DIFFERENCE OF COMPOSITION OPERATORS ON THE MINIMAL MÖBIUS INVARIANT SPACE

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ABSTRACT. We characterize bounded and compact difference of composition operators on the minimal Möbius invariant space. In fact, we provide several equivalent conditions characterizing compact difference of composition operators on the minimal Möbius invariant space.

1. Introduction. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ its boundary, $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ the normalized area measure on $\mathbb{D}$, H^∞ the space of all bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the class of all analytic self-maps of $\mathbb{D}$. For $z \in \mathbb{D}$, let η_z be the conformal automorphism of $\mathbb{D}$ that interchanges 0 and z:

$$\eta_z(w) = \frac{z-w}{1-\overline{z}w},\ w \in \mathbb{D}.$$

The pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\eta_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|.$$

The analytic Besov space denoted by B^1 is the space of all analytic functions f for which

$$f(w) = \sum_{n=0}^{\infty} a_n \eta_{z_n}(w)$$

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for some sequence $\{a_n\} \in l^1$ and $\{z_n\}$ in $\mathbb{D}$. Using this representation of the space B^1, the norm $\|f\|_{B^1}$ is defined by

$$\|f\|_{B^1} = \inf\left\{\sum_{n=0}^{\infty} |a_n| : f(w) = \sum_{n=0}^{\infty} a_n \eta_{z_n}(w)\right\}.$$

It is known that B^1 is minimal among all the Möbius invariant spaces, as it is contained in all Möbius invariant spaces and so also known as minimal Möbius invariant space. It is also known that an analytic function f is in B^1 if and only if $f'' \in A^1$, where A^1 is the Bergman space consisting of analytic functions f such that

$$\|f\|_{A^1} = \int_{\mathbb{D}} |f(w)| dA(w) < \infty.$$

Moreover, B^1 is a Banach space under the norm defined as

$$\|f\|_{B^1} \asymp |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(w)| dA(w).$$

For the detailed study of Besov space B^1 one can refer to **[1, 2, 3, 4, 6, 11, 12, 14, 16, 31, 32]** and references therein. Let B^1_{00} be a subspace of B^1 defined as

$$B^1_{00} = \{f \in B^1 : f(0) = f'(0) = 0\}.$$

Then any function f in B^1_{00} has a nice property first observed by Oscar Blasco in **[3]**. An analytic function f is in B^1_{00} if and only if there exists a complex Borel measure μ of bounded variation on $\mathbb{D}$ such that

$$f(w) = \int_{\mathbb{D}} \eta_z(w) d\mu(z).$$

Moreover,

$$\int_{\mathbb{D}} |f''(w)| dA(w) \asymp \inf\left\{\|\mu\| : f(w) = \int_{\mathbb{D}} \eta_z(w) d\mu(z)\right\}.$$

It is well known that $B^1 \subset H^\infty$. Moreover, for every $w \in \mathbb{D}$, the following growth estimates hold:

$$|f(w)| \le \|f\|_\infty \le \|f\|_{B^1}, \quad \text{and} \quad |f'(w)| \le \frac{\|f\|_{B^1}}{1-|w|} \tag{1.1}$$

for every $f \in B^1$.

Let $\varphi \in S(\mathbb{D})$. Then φ induces a linear operator C_φ defined as

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbb{D})$. This operator is extensively studied on analytic function spaces. An excellent source for the development of the theory of composition operators is **[13]**.

Shapiro and Sundberg **[23]** initiated the study of compact difference $C_\varphi - C_\psi$ of composition operators on H^2, however, no complete characterization of compact difference of composition operators on H^2 exists so far. MacCluer, Ohno and Zhao in **[18]** characterized compact difference of composition operators on H^∞. Moorhouse **[19]** solved the problem of compact difference of composition operators in the Bergman space setting and the corresponding problem in the Hardy space setting was recently solved by Choe, Choi, Koo and Park in **[10]**. For more studies on compact difference of composition operators on analytic function spaces, we refer **[5]** **[7]**-**[9]**, **[15]**-**[30]** and references therein. In this paper, we characterize compact difference of composition operators on the minimal Möbius invariant space. In fact, we provide three equivalent conditions characterizing compact difference of composition operators on the minimal Möbius invariant space.

Throughout this paper, constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \lesssim B$ means that A is less than or equal to a constant multiple of B and $D \gtrsim E$, means that a constant multiple of D is greater than or equal to E. When $A \lesssim B$ as well as $A \gtrsim B$, then we write $A \asymp B$.

2. Boundedness of $C_\varphi - C_\psi : B^1 \to B^1$.

Theorem 1. *Let $\varphi, \psi \in S(\mathbb{D})$. Then $C_\varphi - C_\psi : B^1 \to B^1$ is bounded if and only if $\varphi'' - \psi'' \in A^1$ and the following family of functions*

$$\{(\eta_z'' \circ \varphi)(\varphi')^2 - (\eta_z'' \circ \psi)(\psi')^2 + (\eta_z' \circ \varphi)\varphi'' - (\eta_z'' \circ \psi)\psi'' : z \in \mathbb{D}\}$$

is norm bounded with respect to A^1-norm, that is,

$$\|\varphi'' - \psi''\|_{A^1} = \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)| dA(w) < \infty$$

and

$$M = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \bigg| 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2$$

$$+ \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\bigg| dA(z) < \infty. \tag{2.2}$$

Moreover, the following inequality holds

$$\|\varphi'' - \psi''\|_{A^1} + M \lesssim \|C_\varphi - C_\psi\|_{B^1\to B^1}$$

$$\lesssim 1 + \frac{1}{1-|\varphi(0)|} + \frac{1}{1-|\psi(0)|} + \|\varphi'' - \psi''\|_{A^1} + M. \tag{2.3}$$

Proof. First suppose that (2.2) holds. Let the Maclurian series expansion of $f \in B^1$ be given by

$$f(w) = \sum_{n=0}^{\infty} a_n w^n, \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

Set

$$g(w) = \sum_{n=2}^{\infty} a_n w^n$$

Then $g(w) = f(w) - f(0) - wf'(0)$, $g'(w) = f'(w) - f'(0)$ and $g^{(n)}(w) = f^{(n)}(w)$ for all $n \geq 2$. Since f, and all polynomials are in B^1, so it holds that $g \in B^1$. Moreover, $g(0) = g'(0) = 0$. Thus there exists a complex Borel measure μ of bounded variation on $\mathbb{D}$ such that with $\|\mu\| \asymp \|g\|_{B^1}$ and

$$g(w) = \int_{\mathbb{D}} \eta_z(w) d\mu(z)$$

and

$$\|\mu\| \asymp \|g\|_{B^1} = \int_{\mathbb{D}} |g''(w)| dA(w)$$

$$\lesssim |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(w)| dA(w) = \|f\|_{A^1}.$$

Therefore, it holds that

$$f(w) = f(0) + wf'(0) + \int_{\mathbb{D}} \eta_z(w) d\mu(z).$$

Then, we have that

$$f'(w) = f'(0) + \int_{\mathbb{D}} \eta_z'(w) d\mu(z) \tag{2.4}$$

and

$$f''(w) = \int_{\mathbb{D}} \eta_z''(w) d\mu(z). \tag{2.5}$$

Replacing w in (2.4) by $\varphi(w)$, and multiplying such obtained inequality by $\varphi''(w)$, we obtain

$$f'(\varphi(w))\varphi''(w) = f'(0)\varphi''(w) + \int_{\mathbb{D}} \eta_z'(\varphi(w))\varphi''(w) d\mu(z), \tag{2.6}$$

Similarly, we get

$$f'(\psi(w))\psi''(w) = f'(0)\psi''(w) + \int_{\mathbb{D}} \eta_z'(\psi(w))\psi''(w) d\mu(z), \tag{2.7}$$

Again, replacing w in (2.5) by $\varphi(w)$, and multiplying such obtained inequality by $(\varphi'(w))^2$, we obtain

$$f''(\varphi(w))(\varphi'(w))^2 = \int_{\mathbb{D}} \eta_z''(\varphi(w))(\varphi'(w))^2 d\mu(z). \tag{2.8}$$

Similarly, we have that

$$f''(\psi(w))(\psi'(w))^2 = \int_{\mathbb{D}} \eta_z''(\psi(w))(\varphi'(w))^2 d\mu(z). \tag{2.9}$$

From (2.6), (2.7), (2.8) and (2.9), we obtain that

$$\begin{aligned} &|f''(\varphi(w))(\varphi'(w))^2 - f''(\psi(w))(\psi'(w))^2 + f'(\varphi(w))\varphi''(w) - f'(\psi(w))\psi''(w)| \\ &\ \leq |f'(0)||\varphi''(w) - \psi''(w)| + \int_{\mathbb{D}} |\eta_z''(\varphi(w))(\varphi'(w))^2 - \eta_z''(\psi(w))(\varphi'(w))^2 \\ &\ + \eta_z'(\varphi(w))\varphi''(w) - \eta_z'(\psi(w))\psi''(w)| d|\mu|(z) \end{aligned} \tag{2.10}$$

Integrating (2.10) with respect to $dA(w)$ and applying Fubini's theorem, we have that

$$
\begin{aligned}
&\int_{\mathbb{D}} |f''(\varphi(w))(\varphi'(w))^2 - f''(\psi(w))(\psi'(w))^2 \\
&\qquad + f'(\varphi(w))\varphi''(w) - f'(\psi(w))\psi''(w)|dA(w) \\
&\quad \leq |f'(0)| \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)|dA(w) \\
&\quad + \int_{\mathbb{D}} \left[\int_{\mathbb{D}} |\eta''_z(\varphi(w))(\varphi'(w))^2 - \eta''_z(\psi(w))(\varphi'(w))^2 \right. \\
&\quad \left. + \eta'_z(\varphi(w))\varphi''(w) - \eta'_z(\psi(w))\psi''(w)|dA(w) \right] d|\mu|(z).
\end{aligned} \tag{2.11}
$$

Using the facts that

$$
\eta'_z(w) = -\frac{1-|z|^2}{(1-\overline{z}w)^2} \quad \text{and} \quad \eta''_z(w) = -2\overline{z}\frac{1-|z|^2}{(1-\overline{z}w)^3},
$$

we see that

$$
\begin{aligned}
&\int_{\mathbb{D}} |\eta''_z(\varphi(w))(\varphi'(w))^2 - \eta''_z(\psi(w))(\varphi'(w))^2 \\
&\qquad + \eta'_z(\varphi(w))\varphi''(w) - \eta'_z(\psi(w))\psi''(w)|dA(w) \\
&\quad = \int_{\mathbb{D}} \left| 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 \right. \\
&\qquad \left. + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w) \right| dA(z) \leq M.
\end{aligned} \tag{2.12}
$$

From (2.11) and (2.12), we have that

$$
\begin{aligned}
&\int_{\mathbb{D}} |((C_\varphi - C_\psi)f)''(w)|dA(w) \leq \|f\|_{B^1} \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)|dA(w) \\
&\quad + \int_{\mathbb{D}} \left[\int_{\mathbb{D}} |\eta''_z(\varphi(w))(\varphi'(w))^2 - \eta''_z(\psi(w))(\varphi'(w))^2 \right. \\
&\quad \left. + \eta'_z(\varphi(w))\varphi''(w) - \eta'_z(\psi(w))\psi''(w)dA(w) \right] |d|\mu|(z) \\
&\quad \leq \|f\|_{B^1}\|\varphi'' - \psi''\|_{A^1} + M\int_{\mathbb{D}} |d|\mu|(z) \\
&\quad \lesssim (\|\varphi'' - \psi''\|_{A^1} + M)\|f\|_{B^1}
\end{aligned} \tag{2.13}
$$

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Again, using (1.1) and the fact that

$$
\begin{aligned}
&|((C_\varphi - C_\psi)f)(0)| + |((C_\varphi - C_\psi)f)'(0)| \\
&\le |f(\varphi(0))| + |f(\psi(0))| + |f'(\varphi(0))||\varphi'(0)| + |f'(\psi(0))||\psi'(0)| \\
&\lesssim \left\{1 + \frac{|\varphi'(0)|}{1-|\varphi(0)|} + \frac{|\psi'(0)|}{1-|\psi(0)|}\right\}\|f\|_{B^1}. \qquad (2.14)
\end{aligned}
$$

Combining (2.13) and (2.14), we have that

$$
\|(C_\varphi - C_\psi)f\|_{B^1} \le \left\{1+\frac{|\varphi'(0)|}{1-|\varphi(0)|}+\frac{|\psi'(0)|}{1-|\psi(0)|}+\|\varphi''-\psi''\|_{A^1}+M\right\}\|f\|_{B^1}.
$$

Thus $C_\varphi - C_\psi : B^1 \to B^1$ is bounded and

$$
\|C_\varphi - C_\psi\|_{B^1\to B^1} \le 1 + \frac{|\varphi'(0)|}{1-|\varphi(0)|} + \frac{|\psi'(0)|}{1-|\psi(0)|} + \|\varphi'' - \psi''\|_{A^1} + M. \qquad (2.15)
$$

Conversely, assume that $C_\varphi - C_\psi : B^1 \to B^1$ is bounded. By taking $f(z) = z$, we can easily show that

$$
\|\varphi'' - \psi''\|_{A^1} = \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)| dA(w) < \infty.
$$

Since $C_\varphi - C_\psi : B^1 \to B^1$ is bounded, it follows that $(C_\varphi - C_\psi)f \in B^1$ for every $f \in B^1$. In particular, by taking $\eta_z \in B^1$, we have that $((C_\varphi - C_\psi)\eta_z)'' \in A^1$ for each $z \in \mathbb{D}$. Therefore,

$$
(\eta_z'' \circ \varphi)(\varphi')^2 - (\eta_z'' \circ \psi)(\psi')^2 + (\eta_z' \circ \varphi)\varphi'' - (\eta_z'' \circ \psi)\psi'' \in A^1
$$

for every $z \in \mathbb{D}$. Moreover, $\|\eta_z\|_{B^1} \lesssim 1$ for each $z \in \mathbb{D}$ and

$$
\begin{aligned}
M = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \bigg| & 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 \\
&+ \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\bigg| dA(z) \\
\le{}& \sup_{z\in\mathbb{D}} \left\|(C_\varphi - C_\psi)\eta_z\right\|_{B^1} \\
\le{}& \|C_\varphi - C_\psi\|_{B^1\to B^1} \sup_{z\in\mathbb{D}} \|\eta_z\|_{B^1} \\
\lesssim{}& \|C_\varphi - C_\psi\|_{B^1\to B^1}. \qquad (2.16)
\end{aligned}
$$

Thus (2.2) holds, as desired. Moreover, by (2.15) and (2.16), the inequality (2.3) holds. □

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As a consequence of the above Theorem, we can easily obtain the norm of difference of composition operators on B^1_{00} space.

Corollary 1. *Let $\varphi, \psi \in S(\mathbb{D})$. Then $C_\varphi - C_\psi : B^1_{00} \to B^1_{00}$ is bounded if and only if φ and ψ satisfy* (2.2). *Moreover, the following equality holds*

$$\|C_\varphi - C_\psi\|_{B^1_{00} \to B^1_{00}} = M.$$

3. Compactness of $C_\varphi - C_\psi : B^1 \to B^1$.

We need the following lemma. The proof follows on same lines as the proof of Proposition 3.11 in **[13]**. The details are omitted.

Lemma 1. *Let $\varphi,\ \psi \in S(\mathbb{D})$ such that $C_\varphi - C_\psi : B^1 \to B^1$ is bounded. Then $C_\varphi - C_\psi : B^1 \to B^1$ is compact if and only if for any norm bounded sequence $\{f_j\}$ in B^1 which converges to zero locally uniformly, then we have that $\lim_{j\to\infty} ||(C_\varphi - C_\psi)f_j||_{B^1} \to 0$.*

Theorem 2. *Let $\varphi,\ \psi \in S(\mathbb{D})$ such that $C_\varphi - C_\psi : B^1 \to B^1$ is bounded. Then the following statements are equivalent:*

(1) $C_\varphi - C_\psi : B^1 \to B^1$ is compact;
(2) $\Gamma : \mathbb{D} \to \mathbb{C}$ is a continuous function of z, where

$$\Gamma(z) = \int_{\mathbb{D}} \bigg| 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\bigg| dA(w).$$

(3) For each given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\nu_z(E) < \varepsilon$ for all $z \in \mathbb{D}$, where

$$\nu_z(E) = \int_{E} \bigg| 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\bigg| dA(w),$$

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whenever $A(E) < \delta$. That is, the family of measures $\{\nu_z : z \in \mathbb{D}\}$ is equi-absolutely continuous.

Proof. (1) $\Rightarrow$ (2). Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $z_j \to z$ as $j \to \infty$. Then $\sup_{j\in\mathbb{N}} ||\eta_{z_j}||_{B^1} \asymp 1$ and $\eta_{z_j} \to \eta_z$ uniformly on compact subsets of $\mathbb{D}$. Since $C_\varphi - C_\psi : B^1 \to B^1$ is compact. By Lemma 1, we have

$$\|(C_\varphi - C_\psi)\eta_{z_j} - (C_\varphi - C_\psi)\eta_z\|_{B^1} \to 0$$

as $j \to \infty$. Thus

$$\begin{aligned}
|\Gamma(z_j) - \Gamma(z)| \leq \int_{\mathbb{D}} \Bigg| & 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^3}(\psi'(w))^2 \\
& + \frac{(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^2}\varphi''(w) - \frac{(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^2}\psi''(w) \\
& - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 + 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 \\
& - \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) + \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\Bigg| dA(w) \\
& \leq \|(C_\varphi - C_\psi)\eta_{z_j} - (C_\varphi - C_\psi)\eta_z\|_{B^1} \to 0. \qquad (3.17)
\end{aligned}$$

as $j \to \infty$. Thus $\Gamma(z)$ is a continuous function of $z \in \mathbb{D}$.
(2) $\Rightarrow$ (3). Suppose that (3) does not hold. Then there exists a sequence $\{z_j\}$ in $\mathbb{D}$ such that $z_j \to z$ and a sequence of Borel sets $\{E_j\}$ in $\mathbb{D}$ such that $A(E_j) \to 0$ as $j \to \infty$, but $\nu_{z_j}(E_j) \geq C > 0$ for all $j \in \mathbb{N}$. Note that

$$\begin{aligned}
|\nu_{z_j}(E_j) - \nu_z(E_j)| \leq \int_{E_j} \Bigg|\Bigg| & 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^3}(\psi'(w))^2 \\
& + \frac{(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^2}\varphi''(w) - \frac{(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^2}\psi''(w)\Bigg| \\
& - \Bigg|2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 \\
& + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\Bigg|\Bigg| dA(w).
\end{aligned}$$

Thus we have that

$$
\begin{aligned}
\nu_{z_j}(E_j) \le \int_{E_j} \Bigg| & \Bigg| 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^3}(\psi'(w))^2 \\
& + \frac{(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^2}\varphi''(w) - \frac{(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^2}\psi''(w) \Bigg| \\
& - \Bigg| 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 \\
& + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w) \Bigg| \Bigg| dA(w) + \nu_z(E_j)
\end{aligned}
\tag{3.18}
$$

Boundedness of $C_\varphi - C_\psi : B^1 \to B^1$ asserts that (2.2) holds. Therefore, $\nu_z(E_j) \to 0$ as $j \to \infty$. Moreover, as earlier, the first term in (3.18) is dominated by a constant times

$$
\begin{aligned}
\int_{E_j} \Bigg| & 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z_j}(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^3}(\psi'(w))^2 \\
& + \frac{(1-|z_j|^2)}{(1-\overline{z_j}\varphi(w))^2}\varphi''(w) - \frac{(1-|z_j|^2)}{(1-\overline{z_j}\psi(w))^2}\psi''(w) - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 \\
& + 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}(\psi'(w))^2 - \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) + \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w) \Bigg| dA(w).
\end{aligned}
$$

Therefore, $\nu_{z_j}(E_j) \to 0$ as $j \to \infty$, a contradiction. This shows that (2) $\Rightarrow$ (3).
(3) $\Rightarrow$ (1). Let $\{f_j\}$ be a sequence in B^1 such that $\sup_j ||f_j||_{B^1} \lesssim 1$ and $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$. We have to show that $||(C_\varphi - C_\psi)f_j||_{B^1} \to 0$ as $j \to \infty$. For each j, we can find a complex Borel measure μ_j with $||\mu_j|| \lesssim ||f_j||_{B^1}$ such that $f_j(w) =$

$\int_{\mathbb{D}} \eta_z(w)d\mu_j(z)$. Then as in the proof of Theorem 1, we have that

$$\begin{aligned}
&\int_{\mathbb{D}} |f_j''(\varphi(w))(\varphi'(w))^2 - f_j''(\psi(w))(\psi'(w))^2 \\
&\qquad + f_j'(\varphi(w))\varphi''(w) - f_j'(\psi(w))\psi''(w)|dA(w) \\
&\quad \leq |f_j'(0)| \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)|dA(w) + \int_{\mathbb{D}} \bigg[\int_{\mathbb{D}} |\eta_z''(\varphi(w))(\varphi'(w))^2 \\
&\quad - \eta_z''(\psi(w))(\varphi'(w))^2 + \eta_z'(\varphi(w))\varphi''(w) - \eta_z'(\psi(w))\psi''(w)dA(w)\bigg] |d|\mu_j|(z).
\end{aligned} \tag{3.19}$$

Let $\varepsilon > 0$ be given. Now choose a compact set $K \subset \mathbb{D}$ such that $A(\mathbb{D} \setminus K) < \delta$. Then using the fact that $|f_j'(0)| < \epsilon$ for $j \geq j_0$, we have that

$$\begin{aligned}
&\int_{\mathbb{D}\setminus K} |f_j''(\varphi(w))(\varphi'(w))^2 - f_j''(\psi(w))(\psi'(w))^2 \\
&\qquad + f_j'(\varphi(w))\varphi''(w) - f_j'(\psi(w))\psi''(w)|dA(w) \\
&\quad \leq |f_j'(0)| \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)|dA(w) \\
&\quad + \int_{\mathbb{D}} \bigg[\int_{\mathbb{D}\setminus K} |\eta_z''(\varphi(w))(\varphi'(w))^2 - \eta_z''(\psi(w))(\varphi'(w))^2 \\
&\quad + \eta_z'(\varphi(w))\varphi''(w) - \eta_z'(\psi(w))\psi''(w)dA(w)\bigg] |d|\mu_j|(z) \\
&\quad < \varepsilon \bigg[\|\varphi'' - \psi''\|_{A^1} + \int_{\mathbb{D}} d|\mu_j|(z) \bigg] \lesssim \varepsilon.
\end{aligned} \tag{3.20}$$

On K, $|f_j'(\varphi(w))| < \varepsilon$ and $|f_j''(\psi(w))| < \varepsilon$ as $j \geq j_0$. Thus

$$\begin{aligned}
&\int_K |f_j''(\varphi(w))(\varphi'(w))^2 - f_j''(\psi(w))(\psi'(w))^2 \\
&\qquad + f_j'(\varphi(w))\varphi''(w) - f_j'(\psi(w))\psi''(w)|dA(w) \\
&\quad \leq \int_K |f_j''(\varphi(w))||\varphi'(w)|^2 + |f_j''(\psi(w))||\psi'(w)|^2 \\
&\qquad + |f_j'(\varphi(w))||\varphi''(w)| + |f_j'(\psi(w))||\psi''(w)|dA(w) \\
&\quad < \varepsilon \sup_{w \in K} \left(|\varphi'(w)|^2 + |\psi'(w)|^2 + |\varphi''(w)| + |\psi''(w)| \right) \lesssim \varepsilon
\end{aligned} \tag{3.21}$$

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as $j \geq j_0$. Therefore, using the fact that $|f_j(0)| < \varepsilon$, $|f_j'(0)| < \varepsilon$ for $j \geq j_0$, (3.20) and (3.21), we have that $||(C_\varphi - C_\psi)f_j||_{B^1} \to 0$ as $j \to \infty$. □

Theorem 3. *Let φ, $\psi \in S(\mathbb{D})$ such that both $C_\varphi, C_\psi : B^1 \to B^1$ are bounded, but not compact and* $\min\{|\varphi(w)|, |\psi(w)|\} \to 1$ *as* $|w| \to 1$. *Then $C_\varphi - C_\psi : B^1 \to B^1$ is compact if and only if φ and ψ satisfy the following condition*

$$\lim_{r\to 1}\sup_{z\in\mathbb{D}}\int_{\min\{|\varphi(w)|,|\psi(w)|\}>r}\Bigg|2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}$$

$$\times(\psi'(w))^2 + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\Bigg|dA(w) = 0. \tag{3.22}$$

Proof. Suppose that $C_\varphi - C_\psi : B^1 \to B^1$ is compact. Since $C_\varphi, C_\psi : B^1 \to B^1$ are bounded, so $\varphi, \psi \in B^1$ and $B^1 \subset \mathcal{D}$ so $\varphi, \psi \in \mathcal{D}$ also, where $\mathcal{D}$ is the Dirichlet space. Thus for each $\epsilon > 0$, we can choose $r \in (0,1)$ such that

$$\int_{\min(|\varphi(z)|,|\psi(z)|)>r}|\varphi'(z)|^2 dA(z) < \varepsilon, \quad \int_{\min(|\varphi(z)|,|\psi(z)|)>r}|\psi'(z)|^2 dA(z) < \varepsilon, \tag{3.23}$$

$$\int_{\min(|\varphi(z)|,|\varphi(z)|)>r}|\psi''(z)|dA(z) < \varepsilon \text{ and } \int_{\min(|\varphi(z)|,|\psi(z)|)>r}|\psi''(z)|dA(z) < \varepsilon. \tag{3.24}$$

Let $\mathcal{B}_{B^1}$ be the unit ball of B^1 and $f \in \mathcal{B}_{B^1}$. Let $f_t(z) = f(tz)$, where $t \in (0,1)$. Then for all $t \in (0,1)$, $f_t \in B^1$ and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. Moreover, $\sup_{0<t<1}\|f_t\|_{B^1} \leq \|f\|_{B^1}$. The compactness of $C_\varphi - C_\psi : B^1 \to B^1$ asserts that

$$\|(C_\varphi - C_\psi)f_t - (C_\varphi - C_\psi)f\|_{B^1} \to 0 \quad \text{as } t \to 1.$$

Hence for every $\varepsilon > 0$, there is a $t \in (0,1)$ such that

$$\int_{\mathbb{D}}|((C_\varphi - C_\psi)f_t(w))'' - ((C_\varphi - C_\psi)f(w))''|dA(w) < \varepsilon. \tag{3.25}$$

Inequalities (3.23), (3.24) and (3.25), yield that

$$\begin{aligned}
&\int_{\min(|\varphi(z)|,|\psi(z)|)>r} |((C_\varphi - C_\psi)f)''(z)|^2 dA(z) \\
&\le \int_{\mathbb{D}} |((C_\varphi - C_\psi)f_t)''(z) - ((C_\varphi - C_\psi)f)''(z)| dA(z) \\
&\quad + \int_{\min(|\varphi(z)|,|\psi(z)|)>r} |((C_\varphi - C_\psi)f_t)''(z)| dA(z) \\
&\lesssim \varepsilon(1 + \|f_t'\|_\infty + \|f_t''\|_\infty).
\end{aligned}$$

Thus for every $f \in \mathbf{B}_{B^1}$, the unit ball in B^1, there is a $\delta_0 \in (0,1)$ such that for $r \in (\delta_0, 1)$

$$\begin{aligned}
&\int_{\min(|\varphi(w)|,|\psi(w)|)>r} |f''(\varphi(w))(\varphi'(w))^2 - f''(\psi(w))(\psi'(w))^2 \\
&\qquad + f'(\varphi(w))\varphi''(w) - f'(\psi(w))\psi''(w)| dA(w) < \varepsilon. \qquad (3.26)
\end{aligned}$$

The compactness of $C_\varphi - C_\psi : B^1 \to B^1$, asserts that for every $\varepsilon > 0$ there is a finite collection of functions $f_1, f_2, \ldots, f_n \in \mathbf{B}_{B^1}$ such that for each $f \in \mathcal{B}_{B^1}$, there is a $j \in \{1, 2, \ldots, n\}$ such that

$$\int_{\mathbb{D}} |((C_\varphi - C_\psi)f)''(w) - ((C_\varphi - C_\psi)f_j)''(w)|^2 dA(z) < \varepsilon. \qquad (3.27)$$

On the other hand, from (3.26) it follows that if $\delta := \max\{\delta_j : j = 1, 2, \cdots n\}$, then for $r \in (\delta, 1)$ such that for all $j \in \{1, 2, \cdots, n\}$ we have that

$$\begin{aligned}
&\int_{\min(|\varphi(w)|,|\psi(w)|)>r} |f_j''(\varphi(w))(\varphi'(w))^2 - f_j''(\psi(w))(\psi'(w))^2 \\
&\qquad + f_j'(\varphi(w))\varphi''(w) - f_j'(\psi(w))\psi''(w)| dA(w) < \varepsilon. \qquad (3.28)
\end{aligned}$$

From (3.27) and (3.28), we have that for $r \in (\delta, 1)$ and every $f \in \mathcal{B}_{B^1}$

$$\begin{aligned}
&\int_{\min(|\varphi(w)|,|\psi(w)|)>r} |f''(\varphi(w))(\varphi'(w))^2 - f''(\psi(w))(\psi'(w))^2 \\
&\qquad + f'(\varphi(w))\varphi''(w) - f'(\psi(w))\psi''(w)| dA(w) < 2\varepsilon. \qquad (3.29)
\end{aligned}$$

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Taking $f(w) = \eta_z(w)$, $z \in \mathbb{D}$ in (3.29), we have that

$$\begin{aligned}\int_{\min(|\varphi(w)|,|\psi(w)|)>r} &|\eta_z''(\varphi(w))(\varphi'(w))^2 - \eta_z''(\psi(w))(\psi'(w))^2 \\ &+ \eta_z'(\varphi(w))\varphi''(w) - \eta_z'(\psi(w))\psi''(w)|dA(w) < 2\varepsilon. \qquad (3.30)\end{aligned}$$

From (3.30) we can easily see that (3.22) holds.
Conversely, suppose that (3.22) holds. Let $(f_j)_{j\in\mathbb{N}}$ be a norm bounded sequence in B^1 which converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Then f_j' and f_j'' also converges to 0 uniformly on compact subsets of $\mathbb{D}$, for each $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, we can find a complex measure μ_j with $\|\mu_j\| \lesssim \|f_j\|_{B^1}$ such that

$$\begin{aligned}&\int_{\min\{|\varphi(w)|,|\psi(w)|\}>r} |f_j''(\varphi(w))(\varphi'(w))^2 - f_j''(\psi(w))(\psi'(w))^2 \\ &\qquad + f_j'(\varphi(w))\varphi''(w) - f_j'(\psi(w))\psi''(w)|dA(w) \\ &\leq |f_j'(0)| \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)|dA(w) \\ &+ \int_{\mathbb{D}} \left[\int_{\min\{|\varphi(w)|,|\psi(w)|\}>r} \left| 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3} \right.\right. \\ &\left.\left. \times (\psi'(w))^2 + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\right| dA(w) \right] d|\mu_j|(z). \qquad (3.31)\end{aligned}$$

By the condition in (3.22), we have that for every $\varepsilon > 0$, there is an $r_1 \in (0,1)$ such that for $r \in (r_1, 1)$, we have that

$$\begin{aligned}&\left| \sup_{z\in\mathbb{D}} \int_{\min\{|\varphi(w)|,|\psi(w)|\}>r} \left| 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2 - 2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3} \right.\right. \\ &\left.\left. \times (\psi'(w))^2 + \frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w) - \frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\right| dA(w) \right| < \varepsilon. \qquad (3.32)\end{aligned}$$

Since $C_\varphi, C_\psi : B^1 \to B^1$ are not compact and $\min\{|\varphi(w)|, |\psi(w)|\} \to 1$

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as $|w| \to 1$, so

$$\begin{aligned}||(C_\varphi - C_\psi)f_j\|_{B^1} \leq C\Big(&|f_j(\varphi(0)) - f_j(\psi(0))| + |f_j'(\varphi(0))\varphi'(0) - f_j(\psi(0))\psi'(0)| \\ &+ \int_{\max(|\varphi(w)|,|\psi(w)|)\leq r} |f_j''(\varphi(w))(\varphi'(w))^2 - f_j''(\psi(w))(\psi'(w))^2 \\ &\quad + f_j'(\varphi(w))\varphi''(w) - f_j'(\psi(w))\psi''(w)|dA(w) \\ &+ \int_{\min(|\varphi(w)|,|\psi(w)|)>r} |f_j''(\varphi(w))(\varphi'(w))^2 - f_j''(\psi(w))(\psi'(w))^2 \\ &\quad + f_j'(\varphi(w))\varphi''(w) - f_j'(\psi(w))\psi''(w)|dA(w)\Big).\end{aligned}$$

Moreover,

$$|f_j(\varphi(0))| < \varepsilon,\ |f_j(\psi(0))| < \varepsilon,\ |f_j'(\varphi(0))| < \varepsilon,$$

$$|f_j'(\psi(0))| < \varepsilon,\ \sup_{|w|\leq r} |f_j'(w)| < \varepsilon \ \text{ and } \ \sup_{|w|\leq r} |f_j''(w)| < \varepsilon$$

for sufficiently large m, say $j \geq j_0$. Thus for $j \geq j_0$, we have that

$$\begin{aligned}&\|(C_\varphi - C_\psi)f_j\|_{B^1} \\ &\leq C\Big(|f_j(\varphi(0)) - f_j(\psi(0))| + |f_j'(\varphi(0))\varphi'(0) - f_j(\psi(0))\psi'(0)| \\ &\quad + \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |f_j''(\varphi(z))| \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |\varphi'(w)|^2 \\ &\quad + \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |f_j''(\psi(z))| \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |\psi'(w)|^2 \\ &\quad + \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |f_j'(\varphi(z))| \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |\varphi''(w)| \\ &\quad + \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |f_j'(\psi(z))| \sup_{\max(|\varphi(z)|,|\psi(z)|)\leq r} |\psi''(w)| \\ &\quad + |f_j'(0)| \int_{\mathbb{D}} |\varphi''(w) - \psi''(w)|dA(w)\end{aligned}$$

$$+\int_{\mathbb{D}}\Bigg[\int_{\min\{|\varphi(w)|,|\psi(w)|\}>r}\Bigg|2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\varphi(w))^3}(\varphi'(w))^2-2\frac{\overline{z}(1-|z|^2)}{(1-\overline{z}\psi(w))^3}$$

$$\times(\psi'(w))^2+\frac{(1-|z|^2)}{(1-\overline{z}\varphi(w))^2}\varphi''(w)-\frac{(1-|z|^2)}{(1-\overline{z}\psi(w))^2}\psi''(w)\Bigg|dA(w)\Bigg]|d|\mu_j|(z)\Bigg)$$

$$< C\Bigg(2+|\varphi'(0)|+|\psi'(0)|+\sup_{\max(|\varphi(z)|,|\psi(z)|)\le r}|\varphi'(w)|^2$$

$$+\sup_{\max(|\varphi(z)|,|\psi(z)|)\le r}|\psi'(w)|^2+\sup_{\max(|\varphi(z)|,|\psi(z)|)\le r}|\varphi''(w)|+\|\varphi''-\psi''\|_{A^1}$$

$$+\sup_{\max(|\varphi(z)|,|\psi(z)|)\le r}|\psi''(w)|+\int_{\mathbb{D}}d|\mu_j|(x)\Bigg)\varepsilon < C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, so $C_\varphi - C_\psi : B^1 \to B^1$ is compact. □

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