CHARACTERIZATION AND ENUMERATION OF 3-REGULAR PERMUTATION GRAPHS

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ABSTRACT. A permutation graph is a graph that can be derived from a permutation, where the vertices correspond to letters of the permutation, and the edges represent pairs of letters whose order has been reversed by the permutation. We provide a construction for every 3-regular permutation graph, and we prove that there are infinitely many connected r-regular permutation graphs for $r \geq 3$ as well. Finally, we enumerate all 3-regular permutation graphs on n vertices.

1. INTRODUCTION

The graphs considered here are finite and simple. A graph on n vertices is a *permutation graph* if there is a labeling $v_1, v_2, \ldots, v_n$ of the vertices, and a permutation $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$, such that v_i and v_j are adjacent in G if and only if $i < j$ and $\pi(i) > \pi(j)$. In this case, the ordered pair $(\pi(i), \pi(j))$ is said to be an *inversion* of π. This definition of permutation graphs was given in 1971 by Pneuli et al. [13]. For example, the permutation $\pi = [3, 1, 4, 2, 5]$ has precisely three inversions $\{(3, 1), (3, 2), (4, 2)\}$, and in Figure 1, the permutation graph of π and the corresponding labeling is given. We note that this is different from the "generalized prisms" [17] notion of permutation graphs given by Chartrand and Harary [3].
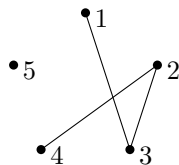

FIGURE 1. A permutation graph corresponding to $\pi = [3, 1, 4, 2, 5]$.

Permutation graphs have received a considerable amount of attention in the literature since their introduction (see, for example, [8, 14, 15]). Many algorithmic problems have efficient solutions on permutation graphs. For example, it was shown in [2] that the longest path problem (which is NP-complete on general graphs) can be solved in linear time on permutation graphs.

There has been interest in enumerating various types of permutation graphs. For instance, in [10], Koh and Ree gave a recurrence relation for the number of

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connected permutation graphs. In [1], the number of permutation trees on n vertices was shown to be $2n - 2$ for $n \geq 2$.

A graph is called *r-regular* if the degree of the each vertex of the graph is equal to r . It is easy to see that the only connected 2-regular permutation graphs are C_3 and C_4 [9], as it is well known that permutation graphs cannot have induced cycles of length five or greater. In this direction, we consider r-regular permutation graphs with $r > 2$. We show that the family is infinite for each $r > 2$ in Section 3.

Theorem 1.1. *For every $r \geq 3$, there are infinitely many connected r-regular permutation graphs.*

We give a complete characterization of 3-regular permutation graphs in Section 4. This will be given in terms of the construction described at the beginning of the section.

An interesting corollary of our construction is that, with the exception of $K_{3,3}$, all 3-regular permutation graphs are planar. The family of permutation graphs is closed under induced subgraphs (see, for example, [4]), but a description in terms of minors, as planarity results are normally stated, is not possible since permutation graphs are not closed under subgraphs.

Corollary 1.2. *Every 3-regular permutation graph except $K_{3,3}$ is planar.*

Finally, we use the characterization of 3-regular permutation graphs to enumerate them with a generating function.

Theorem 1.3. *Let $a(n)$ be the number of 3-regular permutation graphs on n vertices, and let $A(x)$ be the function*

$$\frac{1}{2}\Big(\frac{1}{1 - x^2 - x^3} + \frac{1 + x^2 + x^3}{1 - x^4 - x^6}\Big).$$

(i) If $n \in \{4, 6\}$, then $a(n) = 1$;
(ii) If $m = \frac{n-10}{2}$ is a positive integer, then $a(n)$ is given by the coefficient of x^m in the expansion of $A(x)$;
(iii) Otherwise $a(n) = 0$.

Proofs of Corollary 1.2 and Theorem 1.3 can be found in Sections 4 and 5, respectively.

2. PRELIMINARIES

If G is a permutation graph with corresponding permutation π, we say that π is a *realizer* of G. When discussing a realizer and its graph, we will sometimes refer to a vertex in the graph and an entry in the permutation with the same label. It is well known (for example, in [13]) that G is a permutation graph if and only if its complement $\overline{G}$ is also a permutation graph.

There are many known characterizations of permutation graphs. Recent characterizations include one by Gervacio et al. [7] in terms of cohesive vertex-set orders, and one by Limouzy [11] in terms of Seidel minors. Here we rely on the 1967 characterization by Gallai [6] in terms of forbidden induced subgraphs [12]; namely, since the class of permutation graphs is the intersection of the classes of comparability and cocomparability graphs, it can be completely characterized by a long list of forbidden induced subgraphs given in [5]. All cycle graphs on five or more vertices are forbidden induced subgraphs. We will refer to these as *large holes*. Table 1

illustrates all other forbidden induced subgraphs with maximum degree 3 or less. Although there are forbidden induced subgraphs with larger maximum degree, we will not need them here, as a graph can only be an induced subgraph of a 3-regular graph if its maximum degree does not exceed 3.

TABLE 1. The forbidden induced subgraphs for permutation graphs that have $\Delta \leq 3$ and that are not large holes.
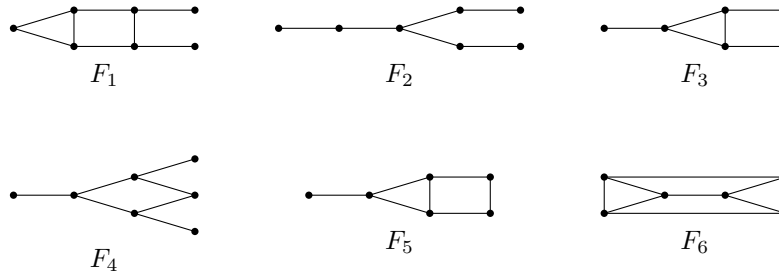
Throughout this paper, we use K_i and I_i to denote the complete graph on i vertices and the empty graph on i vertices, respectively. We also use $\oplus$ to denote graph disjoint union, and $\square$ to denote a Cartesian product of graphs. A cycle of length 3 is referred to as a *triangle*, and a cycle of length 4 is referred to as a *square*, regardless of whether or not the cycle is induced.

Let G be a graph of order n with vertices $v_1, v_2, \ldots, v_n$. When we have n graphs $H_1, H_2, \ldots, H_n$, we define the *composition of $H_1, H_2, \ldots, H_n$ into G*, denoted $G[H_1, H_2, \ldots, H_n]$, as the graph which is obtained from G by replacing the vertex v_i with the graph H_i. More precisely, the vertex set of the graph $G[H_1, H_2, \ldots, H_n]$ is the disjoint union of the vertex sets of every H_i, and uv is an edge of $G[H_1, H_2, \ldots, H_n]$ if and only if either $uv \in E(H_i)$ for some i, or there are distinct indices i and j such that $u \in V(H_i)$, $v \in V(H_j)$ and $v_i v_j \in E(G)$. If each graph H_i is a complete graph or empty graph then $G[H_1, H_2, \ldots, H_n]$ is called a *blow-up* of G, and we say that vertex v_i is *blown up* into H_i, or *replaced* with H_i. Notice that $G = G[H_1, H_2, \ldots, H_n]$, where every H_i is isomorphic to I_1. We call this the *trivial blow-up* of G. We will use *blow-up* to mean non-trivial blow-up for the rest of this paper, and we will use the notation G^* to represent a blow-up of a graph G.

3. INFINITELY MANY r-REGULAR PERMUTATION GRAPHS FOR $r \geq 3$

We will make use of the following lemma to prove that there are infinitely many connected r-regular permutation graphs for every $r \geq 3$.

Lemma 3.1. *[7, Theorem 3.3] Let G be a graph of order n, and let $H_1, H_2, \ldots, H_n$ be arbitrary graphs. Then $G^* = G[H_1, H_2, \ldots, H_n]$ is a permutation graph if and only if G and each of $H_1, H_2, \ldots, H_n$ are permutation graphs.*

Proof of Theorem 1.1. Let $r \geq 3$. For every $n \geq 0$, we construct a connected r-regular permutation graph G_n of order $2nr + r + 1$ by taking a blow-up of a path.

Let $m = 4n + 2$, and take a path graph P_m with vertices $v_1, v_2, \ldots, v_m$ in standard order. Note that P_m is a permutation graph because it is not a cycle, its maximum degree is 2, and it does not have an induced subgraph from Table 1. Replace the first vertex v_1 with K_2 and the last vertex v_m with K_{r-1}. For other vertices v_i with $i \equiv 2 \pmod 4$, replace them with I_{r-1}; with $i \equiv 3 \pmod 4$, replace v_i with I_{r-2}; with $i \equiv 0 \pmod 4$, replace v_i with I_1; and for $i \equiv 1 \pmod 4$, replace v_i with I_2. The resulting graph G_n is r-regular and connected since it is the blow-up of a path. It is also a permutation graph by Lemma 3.1, since complete graphs and empty graphs are permutation graphs. Hence, we obtain an infinite list of connected r-regular permutation graphs

$$G_0 = P_2[K_2, K_{r-1}]$$
$$G_1 = P_6[K_2, I_{r-1}, I_{r-2}, I_1, I_2, K_{r-1}]$$
$$G_2 = P_{10}[K_2, I_{r-1}, I_{r-2}, I_1, I_2, I_{r-1}, I_{r-2}, I_1, I_2, K_{r-1}]$$
$$G_3 = P_{14}[K_2, I_{r-1}, I_{r-2}, I_1, I_2, I_{r-1}, I_{r-2}, I_1, I_2, I_{r-1}, I_{r-2}, I_1, I_2, K_{r-1}]$$
$$\vdots$$

and the result follows.   □

The following corollary of Lemma 3.1 is immediate:

Corollary 3.2. *If G^* and G are permutation graphs, and G^* is a blow-up of the form $G[H_1, H_2, \ldots, H_n]$, then any realizer of G can be used to give a realizer of G^*.*

While we omit the proof, we give an illustrative example. Suppose that $G = P_2$, and let $H_1 = K_3$ and $H_2 = I_3$. Then G is realized by $\pi = [2, 1]$. In G^*, all of the vertices corresponding to H_1 must be adjacent to all of the vertices corresponding to H_2, and therefore the three entries of the realizer π^* of G^* corresponding to H_1 must be greater than the three entries corresponding to H_2. Thus we have $\pi^* = [6, 5, 4, 1, 2, 3]$, noting that the vertices corresponding to $\{6, 5, 4\}$ induce K_3 and that the vertices corresponding to $\{1, 2, 3\}$ induce I_3.

4. CHARACTERIZATION OF 3-REGULAR PERMUTATION GRAPHS

We now show that all 3-regular permutation graphs are isomorphic to K_4, $K_{3,3}$, or a what we will call a *boxcar graph*. We will first define boxcar graphs and then show that any 3-regular graph that is a blow-up of a path is isomporphic to one of these three types of graphs (Lemma 4.1). We will then show that every connected 3-regular graph is a blow-up of a path (Theorem 4.5). These two results together characterize all 3-regular permutation graphs (see Corollary 4.6).

Table 2 shows the induced subgraphs used to construct boxcar graphs. A *boxcar graph* is a graph that can be constructed by the following process.

(A) Let $S_1 = G_1$ from Table 2. Then go to (B).
(B) Choose S_2 to be one of $\{G_2, G_3, G_4\}$ from Table 2. Then go to (C).
(C) Let G be the graph obtained by identifying the rightmost vertex of S_1 with the leftmost vertex of S_2. If $S_2 = G_4$, go to (D); otherwise, set $S_1 = G$ and go to (B).
(D) Stop. G is a boxcar graph.

TABLE 2. The induced subgraphs used in the construction of box-car graphs.
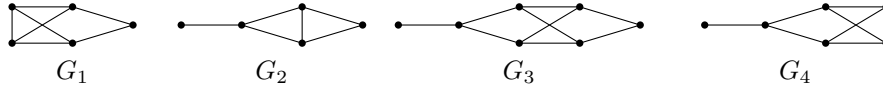
Lemma 4.1. *A 3-regular graph that is a blow-up of a path is isomorphic to K_4, $K_{3,3}$, or a boxcar graph.*

Proof. Let G be a path graph P_n with vertices $(v_1, v_2, \ldots, v_n)$ in standard order, and consider a blow-up $G^* = G[H_1, H_2, \ldots, H_n]$. There are four possibilities for the graph H_1.

Suppose the first vertex v_1 is blown up into K_k or I_k, with $k \geq 4$. If v_2 exists, then the vertices resulting from blowing up v_2 will have degree at least 4. Thus to obtain a 3-regular graph, v_1 must be the only vertex of G, and it must be blown up into K_4.

Now suppose $H_1 \cong K_3$. Then v_2 must be blown up into a graph of order 1 because the vertices from K_3 require one more neighbor to have degree 3. Since all the vertices have degree 3, we see that G must be P_2, and it blows up into $P_2[K_3, K_1] \cong K_4$.

Suppose $H_1 \cong I_k$, where $k \leq 3$. Since the vertices of H_1 require three neighbors, v_2 must be blown up into a graph of order 3. If $H_2 \cong K_3$, then $k = 1$ and we have K_4 as in the case above. If $H_2 \cong I_3$, then the vertices of H_2 have k neighbors on left and they requre $3 - k$ neighbors on the right. In order to not exceed degree 3, we must have $H_3 \cong I_{k-3}$. Thus we obtain $P_2[I_3, I_3]$, $P_3[I_1, I_3, I_2]$, or $P_3[I_2, I_3, I_1]$, all of which are isomorphic to $K_{3,3}$.

The only remaining cases are when H_1 is isomorphic to K_2. If $H_1 \cong K_2$, then H_2 must have order 2. If $H_2 \cong K_2$, then we have $G^* = P_2[K_2, K_2] \cong K_4$. If $H_2 \cong I_2$, then H_3 must have order 1, so $H_3 \cong K_1$ and we see that G^* must begin with G_1 from Table 2. Then $H_4 \cong K_1$. We will use the following sublemma in the remaining cases for the construction of G^*.

4.1.1. *Let G_k^* be the graph G^* restricted to the vertices obtained from blowing up $(v_1, v_2, \ldots, v_k)$ for $k \leq n$. If G_k^* is 3-regular except for a leaf, then the vertices $(v_{k-1}, v_k, v_{k+1}, v_{k+2})$ or $(v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3})$ of G must blow up to induce one of $\{G_2, G_3, G_4\}$ in G^*.*

Assume v_k blows up to be a leaf in G^*. Then $H_{k-1} \cong H_k \cong K_1$, and H_{k+1} must have order 2. If $H_{k+1} \cong K_2$, then H_{k+2} must have order 1, and we have G_2 as an induced subgraph on $\cup_{i=k-1}^{k+2} V(H_i)$. If instead $H_{k+1} \cong I_2$, then we must have either $H_{k+2} \cong I_2$ and $H_{k+3} \cong K_1$, giving us G_3, or $H_{k+2} \cong K_2$, giving us G_4. Thus 4.1.1 holds.

By repeated application of 4.1.1, we see that G^* consists of a series composition of G_1 followed by a sequence of graphs isomorphic to G_2 or G_3 and terminating in G_4. $\qquad\square$

The following lemmas will be useful in our characterization of 3-regular permutation graphs. We say that vertices v_1 and v_2 are *twins* if $N(v_1) - \{v_2\} = N(v_2) - \{v_1\}$, where $N(v_i)$ is the set of vertices that neighbor v_i.

A subsequence $\pi_{i_1}, \ldots, \pi_{i_k}$ of a permutation π is called *consecutive* if

$$\pi_{i_{j+1}} = \pi_{i_j} + 1 \text{ or } \pi_{i_{j+1}} = \pi_{i_j} - 1$$

and it is called *contiguous* if

$$i_{j+1} = i_j + 1$$

for all $j = 1, \ldots, k-1$.

The following lemma appears to be well known in the field of modular decompositions. We include its proof for completeness.

Lemma 4.2. *Every permutation graph G has a realizer π where, for every pair of twins u and v in G, there is a contiguous, consecutive increasing or decreasing subsequence s of π that contains u and v. Moreover, u and v are adjacent in G if and only if s is decreasing.*

Proof. Let π be a realizer of a permutation graph G, and define G_π to be a graph isomorphic to G with vertex labels corresponding to π. Let u and v be twins in G_π with $u < v$. We will first assume u and v are nonadjacent. If u and v are not part of a contiguous, consecutive increasing subsequence of π, then we can obtain another realizer π' of G by removing v from π, shifting all of the entries greater than u and less than v up by 1, and inserting $u+1$ to the immediate right of u. Clearly π and π' realize isomorphic graphs, and if a and b are entries of π that belong to a contiguous, consecutive increasing or decreasing subsequence of π, then this transformation does not separate them.

If we assume instead that u and v are adjacent in G_π, then we apply a similar transformation, ultimately placing $u+1$ to the left of u instead of the right. This results in u and $u+1$ being part of a contiguous, consecutive decreasing subsequence, instead of increasing. $\square$

Lemma 4.3. *If G^* is a permutation graph with maximum degree d, and if G is a permutation graph of minimum order such that G^* is a blow-up of G, then G has no degree d twins.*

Proof. Let G^* be a permutation graph with maximum degree d, and let G be a permutation graph of minimum order such that G^* is a blow-up of G. Recall that from Corollary 3.2, any realizer of G^* can be used to obtain a realizer of G.

By contradiction, assume u and v are degree d twins of G. By Lemma 4.2, G has a realizer π where u and v are adjacent and consecutive. Let $\{u_1, u_2, \ldots, u_j\}$ and $\{v_1, v_2, \ldots, v_k\}$ be the entries of a realizer π^* for G^* obtained by blowing up u and v, respectively. Then u must be blown up into I_j and v must be blown up into I_k, because if they were blown up into K_j or K_k for $k \geq 2$, then we would have vertices with degree exceeding d. Moreover, unless $j = k = 1$, the vertices u and v must be nonadjacent. In the case that $j = k = 1$, u_1 and v_1 are twins in G^*, and they are contiguous and consecutive in π^*, which means that there is a graph such that $\{u, v\}$ is blown up from a single vertex, which contradicts the assumption that G has minimum order. If j and k are not both 1, then $u_1, u_2, \ldots, u_j, v_1, v_2, \ldots, v_k$ are all twins in G^*. Therefore, they are part of a contiguous, consecutive increasing sequence of π^*, so they can also be blown up from a single vertex which, leads to a contradiction in this case as well. $\square$

Recall that a *ladder* is a graph $P_2 \square P_n$, with $2n$ vertices $u_1, \ldots, u_n, v_1, \ldots, v_n$ such that each of $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ induces a P_n, and u_i is adjacent to v_i for each $i = 1, \ldots, n$. Each edge $u_i v_i$ is called a *rung* of the ladder.

Lemma 4.4. *A 3-regular permutation graph G cannot have a ladder with four or more rungs as a subgraph.*

Proof. Suppose G has a ladder as a subgraph, and let u_i and v_i be adjacent vertices on the ith rung of a maximal ladder for i in $\{1, 2, \ldots, k\}$. We will prove the lemma by considering three propositions.

(1) In a ladder with three or more rungs, v_1 and v_k cannot be adjacent in G; by symmetry, nor can u_1 or u_k.

(2) In a ladder with four or more rungs, v_1 and u_k cannot be adjacent in G; by symmetry, nor can u_1 and v_k.

(3) There cannot be a ladder with three or more rungs without an edge between the first and last rung of the ladder.

To prove proposition (1), suppose that $k = 3$. By contradiction, assume v_1 and v_3 are adjacent, and suppose first that u_1 and u_3 are not. Then we have C_5 using vertices $\{v_1, u_1, u_2, u_3, v_3\}$. However, if u_1 and u_3 are also adjacent, then we have F_6. Next suppose $k = 4$. If (v_1, v_4) is an edge and (u_1, u_4) is not, then $\{v_1, u_1, u_3, u_4, v_4\}$ is C_5. If (u_1, u_4) is also an edge, then the graph is isomorphic to a cube $(C_4 \square K_2)$, which has C_6 as an induced subgraph by deleting a pair of opposite vertices. Finally, suppose $k \geq 5$. Then $\{v_1, v_2, \ldots, v_k\}$ is a large hole, i.e. a cycle graph with five or more vertices. In each case, we have demonstrated a forbidden induced subgraph.

Similarly, for proposition (2), if v_1 and u_k are adjacent, we have a large hole using $\{v_1, v_2, u_2, u_3, \ldots, u_k\}$.

Finally, for proposition (3), suppose v_1 and u_1 have a common neighbor v. Then v cannot have v_k or u_k as neighbors, or else we have a large hole. So v has another neighbor v', but this gives us F_5 using $\{v', v, v_1, v_2, u_1, u_2\}$. Suppose instead that the third neighbors of v_1 and u_1 are v and u, respectively, with $v \neq u$. Then we have F_4 using $\{v, v_1, v_2, u, u_1, u_2, u_3\}$. $\square$

We now prove that the graphs from Lemma 4.1 are the only 3-regular permutation graphs.

Theorem 4.5. *Every connected 3-regular permutation graph is the blow-up of a path.*

Proof. Suppose G^* is a 3-regular permutation graph that is not a blow-up of a path. Let G be a permutation graph of minimum order such that G^* is a blow-up of G. Then G is either a cycle or G has a degree 3 vertex.

If G is a cycle, then G must be C_3 or C_4, because larger cycles are forbidden as induced subgraphs. In C_3, since all the vertices have degree 2 and they are all adjacent to each other, only one vertex can be blown up, or else we would have a vertex with degree exceeding 3. Moreover, the vertex must be blown up into K_2 in order for every vertex to have degree 3. The resulting graph is K_4. This contradicts the assumption that C_3 is a permutation graph G of minimum order such that G^* is a blow-up of G, as it is shown in Lemma 4.1 that K_4 is a blow-up of P_2. In C_4, since every vertex has degree 2, at most one of the neighbors of every vertex can be blown up. If a vertex v is blown up into K_2, then everything in the resulting graph

will have degree 3 expect for the vertex that was opposite of v, and it is impossible to use further blow-ups to obtain a 3-regular graph. Thus, the only possibility that gives a 3-regular graph is blowing up each of two adjacent vertices into I_2. This gives a graph isomorphic to $K_{3,3}$; a contradiction, as Lemma 4.1 shows that $K_{3,3}$ is also a blow-up of P_2, and therefore G does not have minimum order as assumed.

Suppose instead that G has a vertex v of degree 3. Note that G has maximum degree 3. We proceed by analyzing the possible induced subgraphs H containing v and its neighborhood. In some cases, H has degree-3 twins, which by Lemma 4.3 contradicts the assumption that G is a minimal-order graph that blows up into G^*. In some of the remaining cases, H can be blown up (perhaps trivially) into an induced subgraph of a boxcar graph. If H cannot be blown up this way, then we give one of two reasons why. Either H is not an induced subgraph of a 3-regular permutation graph and cannot be blown up without creating a vertex of degree 4 or greater, or H has a large hole or forbidden induced subgraph from Table 1.

Let the neighbors of v, $N(v)$, be $\{v_1, v_2, v_3\}$. We will consider the following cases based upon the possible subgraphs induced by $N(v)$:

(1) P_3,
(2) K_3,
(3) I_3, and
(4) $K_2 \oplus I_1$.

In cases (1) and (2), there are twin vertices of degree 3, contradicting Lemma 4.3.

In case (3), $N(v)$ induces I_3, that is, none of the vertices in $N(v)$ are adjacent. Observe that if v is adjacent to a leaf in G, then v must be blown up into I_3 in order to obtain a 3-regular graph. This implies that all the neighbors of v in G must be leaves or else we would have a vertex of degree exceeding 3 upon blowing up v to I_3. The resulting graph of this blow-up is $K_{3,3}$. Thus we may assume that all vertices adjacent to a degree 3 vertex have degree at least 2.

We will proceed by considering the number of squares that contain v as a vertex. If v is not involved in any squares, then the subgraph induced by $N(v)$ and its neighbors has either F_2 or a large hole as an induced subgraph. Suppose instead that $\{v, v_2, v_3\}$ are used in a square. Let v_4 be the remaining vertex of the square. In the case that $v_4 = v_1$, then we have an induced subgraph of G_2 from Table 2; in particular, it is the blow-up of a path. In the remaining case where v_4 is a distinct vertex, if each one of $\{v_2, v_3, v_4\}$ has degree 2, then v_4 can be blown up into K_2 to realize G_4 from Table 2. Similarly, if v_2 and v_3 have degree 2 and v_4 has degree 3, then v_4 can be blown up into I_2 to realize G_3. If, however, only one of $\{v_2, v_3\}$ has degree 3, or they both have degree 3 and v_4 has degree 2, then we cannot perform any more blow-ups without creating a vertex with degree exceeding 3. If all of $\{v_2, v_3, v_4\}$ have degree 3, then depending the configuration of the remaining edges, we either have F_4, F_5, or a large hole as an induced subgraph.

Suppose v is used in two squares. One possibility is for two neighbors of v to be involved in both squares; say $\{v, v_2, v_3\}$ are involved in two distinct squares. This implies that v_2 and v_3 are degree 3 twins. Another possibility is for the two squares to share a single edge, creating a ladder subgraph with at least three rungs. Observe that if the largest ladder subgraph using v has three rungs, and there is an edge between two opposite vertices in a cycle around the ladder, then v is involved in at least three squares. The remaining possibilities for a ladder on three or more rungs using v contradict cases (1), (2), and (3) from Lemma 4.4.

The final possibilities when $N(v)$ induces I_3 are for v to be involved in three or more squares. If v is in exactly 3 squares, either there is an induced 6-cycle around v, or the vertices at distance 2 or less from v induce G_3 as a subgraph. If v is used in more than three squares, then $G \cong K_{3,3}$.

In case (4), $N(v)$ induces $K_2 \oplus I_1$. Suppose that $\{v, v_2, v_3\}$ forms a triangle. If one of $\{v_2, v_3\}$ has degree 2, then the graph cannot be blown up to be 3-regular. If they both have degree 3 and are not in a square with v_1, then either they have a common neighbor other than v, giving us G_2 from Table 2, or they have different neighbors, giving us F_3. Suppose that $\{v, v_2, v_3\}$ is a triangle and $\{v, v_1, v_2, v_4\}$ is a square for some new vertex v_4. If v_3 has degree 2, then G cannot be blown up into a 3-regular graph. If v_3 is adjacent to v_4, then this is isomorphic to G_1. If v_3 is adjacent to a new vertex v_5, then we have F_5 as an induced subgraph.

Finally, let $\{v, v_2, v_3\}$ be a triangle, and suppose there are squares $\{v, v_1, v_2, v_4\}$ and $\{v, v_1, v_3, v_5\}$. If $v_4 = v_5$, then v_4 and v are twins; a contradiction. Suppose $v_4 \neq v_5$. If v_4 and v_5 are nonadjacent, then $\{v_1, v_2, v_3, v_4, v_5\}$ is a large hole, and if they are adjacent, then our induced subgraph is isomorphic to F_6. □

This theorem and Lemma 4.1 immediately imply the following corollary.

Corollary 4.6. *Every 3-regular permutation graph is isomorphic to K_4, $K_{3,3}$, or a boxcar graph.*

Note that this also implies Corollary 1.2, since every boxcar graph has a planar embedding.

Corollary 4.7. *Every 3-regular permutation graph has a Hamiltonian path.*

Proof. Clearly K_4 and $K_{3,3}$ are Hamiltonian. Observe that every graph in the set $\{G_1, G_2, G_3, G_4\}$ from Table 2 also has a Hamiltonian path. When we identify vertices in order to obtain a boxcar graph, we find that Hamiltonian paths of each of the graphs $\{G_1, G_2, G_3, G_4\}$ connected in sequence give a Hamiltonian path for the boxcar graph. □

5. ENUMERATION OF CONNECTED 3-REGULAR PERMUTATION GRAPHS

Using the characterization given in Corollary 4.6 and a generating function to enumerate 3-regular permutation graphs, we prove Theorem 1.3. In what follows, we will use *sequences for m* (where m is a positive integer) to mean equivalence classes of compositions of m into parts of size 2 and 3, where a composition and its reverse are considered to be the same. For example, the integer 11 has five equivalence classes and their representatives are $\gamma_1 = (2, 3, 3, 3)$, $\gamma_2 = (3, 2, 3, 3)$, $\gamma_3 = (3, 2, 2, 2, 2)$, $\gamma_4 = (2, 3, 2, 2, 2)$, $\gamma_5 = (2, 2, 3, 2, 2)$. The sequence $\gamma_1' = (3, 3, 3, 2)$ belongs to the equivalence class of γ_1, as γ_1 and γ_1' are reverses of each other.

Before we present the generating function, we first show that sequences for m are in $1 - 1$ correspondence with boxcar graphs.

Proposition 5.1. *Let $m = \frac{n-10}{2}$. For even n, the number of boxcar graphs on n vertices is equal to the number of sequences for m.*

Proof. We can think of a boxcar graph as starting with G_1, followed by a sequence of G_2 and G_3 subgraphs, and ending with G_4 (see Table 2). Notice that G_1 and G_4 contribute five vertices each to the boxcar graph, while each copy of G_2 and G_3

contribute four and six, respectively, after identification of vertices. Thus the iso-morphism class of the graph is determined by the sequence of G_2 and G_3 subgraphs in the middle. To count boxcar graphs, we wish to find the integer compositions of $n-10$ into parts of size 4 and 6, which is equivalent to the integer compositions of $\frac{n-10}{2}$ into parts of size 2 and 3. Moreover, since the start and end of the sequence of subgraphs in a boxcar graph are not distinguished, we count a composition and its reverse as being the same. $\square$

Techniques used for some omitted computations below may be found in [16].

Proposition 5.2. *The generating function for sequences for m is*

$$A(x) = \frac{1}{2}\Big(\frac{1}{1-x^2-x^3} + \frac{1+x^2+x^3}{1-x^4-x^6}\Big).$$

Proof. Since we want to count a composition and its reverse as the same, we may count all compositions that are symmetric, and half of all the non-symmetric ones. This is equivalent to counting half of all compositions, and adding again half of all the symmetric ones.

Let t_n denote the number of compositions of n into parts of size 2 and 3. Note that these compositions of n are in bijection with the compositions of $n-2$ and $n-3$ into parts of size 2 and 3, because we can obtain such a composition of $n-2$ or $n-3$ by taking such a composition of n and removing the first part, and this operation has a well-defined inverse. Thus $t_n = t_{n-2} + t_{n-3}$. Deriving a generating function $T(x)$ from this recursion relation with $t_0 = t_2 = 1$ and $t_1 = 0$ gives $T(x) = \frac{1}{1-x^2-x^3}$.

Observe that a symmetric composition may have an even number of parts, or it may have an odd number of parts with a 2 or 3 in the middle. Moreover, each part not in the middle must have an identical part on the opposite end of the composition. Therefore the symmetric compositions of n into parts of size 2 and 3 are in bijection with the compositions of $n - k$ into parts of size 4 and 6 for all k in $\{0, 2, 3\}$.

If we let u_n be the number of compositions of n into parts of size 4 and 6, we obtain the recurrence relation $u_n = u_{n-4} + u_{n-6}$. Taking $u_0 = u_4 = 1$ and $u_1 = u_2 = u_3 = u_5 = 0$, we can derive the generating function $U(x) = \frac{1}{1-x^4-x^6}$. Then the generating function for the number of symmetric compositions is $V(x) = U(x) + x^2 U(x) + x^3 U(x) = \frac{1+x^2+x^3}{1-x^4-x^6}$. Thus we have $A(x) = \frac{1}{2}(T(x) + V(x))$. $\square$

The above two propositions are used to complete the proof of Theorem 1.3.

Theorem 1.3. Let $a(n)$ be the number of 3-regular permutation graphs on n vertices, and let $A(x)$ be the function

$$\frac{1}{2}\Big(\frac{1}{1-x^2-x^3} + \frac{1+x^2+x^3}{1-x^4-x^6}\Big).$$

(i) If $n \in \{4, 6\}$, then $a(n) = 1$;
(ii) If $m = \frac{n-10}{2}$ is a positive integer, then $a(n)$ is given by the coefficient of x^m in the expansion of $A(x)$;
(iii) Otherwise $a(n) = 0$.

Proof of Theorem 1.3. Corollary 4.6 tells us that the only 3-regular permutation graphs that are not boxcar graphs are K_4 and $K_{3,3}$, which have 4 and 6 vertices, respectively. By Proposition 5.1, the problem of counting boxcar graphs can be

reduced to the problem of counting sequences for m, which is done in Proposition 5.2. □

6. Conclusion

We have proven that there are infinitely many r-regular permutation graphs for $r \geq 3$ and given a complete characterization of 3-regular permutation graphs in terms of blow-ups of paths. While it is perhaps surprising that all 3-regular permutation graphs are blow-ups of paths, this is not the case for all r-regular graphs in general. In particular, Figure 2 is a counterexample with $r = 4$.
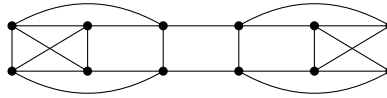

FIGURE 2. A 4-regular permutation graph that is not a blow-up of a path ($\pi = [5, 4, 7, 2, 1, 10, 3, 12, 11, 6, 9, 8]$).

The graph from Figure 2 can be constructed by blowing up a 4-runged ladder. More specifically, if G is the 4-runged ladder whose vertices are labeled as they appear in a Hamiltonian path starting and ending on a degree 2 vertex, then the graph from Figure 2 is $G[K_2, K_1, K_1, K_2, K_2, K_1, K_1, K_2]$. Note that G is a permutation graph with realizer $[3, 5, 1, 7, 2, 8, 4, 6]$. This observation, along with the lemma below, indicates that the permutation graph from Figure 2 is not the blow-up of a path.

Lemma 6.1. *For each graph G, there is unique graph G' of minimal order such that G is a blow-up of G'.*

Proof. Let $P = (p_1, p_2, \ldots, p_m)$ be the partition of $V(G)$ such that two vertices are in the same part if and only if they are twins. We construct an m-vertex graph G', where distinct vertices v_i, v_j of $V(G')$ are adjacent if and only if the members of p_i and p_j are adjacent in G. Then G is a blow-up of G', obtained by replacing each vertex v_i with the vertices of p_i. We know that G' is minimal because if H is a graph such that G is a blow-up of H, and u_1 and u_2 are vertices of G that arise from the same vertex of H, then u_1 and u_2 must be twins. Moreover, G' is unique because P is unique. □

By taking complements of the graphs listed in Corollary 4.6 and then applying Lemma 6.1, we find other counterexamples for r-regularity for certain even values of r. Counterexamples to show that not every r-regular permutation graph is a blow-up of a path for odd values of $r > 4$ are not known.

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