Arithmetic Relations for Overpartitions Modulo 27

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Abstract. Let $\bar{p}(n)$ denote the number of overpartitions of n. In this paper, we establish the generating function of $\bar{p}(96n+12)$ modulo 27 by using elementary dissection techniques. As a consequence, we obtain infinite families of congruences $\bar{p}(96\ell n+96i+12) \equiv 0 \pmod{27}$, where $n \geq 0$, ℓ is an arbitrary odd prime and $0 \leq i < \ell$ is a nonnegative integer such that $\left(\frac{8i+1}{\ell}\right) = -1$. In this way, we find various congruences such as $\bar{p}(288n+204) \equiv 0 \pmod{27}$, $\bar{p}(480n+204, 396) \equiv 0 \pmod{27}$ and $\bar{p}(672n+204, 396, 492) \equiv 0 \pmod{27}$ for $n \geq 0$.

Keywords: Overpartition, Generating Function, Congruence, Dissection

MSC(2020): 05A17, 11P83

1 Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. An overpartition of a positive integer n is a partition of n in which the first occurrence of each distinct part may be overlined. Let $\bar{p}(n)$ denote the number of overpartitions of n. By convention, we set $\bar{p}(0) = 1$ and $\bar{p}(n) = 0$ if $n < 0$.

As noted by Corteel and Lovejoy [5], the generating function of $\bar{p}(n)$ is given by

$$\sum_{n\geq 0} \bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q| < 1$$

is the standard notation in q-series. Throughout this paper, f_k is defined by

$$f_k := (q^k;q^k)_\infty.$$

Recall that Ramanujan's general theta function $f(a,b)$ is defined by:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1. \tag{1.1}$$

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And three Ramanujan's theta functions are given by:

$$\varphi(q) := f(q,q) = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.2}$$

$$\varphi(-q) := f(-q,-q) = \frac{f_1^2}{f_2}, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \frac{f_2^2}{f_1}. \tag{1.4}$$

The arithmetic properties and the divisibility of $\bar{p}(n)$ have been widely investigated in recent years. For powers of 2, Fortin et al. [7] and Hirschhorn and Sellers [8] independently found 2-, 3- and 4-dissections of the generating function for $\bar{p}(n)$ and obtained a number of congruences for $\bar{p}(n)$ modulo 4, 8 and 64. Chen et al. [3] derived several infinite families of congruences for $\bar{p}(n)$ modulo 16 and four congruence relations for $\bar{p}(n)$ modulo $4, 16, 32$, and 64. Mahlburg [10] showed that $\bar{p}(n) \equiv 0 \pmod{64}$ holds for a set of integers of arithmetic density 1. Kim [9] showed that $\bar{p}(n) \equiv 0 \pmod{128}$ holds for a set of integers of arithmetic density 1. For powers of 3, Lovejoy and Osburn [11] obtained some infinite families of congruences for $\bar{p}(n)$ modulo 3. Xia and Yao [13] proved some congruences for $\bar{p}(n)$ modulo 3, 9, and 27 by applying elementary generating function dissection techniques. For powers of 5, Chen et al. [2] and Treneer [12] obtained an infinite family of congruences for $\bar{p}(n)$ modulo 5 by utilizing half-integral weight modular forms. Chern and Dastidar [4] proved $\bar{p}(80n + 8, 52, 68, 72) \equiv 0 \pmod{25}$.

In this paper, we are mainly concerned with arithmetic relations for $\bar{p}(n)$ modulo 27. We first establish the following generating function of $\bar{p}(96n + 12)$ modulo 27 by using elementary dissection techniques due to Xia and Yao [13].

Theorem 1.1 *We have*

$$\sum_{n\geq 0} \bar{p}(96n+12)q^n \equiv -9\psi(q) \pmod{27}.$$

Combining the above generating function with the ℓ-dissection formula of $\psi(q)$ given by Cui and Gu [6], we obtain infinite families of congruences modulo 27.

Corollary 1.2 *Let $\left(\frac{\cdot}{\ell}\right)$ denote the Legendre symbol. Assume that n is a nonnegative integer and ℓ is an arbitrary odd prime. Then we have*

$$\bar{p}(96\ell n + 96i + 12) \equiv 0 \pmod{27},$$

where $0 \leq i < \ell$ is a nonnegative integer such that $\left(\frac{8i+1}{\ell}\right) = -1$.

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Setting $\ell = 3, 5, 7$ and 11 respctively in Corollary 1.2, we find the following specific congruences:

$$\bar{p}(288n + 204) \equiv 0 \pmod{27},$$
$$\bar{p}(480n + 204, 396) \equiv 0 \pmod{27},$$
$$\bar{p}(672n + 204, 396, 492) \equiv 0 \pmod{27},$$
$$\bar{p}(1056n + 204, 492, 684, 780, 876) \equiv 0 \pmod{27},$$

where n is a nonnegative integer.

2 Some Lemmas

In order to establish the generating function of $\bar{p}(96n + 12)$ modulo 27, we need to give some lemmas. For more details, see [1, 13].

Lemma 2.1 [13, Lemma 2.2, Lemma 2.3, Lemma 2.6]

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.1}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.3}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}. \tag{2.4}$$

Lemma 2.2 [13, Lemma 2.4, Lemma 2.5]

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{2.5}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.6}$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \tag{2.7}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.8}$$

Lemma 2.3 *If* $a \equiv b \pmod{3}$, *then* $9a \equiv 9b \pmod{27}$.

Lemmma 2.3 is obvious by using elementary number theory.

Lemma 2.4 For positive integers r and s, we have

$$f_{3r}^{s} \equiv f_r^{3s} \pmod{3}, \tag{2.9}$$

$$f_{3r}^{3s} \equiv f_r^{9s} \pmod{9}, \tag{2.10}$$

$$f_{3r}^{9s} \equiv f_r^{27s} \pmod{27}. \tag{2.11}$$

Lemmma 2.4 is easy to be checked by the binomial theorem.

Lemma 2.5 [1, p.356, Entry 3, p.357, eq.(4.3)]

$$\psi(q) - q\psi(q^9) = f(q^3, q^6) = \frac{f_6 f_9^2}{f_3 f_{18}}, \tag{2.12}$$

$$9\frac{\varphi(-q^3)^3}{\varphi(-q)} - \frac{\varphi(-q)^3}{\varphi(-q^3)} = 8\frac{\psi(q)^3}{\psi(q^3)}. \tag{2.13}$$

Lemma 2.6 [14, eq.(2.6)]

$$\varphi(q)^4 + q\psi(q^2)^4 \equiv 1 \pmod{3}. \tag{2.14}$$

Lemma 2.7 *We have*

$$\frac{\psi(q^3)}{\psi(q)^3} - \frac{\varphi(-q^3)}{\varphi(-q)^3} + 9q\psi(q)^8 \equiv 0 \pmod{27}. \tag{2.15}$$

Proof. Rewriting the left-hand side of (2.15), we obtain that

$$\frac{\psi(q^3)}{\psi(q)^3} - \frac{\varphi(-q^3)}{\varphi(-q)^3} + 9q\psi(q)^8 = \frac{\frac{\varphi(-q)^3}{\varphi(-q^3)} - \frac{\psi(q)^3}{\psi(q^3)}}{\frac{\psi(q)^3}{\psi(q^3)} \cdot \frac{\varphi(-q)^3}{\varphi(-q^3)}} + 9q\psi(q)^8. \tag{2.16}$$

Using (2.13) and Lemma 2.3, we deduce that

$$\begin{aligned} \frac{\varphi(-q)^3}{\varphi(-q^3)} - \frac{\psi(q)^3}{\psi(q^3)} &= 9\frac{\varphi(-q^3)^3}{\varphi(-q)} - 9\frac{\psi(q)^3}{\psi(q^3)} \\ &\equiv 9\left\{\varphi(-q)^8 - 1\right\} \pmod{27}. \end{aligned} \tag{2.17}$$

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Notice that

$$\frac{\psi(q)^3}{\psi(q^3)}\cdot\frac{\varphi(-q)^3}{\varphi(-q^3)}\equiv 1 \pmod 3. \tag{2.18}$$

Applying (2.17), (2.18) and Lemma 2.3, (2.16) becomes

$$\frac{\psi(q^3)}{\psi(q)^3}-\frac{\varphi(-q^3)}{\varphi(-q)^3}+9q\psi(q)^8\equiv 9\left\{\varphi(-q)^8-1+\frac{q\psi(q^9)}{\psi(q)}\right\} \pmod{27}. \tag{2.19}$$

By (2.12) and (1.4), we have

$$\frac{q\psi(q^9)}{\psi(q)}=1-\frac{f_1}{f_2^2}\frac{f_6f_9^2}{f_3f_{18}}. \tag{2.20}$$

Plugging (2.20) into the right-hand side of (2.19) and by (1.3), we yield

$$\frac{\psi(q^3)}{\psi(q)^3}-\frac{\varphi(-q^3)}{\varphi(-q)^3}+9q\psi(q)^8\equiv 9\left\{\frac{f_1^{16}}{f_2^8}-\frac{f_1}{f_2^2}\frac{f_6f_9^2}{f_3f_{18}}\right\} \pmod{27}. \tag{2.21}$$

Using (2.9), it is easy to check that

$$\frac{f_1^{16}}{f_2^8}-\frac{f_1}{f_2^2}\frac{f_6f_9^2}{f_3f_{18}}\equiv 0 \pmod 3. \tag{2.22}$$

Combining (2.21), (2.22) and Lemma 2.3, we complete the proof of Lemma 2.7. ∎

3 Proofs of Theorem 1.1 and Corollary 1.2

In this section, we give a proof of Theorem 1.1 by using elementary generating function dissection techniques, and then give a proof of Corollary 1.2 by using Theorem 1.1 and the ℓ-dissection formula of $\psi(q)$ due to Cui and Gu [6].

Proof of Theorem 1.1. From Hirschhorn and Sellers [8, Thereom 1, eq.(2)], we have

$$\sum_{n\geq 0}\bar{p}(3n)q^n=\frac{f_2^4f_3^6}{f_1^8f_6^3}=\frac{f_2^4}{f_6^3}\left(\frac{f_3}{f_1^3}\right)^3\frac{f_3^3}{f_1}f_1^2. \tag{3.1}$$

Plugging (2.1), (2.6) and (2.8) into the right-hand side of (3.1), we obtain that

$$\sum_{n\geq 0}\bar{p}(3n)q^n=\frac{f_2^4}{f_6^3}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right)^3\left(\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}\right)\left(\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}\right). \tag{3.2}$$

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Expanding the right-hand side of (3.2) and taking modulo 27, we deduce that

$$\sum_{n\geq 0}\bar{p}(3n)q^n \equiv \left\{\frac{f_4^{19}f_6^8f_8^5}{f_2^{24}f_{12}^7f_{16}^2}+7q^2\frac{f_4^{17}f_6^6f_{16}^2}{f_2^{22}f_8f_{12}^3}+9q^2\frac{f_4^{11}f_6^4f_8^5f_{12}}{f_2^{20}f_{16}^2}\right\}$$
$$+q\left\{10\frac{f_4^{15}f_6^6f_8^5}{f_2^{22}f_{12}^3f_{16}^2}-2\frac{f_4^{21}f_6^8f_{16}^2}{f_2^{24}f_8f_{12}^7}+9q^2\frac{f_4^{13}f_6^4f_{12}f_{16}^2}{f_2^{20}f_8}\right\} \pmod{27}. \tag{3.3}$$

By Lemma 2.3 and (2.9), it is easy to check that

$$9q^2\frac{f_4^{11}f_6^4f_8^5f_{12}}{f_2^{20}f_{16}^2}\equiv 9q^2\frac{f_4^{14}f_8^5}{f_2^8f_{16}^2} \pmod{27}. \tag{3.4}$$

Putting (3.4) into the right-hand side of (3.3), and then extracting the terms of q^{2n} on both sides of (3.3), and setting q^2 to q, we derive that

$$\sum_{n\geq 0}\bar{p}(6n)q^n \equiv \frac{f_2^{19}f_3^8f_4^5}{f_1^{24}f_6^7f_8^2}+7q\frac{f_2^{17}f_3^6f_8^2}{f_1^{22}f_4f_6^3}+9q\frac{f_2^{14}f_4^5}{f_1^8f_8^2}$$
$$\equiv \frac{f_2^{19}f_4^5}{f_6^7f_8^2}\left(\frac{f_3}{f_1^3}\right)^8+7q\frac{f_2^{17}f_8^2}{f_4f_6^3}\left(\frac{f_3}{f_1^3}\right)^6\frac{1}{f_1^4}+9q\frac{f_2^{14}f_4^5}{f_8^2}\left(\frac{1}{f_1^4}\right)^2 \pmod{27}. \tag{3.5}$$

Plugging (2.3) and (2.6) into the right-hand side of (3.5), we arrive at

$$\sum_{n\geq 0}\bar{p}(6n)q^n \equiv \frac{f_2^{19}f_4^5}{f_6^7f_8^2}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right)^8$$
$$+7q\frac{f_2^{17}f_8^2}{f_4f_6^3}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right)^6\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)$$
$$+9q\frac{f_2^{14}f_4^5}{f_8^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2 \pmod{27}. \tag{3.6}$$

Expanding the right-hand side of (3.6) and taking modulo 27, and then applying Lemma 2.3, we get

$$\sum_{n\geq 0}\bar{p}(6n)q^n \equiv \left\{\frac{f_4^{53}f_6^{17}}{f_2^{53}f_8^2f_{12}^{16}}+q^2\frac{f_4^{37}f_6^{15}f_8^6}{f_2^{47}f_{12}^{12}}-9q^2\frac{f_4^{21}}{f_2^{10}f_8^2}\right\}+q\left\{4\frac{f_4^{49}f_6^{15}}{f_2^{51}f_8^2f_{12}^{12}}+9\frac{f_4^{33}}{f_2^{14}f_8^{10}}\right\} \pmod{27}. \tag{3.7}$$

By (2.11), it is readily checked that

$$\frac{f_4^{53}f_6^{17}}{f_2^{53}f_8^2f_{12}^{16}}=\frac{f_4^{53}}{f_2^{53}f_8^2}\frac{f_6^{18}}{f_6}\frac{f_{12}^2}{f_{12}^{18}}\equiv\frac{f_2f_{12}^2}{f_4f_6f_8^2} \pmod{27}, \tag{3.8}$$

and

$$\frac{f_4^{37}f_6^{15}f_8^6}{f_2^{47}f_{12}^{12}}=\frac{f_4^{37}f_8^6}{f_2^{47}}\frac{f_6^{18}}{f_6^3}\frac{f_{12}^6}{f_{12}^{18}}\equiv\frac{f_2^7f_8^6f_{12}^6}{f_4^{17}f_6^3} \pmod{27}. \tag{3.9}$$

Plugging (3.8) and (3.9) into the right-hand side of (3.7), and then extracting the terms of q^{2n} on both sides of (3.7), and setting q^2 to q, we derive that

$$\sum_{n\geq 0}\bar{p}(12n)q^n \equiv \frac{f_1f_6^2}{f_2f_3f_4^2}+q\frac{f_1^7f_4^6f_6^6}{f_2^{17}f_3^3}-9q\frac{f_2^{21}}{f_1^{10}f_4^2}$$

$$\equiv \frac{f_6^2}{f_2f_4^2}\Big(\frac{f_1}{f_3}\Big)+q\frac{f_4^6f_6^6}{f_2^{17}}\Big(\frac{f_1^3}{f_3}\Big)^3\frac{1}{f_1^2}-9q\frac{f_2^{21}}{f_4^2}\Big(\frac{1}{f_1^4}\Big)^2\frac{1}{f_1^2} \pmod{27}. \tag{3.10}$$

Putting (2.2)-(2.5) into the right-hand side of (3.10), we are led to

$$\sum_{n\geq 0}\bar{p}(12n)q^n \equiv \frac{f_6^2}{f_2f_4^2}\Big(\frac{f_2f_{16}f_{24}^2}{f_6^2f_8f_{48}}-q\frac{f_2f_8^2f_{12}f_{48}}{f_4f_6^2f_{16}f_{24}}\Big)+q\frac{f_4^6f_6^6}{f_2^{17}}\Big(\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}\Big)^3\Big(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\Big)$$

$$-9q\frac{f_2^{21}}{f_4^2}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^2\Big(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\Big) \pmod{27}. \tag{3.11}$$

Expanding the right-hand side of (3.11) and taking modulo 27, and then applying Lemma 2.3, we obtain that

$$\sum_{n\geq 0}\bar{p}(12n)q^n \equiv \left\{\frac{f_{16}f_{24}^2}{f_4^2f_8f_{48}}+2q^2\frac{f_4^{17}f_6^6f_{16}^2}{f_2^{22}f_8f_{12}^3}+9q^2\frac{f_4^{28}f_{16}^2}{f_2^{12}f_8^9}+9q^4\frac{f_4^4f_8^7f_{16}^2}{f_2^4}\right\}$$

$$+q\left\{\frac{f_4^{15}f_6^6f_8^5}{f_2^{22}f_{12}^3f_{16}^2}-\frac{f_8^2f_{12}f_{48}}{f_4^3f_{16}f_{24}}-9\frac{f_4^{26}}{f_2^{12}f_8^3f_{16}^2}-9q^2\frac{f_4^2f_8^{13}}{f_2^4f_{16}^2}\right\} \pmod{27}. \tag{3.12}$$

By (2.11), we have

$$\frac{f_4^{15}f_6^6f_8^5}{f_2^{22}f_{12}^3f_{16}^2}\equiv\frac{f_2^5f_4^{15}f_8^5}{f_6^3f_{12}^3f_{16}^2} \pmod{27}. \tag{3.13}$$

Plugging (3.13) into the right-hand side of (3.12), and then extracting the terms of q^{2n+1} on both sides of (3.12), and setting q^2 to q, we find that

$$\sum_{n\geq 0}\bar{p}(24n+12)q^n \equiv \frac{f_1^5f_2^{15}f_4^5}{f_3^3f_6^3f_8^2}-\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}}-9\frac{f_2^{26}}{f_1^{12}f_4^3f_8^2}-9q\frac{f_2^2f_4^{13}}{f_1^4f_8^2}$$

$$\equiv \frac{f_2^{15}f_4^5}{f_6^3f_8^2}\Big(\frac{f_1^3}{f_3}\Big)^3\frac{1}{f_1^4}-\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}}-9\frac{f_2^{26}}{f_4^3f_8^2}\Big(\frac{1}{f_1^4}\Big)^3-9q\frac{f_2^2f_4^{13}}{f_8^2}\frac{1}{f_1^4} \pmod{27}. \tag{3.14}$$

Substituting (2.3) and (2.5) into the right-hand side of (3.14) implies that

$$\sum_{n\geq 0}\bar{p}(24n+12)q^n \equiv \frac{f_2^{15}f_4^5}{f_6^3f_8^2}\Big(\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}\Big)^3\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)-\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}}$$

$$-9\frac{f_2^{26}}{f_4^3f_8^2}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^3-9q\frac{f_2^2f_4^{13}}{f_8^2}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big) \pmod{27}. \tag{3.15}$$

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Expanding the right-hand side of (3.15) and taking modulo 27, and then applying Lemma 2.3, we deduce that

$$\sum_{n\geq 0}\bar{p}(24n+12)q^n \equiv \left\{\frac{f_2f_4^{28}}{f_6^3f_8^6f_{12}^3}-\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}}-9\frac{f_4^{39}}{f_2^{16}f_8^{14}}+9q^2\frac{f_4^{15}f_8^2}{f_2^8}\right\}$$
$$+q\left\{4\frac{f_2^5f_4^{16}f_8^2}{f_6^3f_{12}^3}+9\frac{f_4^{27}}{f_2^{12}f_8^6}-9q^2\frac{f_4^3f_8^{10}}{f_2^4}\right\} \pmod{27}. \tag{3.16}$$

Extracting the terms of q^{2n} on both sides of (3.16), and setting q^2 to q, we derive that

$$\sum_{n\geq 0}\bar{p}(48n+12)q^n \equiv \frac{f_1f_2^{28}}{f_3^3f_4^6f_6^3}-\frac{f_2^2f_3f_{12}}{f_1^3f_4f_6}-9\frac{f_2^{39}}{f_1^{16}f_4^{14}}+9q\frac{f_2^{15}f_4^2}{f_1^8}$$
$$\equiv \frac{f_2^{28}}{f_4^6f_6^3}\left(\frac{f_1}{f_3^3}\right)-\frac{f_2^2f_{12}}{f_4f_6}\left(\frac{f_3}{f_1^3}\right)-9\frac{f_2^{39}}{f_4^{14}}\left(\frac{1}{f_1^4}\right)^4+9qf_2^{15}f_4^2\left(\frac{1}{f_1^4}\right)^2 \pmod{27}. \tag{3.17}$$

Plugging (2.7), (2.6) and (2.3) into the right-hand side of (3.17) yields that

$$\sum_{n\geq 0}\bar{p}(48n+12)q^n \equiv \frac{f_2^{28}}{f_4^6f_6^3}\left(\frac{f_2f_4^2f_{12}^2}{f_6^7}-q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}\right)-\frac{f_2^2f_{12}}{f_4f_6}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right)$$
$$-9\frac{f_2^{39}}{f_4^{14}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^4+9qf_2^{15}f_4^2\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2 \pmod{27}. \tag{3.18}$$

Expanding the right-hand side of (3.18) and taking modulo 27, we obtain that

$$\sum_{n\geq 0}\bar{p}(48n+12)q^n \equiv \left\{\frac{f_2^{29}f_{12}^2}{f_4^4f_6^{10}}-\frac{f_4^5f_6^2}{f_2^7f_{12}}-9\frac{f_4^{42}}{f_2^{17}f_8^{16}}-9q^2\frac{f_4^{18}}{f_2^9}-9q^4\frac{f_8^{16}}{f_2f_4^6}\right\}$$
$$-q\left\{\frac{f_2^{31}f_{12}^6}{f_4^8f_6^{12}}+3\frac{f_4f_{12}^3}{f_2^5}\right\} \pmod{27}. \tag{3.19}$$

By (2.11), it is easy to check that

$$\frac{f_2^{29}f_{12}^2}{f_4^4f_6^{10}} \equiv \frac{f_2^2f_{12}^2}{f_4^4f_6} \pmod{27}. \tag{3.20}$$

Putting (3.20) into the right-hand side of (3.19), and then extracting the terms of q^{2n} on both sides of (3.19), and setting q^2 to q, we derive that

$$\sum_{n\geq 0}\bar{p}(96n+12)q^n \equiv \frac{f_1^2f_6^2}{f_2^4f_3}-\frac{f_2^5f_3^2}{f_1^7f_6}-9\frac{f_2^{42}}{f_1^{17}f_4^{16}}-9q\frac{f_2^{18}}{f_1^9}-9q^2\frac{f_4^{16}}{f_1f_2^6}$$
$$\equiv \frac{f_2^2}{f_1}\left\{\frac{f_1^3f_6^2}{f_2^6f_3}-\frac{f_2^3f_3^2}{f_1^6f_6}+9q\frac{f_2^{16}}{f_1^8}-9\left(\frac{f_2^{20}}{f_1^8f_4^8}+q\frac{f_4^8}{f_2^4}\right)^2\right\} \pmod{27}. \tag{3.21}$$

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By (1.2), (1.3) and (1.4), we have

$$\sum_{n\geq 0}\bar{p}(96n+12)q^n \equiv \psi(q)\left\{\frac{\psi(q^3)}{\psi(q)^3}-\frac{\varphi(-q^3)}{\varphi(-q)^3}+9q\psi(q)^8-9\Big(\varphi(q)^4+q\psi(q^2)^4\Big)^2\right\} \pmod{27}. \tag{3.22}$$

Combining Lemma 2.3, Lemma 2.6 and Lemma 2.7, we conclude that

$$\sum_{n\geq 0}\bar{p}(96n+12)q^n \equiv -9\psi(q) \pmod{27}, \tag{3.23}$$

which completes the proof of Theorem 1.1. ∎

Proof of Corollary 1.2. Recall the ℓ-dissection formula of $\psi(q)$ given by Cui and Gu [6, Theorem 2.1], namely, for any odd prime ℓ,

$$\psi(q)=\sum_{k=0}^{\frac{\ell-3}{2}} q^{\frac{k^2+k}{2}} f(q^{\frac{\ell^2+(2k+1)\ell}{2}}, q^{\frac{\ell^2-(2k+1)\ell}{2}}) + q^{\frac{\ell^2-1}{8}}\psi(q^{\ell^2}). \tag{3.24}$$

Notice that $i \equiv \frac{k^2+k}{2} \pmod{\ell}$, $0 \leq k \leq \frac{\ell-3}{2}$ or $i \equiv \frac{\ell^2-1}{8} \pmod{\ell}$ if and only if $8i+1 \equiv (2k+1)^2 \pmod{\ell}$ or $8i+1 \equiv 0 \pmod{\ell}$. That is to say, there are no terms of $q^{\ell n+i}$ such that $0 \leq i < \ell$ and $\left(\frac{8i+1}{\ell}\right) = -1$ on both sides of (3.24). Hence extracting such terms of $q^{\ell n+i}$ on both sides of (3.23), we obtain that

$$\bar{p}(96\ell n+96i+12) \equiv 0 \pmod{27}.$$

This completes the proof of Corollary 1.2. ∎

Acknowledgments. This work was supported by the National Natural Science Foundation of China (No. 11901403).

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