On Spectra of Power Graphs of Finite Cyclic & Dihedral Groups

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Abstract

The power graph $\mathcal{P}(G)$ of a finite group $G$ is defined to be the graph whose vertex set is $G$ and two distinct vertices $u, v \in \mathcal{P}(G)$ are adjacent if and only if $u = v^m$ or $v = u^n$ for some positive integers $m, n$. The distance signless Laplacian matrix of a graph $\mathcal{G}$ denoted by $DQ(\mathcal{G})$ is defined as $DQ(\mathcal{G}) = Tr(\mathcal{G}) + D(\mathcal{G})$, where $D(\mathcal{G})$ is the distance matrix of $\mathcal{G}$ and $Tr(\mathcal{G})$ is the transmission matrix of $\mathcal{G}$. In this paper, we determine the distance signless Laplacian eigenvalues of power graphs of the finite cyclic group $\mathbb{Z}_n$ and the dihedral group $D_n$. We provide upper and lower bounds on the largest eigenvalue of the distance signless Laplacian matrix of $\mathcal{P}(\mathbb{Z}_n)$ and $\mathcal{P}(D_n)$. We also give a short proof of the lower bound obtained on the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ published in (Chattopadhyay, Sriparna, and Pratima Panigrahi, On Laplacian spectrum of power graphs of finite cyclic and dihedral groups, Linear and Multilinear Algebra 63(7) (2015) 1345-1355.).

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1 Introduction

The directed power graph of a semigroup $S$ was defined by Kelarav and Quinn [12] as the digraph $\mathcal{P}(S)$ with vertex set $S$, in which there is an arc from $x$ to $y$ if and only if $x \neq y$
and $y = x^m$ for some positive integer $m$. The **undirected power graph** of a finite group was introduced by Chakrabarty et al. in [6] following which it received a great attention and was extensively studied. The undirected power graph $\mathcal{P}(G)$ of a finite group $G$ is defined to be the graph whose vertex set is $G$ and two distinct vertices $u, v \in \mathcal{P}(G)$ are adjacent if and only if $u = v^m$ or $v = u^n$ for some positive integers $m, n$. The power graph $\mathcal{P}(G)$ of a finite group $G$ is always connected. Also the power graph $\mathcal{P}(G)$ of a finite group $G \neq \{e\}$ is complete if and only if $G \cong \mathbb{Z}_p^m$ for some prime $p$ and for some $m \in \mathbb{N}$. Readers may refer to [1] for a detailed discussion and various interesting results on power graphs of finite groups.

Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** $A(G) = (a_{ij})_{n \times n}$ of $G$ is an $n \times n$ matrix such that $a_{ij} = 1$ when the vertices $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ elsewhere. The **Laplacian matrix** $L(G)$ of $G$ is defined as $L(G) = \text{Deg}(G) - A(G)$ where $\text{Deg}(G)$ is the diagonal matrix of vertex degrees. The matrix $L(G)$ is a real and symmetric matrix and hence all its eigenvalues are real. Also $L(G)$ is known to be a positive semi-definite matrix, and hence all its eigenvalues are non-negative. Since the sum of all entries in a given row of $L(G)$ is 0, the matrix $L(G)$ has determinant 0 and hence 0 is an eigenvalue of $L(G)$ with eigenvector $[1,1,1,\ldots,1]^T$. It is also known that multiplicity of 0 as an eigenvalue of $L(G)$ equals the number of connected components of the graph $G$. The survey paper [13] lists several interesting properties of $L(G)$. The second smallest eigenvalue of $L(G)$ is known as the **algebraic connectivity** of $G$. The term algebraic connectivity was coined by Feidler in [10]. A **separating set** in a connected graph $G$ is a set $S \subset V(G)$ such that $V(G) \setminus S$ has more than 1 connected component. The **vertex connectivity** of $G$ denoted by $\kappa(G)$ is defined as $\kappa(G) = \min\{|S| : S \text{ is a separating set of } G\}$. It is known from [10] that the algebraic connectivity of a non-complete graph is less than or equal to its vertex connectivity. The **signless Laplacian matrix** $Q(G)$ of $G$ introduced in [8] is defined as $Q(G) = \text{Deg}(G) + A(G)$. The **distance** between two vertices $u, v$ of a connected graph $G$ is defined as the number of edges in a shortest path between $u$ and $v$. The **distance matrix** of $G$ denoted by $D(G) = (d_{ij})_{n \times n}$ is an $n \times n$ matrix where $d_{ij}$ is equal to the distance between two vertices $v_i$ and $v_j$. The **transmission** of a vertex $v \in V(G)$ denoted by $Tr(v)$ is the sum of distances from $v$ to all other vertices of $G$. The **transmission matrix** $Tr(G)$ of $G$ is a diagonal matrix whose diagonal entries denote the transmission of a vertex $v \in G$. The **distance Laplacian matrix** and **distance signless Laplacian matrix** of $G$ were introduced by Aouchiche and Hansen in [2]. The distance Laplacian matrix of $G$ denoted by $D^L(G)$ is defined as $D^L(G) = Tr(G) - D(G)$. The distance signless Laplacian matrix of $G$ denoted by $D^Q(G)$ is defined as $D^Q(G) = Tr(G) + D(G)$. Various properties of $D^Q(G)$ have been studied in [3]. The matrix $D^Q(G)$ is a real, symmetric and positive semi-definite matrix, and hence all its eigenvalues are non-negative. The eigenvalues of $A(G), L(G)$ and $Q(G)$ are known as the adjacency, Laplacian and signless Laplacian eigenvalues of $G$, respectively. Similarly, the eigenvalues of $D(G), D^L(G)$ and $D^Q(G)$ are known as the distance, distance Laplacian, and distance signless Laplacian eigenvalues of $G$, respectively. The elements of the finite cyclic group $\mathbb{Z}_n$ are denoted by $0, 1, 2, \ldots, n - 2, n - 1$. For a given positive integer $n > 1$, the number of integers prime to $n$ and less than $n$ is denoted by the **Euler’s totient function** $\phi(n)$. Moreover, let $\ell = \phi(n) + 1$. The dihedral group of order $2n$ is denoted by $D_n$. Two $n \times n$ matrices $A$ and $B$ are said to be **similar** if there exists a non-singular matrix $P$ such that $A = PBP^{-1}$. Let $A$ be a positive semi-definite matrix of order $n$ having eigenvalues as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \cdots \geq \lambda_n \geq 0$. Then $\lambda_n$ is known as **spectral radius** of $A$, and it is denoted
as $\rho(A)$. For any two positive integers $a$ and $b$, we denote the greatest common divisor of $a$ and $b$ by $\gcd(a,b)$ and least common multiple of $a$ and $b$ by $\text{lcm}(a,b)$. The readers may refer to the standard texts [9] and [11], respectively, for undefined terms related to graphs and matrices used in this paper.

1.1 Motivation and Arrangement of Paper

The eigenvalues of Laplacian matrix of $\mathcal{P}(\mathbb{Z}_n)$ and $\mathcal{P}(D_n)$ have been studied by Chattopadhyay and Panigrahi in [7]. They have also investigated the vertex connectivity and algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$. In [14], the authors have studied the adjacency spectra of power graph of various finite groups like $\mathbb{Z}_n$, $D_n$, elementary abelian group, and Mathieu group. In [16] Panda has studied the Laplacian spectra of power graphs of finite cyclic, dicyclic, and $p$-groups in more detail. Motivated by the above works, in this paper we have determined the eigenvalues of distance signless Laplacian matrix of power graph of $\mathbb{Z}_n$ and $D_n$ for $n \geq 2$. Since it is not possible to find all the distance signless Laplacian eigenvalues of $\mathcal{G}$ when the number of vertices of $\mathcal{G}$ is large, it is natural to study the largest eigenvalue of the distance signless Laplacian matrix of $\mathcal{G}$ and provide lower and upper bounds on the same. The largest eigenvalue of the distance signless Laplacian matrix of $\mathcal{G}$ is known as the distance signless Laplacian spectral radius of $\mathcal{G}$ and it is denoted by $\rho(D^Q(\mathcal{G}))$. The distance signless Laplacian spectral radius of bicyclic and tricyclic graphs has been studied in [17] and [19] respectively. Lower and upper bounds on the distance signless Laplacian spectral radius of a graph in terms of its clique number were given in [13]. The above works motivate us to provide lower and upper bounds on the distance signless Laplacian spectral radius of power graphs of $\mathbb{Z}_n$ and $D_n$ in terms of a given $n$. While providing an upper bound on the spectral radius of the distance signless Laplacian matrix of $\mathcal{P}(\mathbb{Z}_n)$, we notice that the lower bound on the algebraic connectivity obtained in [7, Theorem 2.12] can be derived alternately and simply. Hence we provide a shorter proof of the lower bound on the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$.

The paper has been arranged as follows: In Section 2 we provide the preliminary theorems and definitions that have been used in this paper. In Section 3 we study the distance signless Laplacian spectrum of power graph of $\mathbb{Z}_n$. We find the characteristic polynomial of $D^Q(\mathcal{P}(\mathbb{Z}_n))$ and completely determine the spectrum of $D^Q(\mathcal{P}(\mathbb{Z}_n))$ for various forms of $n$. We also provide upper and lower bounds on the spectral radius of $D^Q(\mathcal{P}(\mathbb{Z}_n))$ for large $n$. We further provide a short proof of the lower bound on the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ obtained in [7]. Finally, in Section 4 we find the characteristic polynomial of distance signless Laplacian matrix of power graph of the dihedral group $D_n$. We also find lower and upper bounds on the largest eigenvalue of $D^Q(\mathcal{P}(D_n))$.

2 Preliminaries

In this section, we will provide some definitions and theorems that have been used in the subsequent sections.

**Definition 2.1.** Let $\mathcal{G}_1, \mathcal{G}_2$ be two graphs. The join of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \vee \mathcal{G}_2$, is a graph obtained from $\mathcal{G}_1$ and $\mathcal{G}_2$ by joining each vertex of $\mathcal{G}_1$ to all vertices of $\mathcal{G}_2$. 
Definition 2.2. Let $M$ be a complex matrix of order $n$ described as

$$
\begin{pmatrix}
M_{11} & \cdots & M_{1t} \\
\vdots & \ddots & \vdots \\
M_{t1} & \cdots & M_{tt}
\end{pmatrix}
$$

Here the blocks $M_{ij}$ are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$ and $n = n_1 + \cdots + n_t$. For $1 \leq i, j \leq t$, let $b_{ij}$ denote the average row sum of $M_{ij}$, i.e. $b_{ij}$ is the sum of all entries in $M_{ij}$ divided by the number of rows. Then the matrix $B = (b_{ij})$ is called the quotient matrix of $M$. If, in addition, for each pair $i, j$, $M_{ij}$ has a constant row sum, then $B$ is called the equitable quotient matrix of $M$.

Theorem 2.3. Consider a matrix $M$ and its equitable quotient matrix $B$. Then we have the following:

$P1.$ [5] Lemma 2.3.1] If $\sigma(M)$ and $\sigma(B)$ denote the set of eigenvalues of $M$ and $B$ respectively, then \( \sigma(B) \subseteq \sigma(M) \).

$P2.$ [4] Theorem 2.3.] Let $\rho(M)$ and $\rho(B)$ denote the spectral radius of $M$ and $B$ respectively. If $M$ is a non-negative matrix, then $\rho(M) = \rho(B)$.

Theorem 2.4. [Courant-Weyl Inequalities] Let $A$ and $B$ be $n \times n$ Hermitian matrices. Let $\lambda_i(A), \lambda_i(B)$ and $\lambda_i(A+B)$ denote the eigenvalues of $A$, $B$ and $A+B$, respectively. Then the following hold:

1. $\lambda_i(A+B) \leq \lambda_j(A) + \lambda_{i-j+1}(B)$ ($n \geq i \geq j \geq 1$).
2. $\lambda_i(A+B) \geq \lambda_j(A) + \lambda_{i-j+n}(B)$ ($1 \leq i \leq j \leq n$).

Theorem 2.5. [11] Theorem 8.1.22] Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ non-negative matrix, i.e. $a_{ij} \geq 0$ for $1 \leq i, j \leq n$. If $\rho(A)$ denotes the largest eigenvalue of $A$, then $\rho(A)$ satisfies the following inequality:

$$
\min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}.
$$

Theorem 2.6. [18] Lemma 2.1] Let $G$ be a connected graph with two non-adjacent vertices $u$ and $v$. Then $\rho(D^2(G + uv)) < \rho(D^2(G))$. Here $G + uv$ is the graph obtained from $G$ by adding the edge $uv$.

3 Distance Signless Laplacian Spectrum of $P(\mathbb{Z}_n)$

In this section, we determine the characteristic polynomial and eigenvalues of distance signless Laplacian matrix of $P(\mathbb{Z}_n)$ for various $n$. Throughout the paper, $I_n$ will denote the identity matrix of order $n$, and $J_{m \times n}$ will denote the matrix of order $m \times n$ having all entries 1. For a given matrix $M$, $S(M)$ will denote the sum of all elements of $M$. 
3.1 Characteristic Polynomial of $\mathcal{P}(\mathbb{Z}_n)$

We need the following lemma to arrive at the main result of this section.

**Lemma 3.1.** Let $M$ be an invertible square matrix of order $n$ such that the sum of all elements in each row of $M$ is the same, say $r$. Then $S(M^{-1}) = \frac{n}{r}$.

**Proof.** Since the sum of all the elements in each row of $M$ is $r$, $r$ is an eigenvalue of $M$ with eigenvector $v = (1, 1, \ldots, 1, 1)^T$. Thus $Mv = rv$, which implies that $M^{-1}v = \frac{r}{v}$. Thus the sum of all elements in each row of $M^{-1}$ is $\frac{1}{r}$. Consequently, $S(M^{-1}) = \frac{n}{r}$. \hfill \Box

We now prove the main result of this section.

**Theorem 3.2.** The characteristic polynomial of distance signless Laplacian matrix of $\mathcal{P}(\mathbb{Z}_n)$ is given by:

$$
\mu(D^Q(\mathcal{P}(\mathbb{Z}_n)), x) = (x - n - \ell + 2)(x - n + 2)^{\ell-1} \times \det \left( (xI - T) - \frac{\ell}{(x - n + 2 - \ell)} J_{(n-\ell)\times(n-\ell)} \right).
$$

Here $\ell = \phi(n) + 1$ and $T$ is the matrix whose rows and columns are indexed by the non-generators of $\mathbb{Z}_n$.

**Proof.** The indexing of rows and columns of $D^Q(\mathcal{P}(\mathbb{Z}_n))$ goes as follows: First, we write the element 0, then the $\phi(n)$ generators, and then the $n - \ell$ non-generators of $\mathbb{Z}_n$. Using the above indexing, the distance signless Laplacian matrix of $\mathcal{P}(\mathbb{Z}_n)$ takes the following form:

$$
D^Q(\mathcal{P}(\mathbb{Z}_n)) = \begin{pmatrix} (n - 2)I_{\ell\times\ell} + J_{\ell\times\ell} & J_{\ell\times(n-\ell)} \\ J_{(n-\ell)\times\ell} & T_{(n-\ell)\times(n-\ell)} \end{pmatrix}.
$$

Here $T$ is the matrix whose rows and columns are indexed by the $n - \ell$ non-generators of $\mathbb{Z}_n$. The characteristic polynomial of $D^Q(\mathcal{P}(\mathbb{Z}_n))$ is given by:

$$
\mu(D^Q(\mathcal{P}(\mathbb{Z}_n)), x) = \det \begin{pmatrix} (x - n + 2)I_{\ell\times\ell} - J_{\ell\times\ell} & -J_{\ell\times(n-\ell)} \\ -J_{(n-\ell)\times\ell} & (xI - T)_{(n-\ell)\times(n-\ell)} \end{pmatrix} \\
= \det \left( (x - n + 2)I - J \right) \times \det \left( (xI - T) - J((x - n + 2)I - J)^{-1}J^T \right) \\
= \det \left( (x - n + 2)I - J \right) \times \det N.
$$

Here $N = \left( (xI - T) - J((x - n + 2)I - J)^{-1}J^T \right)$.

Now we notice that,

$$
J \left( (x - n + 2)I - J \right)^{-1}J^T = S \left( ((x - n + 2)I - J)^{-1} \right) \times J.
$$

The sum of the elements of each row of the matrix $((x - n + 2)I - J)_{\ell\times\ell}$ is $x - n + 2 - \ell$, which is constant. Since there are $\ell$ rows in the matrix $((x - n + 2)I - J)_{\ell\times\ell}$ and each row
has sum \( x - n + 2 - \ell \), we find that \( S \left( ((x - n + 2)I - J)_{\ell \times \ell} \right) = (x - n + 2 - \ell)\ell \). Also using Lemma 3.1 we have,

\[
S \left( ((x - n + 2)I - J)^{-1} \right) = \frac{\ell}{(x - n + 2 - \ell)}.
\]

Thus, \( J \left( (x - n + 2)I - J \right)^{-1} J^T = \frac{\ell}{(x - n + 2 - \ell)} \times J \).

Also the eigenvalues of \( ((x - n + 2)I - J) \) are \( (x - n + 2) - \ell \) with multiplicity 1 and \( (x - n + 2) \) with multiplicity \( \ell - 1 \). Since the determinant of a square matrix is a product of its eigenvalues, we get

\[
\det \left( (x - n + 2)I - J \right) = (x - n - \ell + 2)(x - n + 2)^{\ell-1}.
\]

Hence we have

\[
\mu(D^Q(\mathcal{P}(\mathbb{Z}_n)), x) = (x - n - \ell + 2)(x - n + 2)^{\ell-1} \\
\times \det \left( (xI - T) - \frac{\ell}{(x - n + 2 - \ell)} J_{(n-\ell)\times(n-\ell)} \right).
\]

(1)

Thus the result follows.

\[\square\]

**Corollary 3.3.** If \( n = p \) where \( p \) is a prime, then the eigenvalues of distance signless Laplacian matrix of \( \mathcal{P}(\mathbb{Z}_n) \) are \( 2p - 2 \) and \( p - 2 \) with multiplicity 1 and \( p - 1 \), respectively.

**Proof.** If \( n = p \) then \( \ell = n = p \), and hence

\[
\mu(D^Q(\mathcal{P}(\mathbb{Z}_n)), x) = (x - 2p + 2)(x - p + 2)^{p-1}.
\]

Thus the eigenvalues of distance signless Laplacian matrix of \( \mathcal{P}(\mathbb{Z}_n) \) are \( 2p - 2 \) and \( p - 2 \) with multiplicity 1 and \( p - 1 \), respectively.

\[\square\]

**Corollary 3.4.** If \( n = p^m \) where \( p \) is a prime and \( m > 1 \), then the eigenvalues of distance signless Laplacian matrix of \( \mathcal{P}(\mathbb{Z}_n) \) are \( 2p - 2 \) and \( p - 2 \) with multiplicity 1 and \( p - 1 \), respectively.

**Proof.** When \( n = p^m \) where \( p \) is a prime and \( m > 1 \), \( \mathcal{P}(\mathbb{Z}_n) \) is a complete graph \[6\] Theorem
2.12]. Hence the matrix $T$ is of the form $T = \{(n - 2)I + J\}_{n-\ell}$. Thus
\[
\det \left( (xI - T) - \frac{\ell}{x - n + 2 - \ell} J \right) \\
= \det \left( (xI - (n - 2)I - J) - \frac{\ell}{x - n + 2 - \ell} J \right) \\
= \det \left\{ (x - n + 2)I - \left( 1 + \frac{\ell}{x - n + 2 - \ell} \right) J \right\} \\
= \det \left( (x - n + 2)I - \frac{x - n + 2}{x - n + 2 - \ell} J \right) \\
= (x - n + 2)^{n-\ell-1} \times \left( (x - n + 2)(1 - \frac{n - \ell}{x - n + 2 - \ell}) \right) \\
= (x - n + 2)^{n-\ell} \times \left( \frac{x - 2n + 2}{x - n + 2 - \ell} \right).
\]

Hence we have
\[
\mu(D^Q(P(Z_n)), x) = (x - n - \ell + 2) \times (x - n + 2)^{\ell-1} \times (x - n + 2)^{n-\ell} \times \left( \frac{x - 2n + 2}{x - n + 2 - \ell} \right)
\]
\[
= (x - n + 2)^{n-1} \times (x - 2n + 2).
\]

Thus the eigenvalues of distance signless Laplacian matrix of $P(Z_n)$ are $n-2$ with multiplicity $n-1$ and $2n-2$ with multiplicity 1.

\begin{remark}
It is known that $P(Z_n)$ is a complete graph for $n = p^m$, and hence $D^Q(P(Z_n)) = ((n - 2)I + J)_{n \times n}$. The eigenvalues of the matrix $((n - 2)I + J)_{n \times n}$ can be easily determined. However the main intention behind deriving Corollaries 3.3 and 3.4 was to show the readers how to use Equation (1) as a working tool to find the eigenvalues of $D^Q(P(Z_n))$ for any $n$.
\end{remark}

\begin{corollary}
If $n = pq$ where $p, q$ are primes with $p < q$, then the eigenvalues of distance signless Laplacian matrix of $P(Z_n)$ are $pq - 2, pq + p - 3, pq + q - 3$ with multiplicity $\ell, q - 2$ and $p - 2$, respectively. The remaining eigenvalues are roots of a quadratic equation given by
\[
x^2 - x(3pq + p + q - 6) + (2p^2q^2 + 2p^2q + 2pq^2 - 14pq + 2p + 2q + 4) = 0.
\]

\begin{proof}
Let $\langle p \rangle, \langle q \rangle$ denote the cyclic subgroups generated by the elements $p$ and $q$ respectively. It is clear that the eigenvalues of $P(Z_n)$ are completely determined from the eigenvalues of the matrix $T$. We list the rows and columns of the matrix $T$ in the following way:
We first list the members of $\langle p \rangle \setminus \{0\}$ and then the members of $\langle q \rangle \setminus \{0\}$.
The degree of each member of the set $\langle p \rangle \setminus \{0\}$ is $\ell + (q - 2) + 2(p - 1)$ and degree of each member of the set $\langle q \rangle \setminus \{0\}$ is $\ell + 2(q - 1) + (p - 2)$.
Also $\langle p \rangle \cap \langle q \rangle = \langle \text{lcm}(p, q) \rangle = \{pq\} = \{0\}$ and hence we have $\langle p \rangle \cap \langle q \rangle \setminus \{0\} = \emptyset$.
Using the above justifications we find that the matrix $T$ takes the following form:
\[
T = \begin{pmatrix}
(\ell + (q - 2) + 2(p - 1) - 1)I_{(q-1)} + J_{(q-1)} & 2J_{(q-1)\times(p-1)} \\
2J_{(p-1)\times(q-1)} & (\ell + 2(q - 1) + (p - 2) - 1)I_{(p-1)} + J_{(p-1)}
\end{pmatrix}.
\]
\end{proof}
Now we have,

\[xI - T - \left( \frac{\ell}{x - n + 2 - \ell} \right) J = \begin{pmatrix}
(x - pq - p + 3)I - J \left( \frac{x + 2 - n}{x - n + 2 - \ell} \right) & - \left( 2 + \frac{\ell}{x - n + 2 - \ell} \right) J \\
- \left( 2 + \frac{\ell}{x - n + 2 - \ell} \right) J & (x - pq - q + 3)I - J \left( \frac{x + 2 - n}{x - n + 2 - \ell} \right)
\end{pmatrix}.\]  

(2)

We find that the eigenvalues of \(xI - T - \frac{\ell}{x - n + 2 - \ell} J\) are \(x - (pq + p - 3)\) with multiplicity \(q - 2\), \(x - (pq + q - 3)\) with multiplicity \(p - 2\) and the remaining eigenvalues (using P1 of Theorem 2.3) are the eigenvalues of the equitable quotient matrix given below:

\[
\begin{pmatrix}
(x - pq - p + 3 - (q - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right)) & (1 - p)(2 + \frac{\ell}{x - n + 2 - \ell}) \\
(1 - q)(2 + \frac{\ell}{x - n + 2 - \ell}) & x - pq - q + 3 - (p - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right)
\end{pmatrix}.
\]

(3)

From Equation (3) we have,

\[
\det \begin{pmatrix}
x - pq - p + 3 - (q - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right) & (1 - p)(2 + \frac{\ell}{x - n + 2 - \ell}) \\
(1 - q)(2 + \frac{\ell}{x - n + 2 - \ell}) & x - pq - q + 3 - (p - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right)
\end{pmatrix}.
\]

We now apply the row operation \(R_1 \rightarrow R_1 - R_2\) and obtain,

\[
\det \begin{pmatrix}
x - pq - p + q + 2 & -x + pq + q - p - 2 \\
(1 - q)(2 + \frac{\ell}{x - n + 2 - \ell}) & x - pq - q + 3 - (p - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right)
\end{pmatrix} = \left( x - pq - p + q + 2 \right) \left\{ x - pq - q + 3 - (p - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right) \right\}
\]

\[- \left\{ (-x + pq + q - p - 2)(1 - q)\left( 2 + \frac{\ell}{x - n + 2 - \ell} \right) \right\}.
\]

Substituting \(n = pq\) and \(\ell = \phi(pq) + 1 = (pq + 2 - p - q)\) above and expanding we get,

\[
\det \begin{pmatrix}
x - pq - p + 3 - (q - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right) & (1 - p)(2 + \frac{\ell}{x - n + 2 - \ell}) \\
(1 - q)(2 + \frac{\ell}{x - n + 2 - \ell}) & x - pq - q + 3 - (p - 1)\left( \frac{x + 2 - n}{x - n + 2 - \ell} \right)
\end{pmatrix} = \frac{f(p, q, x)}{x - 2pq + p + q}.
\]

Here,

\[
f(p, q, x) = (x + 2 - pq) \times (2p^2q^2 + 2p^2q + 2pq^2 - 3pqx - 14pq - px + 2p - qx + 2q + x^2 + 6x + 4).
\]
From Equation (2) we have,
\[
\det \left( xI - A - \frac{\ell}{x - n + 2 - \ell} F \right) = \left( x - \left( pq + p - 3 \right) \right)^{q-2} \left( x - \left( pq + q - 3 \right) \right)^{p-2} \left( \frac{f(p, q, x)}{x - 2pq + p + q} \right)
\]

(4)

Thus plugging the values from Equation (4) into Equation (1), the characteristic polynomial of \(D^Q(\mathcal{P}(\mathbb{Z}_n))\) becomes,
\[
\mu(D^Q(\mathcal{P}(\mathbb{Z}_n)), x) = (x - n + 2)^{\ell-1}(x - n - \ell + 2) \times \det \left( xI - A - \frac{\ell}{x - n + 2 - \ell} F \right) = (x - n + 2)^{\ell-1}(x - n - \ell + 2) \left( x - \left( pq + p - 3 \right) \right)^{q-2} \left( x - \left( pq + q - 3 \right) \right)^{p-2} \times \left( \frac{f(p, q, x)}{x - 2pq + p + q} \right)
\]
\[
= (x - pq + 2)^{\ell-1}(x - 2pq + p + q) \left( x - \left( pq + p - 3 \right) \right)^{q-2} \left( x - \left( pq + q - 3 \right) \right)^{p-2} \times g(p, q, x).
\]

Here \(g(p, q, x) = x^2 - x(3pq + p + q - 6) + (2p^2q^2 + 2p^2q + 2pq^2 - 14pq + 2p + 2q + 4)\). Thus the result follows.

\[\square\]

### 3.2 Bounds on Distance Signless Laplacian Spectral Radius of \(\mathcal{P}(\mathbb{Z}_n)\)

When \(n = p^m\) and \(n = pq\) where \(p, q\) are primes and \(m \in \mathbb{N} \cup \{0\}\), then the eigenvalues of distance signless Laplacian matrix of \(\mathcal{P}(\mathbb{Z}_n)\) have been obtained in Section 3.1. When \(n \notin \{p^m, pq\}\) the next theorem provides an upper bound on the largest eigenvalue of the distance signless Laplacian matrix of \(\mathcal{P}(\mathbb{Z}_n)\).

**Theorem 3.7.** Let \(n \geq 2\) be a given positive integer. Then the largest eigenvalue of distance signless Laplacian matrix of \(\mathcal{P}(\mathbb{Z}_n)\), denoted by \(\rho(D^Q(\mathcal{P}(\mathbb{Z}_n)))\), satisfies the following inequality:

\[
\rho(D^Q(\mathcal{P}(\mathbb{Z}_n))) \leq \frac{5n}{2} - 3 - \ell + \frac{1}{2} \sqrt{12\ell^2 - 4n(5\ell + 3) + 9n^2 + 16\ell + 4}.
\]

**Proof.** Let \(K_\ell\) denote the complete graph on \(\ell\) vertices and \(\overline{K}_{n-\ell}\) denote the null graph on \(n - \ell\) vertices. Now we find that for a given \(n \in \mathbb{N}\), the matrix \(D^Q(\mathcal{P}(\mathbb{Z}_n))\) satisfies the following matrix equality:

\[
D^Q(K_\ell \lor \overline{K}_{n-\ell}) = D^Q(\mathcal{P}(\mathbb{Z}_n)) + E.
\]
Here $K_\ell \vee \overline{K}_{n-\ell}$ is defined as in Definition 2.1. For a suitable relabeling of the vertices of $P(Z_n)$ it is not difficult to verify that $E$ is a positive semi-definite matrix. Using Theorem 2.4 and the fact that $E$ is a positive semi-definite matrix we obtain,

$$\rho(DQ(K_\ell \vee \overline{K}_{n-\ell})) = \rho(DQ(P(Z_n)) + E) \geq \rho(DQ(P(Z_n))) + \lambda_n(E) \geq \rho(DQ(P(Z_n))).$$  \hspace{1cm} (5)

Here $\lambda_n(E)$ denotes the smallest eigenvalue of $E$. Now the matrix $DQ(K_\ell \vee \overline{K}_{n-\ell})$ is of the form:

$$\begin{pmatrix} (n-2)I + J & J_{(\ell \times (n-\ell))} \\ J^T & ((\ell + 2n - 2\ell - 4)I + 2J)_{(n-\ell) \times (n-\ell)} \end{pmatrix} = \begin{pmatrix} (n-2)I + J & J_{(\ell \times (n-\ell))} \\ J^T & ((2n - \ell - 4)I + 2J)_{(n-\ell) \times (n-\ell)} \end{pmatrix}.$$  \hspace{1cm} (6)

We find that the row sum of each block of the matrix given in Equation (6) is constant, and hence we can use the concept of equitable quotient matrix given in Definition 2.2. The eigenvalues of $DQ(K_\ell \vee \overline{K}_{n-\ell})$ are $n-2$ with multiplicity $\ell-1$, $\ell + 2n - 2\ell - 4 = 2n - \ell - 4$ with multiplicity $n - \ell - 1$ and remaining two eigenvalues (using P1 of Theorem 2.3) are the eigenvalues of its equitable quotient matrix given below:

$$\begin{pmatrix} \ell + n - 2 & n - \ell \\ \ell & 4n - 3\ell - 4 \end{pmatrix}.$$  \hspace{1cm} (7)

The eigenvalues of the matrix given in Equation (7) are

$$\frac{5n}{2} - 3 - \ell \pm \frac{1}{2} \sqrt{12\ell^2 - 4n(5\ell + 3) + 9n^2 + 16\ell + 4}.$$

Since spectral radius of a non-negative matrix and its quotient matrix are equal (using P2 of Theorem 2.3), we obtain

$$\rho(DQ(K_\ell \vee \overline{K}_{n-\ell})) = \frac{5n}{2} - 3 - \ell + \frac{1}{2} \sqrt{12\ell^2 - 4n(5\ell + 3) + 9n^2 + 16\ell + 4}. \hspace{1cm} (8)$$

Using Equations (5) and (8) we obtain

$$\rho(DQ(P(Z_n)))) \leq \frac{5n}{2} - 3 - \ell + \frac{1}{2} \sqrt{12\ell^2 - 4n(5\ell + 3) + 9n^2 + 16\ell + 4}.$$  \hspace{1cm} (9)

Thus the result follows.

The next theorem provides a lower bound on the largest eigenvalue of distance signless Laplacian matrix of $P(Z_n)$ for a given $n \in \mathbb{N}$. 

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Theorem 3.8. For each positive integer \( n \geq 2 \), the largest eigenvalue of distance signless Laplacian matrix of \( \mathcal{P}(Z_n) \) satisfies the inequality \( \rho(D^Q(\mathcal{P}(Z_n))) \geq 2n - 2 \) where equality holds if and only if \( \mathcal{P}(Z_n) \) is complete i.e. \( n = p^m \) where \( p \) is a prime and \( m \in \mathbb{N} \cup \{0\} \).

Proof. Consider the complete graph \( K_n \) on \( n \) vertices. Using Theorem \( 2.6 \) we find that \( \rho(D^Q(\mathcal{P}(Z_n))) \geq \rho(D^Q(K_n)) \). Since \( \rho(D^Q(K_n)) = 2n - 2 \), the result follows. The equality follows from the fact that \( \mathcal{P}(Z_n) = K_n \) if and only if \( n = p^m \) where \( p \) is a prime and \( m \in \mathbb{N} \cup \{0\} \).

\[ \blacksquare \]

3.3 Lower Bound on Algebraic Connectivity of \( \mathcal{P}(Z_n) \)

We now give a simple and alternate proof of \( \blacksquare \) Theorem 2.12 in the next theorem. We provide the proof here as it can be obtained using similar techniques as used in Theorem \( 3.7 \). We would like to mention the fact that here we are dealing with the Laplacian matrix of \( \mathcal{P}(Z_n) \), and it is denoted by \( L(\mathcal{P}(Z_n)) \). Let the eigenvalues of \( L(\mathcal{P}(Z_n)) \) be given by \( 0 = \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \). The eigenvalue \( \lambda_{n-1} \) is known as the algebraic connectivity of \( \mathcal{P}(Z_n) \).

Theorem 3.9. For each positive integer \( n \geq 2 \), the second smallest eigenvalue \( \lambda_{n-1}(L(\mathcal{P}(Z_n))) \) satisfies the inequality \( \lambda_{n-1}(L(\mathcal{P}(Z_n))) \geq \ell \).

Proof. Using \( \blacksquare \) Corollary 3.7 we have

\[
\mu(G_1 \cup G_2, x) = \frac{x(x - n_1 - n_2)}{(x - n_1)(x - n_2)}\mu(G_1, (x - n_2))\mu(G_2, (x - n_1)). \tag{10}
\]

Suppose \( G_1 = K_\ell \), where \( K_\ell \) is the complete graph on \( \ell \) vertices and \( G_2 = \overline{K}_{n-\ell} \) is the null graph on \( n - \ell \) vertices. Clearly,

\[
\mu(G_1, x) = \mu(K_\ell, x) = x(x - \ell)^{\ell-1},
\]

\[
\mu(G_2, x) = \mu(\overline{K}_{n-\ell}, x) = x^{n-\ell}. \tag{11}
\]

Thus using Equation (11) in Equation (10) we obtain

\[
\mu(K_\ell \cup G_2) = \left( \frac{x(x - n)}{(x - \ell)(x - (n - \ell))} \right) \left( x - (n - \ell) \right) (x - n)^{\ell-1}(x - \ell)^{n-\ell} \tag{12}
\]

\[
\implies \mu(K_\ell \cup G_2) = x(x - n)^{\ell}(x - \ell)^{n-\ell-1}.
\]

Now for a proper relabeling of the vertices of \( \mathcal{P}(Z_n) \) we obtain

\[ L(\mathcal{P}(Z_n)) = L(K_\ell \cup G_2) + F. \]

Here \( F \) is a positive semi-definite matrix. Thus using Theorem \( 2.4 \) and the fact that \( F \) is a positive semi-definite matrix, we obtain

\[
\lambda_{n-1}(L(\mathcal{P}(Z_n))) \geq \lambda_{n-1}(L(K_\ell \cup G_2)) + \lambda_n(F) \geq \lambda_{n-1}(L(K_\ell \cup G_2)). \tag{13}
\]

Here \( \lambda_n(F) \) denotes the smallest eigenvalue of \( F \). Using Equations (12) and (13) and the fact that \( 0 < \ell < n \), we obtain

\[
\lambda_{n-1}(L(\mathcal{P}(Z_n))) \geq \ell.
\]

This completes the proof. \[ \blacksquare \]
4 Distance Signless Laplacian Spectra of Power Graph of Dihedral Groups

In this section, we compute the characteristic polynomial of the distance signless Laplacian matrix of the power graph of the dihedral group $D_n$. We also give lower and upper bounds on the largest eigenvalue of distance signless Laplacian matrix of $P(D_n)$. The dihedral group $D_n$ of order $2n$ is given by the following presentation:

$$D_n = \{(r, s) : r^n = s^2 = 1, rs = sr^{-1}\}.$$

4.1 Characteristic Polynomial of $D^Q(P(D_n))$

The rows and columns of $D^Q(P(D_n))$ have been indexed in the following way:

We first list the identity element of $D_n$ i.e. 1. We then list the elements of $D_n$ generated by $r$ i.e. $r, r^2, r^3, \ldots, r^{n-2}, r^{n-1}$. Finally, we list the remaining elements of $D_n$ i.e. $s, sr, sr^2, \ldots, sr^{n-2}, sr^{n-1}$. Using the above indexing we find that the distance signless Laplacian matrix of $P(D_n)$ is given by:

$$D^Q(P(D_n)) = \begin{pmatrix} D^Q(P(Z_n)) + W & V \\ V^T & \end{pmatrix} \cdot ((4n - 5)I + 2J). \quad (14)$$

Here $V = \begin{pmatrix} 1 & 1 & 1 & \ldots & \ldots & 1 \\ 2 & 2 & 2 & \ldots & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 2 & 2 & 2 & \ldots & \ldots & 2 \end{pmatrix}_{n \times n}$ and $W = \begin{pmatrix} n & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 2n & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 2n & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ddots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots & 2n \end{pmatrix}_{n \times n}$.

The characteristic polynomial of $D^Q(P(D_n))$ is given by

$$\mu(D^Q(P(D_n)), x) = \det \begin{pmatrix} xI - D^Q(P(D_n)) \\ V^T \end{pmatrix} = \det \begin{pmatrix} xI - D^Q(P(Z_n)) - W & -V \\ -V^T & \end{pmatrix} \cdot ((x - (4n - 5))I - 2J) \quad (15)$$

$$= \det ((x - 4n + 5)I - 2J) \times \det ((xI - D^Q(P(Z_n)) - W) - V((x - (4n - 5))I - 2J)^{-1}V^T).$$

Now,

$$V((x - (4n - 5))I - 2J)^{-1}V^T = \left( S((x - (4n - 5))I - 2J)^{-1} \right) \times P. \quad (16)$$
Here \( P = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 2 & 4 & 4 & \ldots & 4 \\ 2 & 4 & 4 & \ldots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 4 & 4 & \ldots & 4 \end{pmatrix}^{n \times n} \).

The sum of elements of each row of the matrix \((x - (4n - 5))I - 2J\) is constant i.e. \(x - (4n - 5) - 2n = x - 6n + 5\). Using Lemma 3.1 we find that

\[ S\left(\left((x - (4n - 5))I - 2J\right)^{-1}\right) = \frac{n}{(x - 6n + 5)}. \]

Using Equation (16) we obtain

\[ V((x - (4n - 5))I - 2J)^{-1}V^T = \frac{n}{(x - 6n + 5)} \times P. \] (17)

Here \( P = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 2 & 4 & 4 & \ldots & 4 \\ 2 & 4 & 4 & \ldots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 4 & 4 & \ldots & 4 \end{pmatrix}^{n \times n} \).

Using Equation (17) in Equation (15) we get

\[ \mu(D^Q(P(D_n)), x) = \det\left((x - 4n + 5)I - 2J\right) \times \det\left((xI - D^Q(P(Z_n))) - W\right) - \frac{n}{(x - 6n + 5)} \times P. \]

This gives the characteristic polynomial of distance signless Laplacian matrix of \( P(D_n) \) in terms of the characteristic polynomial of distance signless Laplacian matrix of \( P(Z_n) \) for any \( n \geq 2 \).

### 4.2 Bounds on Distance Signless Laplacian Spectral Radius of \( P(D_n) \)

In this section we provide lower and upper bounds on the largest eigenvalue of \( D^Q(P(D_n)) \). We first provide a lower bound on the largest eigenvalue of \( D^Q(P(D_n)) \).

#### 4.2.1 Lower Bound on Distance Signless Laplacian Spectral Radius of \( P(D_n) \)

**Theorem 4.1.** For each positive integer \( n \geq 2 \), the largest eigenvalue of distance signless Laplacian matrix of \( P(D_n) \) satisfies the inequality \( \rho(D^Q(P(D_n))) \geq 6n - (3 + \frac{1}{n}) \).

**Proof.** For each \( n \geq 2 \), we consider the following graph \( G \) on \( 2n \) vertices where the vertex set \( V(G) \) of \( G \) is given by \( V = \{1, r, r^2, \ldots, r^{n-1}, s, sr^2, \ldots, sr^{n-1}\} \). We now give the adjacency criterion between two vertices of \( G \) i.e. we list below how the members of \( G \) are adjacent to each other.
1. The induced sub-graph of \( G \) on the set of vertices \( \{1, r, r^2, \ldots, r^{n-1}\} \) is the complete graph on \( n \) vertices of \( G \).

2. The induced sub-graph of \( G \) on the set of vertices \( \{s, sr, sr^2, \ldots sr^{n-1}\} \) is the null graph on \( n \) vertices.

3. The vertex 1 of \( G \) is adjacent to every member of the set \( \{s, sr, sr^2, \ldots sr^{n-1}\} \).

4. The vertex \( r^i \) is not adjacent to any member of the set \( \{s, sr, sr^2, \ldots sr^{n-1}\} \) for any \( 1 \leq i \leq n - 1 \).

If \( n = 6 \), then \( G \) takes the following form (Figure 1).

![Figure 1: \( G \) for \( n = 6 \).](image)

Using the adjacency criteria of the graph \( G \) as listed above in 1, 2, 3 and 4, it is evident that \( G \) is a supergraph of \( \mathcal{P}(D_n) \) for any \( n \in \mathbb{N} \) and it is equal to \( \mathcal{P}(D_n) \) if and only if \( n = p^m \) where \( p \) is a prime and \( m \in \mathbb{N} \cup \{0\} \). Using Theorem 2.6 we obtain

\[
\rho(D^Q(\mathcal{P}(D_n))) > \rho(D^Q(G)).
\]

Now we know that when \( n = p^m \) where \( m \in \mathbb{N} \cup \{0\}, \mathcal{P}(\mathbb{Z}_n) \) is complete. Using Equation (14) we find that when \( n = p^m \) where \( m \in \mathbb{N} \cup \{0\},

\[
D^Q(\mathcal{P}(D_n)) = \begin{pmatrix}
D^Q(\mathcal{P}(\mathbb{Z}_n)) + W & V \\
V^T & ((4n - 5)I + 2J)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
Z & V \\
V^T & ((4n - 5)I + 2J)
\end{pmatrix}
\]  

(18)
Here,

\[
Z = \begin{pmatrix}
    n-1 & 1 & 1 & \ldots & \ldots & 1 \\
    1 & n-1 & 1 & \ldots & \ldots & 1 \\
    1 & 1 & n-1 & \ldots & \ldots & 1 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    1 & 1 & 1 & \ldots & n-1 & 1 \\
\end{pmatrix}
+ \begin{pmatrix}
    n & 0 & 0 & \ldots & \ldots & 0 \\
    0 & 2n & 0 & \ldots & \ldots & 0 \\
    0 & 0 & 2n & \ldots & \ldots & 0 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & \ldots & 2n \\
\end{pmatrix}
= \begin{pmatrix}
    2n-1 & 1 & 1 & \ldots & \ldots & 1 \\
    1 & 3n-1 & 1 & \ldots & \ldots & 1 \\
    1 & 1 & 3n-1 & \ldots & \ldots & 1 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    1 & 1 & 1 & \ldots & 3n-1 & 1 \\
\end{pmatrix}.
\]

Using Equation (18) we find that the eigenvalues of \(D^Q(P(D_n))\) are \(3n-2\) with multiplicity \(n-2\), \(4n-5\) with multiplicity \(n-1\) and the remaining 3 eigenvalues (using \(P1\) of Theorem 2.3) are the eigenvalues of the equitable quotient matrix given by

\[
\begin{pmatrix}
    2n-1 & n-1 & n \\
    1 & 4n-3 & 2n \\
    1 & 2(n-1) & 6n-5 \\
\end{pmatrix}.
\]

Using \(P2\) of Theorem 2.3 we find that the spectral radius of the matrix given in Equation (18) and the matrix given in Equation (19) are the same. Notice that the matrix

\[
\begin{pmatrix}
    2n-1 & n-1 & n \\
    1 & 4n-3 & 2n \\
    1 & 2(n-1) & 6n-5 \\
\end{pmatrix}
\]

is similar to the matrix

\[
\begin{pmatrix}
    2n-1 & n-1 & n^2 \\
    1 & 4n-3 & 2n^2 \\
    \frac{1}{n} & 2(1-\frac{1}{n}) & 6n-5 \\
\end{pmatrix}
\]

via the diagonal matrix

\[
\mathcal{D} = \begin{pmatrix}
    n & 0 & 0 \\
    0 & n & 0 \\
    0 & 0 & 1 \\
\end{pmatrix}.
\]

Since similar matrices have the same characteristic polynomial, both the matrices will have the same spectral radius.
Using Theorem 2.5 we obtain

\[
\rho \left( \begin{bmatrix}
2n - 1 & n - 1 & n \\
1 & 4n - 3 & 2n \\
1 & 2(n - 1) & 6n - 5
\end{bmatrix} \right)
\]

\[
= \rho \left( \begin{bmatrix}
2n - 1 & n - 1 & n^2 \\
1 & 4n - 3 & 2n^2 \\
\frac{1}{n} & 2(1 - \frac{1}{n}) & 6n - 5
\end{bmatrix} \right)
\]

\[
\geq 6n - 5 + 2 \left( 1 - \frac{1}{n} \right) + \frac{1}{n}
\]

\[
= 6n - 5 + 2 - \frac{1}{n}
\]

\[
= 6n - 3 - \frac{1}{n}.
\]

Thus we obtain,

\[
\rho(D^Q(P(D_n)))) \geq \rho(D^Q(G)) \geq 6n - 3 - \frac{1}{n}. \tag{20}
\]

Thus we get a lower bound on the distance signless Laplacian spectral radius of \(P(D_n)\).

**Remark 4.2.** We note that a sharper lower bound on the distance signless Laplacian spectral radius of \(P(D_n)\) can be obtained if we find the eigenvalues of the matrix given in Equation [19]. Surely it can be calculated for a fixed \(n \in \mathbb{N}\). Now if we find the eigenvalues as a function of \(n\), the calculations are cumbersome and the eigenvalues obtained are messy. As a result, it gets difficult to obtain a lower bound on the largest eigenvalue of the matrix given in Equation [19]. We thus provided an alternative way to find a somewhat simpler lower bound on the largest eigenvalue of the matrix given in Equation [19] in terms of \(n\). This in turn gave us a lower bound of the distance signless Laplacian spectral radius of \(P(D_n)\).

### 4.2.2 Upper Bound on Distance Signless Laplacian Spectral Radius of \(P(D_n)\)

We now provide an upper bound on the largest eigenvalue of \(D^Q(P(D_n)))\).

**Theorem 4.3.** For each positive integer \(n \geq 2\), the largest eigenvalue of distance signless Laplacian matrix of \(P(D_n)\) denoted by \(\rho(D^Q(P(D_n)))\) satisfies the following inequality:

\[
\rho(D^Q(P(D_n)))) \leq \frac{13n}{2} - \frac{11}{2} - \ell + \frac{1}{2}\sqrt{12\ell^2 - 4n(5\ell + 3) + 9n^2 + 16\ell + 4} + \frac{1}{2}\sqrt{32n^2 - 52n + 25}.
\]

Here \(\ell = \phi(n) + 1\).
**Proof.** Using Equation (14) and Theorem 2.4 we have,

\[
\rho(DQ(P(D_n))) \leq \lambda_1 \left( \begin{bmatrix} DQ(P(Z_n)) + W & 0 \\ 0 & 2J \end{bmatrix} \right) + \lambda_1 \left( \begin{bmatrix} 0 & V \\ V^T & (4n-5)I \end{bmatrix} \right)
\]

\[
= \max \left\{ \lambda_1(DQ(P(Z_n)) + W), 2n \right\} + \lambda_1 \left( \begin{bmatrix} 0 & V \\ V^T & (4n-5)I \end{bmatrix} \right)
\]

\[
\leq \rho(DQ(P(Z_n))) + \lambda_1(W) + \lambda_1 \left( \begin{bmatrix} 0 & V \\ V^T & (4n-5)I \end{bmatrix} \right)
\]

\[
\leq \rho(DQ(P(Z_n))) + 2n + \lambda_1 \left( \begin{bmatrix} 0 & V \\ V^T & (4n-5)I \end{bmatrix} \right).
\]

Note that

\[
\begin{bmatrix} 0 & V \\ V^T & (4n-5)I \end{bmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & \ldots & \ldots & 0 & 1 & 1 & \ldots & \ldots & 1 \\
0 & 0 & 0 & \ldots & \ldots & 0 & 2 & 2 & \ldots & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \ldots & 0 & 2 & 2 & \ldots & \ldots & 2 \\
1 & 2 & 2 & \ldots & \ldots & 2 & 4n-5 & 0 & \ldots & \ldots & 0 \\
1 & 2 & 2 & \ldots & \ldots & 2 & 0 & 4n-5 & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 2 & 2 & \ldots & \ldots & 2 & 0 & 0 & \ldots & \ldots & 4n-5 \\
\end{pmatrix}
\]

(22)

We find that each block of the matrix given in Equation (22) has a constant row sum, and hence we can form the equitable quotient matrix of the given matrix. The equitable quotient matrix of

\[
\begin{pmatrix} 0 & V \\ V^T & (4n-5)I \end{pmatrix}
\]

using Equation (22) becomes

\[
\begin{pmatrix} 0 & 0 & n \\ 0 & 0 & 2n \\ 1 & 2(n-1) & 4n-5 \end{pmatrix}
\]

Now using \textbf{P2} of Theorem 2.3 we find that the largest eigenvalue of

\[
\begin{pmatrix} 0 & V \\ V^T & (4n-5)I \end{pmatrix}
\]

equals the largest eigenvalue of the matrix

\[
\begin{pmatrix} 0 & 0 & n \\ 0 & 0 & 2n \\ 1 & 2(n-1) & 4n-5 \end{pmatrix}
\]

The characteristic polynomial of the matrix

\[
\begin{pmatrix} 0 & 0 & n \\ 0 & 0 & 2n \\ 1 & 2(n-1) & 4n-5 \end{pmatrix}
\]

is given by:

\[
x^3 + (-4n + 5)x^2 + (-4n^2 + 3n)x = x(x^2 + x(5 - 4n) + 3n - 4n^2).
\]
Hence

\[
\lambda_1 \left( \begin{bmatrix} 0 & V \\
V^T & (4n - 5)I \end{bmatrix} \right)
= \lambda_1 \left( \begin{bmatrix} 0 & 0 & n \\
0 & 0 & 2n \\
1 & 2(n - 1) & 4n - 5 \end{bmatrix} \right)
= 2n - \frac{5}{2} + \frac{1}{2}\sqrt{32n^2 - 52n + 25}.
\]  

Using Equations (9) and (23) in Equation (21) we obtain

\[
\rho(D^Q(\mathcal{P}(D_n)))
\leq \rho(D^Q(\mathcal{P}(\mathbb{Z}_n))) + 2n + 2n - \frac{5}{2} + \frac{1}{2}\sqrt{32n^2 - 52n + 25}.
\leq \frac{5n}{2} - 3 - \ell + \frac{1}{2}\sqrt{12\ell^2 - 4n(5\ell + 3) + 9n^2 + 16\ell + 4 + 4n - \frac{5}{2} + \frac{1}{2}\sqrt{32n^2 - 52n + 25}}.
= \frac{13n}{2} - \frac{11}{2} - \ell + \frac{1}{2}\sqrt{12\ell^2 - 4n(5\ell + 3) + 9n^2 + 16\ell + 4} + \frac{1}{2}\sqrt{32n^2 - 52n + 25}.
\]  

Thus Equation (24) gives an upper bound on the distance signless Laplacian spectral radius of \( \mathcal{P}(D_n) \).

5 Conclusion

In this paper, we have discussed the distance signless Laplacian spectrum of power graph of finite cyclic groups and dihedral groups. We first found the characteristic polynomial of distance signless Laplacian matrix of power graph of finite cyclic groups and determined the distance signless Laplacian spectrum when the order of the cyclic group is prime, power of a prime, or multiplication of two distinct primes. We also provide lower and upper bounds on the largest eigenvalue of the distance signless Laplacian matrix of power graph of finite cyclic groups. We then shift our attention to the distance signless Laplacian spectrum of power graph of dihedral groups. We found the characteristic polynomial of distance signless Laplacian matrix of power graph of the dihedral group in terms of distance signless Laplacian matrix of power graph of a finite cyclic group. Finally, we provide lower and upper bounds on the largest eigenvalue of the distance signless Laplacian matrix of the power graph of the dihedral group.

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References


