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ON THE SIMULTANEOUS METRIC DIMENSION OF A GRAPH AND ITS COMPLEMENT

CONG X. KANG, IZTOK PETERIN, AND EUNJEONG YI

ABSTRACT. A set $S \subseteq V(G)$ is a *resolving set* of a graph G if, for any two distinct vertices x and y of G, there exists a vertex $z \in S$ such that $d(x,z) \neq d(y,z)$, where $d(u,v)$ denotes the length of a shortest path between vertices u and v in G. The *metric dimension* $\dim(G)$ of G is the minimum of the cardinalities of all resolving sets of G. Ramírez-Cruz, Oellermann and Rodríguez-Velázquez [Discrete Appl. Math. 198 (2016) 241-250] introduced the notion of a simultaneous resolving set and the simultaneous metric dimension of a graph family. A set $S \subseteq V$ is a *simultaneous resolving set* for a finite collection $\mathscr{C}$ of graphs on a common vertex set V if S is a resolving set for every graph in $\mathscr{C}$; the minimum among the cardinalities of all such S is called the *simultaneous metric dimension* of $\mathscr{C}$, denoted by $\mathrm{Sd}(\mathscr{C})$. In this paper, we focus on the simultaneous metric dimension of a graph and its complement. We characterize graphs G satisfying $\mathrm{Sd}(G,\overline{G}) = 1$ and $\mathrm{Sd}(G,\overline{G}) = |V(G)|-1$, respectively, where $\overline{G}$ denotes the complement of G. We show that $\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\} \neq \{3\}$ implies $\mathrm{Sd}(G,\overline{G}) = \max\{\dim(G), \dim(\overline{G})\}$. We construct a family of self-complementary split graphs G of diameter 3 satisfying $\mathrm{Sd}(G,\overline{G}) > \max\{\dim(G), \dim(\overline{G})\}$. We determine $\mathrm{Sd}(G,\overline{G})$ when G is a tree or a unicyclic graph. We conclude the paper with some open problems.

1. Introduction

Let G be a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *distance* between two vertices $x,y \in V(G)$, denoted by $d_G(x,y)$, is the minimum number of edges on a path connecting x and y in G, should such a path exist; otherwise, we regard the distance as infinity and write $d_G(x,y) = \infty$. The *diameter* $\mathrm{diam}(G)$ of G is $\max\{d_G(x,y) : x,y \in V(G)\}$. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *degree* $\deg_G(u)$ of a vertex u in G is $|N_G(u)|$, the cardinality of $N_G(u)$. An *end vertex* is a vertex of degree one; a *support vertex* is a vertex that is adjacent to an end vertex, and a *major vertex* is a vertex of degree at least three. The *complement* of G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G})$ such that $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$ for any distinct $x,y \in V(G)$. We denote by $P_n, C_n, K_n, K_{a,n-a}$, respectively, the path, the cycle, the complete graph, and the complete bi-partite graph on n vertices. Let $[k] = \{1,2,\ldots,k\}$ and $[k]_0 = [k] \cup \{0\}$.

A vertex $z \in V(G)$ *resolves* a pair of vertices $x,y \in V(G)$ if $d_G(x,z) \neq d_G(y,z)$. For two distinct vertices $x,y \in V(G)$, let $R_G\{x,y\} = \{z \in V(G) : d_G(x,z) \neq d_G(y,z)\}$. A set $S \subseteq V(G)$ is a *resolving set* of G if $|S \cap R_G\{x,y\}| \geq 1$ for every pair of distinct vertices x and y of G. The *metric dimension* $\dim(G)$ of G is the minimum of the cardinalities of all resolving sets of G, and a resolving set of G

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The second author is also with IMFM, Jadranska 19, Ljubljana, Slovenia.
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having cardinality $\dim(G)$ is called a (metric) *basis* of G. Introduced by Slater [17] and by Harary and Melter [7], the metric dimension of graphs has been extensively studied. It is noted in [6] that determining the metric dimension of a graph is an NP-hard problem. One of the motivations for studying metric dimension is robot navigation (see [11]), where a robot determines its location in the network by landmarks or transmitters placed at nodes (vertices of the graph) in the network modeled by a graph; thus, metric dimension is the minimum number of landmarks or transmitters required for the robot to know its location at all nodes in the network.

In 1990, Brigham and Dutton [3] considered a family of spanning subgraphs $\{H_1,\ldots,H_t\}$, called factors, of a given graph; they studied the domination parameter for partitions $\{G,\overline{G}\}$ of K_n. In 2016, for a given family $\mathscr{C}=\{H_1,\ldots,H_t\}$ of connected graphs $H_i=(V,E_i)$ with common vertex set V, Ramírez-Cruz et al. [16] defined a *simultaneous resolving set* for $\mathscr{C}$ to be a set $S\subseteq V$ such that S is a resolving set for every graph H_i for $i\in[t]$, where the edge sets of the graphs from $\mathscr{C}$ are not necessarily disjoint. The *simultaneous metric dimension* of $\mathscr{C}$, denoted by $\mathrm{Sd}(\mathscr{C})$ or $\mathrm{Sd}(H_1,\ldots,H_t)$, is the minimum among the cardinalities of all simultaneous resolving sets of $\mathscr{C}$. For other articles on simultaneous metric dimension or its variations, see [1, 5, 10, 13, 14, 15].

In this paper, we investigate the simultaneous metric dimension for a pair of subgraphs $\{G,\overline{G}\}$ of K_n, where $n\geq 2$. The results in the next section will show that the study of $\mathrm{Sd}(G,\overline{G})$ naturally splits into three cases, according to $(\mathrm{diam}(G),\mathrm{diam}(\overline{G}))$: (i) $(\mathrm{diam}(G),\mathrm{diam}(\overline{G}))=(1,\infty)$; (ii) $(\mathrm{diam}(G),\mathrm{diam}(\overline{G}))=(2,k)$, where $k\geq 2$; (iii) $(\mathrm{diam}(G),\mathrm{diam}(\overline{G}))=(3,3)$. Case (i) is trivial, and any resolving set of G is a resolving set of $\overline{G}$ in case (ii), leaving case (iii) to closer scrutiny.

The paper is organized as follows. In Section 2, we obtain some general results on $\mathrm{Sd}(G,\overline{G})$. Noting that $1\leq\max\{\dim(G),\dim(\overline{G})\}\leq\mathrm{Sd}(G,\overline{G})\leq\min\{\dim(G)+\dim(\overline{G}),|V(G)|-1\}\leq|V(G)|-1$ for any non-trivial graph G, we characterize graphs G satisfying $\mathrm{Sd}(G,\overline{G})=1$ and $\mathrm{Sd}(G,\overline{G})=|V(G)|-1$, respectively. We also show that, if $\{\mathrm{diam}(G),\mathrm{diam}(\overline{G})\}\neq\{3\}$, then $\mathrm{Sd}(G,\overline{G})=\max\{\dim(G),\dim(\overline{G})\}$. In Section 3, we consider graphs G with $\mathrm{diam}(G)=3=\mathrm{diam}(\overline{G})$. We determine $\mathrm{Sd}(G,\overline{G})$ when G is a tree or a unicyclic graph with $\mathrm{diam}(G)=3=\mathrm{diam}(\overline{G})$. We construct a family of self-complementary split graphs G of diameter 3 with $\mathrm{Sd}(G,\overline{G})>\max\{\dim(G),\dim(\overline{G})\}$. In Section 4, we examine $\mathrm{Sd}(G,\overline{G})$ when G is a tree or a unicyclic graph. For any non-trivial tree $T\neq P_4$, we show that $\mathrm{Sd}(T,\overline{T})=\dim(\overline{T})\geq\dim(T)$ and characterize tree T satisfying $\mathrm{Sd}(T,\overline{T})=\dim(T)$. We also show that, for any non-trivial tree T, $\dim(\overline{T})\geq\sigma(T)-1$, where $\sigma(T)$ denotes the number of end vertices of T. For any unicyclic graph G of order $n\geq 3$, we show that $\mathrm{Sd}(G,\overline{G})\in\{\dim(\overline{G}),1+\dim(\overline{G})\}$; moreover, we show that $\mathrm{Sd}(G,\overline{G})=\dim(\overline{G})$ for $n\geq 7$. In Section 5, we state some interesting observations and conclude with some open problems.

2. General results on $\mathrm{Sd}(G,\overline{G})$

In this section, we obtain some general results on $\mathrm{Sd}(G,\overline{G})$. A result from [16] implies that, for any non-trivial graph G, $1\leq\max\{\dim(G),\dim(\overline{G})\}\leq\mathrm{Sd}(G,\overline{G})\leq\min\{\dim(G)+\dim(\overline{G}),|V(G)|-1\}\leq|V(G)|-1$. We characterize graphs G satisfying $\mathrm{Sd}(G,\overline{G})=1$ and $\mathrm{Sd}(G,\overline{G})=|V(G)|-1$, respectively. We also show that, if $\{\mathrm{diam}(G),\mathrm{diam}(\overline{G})\}\neq\{3\}$, then $\mathrm{Sd}(G,\overline{G})=\max\{\dim(G),\dim(\overline{G})\}$.

We begin with some useful observations. Two vertices $u,w\in V(G)$ are called *twin vertices* if $N_G(u)-\{w\}=N_G(w)-\{u\}$.

Observation 2.1. [8] *Two distinct vertices u and w are twin vertices in G if and only if they are twin vertices in $\overline{G}$. If S is a resolving set of either G or $\overline{G}$, then $S \cap \{u,w\} \neq \emptyset$.*

Observation 2.2. *Graphs G and $\overline{G}$ cannot both be disconnected.*

Observation 2.3. [16] *For any family $\mathscr{C} = \{H_1, \ldots, H_t\}$,*

$$\max_{1 \le i \le t} \{\dim(H_i)\} \le \mathrm{Sd}(\mathscr{C}) \le \min \left\{ \sum_{i=1}^{t} \dim(H_i), |V(G)| - 1 \right\}.$$

Observation 2.3 implies the following result.

Corollary 2.4. *For any graph G of order $n \ge 2$,*

$$1 \le \max\{\dim(G), \dim(\overline{G})\} \le \mathrm{Sd}(G, \overline{G}) \le \min\{\dim(G) + \dim(\overline{G}), n-1\} \le n-1.$$

Next, we characterize graphs G satisfying $\mathrm{Sd}(G,\overline{G})$ equals 1 and $|V(G)|-1$, respectively. We begin by recalling some known results. In the following, $G+H$ denotes the join of graphs G and H, which is obtained from the disjoint union of G and H by adding an edge xy for each $x \in V(G)$ and each $y \in V(H)$.

Theorem 2.5. [4] *Let G be a connected graph of order $n \ge 2$. Then*

(a) $\dim(G) = 1$ *if and only if* $G = P_n$;
(b) *for* $n \ge 4$, $\dim(G) = n-2$ *if and only if* $G = K_{s,t}$ $(s,t \ge 1)$, $G = K_s + \overline{K_t}$ $(s \ge 1, t \ge 2)$, *or* $G = K_s + (K_1 \cup K_t)$ $(s,t \ge 1)$;
(c) $\dim(G) = n-1$ *if and only if* $G = K_n$.

Theorem 2.6. [16] *Let $\mathscr{C}$ be a family of connected graphs on a common vertex set. Then*

(a) $\mathrm{Sd}(\mathscr{C}) = 1$ *if and only if $\mathscr{C}$ is a collection of paths that share a common end vertex.*
(b) *If $\mathscr{C}$ is a collection of paths, then* $\mathrm{Sd}(\mathscr{C}) \in \{1,2\}$.

Theorem 2.7. *Let G be a graph of order $n \ge 2$. Then*

(a) $\mathrm{Sd}(G,\overline{G}) = 1$ *if and only if* $G \in \{P_2, \overline{P}_2, P_3, \overline{P}_3\}$.
(b) $\mathrm{Sd}(G,\overline{G}) = n-1$ *if and only if* $G \in \{K_n, \overline{K}_n\}$;

Proof. Let G be a graph of order $n \ge 2$.

(a) ($\Leftarrow$) If $G = P_2$ is given by u_1u_2, then $\{u_1\}$ forms a resolving set for both G and $\overline{G}$; notice that $d_G(u_1,u_2) = 1$ and $d_{\overline{G}}(u_1,u_2) = \infty$. If $G = P_3$ is given by $u_1u_2u_3$, then $\overline{G} = K_1 \cup K_2$ consists of two components, u_2 and u_1u_3, and $\{u_1\}$ forms a resolving set for both G and $\overline{G}$; note that $d_G(u_1,u_2) = 1 < 2 = d_G(u_1,u_3)$ and $d_{\overline{G}}(u_1,u_3) = 1 < \infty = d_{\overline{G}}(u_1,u_2)$.

($\Rightarrow$) Let $\mathrm{Sd}(G,\overline{G}) = 1$. We may, by Observation 2.2, assume G to be connected; then $G = P_n$ by Theorem 2.5(a). If $n \ge 4$, then $\overline{G}$ is connected but not a path which shares a common end vertex with G. By Theorem 2.6(a), $\mathrm{Sd}(G,\overline{G}) \neq 1$.

(b) ($\Leftarrow$) Let $G = K_n$. Since any two vertices of G (and thus $\overline{G}$) are twin vertices, $\mathrm{Sd}(G,\overline{G}) \ge n-1$ by Observation 2.1. On the other hand, $\mathrm{Sd}(G,\overline{G}) \le n-1$ by Corollary 2.4. Thus, $\mathrm{Sd}(G,\overline{G}) = n-1$.

($\Rightarrow$) Let $\mathrm{Sd}(G,\overline{G}) = n-1$. The case $n=2$ is subsumed by Part (a) of this proof. So, let $n \ge 3$ and assume, to the contrary, that $G \notin \{K_n, \overline{K}_n\}$. Then, there exist three distinct vertices, say $x,y,z \in V(G)$, such that $xy \in E(G)$ and $xz \notin E(G)$. Then $V(G) - \{y,z\}$ is a resolving set for both G and $\overline{G}$, since $d_G(x,y) = 1 < d_G(x,z)$ and $d_{\overline{G}}(x,z) = 1 < d_{\overline{G}}(x,y)$. So, $G \notin \{K_n, \overline{K}_n\}$ implies $\mathrm{Sd}(G,\overline{G}) \le n-2$ for $n \ge 3$. $\square$

Next, we show that if $\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\} \neq \{3\}$, then $\mathrm{Sd}(G,\overline{G}) = \max\{\dim(G), \dim(\overline{G})\}$. We begin with the following lemma.

Lemma 2.8. *Let G be a graph with $\mathrm{diam}(G) = 2$. If S is a resolving set for G, then S is a resolving set for $\overline{G}$.*

Proof. Let $S \subseteq V(G)$ be a resolving set of G; note that vertices in S are self-resolved in G and $\overline{G}$. Suppose there are distinct vertices x and y in $V(G) - S = V(\overline{G}) - S$. Then, there exists $z \in S$ such that $d_G(x,z) \neq d_G(y,z)$, say $d_G(x,z) = 1$ and $d_G(y,z) = 2$. Then $d_{\overline{G}}(x,z) \geq 2$ and $d_{\overline{G}}(y,z) = 1$. So, if $z \in S$ resolves x and y in G, then z resolves x and y in $\overline{G}$. □

Proposition 2.9. [2] *If G is a graph with $\mathrm{diam}(G) \geq 4$, then $\mathrm{diam}(\overline{G}) \leq 2$.*

Theorem 2.10. *Let G be a graph of order at least two. Suppose $\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\} \neq \{3\}$ and $\mathrm{diam}(G) \leq \mathrm{diam}(\overline{G})$; then $\mathrm{Sd}(G,\overline{G}) = \dim(G)$.*

Proof. Let G be a graph of order $n \geq 2$. By hypotheses and Proposition 2.9, we find $\mathrm{diam}(G) \leq 2$. If $\mathrm{diam}(G) = 1$, then $G = K_n$ and $\mathrm{Sd}(G,\overline{G}) = n-1 = \dim(G) = \dim(\overline{G})$ by Theorem 2.7(b). If $\mathrm{diam}(G) = 2$, then $\mathrm{Sd}(G,\overline{G}) = \dim(G) = \max\{\dim(G), \dim(\overline{G})\}$ by Lemma 2.8. □

We note that Theorem 2.10 implies that $\mathrm{Sd}(G,\overline{G}) = \dim(G)$ when G is the Petersen graph, a complete multi-partite graph, a wheel graph $K_1 + C_{n-1}$, $n \geq 4$, or a fan graph $K_1 + P_{n-1}$, $n \geq 3$, since those graphs have diameter 2. Also note that if G and $\overline{G}$ both have diameter 2, then $\dim(G) = \dim(\overline{G})$.

3. $\mathrm{Sd}(G,\overline{G})$ with $\mathrm{diam}(G) = 3 = \mathrm{diam}(\overline{G})$

In this section, we consider graphs G with $\mathrm{diam}(G) = 3 = \mathrm{diam}(\overline{G})$. First, we determine $\mathrm{Sd}(T,\overline{T})$ for tress T with $\mathrm{diam}(T) = 3 = \mathrm{diam}(\overline{T})$.

We recall some terminology and notations. For an ordered set $S = \{u_1, \ldots, u_k\} \subseteq V(G)$ of distinct vertices, the *(metric) code* of $v \in V(G)$ with respect to S in G is the k-vector $\mathrm{code}_{G,S}(v) = (d_G(v,u_1), \ldots, d_G(v,u_k))$.

Fix a tree T. An end vertex ℓ is called a *terminal vertex* of a major vertex v if $d_T(\ell,v) < d_T(\ell,w)$ for every other major vertex w in T. The *terminal degree*, $ter_T(v)$, of a major vertex v is the number of terminal vertices of v in T, and an *exterior major vertex* is a major vertex that has a positive terminal degree. We denote by $ex(T)$ the number of exterior major vertices of T, and $\sigma(T)$ the number of end vertices of T. Let $M(T)$ be the set of exterior major vertices of T. Let $M_1(T) = \{w \in M(T) : ter_T(w) = 1\}$ and let $M_2(T) = \{w \in M(T) : ter_T(w) \geq 2\}$; note that $M(T) = M_1(T) \cup M_2(T)$. For each $v \in M(T)$, let T_v be the subtree of T induced by v and all vertices belonging to the paths joining v with its terminal vertices, and let $L_v(T)$ be the set of terminal vertices of v in T.

Theorem 3.1. [4, 11, 12] *For any tree T that is not a path, $\dim(T) = \sigma(T) - ex(T)$.*

Theorem 3.2. [12] *Let T be a tree with $ex(T) = k \geq 1$, and let $v_1, v_2, \ldots, v_k$ be the exterior major vertices of T. For each $i \in [k]$, let $\ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,\sigma_i}$ be the terminal vertices of v_i with $ter_T(v_i) = \sigma_i \geq 1$, and let $P_{i,j}$ be the $v_i - \ell_{i,j}$ path, where $j \in [\sigma_i]$. Let $W \subseteq V(T)$. Then W is a basis of T if and only if W contains exactly one vertex from each of the paths $P_{i,j} - v_i$, where $j \in [\sigma_i]$ and $i \in [k]$, with exactly one exception for each $i \in [k]$ and W contains no other vertices of T.*

6 Jul 2023 16:33:46 PDT
201207-Peterin Version 2 - Submitted to Rocky Mountain J. Math.

Proposition 3.3. *Let T be a tree of order n with $\mathrm{diam}(T) = 3 = \mathrm{diam}(\overline{T})$. Then T satisfies one of the following:*

(a) $T = P_4$;
(b) $ex(T) = 1$ *and* $\sigma(T) = n-2$;
(c) $ex(T) = 2$ *and* $\sigma(T) = n-2$.

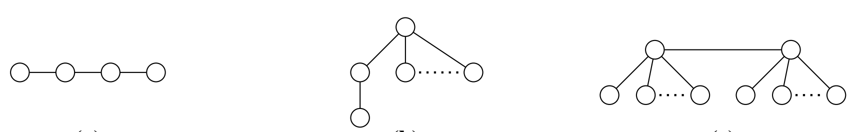
Moreover, $\mathrm{Sd}(T,\overline{T}) = \begin{cases} 2 = \dim(T)+\dim(\overline{T}) & \text{if } T \text{ satisfies (a)},\\ n-3 = \dim(\overline{T}) = \dim(T) & \text{if } T \text{ satisfies (b)},\\ n-3 = \dim(\overline{T}) = \dim(T)+1 & \text{if } T \text{ satisfies (c)}.\end{cases}$

Proof. Let T be a tree of order $n \geq 4$ with $\mathrm{diam}(T) = 3$. Then $ex(T) \in \{0,1,2\}$, since $ex(T) \geq 3$ implies $\mathrm{diam}(T) \geq 4$.

First, let $ex(T) = 0$. Then $\mathrm{diam}(T) = 3$ implies $T = P_4$ (see Figure 1(a)), and $\overline{T} = P_4$. So, $\mathrm{Sd}(P_4,\overline{P_4}) = 2$ by Theorem 2.6.

Second, let $ex(T) = 1$. Let v be the exterior major vertex of T, and let $\ell_1,\ell_2,\ldots,\ell_k$ be the terminal vertices of v in T such that $d_G(v,\ell_1) \geq d_G(v,\ell_2) \geq \cdots \geq d_G(v,\ell_k)$, where $k \geq 3$. Note that $\mathrm{diam}(T) = 3$ implies $d_G(v,\ell_1) = 2 = 1+d_G(v,\ell_2)$; see Figure 1(b). If s is the degree two vertex lying on the $v-\ell_1$ path in T, then $d_{\overline{T}}(v,s) = 3$ and it is easy to see that $\mathrm{diam}(\overline{T}) = 3$. Since $S_1 = \cup_{i=1}^{k-1}\{\ell_i\}$ forms a resolving set for both T and $\overline{T}$ with $|S_1| = k-1 = n-3$, $\mathrm{Sd}(T,\overline{T}) \leq n-3$; we note that, if $S_0 = \{\ell_1,\ell_2\} \subseteq S_1$, then $\mathrm{code}_{\overline{T},S_0}(v) = (1,2)$, $\mathrm{code}_{\overline{T},S_0}(s) = (2,1)$ and $\mathrm{code}_{\overline{T},S_0}(\ell_k) = (1,1)$. By Corollary 2.4 and Theorem 3.1, $\mathrm{Sd}(T,\overline{T}) \geq \dim(T) = k-1 = n-3$; thus $\mathrm{Sd}(T,\overline{T}) = n-3 = \dim(T) = \dim(\overline{T})$.

Third, let $ex(T) = 2$. Let v_1 and v_2 be distinct exterior major vertices of T, let $\ell_1,\ldots,\ell_a$ be the terminal vertices of v_1 and let $\ell'_1,\ldots,\ell'_b$ be the terminal vertices of v_2 in T, where $a,b \geq 2$. Note that the condition $\mathrm{diam}(T) = 3$ implies that $d_T(v_1,v_2) = 1$ and $d_T(v_1,\ell_i) = 1 = d_T(v_2,\ell'_j)$, where $i \in [a]$ and $j \in [b]$ (see Figure 1(c)). Let S be any resolving set for $\overline{T}$. Then $|S \cap (\cup_{i=1}^{a}\{\ell_i\})| \geq a-1$ and $|S \cap (\cup_{i=1}^{b}\{\ell'_i\})| \geq b-1$ by Observation 2.1. If $S_2 = (\cup_{i=1}^{a-1}\{\ell_1\}) \cup (\cup_{i=1}^{b-1}\{\ell'_i\}) \subseteq S$, then S_2 forms a resolving set for T with $|S_2| = \dim(T)$ by Theorems 3.1 and 3.2, and $\mathrm{code}_{\overline{T},S_2}(\ell_a) = \mathbf{1} = \mathrm{code}_{\overline{T},S_2}(\ell'_b)$, where $\mathbf{1}$ is the all-one vector. So, $|S| \geq |S_2|+1 = a+b-1 = n-3 = \dim(T)+1$. Note that $S_2 \cup \{\ell_a\}$ forms a resolving set for both T and $\overline{T}$; if $S' = \{\ell_1,\ell'_1\} \subseteq S_2 \cup \{\ell_a\}$, then $\mathrm{code}_{\overline{T},S'}(v_1) = (2,1)$, $\mathrm{code}_{\overline{T},S'}(v_2) = (1,2)$ and $\mathrm{code}_{\overline{T},S'}(\ell'_b) = (1,1)$. So, $\dim(\overline{T}) \leq n-3$. Thus, $\dim(\overline{T}) = n-3$ and $\mathrm{Sd}(T,\overline{T}) = n-3 = \dim(\overline{T}) = \dim(T)+1$. □



(a) **(b)** **(c)**

FIGURE 1. Trees T with $\mathrm{diam}(T) = 3 = \mathrm{diam}(\overline{T})$.

Second, we determine $\mathrm{Sd}(G,\overline{G})$ for unicyclic graphs G with $\mathrm{diam}(G) = 3 = \mathrm{diam}(\overline{G})$. We begin with the characterization of unicyclic grahs G satisfying $\mathrm{diam}(G) = 3$.

Lemma 3.4. *Let G be a unicyclic graph with $\mathrm{diam}(G) = 3$. Then $G \in \{C_6, C_7\}$ or G is isomorphic to a graph represented in Figure 2.*

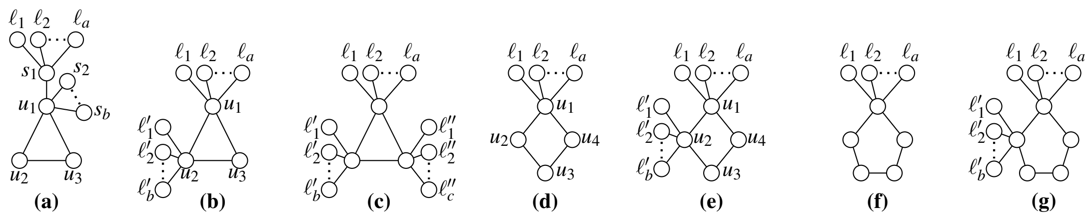


FIGURE 2. Unicyclic graphs $G \notin \{C_6, C_7\}$ with $\operatorname{diam}(G) = 3$, where $a, b, c \geq 1$.

Proof. Let G be a unicyclic graph with $\operatorname{diam}(G) = 3$. Let $\mathscr{C} = C_m$ be the unique cycle of G given by $u_1, u_2, \ldots, u_m, u_1$, where $m \geq 3$. For $w \in V(\mathscr{C})$, let T_w denote the subtree rooted at w in G and let $\tau(G) = \{u_i \in V(\mathscr{C}) : \deg_G(u_i) \geq 3\}$ and let $L_i(G) = \{\ell \in V(T_{u_i}) : \deg_G(\ell) = 1\}$.

Case 1: $\tau(G) = \emptyset$**.** In this case, G is a cycle and $\operatorname{diam}(G) = 3$ implies $G \in \{C_6, C_7\}$.

Case 2: $\tau(G) \neq \emptyset$**.** In this case, $m \in \{3, 4, 5\}$; note that $m \geq 6$ implies $\operatorname{diam}(G) \geq 4$. By relabeling the vertices of $\mathscr{C}$ in G if necessary, let $\deg_G(u_1) \geq 3$.

Subcase 2.1: $m = 3$. First, suppose $\deg_G(u_2) = \deg(u_3) = 2$. In this case, $d_G(\ell, u_1) \leq 2$ for each $\ell \in L_1(G)$, and there exists an end-vertex $\ell' \in L_1(G)$ with $d_G(\ell', u_1) = 2$. So, G is isomorphic to Figure 2(a), where $a \geq 1$ and $b \geq 1$.

Second, suppose $\deg_G(u_2) \geq 3$ or $\deg_G(u_3) \geq 3$. If $\deg_G(u_i) \geq 3$ and $\ell_i \in L_i(G)$, then $d_G(\ell_i, u_i) = 1$, where $i \in [3]$. If $\deg_G(u_2) \geq 3 = 1 + \deg_G(u_3)$ or $\deg_G(u_3) \geq 3 = 1 + \deg_G(u_2)$, then G is isomorphic to Figure 2(b), where $a, b \geq 1$. If $\deg_G(u_2) \geq 3$ and $\deg_G(u_3) \geq 3$, then G is isomorphic to Figure 2(c), where $a, b, c \geq 1$.

Subcase 2.2: $m \in \{4, 5\}$**.** We note the following: (i) for each end-vertex $\ell_i \in L_i(G)$, $d_G(\ell_i, u_i) = 1$, where $i \in [m]$; (ii) $\deg_G(u_3) = 2 = \deg_G(u_{m-1})$. If every vertex in $\{u_1, u_2, u_m\}$ has degree at least three in G, say ℓ_j is a terminal vertex of the major vertex u_j for each $j \in \{1, 2, m\}$, then $d_G(\ell_2, \ell_m) \geq 4$, and thus $\operatorname{diam}(G) \geq 4$. So, $\deg_G(u_1) \geq 3$ and $\operatorname{diam}(G) = 3$ implies that at most one vertex in $\{u_2, u_m\}$ has degree at least three in G. If $\deg_G(u_2) = 2 = \deg_G(u_m)$, then G is isomorphic to Figure 2(d) (when $m = 4$) or G is isomorphic to Figure 2(f) (when $m = 5$), where $a \geq 1$. If $\deg_G(u_2) \geq 3$ or $\deg_G(u_m) \geq 3$, but not both, then G is isomorphic to Figure 2(e) (when $m = 4$) or G is isomorphic to Figure 2(g) (when $m = 5$), where $a, b \geq 1$. □

Proposition 3.5. *Let G be a unicyclic graph of order n with $\operatorname{diam}(G) = 3 = \operatorname{diam}(\overline{G})$. Then G is isomorphic to Figure 2(a)-(b) or Figure 2(d)-(e). Moreover, $\operatorname{Sd}(G, \overline{G})$ equals*

$$\begin{cases} \dim(\overline{G}) + 1 = \dim(G) + 1 & \text{if } G \text{ is isomorphic to Fig. 2(e) with } a = b = 1; \\ \dim(\overline{G}) = \dim(G) & \text{if } G \text{ is isomorphic to Fig. 2(a) with } a = 1 \text{ and } b \geq 1, \\ & \text{or } G \text{ is isomorphic to Fig. 2(b) with } a = b = 1, \\ & \text{or } G \text{ is isomorphic to Fig. 2(d) with } a \geq 1, \\ & \text{or } G \text{ is isomorphic to Fig. 2(e) with } a \geq 2 = 1 + b \text{ or } b \geq 2 = 1 + a; \\ \dim(\overline{G}) = \dim(G) + 1 & \text{if } G \text{ is isomorphic to Fig. 2(a) with } a \geq 2 \text{ and } b \geq 1, \\ & \text{or } G \text{ is isomorphic to Fig. 2(b) with } a \geq 2 = 1 + b \text{ or } b \geq 2 = 1 + a, \\ & \text{or } G \text{ is isomorphic to Fig. 2(e) with } a, b \geq 2; \\ \dim(\overline{G}) = \dim(G) + 2 & \text{if } G \text{ is isomorphic to Fig. 2(b) with } a, b \geq 2. \end{cases}$$

Proof. Let G be a unicyclic graph of order n with $\operatorname{diam}(G) = 3$. By Lemma 3.4, $G \in \{C_6, C_7\}$ or G is isomorphic to one of the graphs in Figure 2. If $G \in \{C_6, C_7\}$, then $\operatorname{diam}(\overline{G}) = 2$. If G is isomorphic to Figure 2(a), then $\operatorname{diam}(\overline{G}) = 3$; note that $d_{\overline{G}}(u_1, s_1) = 3$. If G is isomorphic to

Figure 2(b) or Figure 2(d)-(e), then $\text{diam}(\overline{G}) = 3$; note that $d_{\overline{G}}(u_1, u_2) = 3$. If G is isomorphic to Figure 2(c) or Figure 2(f)-(g), one can easily check that $\text{diam}(\overline{G}) = 2$.

Next, for unicyclic graphs G with $\text{diam}(G) = 3 = \text{diam}(\overline{G})$, we determine $\text{Sd}(G, \overline{G})$. Suppose S and $\overline{S}$ be bases for G and $\overline{G}$ respectively so that $S_0 = S \cap \overline{S}$ is as large as possible.

Case 1: G is isomorphic to Figure 2(a). By Observation 2.1, we have the following: (i) if $a \geq 2$, then $|S_0 \cap (\cup_{i=1}^{a}\{\ell_i\})| \geq a-1$; (ii) $|S_0 \cap \{u_2, u_3\}| \geq 1$; (iii) if $b \geq 3$, then $|S_0 \cap (\cup_{i=2}^{b}\{s_i\})| \geq b-2$.

First, suppose $a = 1$ and $b \in \{1, 2\}$. Then $\{u_2, s_1\}$ forms a basis for G and $\overline{G}$, and thus $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = 2 = \dim(G)$; note that $\text{Sd}(G, \overline{G}) = n-3$ if $a = b = 1$, and $\text{Sd}(G, \overline{G}) = n-4$ if $a = 1$ and $b = 2$.

Second, suppose $a = 1$ and $b \geq 3$. From (ii) and (iii), we may assume that $R_0 = \{u_2\} \cup (\cup_{i=3}^{b}\{s_i\}) \subseteq S_0$ with $|R_0| = b-1 = n-5$. Since $\text{code}_{G,R_0}(s_1) = \text{code}_{G,R_0}(s_2)$ and $\text{code}_{\overline{G},R_0}(s_1) = \text{code}_{\overline{G},R_0}(s_2)$, $|S| \geq |R_0|+1$ and $|\overline{S}| \geq |R_0|+1$. On the other hand, $R_0 \cup \{s_1\}$ forms a resolving set for G and $\overline{G}$, and hence $|S| \leq |R_0|+1$ and $|\overline{S}| \leq |R_0|+1$. Thus $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = n-4 = \dim(G)$.

Third, suppose $a \geq 2$ and $b \in \{1, 2\}$. From (i) and (ii), we may assume that $R_1 = \{u_2\} \cup (\cup_{i=2}^{a}\{\ell_i\}) \subseteq S_0$ with $|R_1| = a$. Then R_1 forms a resolving set for G, but $\text{code}_{\overline{G},R_1}(u_1) = \text{code}_{\overline{G},R_1}(u_3)$ and $R_1 \cup \{s_1\}$ forms a resolving set for $\overline{G}$. Thus, $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = a+1 = \dim(G)+1$; we note that $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = n-3 = \dim(G)+1$ if $a \geq 2$ and $b = 1$, and $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = n-4 = \dim(G)+1$ if $a \geq 2$ and $b = 2$.

Fourth, suppose $a \geq 2$ and $b \geq 3$. From (i), (ii) and (iii), we may assume that $R_2 = \{u_2\} \cup (\cup_{i=2}^{a}\{\ell_i\}) \cup (\cup_{i=3}^{b}\{s_i\}) \subseteq S_0$ with $|R_2| = a+b-2 = n-5$. Then R_2 forms a resolving set for G, but $\text{code}_{\overline{G},R_2}(\ell_1) = \text{code}_{\overline{G},R_2}(s_2)$ and $R_2 \cup \{s_1\}$ forms a resolving set for $\overline{G}$; thus $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = n-4 = \dim(G)+1$.

Case 2: G is isomorphic to Figure 2(b). First, suppose $a = b = 1$. Then $\{\ell_1, \ell'_1\}$ forms a basis for G and $\overline{G}$; thus $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = \dim(G) = 2 = n-3$.

Second, suppose $a \geq 2$ and $b = 1$, or $a = 1$ and $b \geq 2$, say the former by relabeling the vertices of G if necessary; then $|S_0 \cap (\cup_{i=1}^{a}\{\ell_i\})| \geq a-1$ by Observation 2.1. We may assume that $W_0 = \cup_{i=2}^{a}\{\ell_i\} \subseteq S_0$ with $|W_0| = a-1 = n-5$. Then $\text{code}_{G,W_0}(\ell_1) = \text{code}_{G,W_0}(u_2) = \text{code}_{G,W_0}(u_3)$ and $\text{code}_{\overline{G},W_0}(\ell_1) = \text{code}_{\overline{G},W_0}(\ell'_1) = \text{code}_{\overline{G},W_0}(u_2) = \text{code}_{\overline{G},W_0}(u_3)$; moreover, for any $v \in V(G) - W_0$, $W_0 \cup \{v\}$ fails to be a resolving set for $\overline{G}$. So, $\dim(G) \geq |W_0|+1$ and $\dim(\overline{G}) \geq |W_0|+2$. Since $W_0 \cup \{u_2\}$ forms a resolving set for G and $W_0 \cup \{u_2, u_3\}$ forms a resolving set for $\overline{G}$ (as well as for G), $\dim(G) \leq |W_0|+1 = n-4$ and $\dim(\overline{G}) \leq |W_0|+2 = n-3$. Thus, $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = n-3 = \dim(G)+1$.

Third, suppose $a \geq 2$ and $b \geq 2$; then $|S_0 \cap (\cup_{i=1}^{a}\{\ell_i\})| \geq a-1$ and $|S_0 \cap (\cup_{i=1}^{b}\{\ell'_i\})| \geq b-1$ by Observation 2.1. We may assume that $W_1 = (\cup_{i=2}^{a}\{\ell_i\}) \cup (\cup_{i=2}^{b}\{\ell'_i\}) \subseteq S_0$ with $|W_1| = a+b-2 = n-5$. Then W_1 is a resolving set for G, but $\text{code}_{\overline{G},W_1}(\ell_1) = \text{code}_{\overline{G},W_1}(\ell'_1) = \text{code}_{\overline{G},W_1}(u_3)$ and $W_1 \cup \{v\}$ fails to be a resolving set for $\overline{G}$ for any $v \in V(G) - W_1$. So, $\dim(G) = |W_1| = n-5$ and $\dim(\overline{G}) \geq |W_1|+2$. Since $W_1 \cup \{u_2, u_3\}$ forms a resolving set for $\overline{G}$, $\dim(\overline{G}) \leq |W_1|+2$. Thus, $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = n-3 = \dim(G)+2$.

Case 3: G is isomorphic to Figure 2(d). If $a = 1$, then $\{u_2, u_3\}$ forms a basis for G and $\overline{G}$; thus, $\text{Sd}(G, \overline{G}) = \dim(\overline{G}) = \dim(G) = 2 = n-3$. So, suppose $a \geq 2$; then $|S_0 \cap (\cup_{i=1}^{a}\{\ell_i\})| \geq a-1$ and $|S_0 \cap \{u_2, u_4\}| \geq 1$ by Observation 2.1. We may assume that $R = \{u_2\} \cup (\cup_{i=2}^{a}\{\ell_i\}) \subseteq S_0$ with $|R| = a = n-4$. Then $\text{code}_{G,R}(\ell_1) = \text{code}_{G,R}(u_4)$ and $\text{code}_{\overline{G},R}(\ell_1) = \text{code}_{\overline{G},R}(u_4)$; thus,

$\dim(G) \geq |R|+1$ and $\dim(\overline{G}) \geq |R|+1$. Since $R \cup \{u_4\}$ forms a resolving set for both G and $\overline{G}$, $\dim(G) \leq |R|+1$ and $\dim(\overline{G}) \leq |R|+1$. So, $\text{Sd}(G,\overline{G}) = \dim(\overline{G}) = \dim(G) = n-3$.

Case 4: G is isomorphic to Figure 2(e). First, suppose $a = b = 1$. It is easy to check that $\dim(\overline{G}) = 2 = \dim(G)$, but neither S nor $\overline{S}$ forms a resolving set for $\text{Sd}(G,\overline{G})$. There are exactly four following bases for G: $\{\ell_1,\ell'_1\},\{\ell_1,u_4\},\{\ell'_1,u_3\}$ and $\{u_3,u_4\}$. Similarly, there are exactly four following bases for $\overline{G}$: $\{u_1,u_3\},\{u_1,u_4\},\{u_2,u_3\}$ and $\{u_2,u_4\}$. Since $\{u_1,u_3,u_4\}$ forms a resolving set for both G and $\overline{G}$, $\text{Sd}(G,\overline{G}) = \dim(\overline{G})+1 = \dim(G)+1 = 3 = n-3$.

Second, suppose $a \geq 2$ and $b = 1$, or $a = 1$ and $b \geq 2$, say the former by relabeling the vertices of G if necessary; then $|S_0 \cap (\cup_{i=1}^{a}\{\ell_i\})| \geq a-1$ by Observation 2.1. We may assume that $W_0 = \cup_{i=2}^{a}\{\ell_i\} \subseteq S_0$ with $|W_0| = a-1 = n-6$. Note that, for any $v \in V(G)-W_0$, $W_0 \cup \{v\}$ fails to be a resolving set for either G or $\overline{G}$; thus, $|S| \geq |W_0|+2$ and $|\overline{S}| \geq |W_0|+2$. Since $W_0 \cup \{u_2,u_3\}$ forms a resolving set for both G and $\overline{G}$, $|S| \leq |W_0|+2$ and $|\overline{S}| \leq |W_0|+2$. Thus, $\text{Sd}(G,\overline{G}) = \dim(\overline{G}) = n-4 = \dim(G)$.

Third, suppose $a \geq 2$ and $b \geq 2$; then $|S_0 \cap (\cup_{i=1}^{a}\{\ell_i\})| \geq a-1$ and $|S_0 \cap (\cup_{i=1}^{b}\{\ell'_i\})| \geq b-1$ by Observation 2.1. We may assume that $W = (\cup_{i=2}^{a}\{\ell_i\}) \cup (\cup_{i=2}^{b}\{\ell'_i\}) \subseteq S_0$ with $|W| = a+b-2 = n-6$. Then $\text{code}_{G,W}(\ell_1) = \text{code}_{G,W}(u_4)$, $\text{code}_{G,W}(\ell'_1) = \text{code}_{G,W}(u_3)$, and $\text{code}_{\overline{G},W}(\ell_1) = \text{code}_{\overline{G},W}(\ell'_1) = \text{code}_{\overline{G},W}(u_3) = \text{code}_{\overline{G},W}(u_4)$; moreover, for any $v \in V(G)-W$, $W \cup \{v\}$ fails to be a resolving set for $\overline{G}$. So, $\dim(G) \geq |W|+1$ and $\dim(\overline{G}) \geq |W|+2$. Since $W \cup \{u_3\}$ forms a resolving set for G and $W \cup \{u_3,\ell'_1\}$ forms a resolving set for $\overline{G}$, $\dim(G) \leq |W|+1$ and $\dim(\overline{G}) \leq |W|+2$. Thus, $\text{Sd}(G,\overline{G}) = \dim(\overline{G}) = n-4 = \dim(G)+1$. □

A graph G is a *split graph* if $V(G)$ can be partitioned into a clique and an independent set. It is easy to see that the complement of a split graph is a split graph. A *self-complementary graph* is a graph that is isomorphic to its own complement. Next, we construct a family of self-complementary split graphs G with $\text{diam}(G) = 3$. For each integer $k \geq 1$, let G_k be a graph of order $4k$ with $V(G_k) = V_1 \cup V_2$, where $|V_1| = 2k = |V_2|$, such that the edge set of G_k is specified as follows: (i) $V_1 = \cup_{i=0}^{2k-1}\{a_i\}$ induces K_{2k} in G_k; (ii) $V_2 = \cup_{i=0}^{2k-1}\{b_i\}$ induces $\overline{K}_{2k}$ in G_k; (iii) for each $i \in [2k-1]_0$, a_i is adjacent to $b_i,\ldots,b_{i+k-1}$, where the subscript is taken modulo $2k$; (iv) there are no other edges. See Figure 3 for G_3 and the labeling of its vertices.

We recall the concept of a resolving function, which will be used in proving Theorem 3.6. Let g be a function on $V(G)$ with codomain $[0,1]$; for $S \subseteq V(G)$, let $g(S) = \sum_{u \in S} g(u)$. If $g(R_G\{x,y\}) \geq 1$ for any distinct $x,y \in V(G)$, then g is called a *resolving function* of G. We note that $\dim(G) \geq \min\{g(V(G)) : g \text{ is a resolving function of } G\}$; in fact, the preceding inequality would be an equality if the codomain of g is taken to be $\{0,1\}$ in lieu of $[0,1]$.

Theorem 3.6. *For $k \geq 1$, let G_k be the family of split graphs defined above. Then we have the following:*

- (a) *G_k is a self-complementary graph with $\text{diam}(G_k) = 3 = \text{diam}(\overline{G}_k)$;*
- (b) *$\dim(G_k) = \dim(\overline{G}_k) = k$;*
- (c) *$\text{Sd}(G_k,\overline{G}_k) \leq 2k-1$ for $k \geq 2$;*
- (d) *$\text{Sd}(G_k,\overline{G}_k) > k$.*

Proof. Let $k \geq 1$.

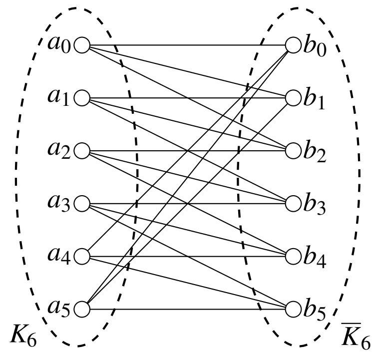


FIGURE 3. G_3 satisfying $\mathrm{Sd}(G_3,\overline{G}_3) > 3 = \dim(G_3) = \dim(\overline{G}_3)$.

(a) First, we show that G_k is self-complementary. Let $\phi : V(G_k) \to V(\overline{G_k})$ be a map such that $\phi(a_i) = b_i$ and $\phi(b_i) = a_{i+1}$, where the subscript is taken modulo $2k$. It is easily checked that ϕ is a graph isomorphism. Second, we show that $\mathrm{diam}(G_k) = 3 = \mathrm{diam}(\overline{G}_k)$. Since G_k is self-complementary, it suffices to show that $\mathrm{diam}(G_k) = 3$. We note the following: (i) $d_G(a_i,a_j) = 1$ for any distinct $i, j \in [2k-1]_0$; (ii) $d_G(a_i,b_j) \in \{1,2\}$ for any $i, j \in [2k-1]_0$; (iii) $d_G(b_i,b_j) \in \{2,3\}$ for any distinct $i, j \in [2k-1]_0$. Since $d_G(b_0,b_k) = 3$, $\mathrm{diam}(G_k) = 3$.

(b) Since G_k is isomorphic to $\overline{G}_k$, it suffices to show that $\dim(G_k) = k$. First, we show that $\dim(G_k) \le k$. Let $S = \{b_0,b_1,\ldots,b_{k-1}\}$ with $|S| = k$. We note the following: (i) $\mathrm{code}_{G_k,S}(a_0)$ has 1 in all of its entries; (ii) for $i \in [k-1]$, $\mathrm{code}_{G_k,S}(a_i)$ has 2 in the first i entries and 1 in the rest of its entries; (iii) $\mathrm{code}_{G_k,S}(a_k)$ has 2 in all of its entries; (iv) for $j \in [k-1]$, $\mathrm{code}_{G_k,S}(a_{k+j})$ has 1 in the first j entries and 2 in the rest of its entries; (v) for $i \in [k-1]_0$, $\mathrm{code}_{G_k,S}(b_{k+i})$ has 3 in the $(i+1)$th entry and 2 in the rest of its entries; (vi) for $i \in [k-1]_0$, $\mathrm{code}_{G_k,S}(b_i)$ has 0 in the ith entry and 2 in the rest of its entries. Thus, S is a resolving set for G_k, and thus $\dim(G_k) \le k$. Next, we show that $\dim(G_k) \ge k$. Let $g : V(G_k) \to [0,1]$ be a minimum resolving function of G_k. Since $R_{G_k}\{a_i,a_{i+1}\} = \{a_i,a_{i+1},b_i,b_{i+k}\}$, we have $g(R_{G_k}\{a_i,a_{i+1}\}) = g(a_i)+g(a_{i+1})+g(b_i)+g(b_{i+k}) \ge 1$ for each $i \in [2k-1]_0$, where the subscript is taken modulo $2k$. By summing over $2k$ such inequalities, we have $2\sum_{i=0}^{2k-1}(g(a_i)+g(b_i)) \ge 2k$, i.e., $g(V(G_k)) \ge k$. So, $\dim(G_k) \ge k$.

(c) Let $k \ge 2$ for this part. It is enough to show that $S = V_1 - \{a_{2k-1}\}$ resolves both G_k and $\overline{G}_k$. Clearly, S resolves $\overline{G}_k$ by part (b). With respect to resolving G_k, notice (1) each pair of vertices of V_1 is resolved by S; (2) each vertex $x \in V_1$ is resolved from any vertex $y \in V_2$ by S since $\mathrm{code}_{G_k,S}(x)$ does not contain 2 as a component, whereas $\mathrm{code}_{G_k,S}(y)$ does; (3) the map $\mathrm{code}_{G_k,S'}$ is already injective on V_2, where $S' = \{a_0,\ldots,a_{k-1}\} \subset S$.

(d) Since $G_1 = P_4$, $\mathrm{Sd}(G_1,\overline{G}_1) = 2 > 1$ by Theorem 2.6. Put $V = V(G_k) = V(\overline{G}_k)$. Let $k \ge 2$ and $S \subseteq V$ be a basis of G_k with $|S_2| = |S \cap V_2| = \beta \le k$; we will show that $\beta = k$. Consider the partition $\mathscr{P}$ of V_1 given by the map code_{G_k,S_2}; i.e, two vertices of V_1 belong to the same cell of $\mathscr{P}$ exactly when they are mapped to the same vector under code_{G_k,S_2}. By the adjacency relation of $b_i \in V_2(G_k)$, it is easily seen that $|\mathscr{P}| = 2\beta$. Since the subgraph induced by V_1 in G_k is a clique, all but one vertex in each cell C_i of $\mathscr{P}$ must belong to S. Thus, we have the inequality

$\sum_{i=1}^{2\beta}(|C_i|-1) \le k-\beta$. Since the left side of the preceding inequality equals $2(k-\beta)$, we find $\beta = k$. It follows that a basis of G_k intersects trivially with V_1 and thus does not resolve $\overline{G}_k$. By symmetry, a basis of $\overline{G}_k$ contains no vertex of V_2 and thus cannot resolve G_k. Therefore, $\mathrm{Sd}(G_k, \overline{G}_k) \ge k+1$ for $k \ge 1$. □

Noting $\mathrm{Sd}(G_1, \overline{G}_1) = 2$ and $\mathrm{Sd}(G_2, \overline{G}_2) = 3$ (as is easily checked), there is some empirical evidence to suggest the following.

Conjecture 3.7. *For $k \ge 3$, $\mathrm{Sd}(G_k, \overline{G}_k) = 2k-1$.*

Towards Conjecture 3.7, the following is an improvement to part (d) of Theorem 3.6, and we thank an anonymous referee for providing both the statement and its proof.

Proposition 3.8. *Let G_k be the family of split graphs defined in Theorem 3.6, where $k \ge 1$. Then $\mathrm{Sd}(G_k, \overline{G}_k) \ge \frac{4}{3}k$.*

Proof. Note that $\mathrm{Sd}(G_1, \overline{G}_1) = 2 \ge \frac{4}{3}$ and $\mathrm{Sd}(G_2, \overline{G}_2) = 3 \ge \frac{8}{3}$. So, let $k \ge 3$ and S be a minimum simultaneous resolving set for both G_k and $\overline{G}_k$. Let $\alpha = |S \cap V_1|$ and $\beta = |S \cap V_2|$. Without loss of generality, assume $\alpha \le \beta$; then $\alpha \le k-1$ by Theorem 3.6(c). Based on the proof for the part (d) of Theorem 3.6, we observe that $\alpha \le k$ implies $\beta \ge 2(k-\alpha)$. If $\alpha \ge \frac{2}{3}k$, then $|S| = \alpha+\beta \ge \frac{4}{3}k$. If $\alpha < \frac{2}{3}k$, then $|S| = \alpha+\beta \ge \alpha+2(k-\alpha) \ge 2k-\frac{2}{3}k = \frac{4}{3}k$. In each case, $\mathrm{Sd}(G_k, \overline{G}_k) \ge \frac{4}{3}k$. □

As an immediate consequence of Theorem 3.6(b) and Proposition 3.8, we have the following.

Remark 3.9. *Let G_k be the family of split graphs defined in Theorem 3.6. Then $\mathrm{Sd}(G_k, \overline{G}_k) - \max\{\dim(G_k), \dim(\overline{G}_k)\} \ge \frac{4}{3}k - k = \frac{k}{3} \to \infty$ as $k \to \infty$.*

4. $\mathrm{Sd}(G, \overline{G})$ when G is a tree or a unicyclic graph

In this section, we examine $\mathrm{Sd}(G, \overline{G})$ when G is a tree or a unicyclic graph.

Trees. For any tree $T \ne P_4$, we show that $\mathrm{Sd}(T, \overline{T}) = \dim(\overline{T}) = \max\{\dim(T), \dim(\overline{T})\}$ and characterize trees T satisfying $\mathrm{Sd}(T, \overline{T}) = \dim(\overline{T}) = \dim(T)$. We also show that, for any tree T that is not a path, $\dim(\overline{T}) \ge \dim(T) + ex(T) - 1 = \sigma(T) - 1$.

We first consider $\mathrm{Sd}(T, \overline{T})$ for $T = P_n$, where $n \ge 2$. We recall the adjacency resolving set and adjacency dimension introduced by Jannesari and Omoomi in [9]. A set $W \subseteq V(G)$ is an *adjacency resolving set* of G if, for any distinct vertices x and y in G, there exists a vertex $z \in W$ such that z is adjacent to exactly one of the two vertices x and y in G. The *adjacency dimension*, $\mathrm{adim}(G)$, of G is the minimum of the cardinalities of all adjacency resolving sets of G.

Proposition 4.1. [9]

(a) *For every graph G, $\mathrm{adim}(G) = \mathrm{adim}(\overline{G})$.*
(b) *If G is a graph with $\mathrm{diam}(G) = 2$, then $\dim(G) = \mathrm{adim}(G)$.*

Proposition 4.2. [9] *For $n \ge 4$, $\mathrm{adim}(C_n) = \mathrm{adim}(P_n) = \lfloor \frac{2n+2}{5} \rfloor$.*

Corollary 4.3. *For $n \ge 2$,*

$$\mathrm{Sd}(P_n, \overline{P}_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor = \begin{cases} 2 = \dim(P_4) + \dim(\overline{P}_4) & \text{if } n = 4, \\ \dim(\overline{P}_n) & \text{otherwise.} \end{cases}$$

Proof. If $n \in \{2,3\}$, then $\mathrm{Sd}(P_n, \overline{P}_n) = 1 = \dim(P_n) = \dim(\overline{P}_n)$ by Theorem 2.7(a). If $n = 4$, then P_4 and $\overline{P_4} \cong P_4$ do not share a common end vertex. Thus, $\mathrm{Sd}(P_4, \overline{P}_4) = 2 = \dim(P_4) + \dim(\overline{P}_4)$ by Theorem 2.6. So, suppose $n \geq 5$; then $\mathrm{diam}(P_n) \geq 4$. Since $\overline{P}_n \ncong K_n$, $\mathrm{diam}(\overline{P}_n) = 2$ by Proposition 2.9. Thus, $\mathrm{Sd}(P_n, \overline{P}_n) = \dim(\overline{P}_n) = \mathrm{adim}(\overline{P}_n) = \mathrm{adim}(P_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 5$, by Theorem 2.10, Proposition 4.1, and Proposition 4.2. □

Theorem 4.4. *For any non-trivial tree $T \neq P_4$, $\mathrm{Sd}(T, \overline{T}) = \dim(\overline{T})$.*

Proof. Let T be a tree of order $n \geq 2$ and let $T \neq P_4$. If $\mathrm{diam}(T) = 1$, then $T = P_2$ and $\mathrm{Sd}(P_2, \overline{P_2}) = 1 = \dim(\overline{P_2}) = \dim(P_2)$. If $\mathrm{diam}(T) = 2$, then $T = K_{1,n-1}$, where $n \geq 3$, and $\mathrm{Sd}(T, \overline{T}) = \dim(T) = n - 2 = \dim(\overline{T})$ by Theorem 2.10, Theorem 3.1 and Observation 2.1. If $\mathrm{diam}(T) = 3$, then $\mathrm{diam}(\overline{T}) \in \{2,3\}$ by Proposition 2.9. If $\mathrm{diam}(\overline{T}) = 2$, then $\mathrm{Sd}(T, \overline{T}) = \dim(\overline{T})$ by Theorem 2.10. If $\mathrm{diam}(\overline{T}) = 3$, then $\mathrm{Sd}(T, \overline{T}) = \dim(\overline{T})$ by Proposition 3.3. If $\mathrm{diam}(T) \geq 4$, then $\mathrm{Sd}(T, \overline{T}) = \dim(\overline{T})$ by Proposition 2.9 and Theorem 2.10. □

Next, we characterize trees T satisfying $\mathrm{Sd}(T, \overline{T}) = \dim(T) = \dim(\overline{T})$. We begin with the following result.

Theorem 4.5. *If T is a tree that is not a path, then* $\dim(\overline{T}) \geq \dim(T) + ex(T) - 1$. *More generally, every non-trivial tree T satisfies the inequality*

$$\dim(\overline{T}) \geq \sigma(T) - 1. \tag{1}$$

Proof. The inequality (1) trivially holds for a path T.

Next, consider a tree T with $ex(T) \geq 2$. Let A denote the set of all paths leading from each $v \in M_1(T)$ to its terminal vertex. Let $B = \bigcup B_v$, where B_v is the set of all paths leading from a major vertex $v \in M_2(T)$ to the terminal vertices associated with v. For each $v \in M_2(T)$, also let $B'_v = \{P - \{v\} : P \in B_v\}$. Assume, for contradiction, that there is a resolving set S of $\overline{T}$ with $|S| \leq \sigma(T) - 2$.

If there exists $v \in S \cap M_2(T)$, then $|S \cap V(T_v)| \geq ter_T(v)$. To see this, assume $|S \cap V(T_v)| < ter_T(v)$ and let x and y be two neighbors of v in two paths of B'_v omitted by S. Note $d_{\overline{T}}(s,x) = 1 = d_{\overline{T}}(s,y)$ for $v \neq s \in S$. Since $ex(T) > 1$, there exists a vertex $z \notin N_T(v) \cup N_T(x) \cup N_T(y)$, and thus z is adjacent to all three vertices x, y, v in $\overline{T}$. Since $x, y \in N_T(v)$, $d_{\overline{T}}(x,v) > 1$ and $d_{\overline{T}}(y,v) > 1$. It follows that $d_{\overline{T}}(x,v) = 2 = d_{\overline{T}}(y,v)$. Therefore, x and y are not resolved by S in $\overline{T}$.

Let the set of vertices $\{u,v,w\} \subseteq M(T)$ such that u is distinct from v and from w, whereas v may equal w, be given. We note the following observation:

No vertex of T_u can resolve vertices x and y in $\overline{T}$ for $\{x,y\} \subseteq V(T_v) \cup V(T_w) - \{v,w\}$. (♣)

Now, if S contains a vertex v_1 with $ter_T(v_1) \geq 2$, we will pass from the triple (S,B,A) to the triple (S_1, B_1, A), where $S_1 = S - (S \cap V(T_{v_1}))$, with $|S_1| \leq \sigma(T) - 2 - ter_T(v_1)$, and $B_1 = B - B_{v_1}$. Through this "descent process", we reach the triple (S_0, B_0, A) where $|S_0| \leq |B_0| + |A| - 2$ and $S_0 \cap M_2(T) = \emptyset$.

Since S_0 omits at least two paths from collection $B_0 \cup A$, S_0 fails in $\overline{T}$ to resolve u' from v', neighbors of u and v along two omitted paths to their end vertices; note that u and v may denote the same major vertex. Now, by observation (♣), the failure of S_0 to resolve u' from v' implies the failure of S to do the same.

Finally, let a tree T with $ex(T)=1$ be given. Assume, to the contrary, that $\dim(\overline{T})<\sigma(T)-1$. Let v be the exterior major vertex of T with $ter_T(v)=\alpha\geq 3$, and let $L_v(T)=\{\ell_1,\ldots,\ell_\alpha\}$. For each $i\in[\alpha]$, let P^i denote the $v-\ell_i$ path excluding v in T. If $v\notin S$, then $S\cap(V(P^i)\cup V(P^j))=\emptyset$ for distinct $i,j\in[\alpha]$; thus, $\text{code}_{\overline{T},S}(\ell_i)=\text{code}_{\overline{T},S}(\ell_j)$, contradicting the assumption that S is a resolving set for $\overline{T}$. So, suppose $v\in S$. Then $S\cap(V(P^a)\cup V(P^b)\cup V(P^c))=\emptyset$ for three distinct $a,b,c\in[\alpha]$; let $d_T(v,\ell_a)\geq d_T(v,\ell_b)\geq d_T(v,\ell_c)$ by relabeling the vertices of T if necessary. If $d_T(v,\ell_b)\geq 2$, then $\text{code}_{\overline{T},S}(\ell_a)=\mathbf{1}=\text{code}_{\overline{T},S}(\ell_b)$, where $\mathbf{1}$ is the all-one vector. If $d_T(v,\ell_b)=1$, then $\text{code}_{\overline{T},S}(\ell_b)=\text{code}_{\overline{T},S}(\ell_c)$, since ℓ_b and ℓ_c are twins in T and $\overline{T}$. In each case, S fails to be a resolving set of $\overline{T}$, contradicting the assumption. Thus, (1) holds for $ex(T)=1$. $\square$

Theorem 4.6. *Let T be a non-trivial tree. Then $\text{Sd}(T,\overline{T})=\dim(\overline{T})=\dim(T)$ if and only if T satisfies one of the following:*

(a) $T\in\{P_2,P_3\}$;
(b) $ex(T)=1$ *with* $v\in M_2(T)$ *such that* $N_T(v)\cap L_v(T)\neq\emptyset$ *and* $d_T(v,\ell)\leq 2$ *for each* $\ell\in L_v(T)$.

Proof. Let T be a tree of order $n\geq 2$. By Corollary 4.3 and Theorem 4.4, it suffices to characterize trees T satisfying $\text{Sd}(T,\overline{T})=\dim(T)$.

($\Leftarrow$) If $T\in\{P_2,P_3\}$, then $\text{Sd}(T,\overline{T})=\dim(T)=1$ by Theorem 2.7(a). If T satisfies (b) of the current theorem such that $v\in M_2(T)$ and $d_T(v,\ell)=1$, where $\ell\in L_v(T)$, then $N_T(v)-\{\ell\}$ forms a basis for both T and $\overline{T}$; thus, $\text{Sd}(T,\overline{T})=\dim(T)$.

($\Rightarrow$) If $ex(T)=0$, then $T\cong P_n$ and $\text{Sd}(P_n,\overline{P}_n)=\dim(P_n)$ implies $n\in\{2,3\}$ by Theorem 2.5(a) and Theorem 2.7(a). If $ex(T)\geq 2$, then $\text{Sd}(T,\overline{T})=\dim(\overline{T})>\dim(T)$ by Theorem 4.4 and Theorem 4.5. So, suppose $ex(T)=1$. Let v be the exterior major vertex of T with $ter_T(v)=k\geq 3$, and let $L_v(T)=\{\ell_1,\ldots,\ell_k\}$; further, let $d_T(v,\ell_1)\geq d_T(v,\ell_2)\geq\cdots\geq d_T(v,\ell_k)$ by relabeling the vertices of T if necessary. For each $i\in[k]$, let $s_i\in N_T(v)$ such that s_i lies on the $v-\ell_i$ path in T, and let P^i denote the $s_i-\ell_i$ path in T. Let S be any basis for $\overline{T}$. Since $\text{Sd}(T,\overline{T})=\dim(T)$ by the hypothesis, there exists some $j\in[k]$ with $S\cap V(P^j)=\emptyset$ by Theorem 3.2 and $d_T(v,\ell_j)=1$; if $d_T(v,\ell_j)\geq 2$, then $\text{code}_{\overline{T},S}(s_j)=\text{code}_{\overline{T},S}(\ell_j)$, contradicting the assumption that S is a resolving set for $\overline{T}$. So, $N_T(v)\cap L_v(T)\neq\emptyset$ and $d_T(v,\ell_k)=1$. We may assume, without loss of generality, that $|S\cap V(P^i)|=1$ for each $i\in[k-1]$ and $S\cap V(P^k)=\emptyset$. It suffices to show that $d_T(v,\ell_1)\leq 2$. Assume, to the contrary, that $d_T(v,\ell_1)=d\geq 3$ and let $v-\ell_1$ path be given by $v=t_0,s_1=t_1,t_2,\ldots,t_d=\ell_1$. If $t_1\in S$, then $\text{code}_{\overline{T},S}(\ell_1)=\text{code}_{\overline{T},S}(\ell_k)$. If $t_2\in S$, then $\text{code}_{\overline{T},S}(t_1)=\text{code}_{\overline{T},S}(t_3)$. If $S\cap(\cup_{i=3}^{d}\{t_i\})\neq\emptyset$, then $\text{code}_{\overline{T},S}(t_1)=\text{code}_{\overline{T},S}(\ell_k)$. So, $d_T(v,\ell_1)\leq 2$. $\square$

Unicyclic graphs. For any unicyclic graph G of order $n\geq 3$, we show that $\text{Sd}(G,\overline{G})\in\{\dim(\overline{G}),1+\dim(\overline{G})\}$; moreover, we show that $\text{Sd}(G,\overline{G})=\dim(\overline{G})$ for $n\geq 7$.

First, we determine $\text{Sd}(C_n,\overline{C}_n)$ for $n\geq 3$. It is well known that $\dim(C_n)=2$, where $n\geq 3$.

Proposition 4.7. *For $n\geq 3$,*

$$\text{Sd}(C_n,\overline{C}_n)=\begin{cases}2 & \text{if } n=3,\\ \lfloor\frac{2n+2}{5}\rfloor & \text{if } n\geq 4.\end{cases}$$

Proof. If $n=3$, then $\text{Sd}(C_3,\overline{C}_3)=2$ by Theorem 2.7(b). If $n\in\{4,5\}$, then $\text{diam}(C_n)=2$; thus, $\text{Sd}(C_n,\overline{C}_n)=\dim(C_n)=2$ by Theorem 2.10. If $n\geq 6$, then $\text{diam}(\overline{C}_n)=2$, and thus

$\text{Sd}(C_n,\overline{C}_n)=\dim(\overline{C}_n)=\text{adim}(\overline{C}_n)=\text{adim}(C_n)=\lfloor\frac{2n+2}{5}\rfloor$ by Theorem 2.10, Proposition 4.1 and Proposition 4.2. □

Next, we consider $\text{Sd}(G,\overline{G})$ for an arbitrary unicyclic graph G.

Lemma 4.8. *Let G be a unicyclic graph of order $n\geq 4$ with $\text{diam}(G)=2$. Then $G\in\{C_4,C_5\}$, or G is isomorphic to the graph H obtained from $K_{1,n-1}$ by joining two end vertices by an edge. Moreover,*

$$\text{Sd}(G,\overline{G})=\dim(G)=\begin{cases}\dim(\overline{G}) & \text{if } G\in\{C_4,C_5\} \text{ or } G{=}H \text{ with } n\geq 5,\\ \dim(\overline{G})+1 & \text{if } G=H \text{ with } n=4.\end{cases}$$

Proof. Let G be a unicyclic graph of order $n\geq 4$ with $\text{diam}(G)=2$, and let $\mathscr{C}=C_m$ be the unique cycle of a unicyclic graph G given by $u_1,u_2,\ldots,u_m,u_1$, where $m\geq 3$. Then $m\in\{3,4,5\}$, since $m\geq 6$ implies $\text{diam}(G)\geq 3$.

First, suppose $m=3$. Since $\text{diam}(G)=2$, $\mathscr{C}$ must contain exactly one major vertex with all its terminal vertices adjacent to it (note that G is isomorphic to H); let $\ell_1,\ell_2,\ldots,\ell_{n-3}$ be the terminal vertices of its major vertex, say u_1, in G with $d_G(u_1,\ell_i)=1$, where $i\in[n-3]$. If $n=4$, then $\{u_2,u_3\}$ forms a basis for G, wherase $\{u_2\}$ forms a basis for $\overline{G}$; thus $\text{Sd}(G,\overline{G})=\dim(G)=2=1+\dim(\overline{G})$. If $n\geq 5$, then $\text{Sd}(G,\overline{G})=\dim(G)=n-3=\dim(\overline{G})$: (i) for any resolving set S of either G or $\overline{G}$, $|S\cap\{u_2,u_3\}|\geq 1$ and $|S\cap(\cup_{i=1}^{n-3}\{\ell_i\})|\geq n-4$ by Observation 2.1; (ii) $\{u_2\}\cup(\cup_{i=2}^{n-3}\{\ell_i\})$ forms a resolving set for both G and $\overline{G}$.

Next, suppose $m\in\{4,5\}$. Since $\text{diam}(G)=2$, $G=\mathscr{C}$; thus $G\in\{C_4,C_5\}$. Since any two adjacent vertices of G form a resolving set for both G and $\overline{G}$, $\text{Sd}(G,\overline{G})=\dim(G)=2=\dim(\overline{G})$. □

Proposition 4.9. *Let G be a unicyclic graph of order at least three. Let H_1 be the graph obtained from $K_{1,3}$ by joining two end vertices by an edge, and let H_2 be the graph obtained from P_6 by adding an edge between the two support vertices. Then*

$$\text{Sd}(G,\overline{G})=\begin{cases}\dim(G)=\dim(\overline{G})+1=2 & \text{if } G=H_1,\\ \dim(G)+1=\dim(\overline{G})+1 & \text{if } G=H_2,\\ \dim(\overline{G}) & \text{otherwise}.\end{cases}$$

Proof. Let G be a unicyclic graph. We consider three cases as follow: (i) $\text{diam}(G)=1$, (ii) $\text{diam}(G)=2$ or $\text{diam}(\overline{G})=2$, and (iii) $\text{diam}(G)=3=\text{diam}(\overline{G})$.

First, suppose $\text{diam}(G)=1$; then $G=C_3$ and $\text{Sd}(C_3,\overline{C}_3)=2=\dim(\overline{C}_3)=\dim(C_3)$ by Theorem 2.7(b).

Second, suppose $\text{diam}(G)=2$ or $\text{diam}(\overline{G})=2$. If $\text{diam}(G)=2$, then $\text{Sd}(H_1,\overline{H}_1)=2=\dim(H_1)=1+\dim(\overline{H}_1)$ and $\text{Sd}(G,\overline{G})=\dim(\overline{G})=\dim(G)$ for $G\neq H$ by Lemma 4.8. If $\text{diam}(\overline{G})=2$, then $\text{Sd}(G,\overline{G})=\dim(\overline{G})$ by Theorem 2.10.

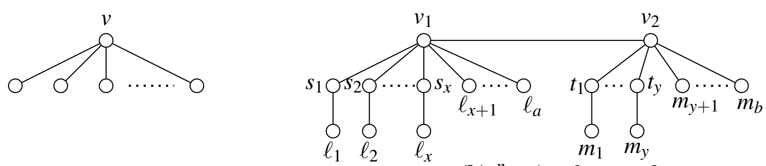
Third, suppose $\text{diam}(G)=3=\text{diam}(\overline{G})$. Then $\text{Sd}(H_2,\overline{H}_2)=\dim(\overline{H}_2)+1=\dim(H_2)+1$ and $\text{Sd}(G,\overline{G})=\dim(\overline{G})$ for $G\neq H_2$ by Proposition 3.5. □

5. Some remarks and open problems

In this section, we provide a realization result for $\text{Sd}(G,\overline{G})$. In view of Corollary 2.4, we provide examples showing that $\min\{\dim(G)+\dim(\overline{G}),|V(G)|-1\}-\text{Sd}(G,\overline{G})$ can be arbitrarily large. We conclude this paper with some open problems.

Theorem 5.1. *For integers n,k with $\frac{n}{2}-1 \le k \le n-1$, there exists a connected graph G of order n with* $\mathrm{Sd}(G,\overline{G}) = k$.

Proof. The case $k = n-1$ is addressed by part (b) of Theorem 2.7. Let a pair of integers (n,k) with $n \ge 5$ and $\frac{n}{2}-1 \le k \le n-2$ be given. If G is the tree in Figure 4(a), then all but one vertex in $N_G(v)$ forms a basis for both G and $\overline{G}$, and thus $k = n-2$. Now, suppose G is the tree in Figure 4(b) with $ex(T) = 2$, $ter_G(v_1) = a \ge 2$ and $ter_G(v_2) = b \ge 2$, where $0 \le x \le a-1$ and $0 \le y \le b-1$. If $x = 1$, let $S_1 = N_G(v_1) \cup N_G(v_2) - \{v_2, \ell_a, m_b\}$; if $x \ne 1$, let $S_2 = N_G(v_1) \cup N_G(v_2) - \{v_1, v_2, \ell_a\}$. Note that $|S_1| = |S_2| = \sigma(G) - 1$. Then $k = \dim(\overline{G}) = \sigma(G) - 1$, where $\frac{n}{2} - 1 \le k = n - (x+y+3) \le n-3$, by Theorem 4.4, Theorem 4.5, and the fact that S_1 (when $x = 1$) or S_2 (when $x \ne 1$) is a resolving set for $\overline{G}$. ☐



(a) $k = n-2$ **(b)** $\frac{n}{2} - 1 \le k \le n-3$

FIGURE 4. Realization graphs G such that $|V(G)| = n$ and $\mathrm{Sd}(G,\overline{G}) = k$, where $\frac{n}{2} - 1 \le k \le n-2$.

Proposition 5.2. *There is a family of graphs G such that* $\min\{\dim(G) + \dim(\overline{G}), |V(G)| - 1\} - \mathrm{Sd}(G,\overline{G})$ *can be arbitrarily large.*

Proof. Let G be the tree in Figure 5 with $k \ge 2$ exterior major vertices $v_1, \ldots, v_k$ such that $ter_G(v_i) = 2$ for each $i \in [k]$. Then $\dim(G) = k$ by Theorem 3.1 and $\mathrm{Sd}(G,\overline{G}) = \dim(\overline{G}) = 2k-1$ by Theorem 4.4, Theorem 4.5 and the fact that $(\cup_{i=1}^{k-1} L_{v_i}(G)) \cup \{\ell_k\}$ forms a resolving set for $\overline{G}$. So, $\dim(G) + \dim(\overline{G}) = 3k - 1 = |V(G)| - 1$ and $\min\{\dim(G) + \dim(\overline{G}), |V(G)| - 1\} - \mathrm{Sd}(G,\overline{G}) = 3k - 1 - (2k-1) = k$ can be arbitrarily large, as $k \to \infty$. ☐

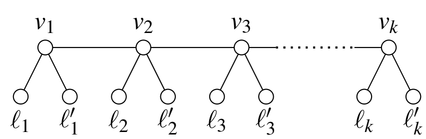


FIGURE 5. A graph G such that $\min\{\dim(G) + \dim(\overline{G}), |V(G)| - 1\} - \mathrm{Sd}(G,\overline{G})$ can be arbitrarily large, where $k \ge 2$.

We conclude this paper with some open problems.

Question 1. Let G and $\overline{G}$ be connected graphs of order $n \ge 4$. Then $2 \le \mathrm{Sd}(G,\overline{G}) \le n-2$ by Corollary 2.4 and Theorem 2.7. Can we characterize graphs G satisfying $\mathrm{Sd}(G,\overline{G})$ equals 2 and $n-2$, respectively?

Question 2. Note that $\mathrm{Sd}(P_4, \overline{P}_4) = \dim(P_4) + \dim(\overline{P}_4)$. Is there any other graph G such that $\mathrm{Sd}(G, \overline{G}) = \dim(G) + \dim(\overline{G})$?

Question 3. If $\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\} \neq \{3\}$, then $\mathrm{Sd}(G, \overline{G}) = \max\{\dim(G), \dim(\overline{G})\}$ by Theorem 2.10. Can we characterize graphs G such that $\mathrm{diam}(G) = 3 = \mathrm{diam}(\overline{G})$ and $\mathrm{Sd}(G, \overline{G}) = \max\{\dim(G), \dim(\overline{G})\}$?

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TEXAS A&M UNIVERSITY AT GALVESTON, GALVESTON, TX 77553, USA
Email address: kangc@tamug.edu

UNIVERSITY OF MARIBOR, FEECS KOROŠKA C. 46, 2000 MARIBOR, SLOVENIA
Email address: iztok.peterin@um.si

TEXAS A&M UNIVERSITY AT GALVESTON, GALVESTON, TX 77553, USA
Email address: yie@tamug.edu