

Let M be a complex n-dimensional Kähler manifold with Kähler connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor(resp. the Ricci tensor or the scalar curvature) on M.

Now, let G be the *projective curvature tensor* with components $G_{\bar{i}jk\bar{l}}$ on M, which is defined by

$$G_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{1}{n+1}(S_{j\bar{i}}\delta_{kl} + S_{k\bar{i}}\delta_{jl}) \ . \tag{7.1}$$

This is introduced by Bochner ([3]) as a formal analogue to the Weyl projective curvature tensor on the Riemannian manifold. Of course, it becomes a curvature-like tensor defined in section 4. As is easily seen, G is the curvature-like tensor on M. Let $\Phi = \Phi(G)$ (resp. $\Psi = \Psi(G)$, $\phi = \phi(G)$ or $\psi = \psi(G)$) be the curvature-like form (resp. the curvature-like 4-form, the Ricci-like form or the Ricci-like 2-form) for G. Let $Ric(G)$ (resp. $\psi = \psi(G)$) be the Ricci-like tensor (resp. Ricci-like 2-form) of G. Then $Ric(G)$ is said to be *closed* if $d\psi(G) = 0$.

Let $G_{\bar{i}j}$ be components of $Ric(G)$ for G. By (7.1) the Ricci-like tensor $Ric(G)$ vanishes identically on M, because

$$G_{\bar{i}j} = \sum\nolimits_l R_{\bar{i}jl\bar{l}} - S_{j\bar{i}} = S_{j\bar{i}} - S_{j\bar{i}} = 0.$$

This means that $Ric(G)$ is always closed.

Remark 7.1. Let M be a Einstein Kähler manifold. Then (2.7) holds on M. So the projective curvature tensor G in (7.1) becomes

$$G_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{r}{2n(n+1)}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) \ .$$

That is, $G = Z$ in (6.1) for a Einstein Kähler manifold. By Theorem 6.1, the second Bianchi equation

$$d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$$

holds for the curvature-like forms $\Phi = \Phi(G)$ of Einstein Kähler manifolds.

If the projective curvature tensor $G = 0$ on M, then M becomes a space of constant holomorphic sectional curvature (see Wolf [29]).

The components $G_{\bar{i}jk\bar{l}r}$ and $G_{\bar{i}jk\bar{l}\bar{r}}$ of the covariant derivative ∇G of G are defined by

$$\begin{aligned}&\sum_r (G_{\bar{i}jk\bar{l}r}\omega_r + G_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r)\\ &= dG_{\bar{i}jk\bar{l}} - \sum_r (G_{\bar{r}jk\bar{l}}\overline{\omega}_{ri} + G_{\bar{i}rk\bar{l}}\omega_{rj} + G_{\bar{i}jr\bar{l}}\omega_{rk} + G_{\bar{i}jk\bar{r}}\overline{\omega}_{rl}).\end{aligned}$$

The projective curvature form $\Phi = \{\Phi_{ij}\}$ is defined by the curvature-like form for the projective curvature tensor G, namely, it is defined by

$$\Phi_{ij} = \sum_{k,l} G_{\bar{i}jk\bar{l}}\omega_k \wedge \overline{\omega}_l \ .$$

And the curvature-like 4-form $\Psi = \Psi(G)$ is defined by

$$\Psi = \sum_{i,j} \Phi_{ij}\overline{\omega}_i \wedge \omega_j.$$

Now, for the projective curvature tensor G let us denote by $\Phi(G)$ the projective curvature-like form. Then the mapping $\delta : \otimes^4 T^*M \to \otimes^3 T^*M$ is defined by the divergence : $\delta(\Phi(G)) = -C_{15}(\nabla G)$, where C_{ab} is the metric contraction functioned by $C_{ab} : T^r_s M \to T^r_{s-2}M$. This is a so-called *codifferential operator*, and it is a generalization of the well known differential operators on R^3. For the unitary frame $\{U_j\}$, in terms of coordinates, the components of $\delta(\Phi(G))$ is given by

$$\delta(\Phi(G))_{i\bar{k}j} = -\sum_r G_{\bar{r}ij\bar{k}r.}$$

Then the *curvature-like 2-form* $\Phi(G)$ is said to be *coclosed* if its cooperator δ satisfies

$$\delta(\Phi(G))_{i\bar{k}j} = 0.$$

Moreover, the projective curvature-like 2-form $\Phi = \Phi(G)$ is *coclosed* if and only if $\nabla S \in C^\infty(\mathscr{I})$ in (3.4), that is, $i(\xi) = \nabla S$ and $c(\xi) = r$ in (3.5), because

$$\begin{aligned} 0 =& \delta(\Phi(G))_{i\bar{k}j} \\ =& -\sum_r G_{\bar{r}ij\bar{k}r} \\ =& -\sum_r \{R_{\bar{r}ij\bar{k}r} - \frac{1}{n+1}(S_{i\bar{r}r}\delta_{jk} + S_{j\bar{r}r}\delta_{ik})\} \\ =& -S_{i\bar{k}j} + \frac{1}{2(n+1)}(r_i\delta_{jk} + r_j\delta_{ik}), \end{aligned} \tag{7.2}$$

where we have put $U = U_i$, $\bar{V} = \bar{U}_k$, and $W = U_j$ in (3.4), and the vector subbundle $\mathscr{I}$ is defined in section 3.

Remark 7.2. The locally symmetric complex submanifolds of a complex projective space $P_n(\mathbb{C})$ are completely classified by Nakagawa and Takagi [18]. They are Kähler manifolds with *parallel projective curvature tensor* and hence its projective curvature tensor satisfies the second Bianchi identity.

Related to these remarks, Proposition 4.1 and Remark 5.3, we investigate a Kähler manifold with closed projective curvature tensor G as follows:

Theorem 7.1. *For the projective curvature tensor G on a complex $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The projective curvature tensor G is closed, that is, $d\Psi = 0$,*
(b) *The projective curvature-like 2-form $\Phi = \Phi(G)$ satisfies the second Bianchi equation $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$,*
(c) *$\nabla S \in C^\infty(\mathscr{I})$, that is, the projective curvature-like 2-form $\Phi(G)$ is coclosed,*
(d) *The Ricci tensor S on M is parallel.*

Proof. By Theorem 4.2, the assertions (a) and (b) are equivalent.

Next we show (b)⇒(c). We suppose that the projective curvature tensor G of M satisfies the second Bianchi identity. Namely, the projective curvature-like form $\Phi = \Phi(G)$ satisfies $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$, where ω is the connection form on M. By (7.1) the components $G_{\bar{i}jk\bar{l}h}$ of the covariant derivative ∇G of G are given by

$$G_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h} - \frac{1}{n+1}(S_{j\bar{i}h}\delta_{kl} + S_{k\bar{i}h}\delta_{jl}) \ . \tag{7.3}$$

By (2.13) and (7.3), we have

$$G_{\bar{i}jk\bar{l}h} - G_{\bar{i}jh\bar{l}k} = R_{\bar{i}jk\bar{l}h} - R_{\bar{i}jh\bar{l}k} - \frac{1}{n+1}(S_{j\bar{i}h}\delta_{kl} - S_{j\bar{i}k}\delta_{hl}) \ . \tag{7.4}$$

Accordingly, by Theorem 4.2 under the condition (b), we have

$$S_{j\bar{l}h}\delta_{ik} = S_{j\bar{l}k}\delta_{ih} \ . \tag{7.5}$$

Putting $k = l$ in (7.5) and summing up with respect to k, we get

$$2S_{\bar{i}jk} = r_j\delta_{ik} \ , \tag{7.6}$$

which implies by (2.13) under the symmetry of the indices j and k

$$r_j\delta_{ik} = r_k\delta_{ij} \ . \tag{7.7}$$

Now, we reform (7.4) as follows :

$$\begin{aligned} &G_{\bar{i}jk\bar{l}h} - G_{\bar{i}jh\bar{l}k} \\ =&R_{\bar{i}jk\bar{l}h} - R_{\bar{i}jh\bar{l}k} - \frac{1}{n+1}\big[\{S_{j\bar{i}h} - \alpha(r_j\delta_{ih} + r_h\delta_{ij})\}\delta_{lk} \\ &\qquad\qquad - \{S_{j\bar{i}k} - \alpha(r_j\delta_{ki} + r_k\delta_{ij})\}\delta_{lh}\big] \\ &+ \frac{1}{n+1}\alpha\{r_j(\delta_{ki}\delta_{lh} - \delta_{ih}\delta_{kl}) + \varepsilon_j\delta_{ji}(r_k\delta_{lh} - r_h\delta_{lk})\}, \end{aligned}$$

where $\alpha = \frac{1}{2(n+1)}$. Taking account of (7.7), we see that the last term can be vanished by the following properties

$$r_j\delta_{ki}\delta_{lh} = r_k\delta_{ji}\delta_{lh} \ ,$$
$$r_j\delta_{hi}\delta_{lk} = r_h\delta_{ji}\delta_{lk} = r_k\delta_{ji}\delta_{lh} \ .$$

This implies that the fourth term vanishes identically. Thus we have

$$\begin{aligned} &G_{\bar{i}jk\bar{l}h} - G_{\bar{i}jh\bar{l}k} \\ =&R_{\bar{i}jk\bar{l}h} - R_{\bar{i}jh\bar{l}k} - \frac{1}{n+1}\big[\{S_{j\bar{i}h} - \alpha(r_j\delta_{ih} + r_h\delta_{ij})\}\delta_{lk} \\ &\qquad\qquad - \{S_{j\bar{i}k} - \alpha(r_j\delta_{ki} + r_k\delta_{ij})\}\delta_{lh}\big], \end{aligned} \tag{7.8}$$

where α denotes $\frac{1}{2(n+1)}$. Under the situation of (b) in Theorem 4.2, we have

$$\{S_{j\bar{i}h} - \alpha(r_j\delta_{ih} + r_h\delta_{ij})\}\delta_{lk} - \{S_{j\bar{i}k} - \alpha(r_j\delta_{ki} + r_k\delta_{ij})\}\delta_{lh} = 0.$$

Putting $l = k$, and summing up with respect to k we have

$$(n-1)\{S_{j\bar{i}h} - \frac{1}{2(n+1)}(r_j\delta_{hi} + r_h\delta_{ji})\} = 0.$$

This shows that if $n \geq 2$, then we have

$$S_{j\bar{i}k} = \frac{1}{2(n+1)}(r_j\delta_{ki} + r_k\delta_{ji}).$$

From this together with (3.3) it follows that ∇S is the section on the subbundle $\mathscr{I}$. Then by (7.2) we know that $\delta(\Phi(G)) = 0$. That is, the curvature-like form $\Phi(G)$ is coclosed. It means that (b)$\Rightarrow$(c).

Now we show that (b)⇒(d). Then by (7.7) the scalar curvature r is constant on M, because of $nr_j = r_j$, and then by (7.6) the Ricci tensor is parallel. It means that (b)⇒(d).

Next let us show that (c)⇒(a). The Ricci tensor S satisfies $\nabla S \in C^\infty(\mathscr{I})$. Then by (3.3), (7.8) is reformed as $G_{\bar{i}jk\bar{l}h} = G_{\bar{i}jh\bar{l}k}$, because the Riemannian curvature tensor R is closed, from which it follows that G is also closed. It means that (c)⇒(a) holds on M.

Finally we show that (d)⇒(b). In order to do this we suppose that the Ricci tensor S is parallel. Then (7.3) is reformed as $G_{\bar{i}jk\bar{l}h} = R_{\bar{i}jh\bar{l}k}$, from which together with the fact that the Riemannian curvature tensor R is closed by Theorem 4.2, it follows that G is also closed. □

Remark 7.3. By virtue of Theorem 7.1 we are able to assert that the projective curvature tensor G is *harmonic*. That is, the curvature-like 2-form $\Phi(G)$ becomes a harmonic curvature-like 2-form as follows:

Indeed by (b) and (c) in Theorem 7.1 the Laplacian of the curvature-like 2-form $\Phi(G)$ of the projective curvature tensor G is given by

$$\begin{aligned}\triangle\Phi(G) =&(d\delta+\delta d)\Phi(G) = \delta d\Phi(G)\\ =&\delta(\Phi(G)\wedge\omega - \omega\wedge\Phi(G))\\ =&\delta\Phi(G)\wedge\omega + \Phi(G)\wedge\delta\omega - \delta\omega\wedge\Phi(G) + \omega\wedge\delta\Phi(G)\\ =&0,\end{aligned} \tag{7.9}$$

where in the second equality we have used the co-closedness $\delta(\Phi(G)) = 0$ of the projective curvature-like 2-form $\Phi(G)$ and in the 3rd equality the property of the second-Bianchi equation (b) is used. In the 4th equality we have just used the property of coderivative δ and 2-form Φ, and 1-form ω. Finally, in the 5th equality we used $\delta\Phi = 0$ and the fact that $\delta\omega$ just a function on the manifold M.

Now let us consider a special case of Theorem 7.1 when the projective curvature tensor G is parallel. So let M be a Kähler manifold with parallel projective curvature tensor G. Substituting (7.6) into (7.3), we can obtain

$$G_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h} - \frac{1}{2(n+1)}(r_j\delta_{ih}\delta_{kl} + r_k\delta_{ih}\delta_{jl}). \tag{7.10}$$

Since G is parallel, its curvature-like form satisfies the second Bianchi equation by Theorem 4.2 and hence by Theorem 7.1 the Ricci tensor is parallel. Accordingly, the scalar curvature r of M is constant, which yields that the Riemannian curvature tensor R is parallel. Thus we can prove

Theorem 7.2. *Let M be a complex $n(\geq 2)$-dimensional Kähler manifold. If the projective curvature tensor G is parallel, then it is locally symmetric.*

Remark 7.4. In Example 6.1, the second fundamental form is not parallel and not locally symmetric provided that $p \geq 3$ and furthermore if $|\lambda| = 1$, then M is Ricci flat. So the scalar curvature r is zero on the indefinite complex hypersurface, which means that the projective curvature tensor coincides with the Riemannian curvature tensor by (7.1). Accordingly, we see that the projective curvature tensor of such an example is not parallel, but *harmonic*.

8. The conformal curvature tensor H

This section is devoted to investigate some equivalent assertions for a Kähler manifold with closed conformal curvature tensor H.

Let M be a complex n-dimensional Kähler manifold with the Kähler connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor(resp. the Ricci tensor or the scalar curvature) on M.

Now, let us consider the *conformal curvature tensor* H with components $H_{\bar{i}jk\bar{l}}$ defined by

$$H_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}(S_{\bar{i}j}\delta_{kl} + S_{\bar{i}k}\delta_{jl} + S_{\bar{l}k}\delta_{ij} + S_{\bar{l}j}\delta_{ik}) \ . \tag{8.1}$$

This is introduced by Bochner [3] as a formal analogue to the Weyl conformal curvature tensor on a Riemannian manifold(see also [28] and [30]). As is easily seen, H is the curvature-like tensor on M. Then by the same method as in previous section we are able to define $\Phi = \Phi(H)$ (resp. $\Psi = \Psi(H)$, $\phi = \phi(H)$ or $\psi = \psi(H)$) the curvature-like 2-form (resp. the curvature-like 4-form, the Ricci-like form or the Ricci-like 2-form) for H.

Remark 8.1. When M is Einstein, by (2.7) the conformal curvature tensor H can be given by

$$H_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{r}{2n(n+1)}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) \ .$$

That is, $G = Z$ in (6.1) for a Einstein Kähler manifold. By Theorem 6.1, the second Bianchi equation

$$d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$$

holds for the curvature-like forms $\Phi = \Phi(H)$ of Einstein Kähler manifolds.

If the conformal curvature tensor $H = 0$ on M, then M becomes a space of constant holomorphic sectional curvature (see Wolf [29]).

The components $H_{\bar{i}jk\bar{l}r}$ and $H_{\bar{i}jk\bar{l}\bar{r}}$ of the covariant derivative of H are defined by

$$\begin{aligned} &\sum_r (H_{\bar{i}jk\bar{l}r}\omega_r + H_{\bar{i}jk\bar{l}\bar{r}}\overline{\omega}_r) \\ &= dH_{\bar{i}jk\bar{l}} - \sum_r (H_{\bar{r}jk\bar{l}}\overline{\omega}_{ri} + H_{\bar{i}rk\bar{l}}\omega_{rj} + H_{\bar{i}jr\bar{l}}\omega_{rk} + H_{\bar{i}jk\bar{r}}\overline{\omega}_{rl}). \end{aligned} \tag{8.2}$$

The conformal curvature-like form $\Phi = \Phi(H) = \{\Phi_{ij}\}$ which is the curvature-like form for the conformal curvature tensor H is defined by

$$\Phi_{ij} = \sum_{k,l} H_{\bar{i}jk\bar{l}}\omega_k \wedge \overline{\omega}_l$$

and the curvature-like 4-form $\Psi = \Psi(H)$ is defined by

$$\Psi = \textstyle\sum_{i,j} \Phi_{ij}\overline{\omega}_i \wedge \omega_j.$$

On the other hand, for the conformal curvature tensor H we denote by $\Phi(H)$ the conformal curvature-like 2-form. For the unitary frame $\{U_j\}$, in terms of coordinates, as in section 7 the components of $\delta(\Phi(H))$ can be given by

$$\delta(\Phi(H))_{i\bar{k}j} = -\sum_r H_{\bar{r}ij\bar{k}r.}$$

Then the *curvature-like 2-form* $\Phi(H)$ is said to be *coclosed* if its cooperator δ satisfies

$$\delta(\Phi(H))_{i\bar{k}j} = 0.$$

Related to these remarks with Proposition 4.1 and Remark 5.3, in this section we want to investigate a Kaehler manifold whose conformal curvature tensor satisfies the second Bianchi equation as follows:

Theorem 8.1. *For the conformal curvature tensor H on an $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The conformal curvature tensor H is closed ; $d\Psi(H) = 0$,*

(b) *The conformal curvature-like 2-form $\Phi = \Phi(H)$ satisfies the second Bianchi equation $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$,*

(c) *The Ricci tensor S on M is parallel,*

(d) *The conformal curvature-like 2-form $\Phi(H)$ is coclosed.*

Proof. By Theorems 4.2, the assertions (a) and (b) are equivalent.

Next we show that (b)⇒(c). We suppose that the conformal curvature tensor H of M satisfies the second Bianchi identity. In other words, the curvature-like form $\Phi = \Phi(H)$ satisfies $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$, where ω is the connection form on M. Then by (8.1) the components $H_{\bar{i}jk\bar{l}h}$ of the covariant derivative ∇H of H are given as

$$H_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h} - \frac{1}{2(n+1)}(S_{\bar{i}jh}\delta_{kl} + S_{\bar{i}kh}\delta_{jl} + S_{\bar{l}kh}\delta_{ij} + S_{\bar{l}jh}\delta_{ik}). \tag{8.3}$$

By (2.13) and (8.3), we have

$$\begin{aligned} H_{\bar{i}jk\bar{l}h} - H_{\bar{i}jh\bar{l}k} &= R_{\bar{i}jk\bar{l}h} - R_{\bar{i}jh\bar{l}k} \\ &- \frac{1}{2(n+1)}\{(S_{\bar{i}jh}\delta_{kl} + S_{j\bar{l}h}\delta_{ik}) - (S_{\bar{i}jk}\delta_{lh} + S_{j\bar{l}k}\delta_{ih})\}\ . \end{aligned} \tag{8.4}$$

Accordingly by (2.12) and Theorem 4.2 we have

$$(S_{\bar{i}jh}\delta_{kl} + S_{j\bar{l}h}\delta_{ik}) - (S_{\bar{i}jk}\delta_{lh} + S_{j\bar{l}k}\delta_{ih}) = 0. \tag{8.5}$$

Putting $k = l$ in (8.5) and summing up with respect to k and using (2.6) for the scalar curvature $r = 2\sum_j S_{j\bar{j}}$, we get

$$2S_{\bar{i}jk} = \frac{1}{n} r_j \delta_{ik}\ . \tag{8.6}$$

Then by (2.13) we have $r_j\delta_{ki} = r_k\delta_{ji}$ and contracting k and i implies $nr_j = r_j$. From this, the scalar curvature r is constant on M, so the Ricci tensor is parallel by (8.6). It implies that (b)⇒(c).

Now we want to show that (c)⇒(d) holds on M.

In fact, the notion of parallel Ricci tensor gives $S_{i\bar{j}k} = 0$. From this we know the scalar curvature r is constant on M, that is, $r_j = 0$. So we assert

$$\begin{aligned} \delta(\Phi(H))_{i\bar{k}j} &= -\sum_r H_{\bar{r}ij\bar{k}r} \\ &= -\sum_r R_{\bar{r}ij\bar{k}r} + \frac{1}{4(n+1)}(r_i\delta_{jk} + r_j\delta_{ik} \\ &\quad + 2S_{\bar{k}ji} + 2S_{\bar{k}ij}) \\ &= -S_{i\bar{k}j} + \frac{1}{4(n+1)}(r_i\delta_{jk} + r_j\delta_{ik}) \\ &\quad + \frac{1}{2(n+1)}(S_{\bar{k}ji} + S_{\bar{k}ij}) \\ &= 0. \end{aligned}$$

Conversely, we are going to assert (d)⇒(c).

In fact, the assumption (d) above and the property $S_{j\bar{i}k} = S_{k\bar{i}j}$ implies

$$nS_{i\bar{k}j} = \frac{1}{4}(r_i\delta_{jk} + r_j\delta_{ik}).$$

Then by putting $i = k$ we know

$$nr_j = 2n\sum_k S_{k\bar{k}j} = \frac{1}{2}(n+1)r_j.$$

This implies $r_j = 0$ for any j. From this together with the above formula, we know that the Ricci tensor S is parallel. That is, $S_{i\bar{k}j} = 0$.

Finally we show that (c)⇒(a). In order to do this we suppose that the Ricci tensor S is parallel. Then (8.3) is reformed as $H_{\bar{i}jk\bar{l}h} = R_{\bar{i}jh\bar{l}k}$, from which together with the fact that the Riemannian curvature tensor R is closed by Theorem 4.2, it follows that H is also closed. This completes the proof. □

Remark 8.2. By virtue of Theorem 8.1 we are able to assert that the conformal curvature tensor H is *harmonic*. That is, the curvature-like 2-form $\Phi(H)$ becomes a harmonic curvature-like 2-form as follows:

Indeed by (b) and (d) in Theorem 8.1 the Laplacian of the curvature-like 2 form $\Phi(H)$ of the conformal curvature tensor H is given by

$$\begin{aligned}\triangle\Phi(H) =&(d\delta+\delta d)\Phi(H) = \delta d\Phi(H)\\ =&\delta(\Phi(H)\wedge\omega-\omega\wedge\Phi(H))\\ =&\delta\Phi(H)\wedge\omega+\Phi(H)\wedge\delta\omega-\delta\omega\wedge\Phi(H)+\omega\wedge\delta\Phi(H)\\ =&0,\end{aligned} \tag{8.7}$$

where in the second equality we have used the coclosedness $\delta(\Phi(H)) = 0$ in (d) for the conformal curvature-like 2-form $\Phi(H)$ and in the 3rd equality the property of the second-Bianchi equation (b) is used, and in the last equality $\delta\omega$ just denotes a function on the Kähler manifold M.

Now, as a special case of Theorem 8.1, the parallelism of the conformal curvature tensor is considered. Let M be a Kähler manifold with parallel conformal curvature tensor H. Then (8.6) holds. Substituting (8.6) into (8.3), we can obtain

$$\begin{aligned}H_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h} - \frac{1}{4n(n+1)}\{&(r_j\delta_{ih}\delta_{kl}+r_k\delta_{ih}\delta_{lj})\\ &+r_h(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl})\}.\end{aligned} \tag{8.8}$$

By (2.13) and (8.6), the scalar curvature r is constant on M and hence the Riemannian curvature tensor R on M is parallel by (8.7). Thus we can prove

Theorem 8.2. *Let M be a complex $n(\geq 2)$-dimensional Kähler manifold whose conformal curvature tensor H is parallel. Then it is locally symmetric.*

Remark 8.3. By virtue of Theorem 8.1 we know that the conformal curvature tensor H is closed if and only if its curvature-like form $\Phi = \Phi(H)$ is *coclosed*. So naturally we assert that the conformal curvature tensor H on a Kähler manifold is *harmonic*.

Remark 8.4. Example 6.1 gives us the existence of indefinite Kaehler manifold whose conformal curvature tensor is not parallel, but harmonic.

9. The Bochner curvature tensor B

In this section we also investigate some equivalent assertions for a Kähler manifold with closed Bochner curvature tensor B.

Let M be a complex n-dimensional Kaehler manifold with the Kaehler connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor(resp. the Ricci tensor or the scalar curvature) on M.

Now, let B be the *Bochner curvature tensor* with components $B_{\bar{i}jk\bar{l}}$ on M, which is defined by

$$B_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - (\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}} + \delta_{ki}S_{j\bar{l}} + \delta_{kl}S_{j\bar{i}})/(n+2) + r(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki})/2(n+1)(n+2). \tag{9.1}$$

This is introduced by Bochner [3] as a complex analogue of the Weyl conformal curvature tensor on a Riemannian manifold.

Let $\Phi = \Phi(B)$ (resp. $\Psi = \Psi(B)$) be a curvature-like 2-form (resp. the curvature-like 4-form) for B. The Bochner curvature tensor B is said to *satisfy the second Bianchi identity* or to be *closed* if it satisfies $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$ or $d\Psi(B) = 0$, where ω denotes the connection form on M. Let $Ric(B)$ (resp. $\psi = \psi(B)$) be the Ricci-like tensor (resp. Ricci-like 2-form) of B. Then $Ric(B)$ is said to be *closed* if $d\psi(B) = 0$.

Let $B_{\bar{i}j}$ be components of $Ric(B)$ for B. By (9.1) we have $B_{\bar{i}j} = 0$, which means that the Ricci-like tensor $Ric(B)$ is always closed.

Remark 9.1. When M is Einstein, by (2.7) the Bochner curvature tensor B can be given by

$$B_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{r}{2n(n+1)}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) \ .$$

That is, $B = Z$ in (6.1) for a Einstein Kähler manifold. By Theorem 6.1, the second Bianchi equation

$$d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$$

holds for the curvature-like forms $\Phi = \Phi(B)$ of Einstein Kähler manifolds.

If the Bochner curvature tensor $B = 0$ on M, then M becomes a space of constant holomorphic sectional curvature (see Wolf [29]).

Remark 9.2. The Bochner curvature-like 2-form $\Phi = \Phi(B)$ is *coclosed* if and only if $\nabla S \in C^\infty(\mathscr{I})$, because

$$\begin{aligned} 0 &= \delta(\Phi(B))_{i\bar{k}j} \\ &= -\sum\nolimits_r B_{\bar{r}ij\bar{k}r} \\ &= -\sum\nolimits_r \{R_{\bar{r}ij\bar{k}r} - \frac{1}{n+1}(S_{j\bar{r}r}\delta_{ik} + S_{i\bar{r}r}\delta_{jk})\} \\ &= -S_{i\bar{k}j} + \frac{1}{2(n+1)}(r_j\delta_{ik} + r_i\delta_{jk}). \end{aligned} \tag{9.2}$$

Summing up above contents, together with Proposition 4.1 and Remark 5.3, we want to investigate a Kähler manifold whose Bochner curvature tensor satisfies the second Bianchi equation as follows:

Theorem 9.1. *For the Bochner curvature-like tensor B on an $n(\geq 2)$-dimensional Kähler manifold M we have the following equivalent assertions:*

(a) *The Bochner curvature tensor B is closed ; $d\Psi(B) = 0$,*
(b) *The Bochner curvature-like 2-form $\Phi = \Phi(B)$ satisfies the second Bianchi equation $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$,*
(c) *$\nabla S \in C^\infty(\mathscr{I})$, that is, the Bochner curvature-like 2-form $\Phi(B)$ is coclosed.*

Proof. By Theorem 4.2, the assertions (a) and (b) are equivalent. Next we show (b) $\Rightarrow$ (c). We suppose that the Bochner curvature tensor B of M satisfies the second Bianchi identity. Namely, the

Bochner curvature-like form $\Phi = \Phi(B)$ satisfies $d\Phi = \Phi \wedge \omega - \omega \wedge \Phi$, where ω is the connection form on M. By (9.1) the components $B_{\bar{i}jk\bar{l}h}$ of the covariant derivative ∇B of B are given by

$$\text{(9.3)} \qquad \begin{aligned} B_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h} - (\delta_{ji}S_{k\bar{l}h} + \delta_{jl}S_{k\bar{i}h} + \delta_{ki}S_{j\bar{l}h} + \varepsilon_k\delta_{kl}S_{j\bar{i}h})/(n+2) \\ + r_h(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki})/2(n+1)(n+2), \end{aligned}$$

which implies that

$$\text{(9.4)} \qquad \begin{aligned} B_{\bar{i}jk\bar{l}h} - B_{\bar{i}jh\bar{l}k} = R_{\bar{i}jk\bar{l}h} - R_{\bar{i}jh\bar{l}k} &- (\delta_{ki}S_{j\bar{l}h} + \delta_{kl}S_{j\bar{i}h})/(n+2) \\ &+ r_h(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki})/2(n+1)(n+2) \\ &+ (\delta_{hi}S_{j\bar{l}k} + \delta_{hl}S_{j\bar{i}k})/(n+2) \\ &- r_k(\delta_{ji}\delta_{hl} + \delta_{jl}\delta_{hi})/2(n+1)(n+2). \end{aligned}$$

Accordingly, by (2.12) and the assumption of (b) we have

$$\text{(9.5)} \qquad \begin{aligned} 0 = &- (\delta_{ki}S_{j\bar{l}h} + \delta_{kl}S_{j\bar{i}h})/(n+2) + r_h(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki})/2(n+1)(n+2) \\ &+ (\delta_{hi}S_{j\bar{l}k} + \delta_{hl}S_{j\bar{i}k})/(n+2) - r_k(\delta_{ji}\delta_{hl} + \delta_{jl}\delta_{hi})/2(n+1)(n+2). \end{aligned}$$

Putting $k = l$ in (9.5) and summing up with respect to k, we get

$$\text{(9.6)} \qquad 2(n+1)S_{j\bar{i}h} = r_h\delta_{ji} + r_j\delta_{hi}.$$

Then by virtue of (3.3) and (3.4), the assertion (c) holds.

Next we suppose the condition (c). By (2.12) and (9.4) we have

$$\begin{aligned} B_{\bar{i}jk\bar{l}h} - B_{\bar{i}jh\bar{l}k} = &- (\delta_{ki}S_{j\bar{l}h} + \delta_{kl}S_{j\bar{i}h})/(n+2) \\ &+ r_h(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki})/2(n+1)(n+2) \\ &+ (\delta_{hi}S_{j\bar{l}k} + \delta_{hl}S_{j\bar{i}k})/(n+2) \\ &- r_k(\delta_{ji}\delta_{hl} + \delta_{jl}\delta_{hi})/2(n+1)(n+2), \end{aligned}$$

which implies that

$$\begin{aligned} B_{\bar{i}jk\bar{l}h} - B_{\bar{i}jh\bar{l}k} = &- \frac{1}{n+2}\{\delta_{ki}(S_{j\bar{l}h} - \alpha\delta_{jl}r_h - \alpha\delta_{hl}r_j)\} \\ &- \frac{1}{n+2}\{\delta_{kl}(S_{j\bar{i}h} - \alpha\delta_{ji}r_h - \alpha\delta_{ih}r_j)\} \\ &+ \frac{1}{n+2}\{\delta_{hi}(S_{j\bar{l}k} - \alpha\delta_{jl}r_k - \alpha\delta_{kl}r_j)\} \\ &+ \frac{1}{n+2}\{\delta_{hl}(S_{j\bar{i}k} - \alpha\delta_{ji}r_k - \alpha\delta_{ki}r_j)\} \\ =&0, \end{aligned}$$

where α denotes $\frac{1}{2(n+1)}$. Under the situation of (c), the Bochner curvature tensor B satisfies the second Bianchi identity. Hence the assertion (b) holds. It completes the proof. □

Remark 9.3. By virtue of Theorem 9.1 we are able to assert that the Bochner curvature tensor B is *harmonic*. That is, the curvature-like 2-form $\Phi(B)$ becomes a harmonic curvature-like 2-form as follows:

In fact, by (b) and (c) in Theorem 9.1 the Laplacian of the curvature-like 2-form $\Phi(B)$ of the Bochner curvature tensor B is given by

$$
\begin{aligned}
\triangle\Phi(B) =&(d\delta+\delta d)\Phi(B) = \delta d\Phi(B)\\
=&\delta(\Phi(B)\wedge\omega-\omega\wedge\Phi(B))\\
=&\delta\Phi(B)\wedge\omega+\Phi(B)\wedge\delta\omega-\delta\omega\wedge\Phi(B)+\omega\wedge\delta\Phi(B)\\
=&0,
\end{aligned}
\tag{9.7}
$$

where in the second equality we have used the coclosedness $\delta(\Phi(B)) = 0$ in (c) for the Bochner curvature-like 2-form $\Phi(B)$ and in the 3rd equality the property of the second-Bianchi equation (b) in Theorem 9.1 is used, and the fact that $\delta\omega$ just denotes a function on the manifold is used in the last equality.

Now let us consider a special case of Theorem 9.1 when the scalar curvature r is constant on M. Let M be a Kähler manifold with constant scalar curvature r. Then the Ricci tensor S is parallel by (9.6). On the other hand, if $\nabla S = 0$, we have $B_{\bar{i}jk\bar{l}h} = R_{\bar{i}jk\bar{l}h}$ by (9.3) and hence $B_{\bar{i}jk\bar{l}h} = B_{\bar{i}jh\bar{l}k}$ by (2.12). Then, by Theorem 4.2, The curvature-like 2-form $\Phi = \Phi(B)$ satisfies the second Bianchi equation. Furthermore, we also obtain the curvature-like 2-form $\Phi(B)$ is coclosed by (9.2). Thus we can prove

Corollary 9.2. *Let M be an $n(\geq 2)$-dimensional Kähler manifold with constant scalar curvature r. Then we have the following equivalent assertions:*

(a) *The Bochner curvature tensor B is closed ; $d\Psi = 0$,*
(b) *The Bochner curvature-like 2-form $\Phi = \Phi(B)$ satisfies the second Bianchi equation $d\Phi = \Phi\wedge\omega-\omega\wedge\Phi$,*
(c) *The Ricci tensor S on M is parallel,*
(d) *$\nabla S \in C^\infty(\mathscr{I})$, that is, the Bochner curvature-like 2-form $\Phi(B)$ is coclosed.*

Remark 9.4. In Example 6.1, the second fundamental form is not parallel and not locally symmetric provided that $p \geq 3$ and furthermore if $|\lambda| = 1$, then M is Ricci flat. So the scalar curvature r is zero on the indefinite complex hypersurface, which means that the Bochner curvature tensor coincides with the Riemannian curvature tensor by (9.1). Accordingly, we see that the Bochner curvature tensor is not parallel, but *harmonic*.

Remark 9.5. In a paper due to [15], Matsumoto and Tanno proved that if M is a Kähler manifold with parallel Bochner curvature tensor, then M is locally symmetric or the Bochner curvature tensor vanishes identically.

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