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FAMILIES OF (3,3)-SPLIT JACOBIANS

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ABSTRACT. We analyze curves of genus two that admit a morphism of degree three to an elliptic curve and we give formulas for the Igusa–Clebsch invariants and for plane models of curves of genus two whose Jacobians are $(3,3)$-isogenous to the product of two given elliptic curves from the Hesse pencil.

1. Introduction

If C is a curve of genus two that is equipped with a (maximal) covering $\phi\colon C \to E$ of degree n, where E is an elliptic curve, then there exists an elliptic curve E' such that the Jacobian $\mathrm{Jac}(C)$ of C is isogenous to $E \times E'$ via an isogeny of degree n^2. Such a Jacobian is said to be (n,n)-split. The classical treatment of cases $n \leqslant 4$ can be found in e.g. [2, 3, 5, 18, 20]. A modern treatment can be found in e.g. [6, 8, 15, 16, 21, 22, 23, 24, 29], dealing with various cases with $n \leqslant 11$. The problem of finding the curve E', given the map ϕ, was considered in [22]. Explicit examples appear in [6, 22, 29] for $n = 3$, which is the case that is the topic of this paper. In the first three sections, we review known results and offer corrections and clarifications. Specifically, we correct [29] regarding the number of isomorphism classes of curves with degree-3 coverings with "special" ramification (Proposition 3.4), thus vindicating [22]. We show subtleties in the approach of [22] when dealing with curves C with additional involutions. In §3.3 we describe in detail the cases in which C has additional involutions and the cases in which E and E' are twists. In the remaining sections, we consider a similar problem: given elliptic curves E and E', find a curve C of genus two such that $\mathrm{Jac}(C)$ is $(3,3)$-isogenous to $E \times E'$. Algorithms for constructing such curves can be found in [6, 7]. We go a step further and give parametrizations of the Igusa–Clebsch invariants and of an affine plane model of C in terms of two parameters that define a pair of elliptic curves from the Hesse pencil (Theorem 5.6). This can be seen as an analogue of the results in [9]. The Hesse pencil is a natural family of elliptic curves to consider when analyzing this problem because it minimizes the number of parameters required to describe the modular invariants of C and allows for a simple description of the 3-torsion and a fixed level-3-structure. Gröbner bases computations and interpolations were performed using the computational algebra system MAGMA [4]. Some details regarding this will be omitted, but the reader can find the code used in [14].

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Notation and convention. Throughout the paper, K denotes a field of characteristic $\mathrm{char}(K) \neq 2$ and $\overline{K}$ denotes an algebraic closure of K. Unless otherwise specified, all varieties are projective and defined over K and isomorphism classes and automorphism groups refer to isomorphisms and automorphisms over $\overline{K}$. The affine and the projective space of dimension n are denoted by $\mathbf{A}^n$ and $\mathbf{P}^n$, respectively. By a *covering map* (or simply *covering*) we mean a finite, surjective, separable morphism. Given a divisor D on a K-variety, we denote by $[D]$ its linear equivalence class and we denote by $L(D)$ the K-vector space of global sections of the invertible sheaf $\mathscr{L}(D)$ associated to D. Given a commutative ring R and polynomials $F, G \in R[x]$, the resultant of F and G is denoted by $\mathrm{Res}_x(F,G)$ and the discriminant of F is denoted by $\mathrm{Disc}_x(F)$. The *Igusa–Clebsch invariants* of a curve of genus two are the invariants A', B', C', D', defined in [25, p. 319]. The *Igusa invariants* are the invariants J_2, J_4, J_6, J_8, J_{10}, defined in [25, p. 324]. By the corresponding *absolute invariants* we mean the values

$$j_1 = \frac{J_2^5}{J_{10}}, \quad j_2 = \frac{J_2^3 J_4}{J_{10}}, \quad j_3 = \frac{J_2^2 J_6}{J_{10}}, \quad j_4 = \frac{J_2 J_8}{J_{10}}, \quad j_5 = \frac{J_4 J_6}{J_{10}}.$$

Throughout the paper, we will rely on various explicit computations of Gröbner bases and eliminations ideals. A good introductory text on these techniques is [12]. We will also rely on the following well known result and its generalization to the projective line and homogeneous polynomials.

Lemma 1.1. *Let k be a field, let $\bar{k}$ be an algebraic closure of k, let $\mathbf{A}^1$ denote the affine line over k, let $P(x), Q(x), F(x), G(x) \in k[x]$ with $\gcd(F(x), G(x)) = 1$, let $f(x) = F(x)/G(x) \in k(x)$, and let $D \in \mathrm{Div}(\mathbf{A}^1)$ denote the divisor that is the zero locus of $P(x)$. Then the following hold:*

(1) The polynomials $P(x)$ and $Q(x)$ have a common root in $\bar{k}$ if and only if $\mathrm{Res}_x(P(x), Q(x)) = 0$.
(2) The polynomial $P(x)$ has a multiple root in $\bar{k}$ if and only if $\mathrm{Disc}_x(P(x)) = 0$.
(3) The divisor $f_*(D)$ is the zero locus of $\mathrm{Res}_x(yG(x) - F(x), P(x))$.
(4) The divisor $f^*(f_*(D))$ is the zero locus of $\mathrm{Res}_x(F(y)G(x) - F(x)G(y), P(x))$.

The polynomials in statements *(3)* and *(4)* of Lemma 1.1 are interpreted as elements of $k[y][x]$ so that the resultants are elements of $k[y]$. Analogous statements that hold for homogeneous polynomials, rational maps $f\colon \mathbf{P}^1 \to \mathbf{P}^1$, and divisors on $\mathbf{P}^1$ can be obtained by homogenization.

2. A curve of genus two covering a curve of genus one

We begin with an overview of the setup, following [22]. Let C be a curve of genus two, defined over K and equipped with a covering $\phi\colon C \to E$ of degree n, where E is a curve of genus one. The curve C is hyperelliptic since the linear system defined by its canonical divisor K_C defines a 2-to-1 map to $\mathbf{P}^1$, by Riemann–Roch. Let ι denote the hyperelliptic involution on C. For a Weierstraß point $W \in C(\overline{K})$, one can define the Abel–Jacobi map $\varepsilon\colon C \hookrightarrow \mathrm{Jac}(C)$, given by $P \mapsto [P - W]$, and embed C into its Jacobian. The induced isomorphism $E \cong \mathrm{Jac}(E)$, given by $P \mapsto [P - \phi(W)]$, endows E with the structure of an elliptic curve. The morphism $\phi : C \to E$ induces a group morphism $\phi_* : \mathrm{Jac}(C) \to E$ so that $\phi_* \circ \varepsilon = \phi$. We therefore have the following commutative diagram:

$$\begin{array}{ccc} C & \xhookrightarrow{\varepsilon} & \mathrm{Jac}(C) \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi_*} \\ E & \xrightarrow{\sim} & \mathrm{Jac}(E) \end{array}$$

(Over $K(W)$, the curve E is elliptic so ϕ factors through $\mathrm{Jac}(C)$, by the Albanese property.) Since $\varepsilon \circ \iota = [-1] \circ \varepsilon$ and the group morphism ϕ_* commutes with $[-1]$, it follows that there is an involution on E, also defined over K, that is respected by ϕ. We denote this involution by ι as well. Since ϕ respects the involutions of C and E, it induces a map $f\colon C/\iota \to E/\iota$ such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\pi_C} & C/\iota \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle f} \\ E & \xrightarrow{\pi_E} & E/\iota \end{array}$$

Here π_C and π_E denote the canonical maps.

2.1. ***Ramification analysis.*** Kuhn [22] analyzed the ramification of the map f. We recall the main results. The map π_C has six geometric ramification points, whereas π_E has four, by Riemann–Hurwitz. These are, of course, the points fixed by the corresponding involutions. Let $W_1,\ldots,W_6$ denote the ramification points of π_C, i.e. the Weierstraß points, and let $T_1,\ldots,T_4$ denote the ramification points of π_E. Let $w_1,\ldots,w_6$ and $t_1,\ldots,t_4$ denote their respective images under the corresponding canonical maps π_C and π_E. It is clear from the above that $\{\phi(W_i)\} \subseteq \{T_j\}$ and $\{f(w_i)\} \subseteq \{t_j\}$. By Riemann–Hurwitz, the ramification divisor of ϕ is of the form $R+\iota(R)$ for some point $R \in C(\overline{K})$, meaning that ϕ either ramifies at two double points or at one triple, Weierstraß point. We distinguish two cases – either the ramification of ϕ occurs above one of the T_j or it does not. These are referred to as the "special" case and the "generic" case, respectively. In the generic case, the map $\pi_E \circ \phi = f \circ \pi_C$ is doubly ramified at each of the $4n$ points that lie above the t_j. Since π_C ramifies at six double points, we conclude that f is doubly ramified at $\frac{1}{2}(4n-6) = 2n-3$ points that lie above the t_j, none of which is any of the w_i. By Riemann–Hurwitz, f has ramification degree $2n-2$, which means that f is also doubly ramified at a point that does not lie above the t_j. In the special case, all of the ramification of f occurs above the t_j and we distinguish two cases – either ϕ ramifies at a Weierstraß point or it does not. Suppose that W_k is a triple ramification point of ϕ. Then there are $4n-3$ double ramification points of $\pi_E \circ \phi$ above the t_j. Accounting for the five Weierstraß points at which ϕ does not ramify, we conclude that f is doubly ramified at $\frac{1}{2}(4n-3-5) = 2n-4$ points above the t_j, none of which is any of the w_i, and triply ramified at w_k. Note that this special case cannot occur if $\mathrm{char}(K) = 3$ because then ϕ cannot be wildly ramified. Suppose instead that ϕ ramifies at two distinct points R and $\iota(R)$ above some T_k. Then f is ramified away from the w_i, at $2n-5$ double points above the t_j and at the quadruple point $\pi_C(R)$ that is above t_k.

Lemma 2.1. *If n is odd (respectively even) then each of the fibres $f^{-1}(t_j)$ contains an odd (respectively even) number of the w_i.*

Proof. As f is separable, every fibre of f contains n points, counting with multiplicity. Since all points in $f^{-1}(t_j)$ other than the w_i have multiplicity two, the parity of the number of the w_i in these fibres must match the parity of n. See also the lemma in [22, §1] or Lemma 2.1 in [16]. □

Corollary 2.2. *One of the points T_j is K-rational and therefore E is an elliptic curve over K.*

Proof. The ramification of the map f is restricted by Lemma 2.1 and forces the K-rationality of the point $t \in \{t_1, \dots, t_4\}$ with a unique number of the w_i in $f^{-1}(t)$. The corresponding point T such that $\pi_E(T) = t$ is therefore also K-rational. Details can be found in [22, pp. 44–45]. □

2.2. *Maximal coverings.*

From now on we will focus on covering maps $\phi\colon C \to E$ that are maximal, which is to say that they do not factor through a non-trivial isogeny over $\overline{K}$. Such covering maps are also called *minimal* [21] or *optimal* [22]; our choice of name is consistent with [28, Ch.VI §3]. Maximal coverings come in pairs, as the following result shows (see e.g. [16, 22]).

Lemma 2.3. *Let C be a curve of genus two and let $\phi_1\colon C \to E_1$ be a maximal covering of degree n of an elliptic curve E_1. Then, after extending the base field if necessary, there exists an elliptic curve E_2, a maximal covering $\phi_2\colon C \to E_2$ of degree n, and a polarized isogeny $\varphi\colon E_1 \times E_2 \to \mathrm{Jac}(C)$ whose kernel $\mathrm{Ker}(\varphi)$ is canonically isomorphic to $E_1[n]$ and $E_2[n]$. Here the abelian surfaces $E_1 \times E_2$ and $\mathrm{Jac}(C)$ are equipped with the usual principal polarizations.*

Proof. The covering map ϕ_1 induces an embedding ${\phi_1}^*\colon E_1 \hookrightarrow \mathrm{Jac}(C)$, with respect to an isomorphism $E_1 \cong \mathrm{Jac}(E_1)$. The elliptic curve E_2 is given as $\mathrm{Ker}(\phi_{1*}) \subset \mathrm{Jac}(C)$, which is connected because ϕ_1 is maximal. Let $\varepsilon\colon C \hookrightarrow \mathrm{Jac}(C)$ be an embedding, not necessarily defined over K. Recalling that Jacobians are (canonically) self-dual, let $\eta\colon \mathrm{Jac}(C) \to E_2$ denote the map dual to the inclusion $E_2 \hookrightarrow \mathrm{Jac}(C)$. The covering map $\phi_2\colon C \to E_2$ is then obtained as the composition $\eta \circ \varepsilon$. The isogeny $\varphi\colon E_1 \times E_2 \to \mathrm{Jac}(C)$ is given by $\varphi = {\phi_1}^* + {\phi_2}^*$ and its kernel is the image of $E_i[n]$ under the embedding $\phi_i^*\colon E_i \hookrightarrow \mathrm{Jac}(C)$ for $i \in \{1,2\}$, which induces a canonical isomorphism $\alpha\colon E_1[n] \xrightarrow{\sim} E_2[n]$. The dual isogeny is $\varphi^\vee = (\phi_{1*}, \phi_{2*})$ and we have $\varphi^\vee \circ \varphi = [n]$. For details, see the Lemma in [22, §2] or [16, §1] or Lemma 1.6 in [13]. □

Definition. With the assumptions of Lemma 2.3, we say that the principally polarized abelian surfaces $\mathrm{Jac}(C)$ and $E_1 \times E_2$ are *(n,n)-isogenous*. The Jacobian $\mathrm{Jac}(C)$ is said to be *(n,n)-split*, while the elliptic curves E_1 and E_2, considered as subgroups of $\mathrm{Jac}(C)$, are said to be *glued along the n-torsion*. We say that E_1 and ϕ_1 are *complementary* to E_2 and ϕ_2, respectively. The induced maps $f_i\colon C/\iota \to E_i/[-1]$ are also referred to as complementary.

Remark. Constructions in Lemma 2.3 depend on the choice of the embedding $\varepsilon\colon C \hookrightarrow \mathrm{Jac}(C)$, which need not be K-rational.

In the following two subsections we recall additional results regarding the distribution of the Weierstraß points of C in the fibres of two complementary coverings (see [22, §§4–5] and [16, §2]). These results allow one to determine a complementary covering from a given one, in principle.

2.2.1. *Maximal coverings of odd degree.*

Let $\phi_1\colon C \to E_1$ be a maximal covering of odd degree n. Let $\pi_1\colon E_1 \to \mathbf{P}^1$ be the canonical map, let T_j be the geometric ramification points of π_1, and let $t_j = \pi_1(T_j)$. It follows from Lemma 2.1 that there is a unique ramification point of π_1, say T_4, such that exactly three of the Weierstraß points W_i map to it under ϕ_1. Moreover, there is exactly one W_i above each point in $\{T_1, T_2, T_3\}$. We index the points so that W_1, W_2, W_3 lie above T_4. Since ϕ_1 is a K-rational map and the fibre above T_4 is the unique fibre containing three Weierstraß points, this fibre is fixed by the absolute Galois action and therefore the divisor $W_1+W_2+W_3$ is K-rational. Consequently, so is $W_4+W_5+W_6$. The point T_4 is also K-rational, as the image of $\{W_1, W_2, W_3\}$. Analogous statements hold for the points w_i and t_j. Thus we conclude that the curve C admits an affine plane model $y^2 = P(x)Q(x)$, where $P(x), Q(x) \in K[x]$ are cubics whose roots are $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$, respectively. Since the class of the canonical divisor $K_C \sim 2W_i$ is K-rational, so is the class of the divisor $W_1 - W_2 + W_3$. Note that this divisor class equals $[W_i - W_j + W_k]$ for $\{i,j,k\} = \{1,2,3\}$ or $\{i,j,k\} = \{4,5,6\}$. Since $W_1 - W_2 + W_3$ is a divisor of degree one that is fixed by the hyperelliptic involution, it follows that ϕ_1 induces a canonical K-rational embedding $C \hookrightarrow \mathrm{Jac}(C)$, given by

$$\varepsilon(P) = [P - W_1 + W_2 - W_3], \tag{2.1}$$

which is compatible with the isomorphism $E_1 \cong \mathrm{Jac}(E_1)$, given by $P \mapsto [P - T_4]$, and the involutions. We summarize with the following two lemmas from [22].

Lemma 2.4. *There is a canonical and K-rational choice for the complementary map $\phi_2 : C \to E_2$ and the induced map $f_2 : C/\iota \to E_2/[-1]$.*

Lemma 2.5. *The roles of the divisors $W_1+W_2+W_3$ and $W_4+W_5+W_6$ are exchanged between canonically complementary coverings ϕ_1 and ϕ_2, which is to say that $\phi_{1*}(W_1+W_2+W_3) = 3O_1$ and $\phi_{1*}(W_4+W_5+W_6) = E_1[2] \setminus \{O_1\}$ implies $\phi_{2*}(W_4+W_5+W_6) = 3O_2$, and hence also $\phi_{2*}(W_1+W_2+W_3) = E_2[2] \setminus \{O_2\}$, where O_1 and O_2 are the identity elements on E_1 and E_2, respectively.*

Proof. It is readily seen that if $E_1[2](\overline{K}) = \{O_1, T_1, T_2, T_3\}$ then

$$\{{\phi_1}^*([T_j - O_1])\}_{j=1,2,3} = \{\varepsilon(W_k)\}_{k=4,5,6} = \{[W_4 - W_5], [W_4 - W_6], [W_5 - W_6]\}.$$

Since ϕ_2 is defined as the composition $C \hookrightarrow \mathrm{Jac}(C) \to \mathrm{Jac}(C)/E_1$, we have $\phi_2(W_k) = O_2$ for $k \in \{4,5,6\}$ and the claim follows. An alternative proof can be found in [22, §4]. □

2.2.2. *Maximal coverings of even degree.* Even though our focus is coverings of degree three, we include the analogue of Lemma 2.5 for maximal coverings of even degree, both for the sake of completeness and because coverings of degree two will play an important role later on, due to the subtleties involved when dealing with genus-2 curves with additional involutions.

Lemma 2.6. *Let $\phi_1 : C \to E_1$ and $\phi_2 : C \to E_2$ be complementary maximal coverings of even degree n. Then both ϕ_1 and ϕ_2 have three fibres such that each one contains exactly two Weierstraß points. Moreover, if three fibres of ϕ_1 respectively contain $\{W_1, W_2\}$, $\{W_3, W_4\}$, $\{W_5, W_6\}$ then the same is true of ϕ_2, which is to say that same pairs of Weierstraß points appear in the fibres of both coverings.*

Proof. The proof follows the argument of [22, §5]. We consider E_1 and E_2 as subgroups of $\mathrm{Jac}(C)$, embedded via ${\phi_1}^*$ and ${\phi_2}^*$, respectively. Both elliptic curves contain four (geometric) points of $\mathrm{Jac}(C)[2]$, namely the identity O and three points of order two, of the form $[W_i - W_j]$ for some $i,j \in \{1,\dots,6\}$. These three points are the same for both curves because $E_1[n]$ and $E_2[n]$ coincide in $\mathrm{Jac}(C)$ by Lemma 2.3 and n is even. By indexing the W_i accordingly, we suppose that one of the three points is $[W_1 - W_2]$. If $[W_1 - W_k]$ is another point of $E_1[2]$ then by factoring ϕ_1 as $C \hookrightarrow \mathrm{Jac}(C) \to \mathrm{Jac}(C)/E_2$, where the embedding is given by $P \mapsto [P - W_1]$, we conclude that ${\phi_1}^{-1}(O)$ contains exactly three Weierstraß points, namely W_1, W_2, W_k, which contradicts Lemma 2.1. This implies that the three points of order two of $E_1[2]$ and $E_2[2]$ in $\mathrm{Jac}(C)$ are $[W_1 - W_2]$, $[W_3 - W_4]$, and $[W_5 - W_6]$, after indexing the W_i accordingly. By factoring the coverings as $C \hookrightarrow \mathrm{Jac}(C) \to \mathrm{Jac}(C)/E_1$ and $C \hookrightarrow \mathrm{Jac}(C) \to \mathrm{Jac}(C)/E_2$, using the embeddings $P \mapsto [P - W_i]$, we conclude that both coverings have three fibres that respectively contain $\{W_1, W_2\}$, $\{W_3, W_4\}$, and $\{W_5, W_6\}$. □

Corollary 2.7. *A curve of genus two has a $(2,2)$-split Jacobian if and only if it admits an involution that is not hyperelliptic. Such a curve admits an affine plane model of the form*

$$C\colon y^2 = ax^6 + bx^4 + cx^2 + d, \tag{2.2}$$

for some $a,b,c,d \in K$ such that $ad(b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2) \neq 0$. On this model the additional involution is given by $(x,y) \mapsto (-x,y)$, while $(x,y) \mapsto (x^2,y)$ and $(x,y) \mapsto (1/x^2, y/x^3)$ are complementary coverings of degree two to elliptic curves defined by $E_1\colon y^2 = ax^3 + bx^2 + cx + d$ and $E_2\colon y^2 = dx^3 + cx^2 + bx + a$, respectively.

Proof. Let us first suppose that C admits complementary coverings ϕ_1 and ϕ_2 of degree two. After applying suitable isomorphisms, we can assume that the maps $f_i\colon C/\iota \to E_i/\iota$ ramify above 0 and ∞, and we can also assume $f_1(x) = x^2$. By Lemma 2.6, this implies $f_2(x) = 1/x^2$, up to multiplication by a non-zero constant. After indexing the points $w_1,\dots,w_6$ accordingly, we have $f_i^{-1}(t_j) = \{\pm w_j\}$ for $j \in \{1,2,3\}$, which means that C admits an affine plane model of the form $y^2 = a(x^2 - t_1)(x^2 - t_2)(x^2 - t_3)$, i.e. of the form (2.2). Now let us suppose that the curve C admits a non-hyperelliptic involution over K. Then there exists an affine plane model of C on which the involution is given by $(x,y) \mapsto (\mu(x), \nu(x)y)$, where $\mu(x) = (ux+v)/(wx-u)$ for some $u,v,w \in K$ such that $u^2 + vw$ is non-zero and a square in K. A K-rational fractional linear transformation $\rho(x)$ exists that satisfies $-\rho(x) = \rho(\mu(x))$ and therefore induces an isomorphism to an affine plane model of C on which the involution is given by $(x,y) \mapsto (-x,y)$. Such a model is of the form (2.2). That curves of genus two with additional involutions admit such a model was first shown by Bolza [1]. □

Curves of genus two with a $(2,2)$-split Jacobian are classically known; they can be traced to the work of Legendre and Jacobi [20] on hyperelliptic integrals.

3. Covering maps of degree three

In this section we deal with the case $n = 3$. Our goal is to describe the genus-2 curves C with a $(3,3)$-split Jacobian and the corresponding elliptic curves that are complementary in $\mathrm{Jac}(C)$, up to isomorphisms. To that end, we will use the canonical embedding (2.1) and apply Lemmas 2.1 and 2.5. The corresponding restrictions on the maps $f_i\colon C/\iota \to E_i/[-1]$ will lead to a parametrization of the family of such curves C and the corresponding elliptic curves E_1 and E_2. We will make repeated use of Lemma 1.1 in this section.

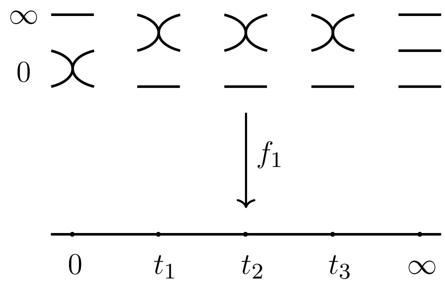

FIGURE 3.1. Ramification of the map f_1 in the generic case.

3.1. ***Generic coverings.*** The main results in this subsection appear in [22] without proof, but here we give the derivation in detail. Let $\phi_1\colon C \to E_1$ be a generic covering of degree 3. Then the map f_1 is doubly ramified at a K-rational point that is not in any of the fibres ${f_1}^{-1}(t_j)$. Let us denote the image of this point under f_1 by t_0. Since t_0 and t_4 are K-rational, we may and do assume that $t_0 = 0$, $t_4 = \infty$, and ${f_1}^*(0) = 2\cdot(0) + (\infty)$. In other words, we assume that the ramification of f_1 is as depicted in Figure 3.1, where the unramified points above $t_1, \dots, t_4$ are the w_i (depicted are the ramification indices of the points in the fibres of f_1 above the t_j). That is, we assume that, up to multiplication by a non-zero constant,

$$f_1(x) = \frac{x^2}{P(x)},$$

where $P(x) = x^3 + ax^2 + bx + c \in K[x]$ is the polynomial that has $w_1, w_2, w_3 \in \overline{K}$ as its roots. The w_i are pairwise distinct and none of them equals zero. This can be expressed as

$$\begin{aligned} \mathrm{Res}_x(x, P(x)) &= c \neq 0, \\ \mathrm{Disc}_x(P(x)) &= a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2 \neq 0. \end{aligned}$$

The pullback of the divisor $t_1 + t_2 + t_3$ corresponds to the roots of $D(x)^2Q(x)$, where $D(x)$ and $Q(x)$ are cubic polynomials in $K[x]$. Furthermore, the roots of $D(x)$ are the ramification points distinct from 0, and the roots of $Q(x)$ are w_4, w_5, w_6. Since

$$\frac{\mathrm{d}f_1}{\mathrm{d}x}(x) = -\frac{x(x^3 - bx - 2c)}{P(x)^2}$$

and the roots of the numerator correspond precisely to the doubly ramified points of f_1, we can take $D(x) = x^3 - bx - 2c$. The ramification points are again pairwise distinct, so we must have

$$\mathrm{Disc}_x(D(x)) = 4(b^3 - 27c^2) \neq 0. \tag{3.1}$$

From this we can calculate the nonic polynomial $D(x)^2Q(x)$ whose roots correspond to the divisor ${f_1}^* \circ {f_1}_*(d_1 + d_2 + d_3)$, where the d_i are the roots of $D(x)$. In particular, we have the following equality, up to multiplication by a non-zero constant:

$$\mathrm{Res}_y(x^2P(y) - y^2P(x), D(y)) = D(x)^2Q(x).$$

This resultant is easily found to be

$$c(x^3-bx-2c)^2(4cx^3+b^2x^2+2bcx+c^2),$$

so we can take $Q(x)=4cx^3+b^2x^2+2bcx+c^2$. It follows that, up to twists, the curve C admits an affine plane model given by

$$y^2=P(x)Q(x)=(x^3+ax^2+bx+c)(4cx^3+b^2x^2+2bcx+c^2). \tag{3.2}$$

The discriminant of (3.2) is $4096c^{12}(b^3-27c^2)(a^2b^2-4b^3-4a^3c+18abc-27c^2)^3\neq 0$. Note that the point (a,b,c) defines the same isomorphism class of C as $(a\lambda,b\lambda^2,c\lambda^3)$ for all $\lambda\in K\setminus\{0\}$ so we can also think of it as the point $[a:b:c]$ in the weighted projective space $\mathbf{P}(1,2,3)$.

By Lemma 2.5, we may assume that the corresponding complementary map is given by

$$f_2(x)=\frac{(x+d)^2(x+e)}{4cx^3+b^2x^2+2bcx+c^2}$$

for some $d,e\in K$, provided that ∞ is not a zero of f_2. To determine d and e, we apply the procedure used to obtain $Q(x)$ from f_1 to the map f_2. In doing so, we must ultimately obtain a cubic polynomial $R(x)$ that is a multiple of $P(x)$, by Lemma 2.5. Working over the field $K(a,b,c,d,e)$, we can compute the polynomial $R(x)$ and perform Euclidean division on $P(x)$ and $R(x)$. By the argument above, the remainder must equal zero, which defines three polynomial equations over the ring $K[a,b,c,d,e]$. More details can be found in [13], where the computations are performed over $K(a,b,c)[d,e]$. The argument here is only slightly different. Let $I\subset K[z,a,b,c,d,e]$ denote the ideal generated by

$$1-z\cdot P(0)\cdot Q(-d)\cdot Q(-e)\cdot \mathrm{Disc}_x(P(x))\cdot \mathrm{Disc}_x(Q(x))$$

and the coefficients of the remainder obtained by dividing $R(x)$ by $P(x)$. Eliminating the variable z and computing the primary decomposition of the corresponding elimination ideal gives two prime ideals. Among the generators of the first ideal we find the following two equations:

$$bd-3c=0,\quad acd-4ace+b^2e-bc+3cde=0.$$

We therefore take

$$f_2(x)=\frac{(bx+3c)^2((b^3-4abc+9c^2)x+b^2c-3ac^2)}{4cx^3+b^2x^2+2bcx+c^2}. \tag{3.3}$$

For $b=0$ (resp. $b^3-4abc+9c^2=0$) we have that ∞ is a double (resp. simple) zero of f_2, so no cases are omitted. The map f_2 has a triple zero if and only if $2b^3-9abc+27c^2=0$; this case is treated in §3.2. The second ideal in the decomposition is of a lower dimension and does not yield any cases that are not already covered by (3.3). Indeed, eliminating the variables d and e from this ideal yields the equation (3.20), which corresponds to a family of curves described in §3.3.2.

Now we have the information required to determine the modular invariants of E_1 and E_2. An affine plane model for E_1 can be determined, up to twists, by requiring that the set of branch points of the canonical map π_1 is $\{t_1,t_2,t_3,\infty\}$, i.e. ∞ and the image under f_1 of the three roots of $Q(x)$. Likewise, an affine plane model for E_2 can be determined, up to twists, by requiring that the canonical map $\pi_2\colon E_2\to\mathbf{P}^1$ ramifies above ∞ and the image under f_2 of the three roots of $P(x)$. The corresponding cubic polynomials can be obtained from $\mathrm{Res}_y(xP(y)-y^2,Q(y))$ and $\mathrm{Res}_y(xQ(y)-(y+d)^2(y+e),P(y))$. The j-invariants of the two elliptic curves can then be obtained from these cubics by direct computation. This yields the following two expressions that appear in [22, §6]:

$$\begin{aligned} j(E_1) &= \frac{16(a^2b^4+216a^2bc^2-126ab^3c+12b^5-972ac^3+405b^2c^2)^3}{(b^3-27c^2)^3(a^2b^2-4a^3c+18abc-4b^3-27c^2)^2}, \\ j(E_2) &= \frac{256(a^2-3b)^3}{a^2b^2-4a^3c+18abc-4b^3-27c^2}. \end{aligned} \tag{3.4}$$

One consequence of these formulas is the following.

Proposition 3.1. *If* $\mathrm{char}(K)\notin\{3,5,7\}$ *then there are exactly two isomorphism classes (over* $\overline{K}$*) of curves* C *such that at least one of the complementary 3-to-1 covering maps* $C\to E_1$ *and* $C\to E_2$ *is generic and* $j(E_1)=j(E_2)=j\in\{0,1728\}$*. If* $\mathrm{char}(K)=3$ *then there is a one-dimensional family of isomorphism classes of such* C*. If* $\mathrm{char}(K)=5$ *(respectively* $\mathrm{char}(K)=7$*) then there is only one isomorphism class of such* C*, with* $j=0$ *(respectively* $j=1728$*). In all cases mentioned, both coverings are generic.*

Proof. If we suppose that $\mathrm{char}(K)=3$ then from (3.4) we find that $j(E_1)=j(E_2)=0$ if and only if $a=0$ (and $bc\neq 0$), which defines a one-dimensional family of curves C. This family will be revisited in §3.3.1, specifically (3.19). Let us suppose instead that $\mathrm{char}(K)\neq 3$. Equating both expressions in (3.4) with zero and solving for a,b,c leads to the two isomorphism classes of C defined by $[a:b:c]=[0:0:1]$ and $[a:b:c]=[6:12:10]$, unless $\mathrm{char}(K)=5$, in which case the latter point defines a singular curve. Equating the expressions in (3.4) with 1728 leads to the two isomorphism classes defined by $[a:b:c]=[3:3\pm 3\sqrt{-3}):1\pm 3\sqrt{-3}]$, unless $\mathrm{char}(K)=7$, in which case exactly one of the two points defines a singular curve. In all cases listed, we have $2b^3-9abc+27c^2\neq 0$, so both coverings are generic by (3.3) and the comments below it. These isomorphism classes are treated in Examples 3.2, 3.3, and 5.8–5.10. □

The curves E_1 and E_2 can also be computed. Consider the morphisms $C\to\mathbf{A}^2$ given by

$$\phi_1(x,y)=\left(f_1(x),\,\frac{y}{x}{f_1}'(x)\right),\quad \phi_2(x,y)=\left(f_2(x),\,\frac{y}{bx+3c}{f_2}'(x)\right), \tag{3.5}$$

where $f_i'(x)=\frac{\mathrm{d}}{\mathrm{d}x}f_i(x)$. Then ϕ_1 and ϕ_2 are of degree three and their images are respectively the elliptic curves

$$\begin{aligned} E_1\colon y^2 &= \Delta_1x^3-2(ab^2-6a^2c+9bc)x^2+(b^2-12ac)x+4c, \\ E_2\colon \Delta_2y^2 &= x^3+(ab^3-27b^2c+54ac^2)x^2+(b^7-18ab^5c+54a^2b^3c^2 \\ &\quad +189b^4c^2-972ab^2c^3+729a^2c^4+729bc^4)x-c(2b^3-9abc+27c^2)^3, \end{aligned}$$

where $\Delta_1=a^2b^2-4b^3-4a^3c+18abc-27c^2$ and $\Delta_2=b^3-27c^2$. In many examples throughout the paper, the maps defined by (3.5) will be composed with isomorphisms of elliptic curves in order to simplify the models of E_1 and E_2.

Remark. Our choices of ramification points and parametrizations in this section are the same as those made by Kuhn [22]. However, in terms of elegance and simplicity, the parametrization given in [6] is arguably superior. This is particularly evident when it comes to explicitly writing down models of the elliptic curves.

Example 3.2. Suppose that $\mathrm{char}(K) \neq 3$ and consider the genus-2 curve defined by the affine plane model

$$C\colon y^2 = (x^3+1)(4x^3+1).$$

The curve C admits complementary 3-to-1 covering maps

$$\phi_1(x,y) = \left(-\frac{3x^2}{x^3+1}, \frac{(x^3-2)y}{(x^3+1)^2}\right), \quad \phi_2(x,y) = \left(-\frac{3x}{4x^3+1}, \frac{(8x^3-1)y}{(4x^3+1)^2}\right), \tag{3.6}$$

whose images are elliptic curves, respectively defined by affine plane models

$$E_1\colon y^2 = x^3+4,$$
$$E_2\colon y^2 = x^3+1.$$

Note that $j(E_1) = j(E_2) = 0$. The map ϕ_1 ramifies at $(0,\pm1)$, while the map ϕ_2 ramifies at $\pm\infty$. The ramification points lie above rational order-3 points of the corresponding elliptic curves; these are the points $(0,\pm2)$ and $(0,\pm1)$, respectively. In particular, the ramification does not occur above 2-torsion points. The two elliptic curves are isomorphic over $K(q)$, where $q \in \overline{K}$ is such that $q^3 = 4$; an isomorphism $\epsilon\colon E_2 \xrightarrow{\sim} E_1$ is given by $\epsilon(x,y) = (qx, 2y)$. The curve C admits an additional automorphism of order two over $K(q)$, i.e. an involution that is not the hyperelliptic one. One such automorphism is $\eta(x,y) = (1/(qx), -y/(2x^3))$. Note that $\phi_1 \circ \eta = \epsilon \circ \phi_2$, meaning that each of the coverings is induced by its complementary covering and an involution of C. Over $K(q)$, the curve C also admits a pair of complementary coverings of degree two. Indeed, after applying the isomorphism $(x,y) \mapsto ((2x-q)/(2x+q), 32y/(2x+q)^3)$, we obtain a new model for C, namely

$$\tilde{C}\colon -2y^2 = (x^3-9x^2+3x-3)(x^3+9x^2+3x+3) = x^6-75x^4-45x^2-9,$$

on which the involution η is given by $(x,y) \mapsto (-x,-y)$. The morphisms $\psi_1(x,y) = (-2x^2, 4y)$ and $\psi_2(x,y) = (18/x^2, 36y/x^3)$ are complementary degree-2 coverings from $\widetilde{C}$ to elliptic curves

$$\tilde{E}_1\colon y^2 = x^3+150x^2-180x+72,$$
$$\tilde{E}_2\colon y^2 = x^3+90x^2+2700x-648,$$

respectively. These two elliptic curves are 9-isogenous over K and their modular invariants are $j(\tilde{E}_1) = -12288000$ and $j(\tilde{E}_2) = 0$. If $r \in \overline{K}$ is such that $r^2 = -3$ then $\tilde{E}_1$ and $\tilde{E}_2$ are 3-isogenous over $K(r)$, while E_1, E_2, and $\tilde{E}_2$ are isomorphic over $K(q,r)$.

Example 3.3. Suppose that $\mathrm{char}(K) \notin \{3,5\}$ and consider the genus-2 curve defined by

$$C\colon y^2 = (x^3+6x^2+12x+10)(10x^3+36x^2+60x+25).$$

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The curve C admits complementary 3-to-1 coverings

$$\phi_1(x,y) = \left(-\frac{3x^2}{x^3+6x^2+12x+10}, \frac{(x^3-12x-20)y}{(x^3+6x^2+12x+10)^2}\right),$$

$$\phi_2(x,y) = \left(\frac{3(2x+5)^2(7x+10)}{10x^3+36x^2+60x+25}, \frac{27(44x^3+120x^2+150x+125)y}{(10x^3+36x^2+60x+25)^2}\right),$$

whose images are 3-isogenous elliptic curves, respectively given by

$$E_1\colon y^2 = x^3+12x^2+48x+10,$$
$$E_2\colon y^2 = x^3-36x^2+432x-270,$$

with $j(E_1) = j(E_2) = 0$. Both maps ramify above the points with $x=0$, which are not of order two. Just as in the previous example, up to isomorphism of elliptic curves over a suitable field extension of K, the two complementary coverings can be obtained from one another by pre-composing with an involution of C, namely

$$(x,y) \mapsto \left(-\frac{5(2x+5)}{7x+10}, \pm\frac{375\sqrt{-3}y}{(7x+10)^3}\right). \tag{3.7}$$

The involution (3.7) induces a pair of complementary degree-2 coverings over $K(\sqrt{-3})$ to elliptic curves $\tilde{E}_1$ and $\tilde{E}_2$ such that $j(\tilde{E}_1) = j(\tilde{E}_2) = -12288000$. Furthermore, there exist isomorphisms $E_1 \xrightarrow{\sim} E_2$ and $\tilde{E}_1 \xrightarrow{\sim} \tilde{E}_2$ and a 3-isogeny $E_1 \to \tilde{E}_1$, all defined over $K(\sqrt{-3})$.

3.2. ***Special coverings.*** In this subsection we will deal with the cases in which at least one of the maps f_i is special, i.e. it has a triple ramification point above the branch locus of π_i (the other special case mentioned in §2.1 cannot occur for coverings of degree three). For the remainder of the subsection, we assume that $\mathrm{char}(K) \neq 3$.

Let us first suppose that f_1 is generic and f_2 is special. By the reasoning in §3.1, we may and do assume that

$$f_1(x) = \frac{x^2}{(bx+3c)(9cx^2+2b^2x+3bc)}, \quad f_2(x) = \frac{(bx+3c)^3}{4cx^3+b^2x^2+2bcx+c^2}, \tag{3.8}$$

where $bc(b^3-27c^2) \neq 0$, so that a twist of C admits an affine plane model given by

$$y^2 = (bx+3c)(9cx^2+2b^2x+3bc)(4cx^3+b^2x^2+2bcx+c^2). \tag{3.9}$$

The discriminant of (3.9) is $2359296bc^{16}(b^3-27c^2)^{10} \neq 0$ and the maps given in (3.5) are degree-3 coverings of elliptic curves defined by

$$E_1\colon 9by^2 = 4\Delta^3x^3+12\Delta^2x^2-3(5b^3+108c^2)x+4,$$

$$E_2\colon y^2 = cx^3+2\Delta x^2-27\Delta cx,$$

where $\Delta = b^3-27c^2$. The j-invariants of E_1 and E_2 are easily found to be

$$j(E_1) = \frac{64b^3}{c^2}, \quad j(E_2) = \frac{64(4b^3-27c^2)^3}{729b^3c^4}. \tag{3.10}$$

Note that $j(E_2)$ is uniquely determined by $j(E_1)$ because we have $F(j(E_1), j(E_2)) = 0$, where

$$F(X,Y) = (X-432)^3 - 729XY. \tag{3.11}$$

Let us now suppose that f_1 and f_2 are both special. Without loss of generality, we can assume

$$f_1(x) = \frac{x^3}{x^2+ax+b}, \quad f_2(x) = \frac{1}{Q(x)},$$

where $b \neq 0$ and $a^2 - 4b \neq 0$. The cubic $Q(x)$ can be obtained from $f_1(x)$, using the same method as in §3.1. This yields

$$Q(x) = (a^2-4b)x^3 - 2abx^2 - 3b^2x.$$

Applying the method to f_2, to re-obtain the denominator of f_1, we conclude that x^2+ax+b divides the polynomial

$$3(a^2-4b)^2x^2 - 4ab(a^2-4b)x - 16b^2(a^2-3b).$$

Dividing the latter by the former gives the remainder $-a(3a^2-8b)(a^2-4b)x - a^2b(3a^2-8b)$. Given that $a^2-4b \neq 0$ and $b \neq 0$, the remainder is identically zero if and only if $a = 0$ or $b = 3a^2/8$. For $a = 0$ we obtain

$$f_1(x) = \frac{x^3}{x^2+b}, \quad f_2(x) = \frac{1}{4x^3+3bx},$$
$$j(E_1) = j(E_2) = 1728.$$

For $b = 3a^2/8$ we obtain

$$f_1(x) = \frac{x^3}{8x^2+8ax+3a^2}, \quad f_2(x) = \frac{1}{32x^3+48ax^2+27a^2x},$$
$$j(E_1) = j(E_2) = -\frac{873722816}{59049} = -\frac{2^6 \cdot 239^3}{3^{10}}.$$

This shows that there are exactly two pairs of isomorphism classes of E_1 and E_2 such that two covering maps $C \to E_i$ of degree three have a triple ramification point and such that the fibre with three Weierstraß points of one covering has no points in common with the fibre with three Weierstraß points of the other covering. However, only the case $j(E_1) = j(E_2) = 1728$ corresponds to a pair of complementary curves and coverings in the sense of Definition 2.2. This case is described in Example 3.5. In the other case we have $E_1 = E_2 = E$ and one special covering is obtained from the other one by pre-composing with an automorphism of C. These two special coverings are not complementary because they are the only degree-3 maps $C \to E$ and both can be obtained as complements of a generic degree-3 covering $C \to E'$, where E' is an elliptic curve such that $j(E) \neq j(E')$. Indeed, a complementary pair of curves with $j(E_2) = -873722816/59049$ and a special covering $C \to E_2$ is obtained by putting $c^2 = 9b^3$ in (3.8)–(3.10), which yields a generic covering $C \to E_1$ with $j(E_1) = 64/9$. This is expounded in Examples 3.6 and 5.11.

We therefore conclude that there is a unique isomorphism class of C whose complementary degree-3 coverings are both special. There have been conflicting claims in the literature about the number of these special cases (cf. [22, 29]) and we hope that our detailed exposition settles

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the matter – the statement in [22, §6] is correct. The following proposition contains a summary analysis of the special cases, which are treated in detail in Examples 3.5–3.9.

Proposition 3.4. *Let C be a curve of genus two that covers elliptic curves E_1 and E_2 via complementary degree-3 covering maps ϕ_1 and ϕ_2, respectively. Then the following hold:*

(1) If ϕ_1 and ϕ_2 are both special then $j(E_1) = j(E_2) = 1728$.
(2) If ϕ_1 is generic and $j(E_1) \in \{0, 1728\}$ then ϕ_2 is also generic.
(3) Suppose that ϕ_1 is generic and ϕ_2 is special. Then $j(E_2) = (j(E_1) - 432)^3/(729 j(E_1))$. In particular, we have $j(E_2) = 0$ if and only if $j(E_1) = 432$ and we have $j(E_2) = 1728$ if and only if $j(E_1) = -216$. Moreover, if $j(E_1) = j(E_2) = j$ then $j = (297 \pm 81\sqrt{-15})/2$.

Proof. Claim *(1)* follows from the discussion above – the isomorphism class of C that is a triple cover of $j(E_1) = -873722816/59049$ is excluded because its complement is generic. Claim *(2)* follows from Proposition 3.1, while claim *(3)* follows from (3.10) and (3.11). □

Remark. The j-invariant pairs that are given in [29] can be obtained as the points of intersection of the curves $F(X,Y) = 0$ and $F(Y,X) = 0$, where $F(X,Y)$ is defined by (3.11). However, these points do not correspond to the j-invariant pairs of elliptic curves whose 3-to-1 coverings by a curve of genus two are both special because (3.11) is obtained under the assumption that one of the coverings is generic. For example, we have $F(1728, 1728) = 0$. However, setting $j(E_1) = j(E_2) = 1728$ in (3.10) forces $b^3 - 27c^2 = 0$, which implies that the sextic in (3.9) has discriminant zero, so (3.9) does not define a curve of genus two in this case.

Example 3.5. Let C be the genus-2 curve defined by the affine plane model

$$y^2 = x(x^2+1)(4x^2+3).$$

Then C admits complementary 3-to-1 coverings

$$\phi_1(x,y) = \left(\frac{1}{x(4x^2+3)}, \frac{(4x^2+1)y}{x^2(4x^2+3)^2}\right), \quad \phi_2(x,y) = \left(\frac{4x^3}{x^2+1}, \frac{4x(x^2+3)y}{(x^2+1)^2}\right),$$

whose images are respectively the elliptic curves defined by the affine plane models

$$E_1\colon y^2 = x^3 + x,$$
$$E_2\colon y^2 = x^3 + 108x.$$

We have $j(E_1) = j(E_2) = 1728$. Moreover, ∞ is a triple ramification point for ϕ_1 and $(0,0)$ is a triple ramification point for ϕ_2. Both points lie above $(0,0)$, which is a point of order two on both E_1 and E_2. In fact, over a field extension of K that contains an element q such that $q^4 = 3/4$, the two complementary coverings can be obtained from one another by pre-composing with the involution of C given by $(x,y) \mapsto (q^2/x,\, q^3y/x^3)$. This involution induces complementary coverings of degree two, for example

$$\psi_1(x,y) = \left(\frac{(x-q)^2}{(x+q)^2}, \frac{8y}{(x+q)^3}\right), \quad \psi_2(x,y) = \left(\frac{(x+q)^2}{(x-q)^2}, \frac{8y}{(x-q)^3}\right),$$

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whose images are respectively the elliptic curves given by

$$\tilde{E}_1\colon (6q-7q^3)y^2 = (x-1)(x^2-(386-448q^2)x+1),$$
$$\tilde{E}_2\colon (7q^3-6q)y^2 = (x-1)(x^2-(386-448q^2)x+1).$$

The two elliptic curves are twists and their j-invariant is $j(\tilde{E}_1) = j(\tilde{E}_2) = 76771008 - 88660992q^2$. Let $i \in \overline{K}$ be such that $i^2 = -1$. The curves C, E_1, E_2 all admit an automorphism over $K(i)$, given by $(x,y) \mapsto (-x, iy)$. Additionally, over the field $K(q,i)$, we have the following. The elliptic curves $\tilde{E}_1$ and $\tilde{E}_2$ are isomorphic. The curve C admits another involution, namely $(x,y) \mapsto (-q^2/x, iq^3y/x^3)$, that induces another pair of coverings. For example,

$$\chi_1(x,y) = \left(\frac{(x-iq)^2}{(x+iq)^2}, \frac{4(1+i)y}{(x+iq)^3}\right), \quad \chi_2(x,y) = \left(\frac{(x+iq)^2}{(x-iq)^2}, \frac{4(1-i)y}{(x-iq)^3}\right),$$

are complementary 2-to-1 coverings whose image is the elliptic curve

$$E'\colon 2(6q+7q^3)y^2 = (x-1)(x^2-(386+448q^2)x+1),$$

whose j-invariant is $j(E') = 76771008 + 88660992q^2$. Moreover, E' is 2-isogenous to $\tilde{E}_1$ and $\tilde{E}_2$.

Example 3.6. Let C be the genus-2 curve defined by the affine model

$$y^2 = x(2x^2+4x+3)(3x^2+4x+2).$$

Then C admits 3-to-1 coverings

$$\phi_1\colon C \to E_1, \quad (x,y) \mapsto \left(\frac{18x^3}{3x^2+4x+2}, \frac{18x(3x^2+8x+6)y}{(3x^2+4x+2)^2}\right),$$

$$\tilde{\phi}_1\colon C \to E_1, \quad (x,y) \mapsto \left(\frac{18}{x(2x^2+4x+3)}, \frac{18(6x^2+8x+3)y}{x^2(2x^2+4x+3)^2}\right),$$

where $E_1\colon y^2 = x(x^2+44x+486)$. We have $j(E_1) = -873722816/59049$. Note that $\tilde{\phi}_1 = \phi_1 \circ \eta$, where $\eta \in \mathrm{Aut}(C)$ is given by $\eta(x,y) = (1/x, y/x^3)$. Both covering maps ramify only above $(0,0)$, which is a 2-torsion point, and send disjoint sets of three Weierstraß points to ∞, i.e. the identity point. However, as already mentioned, these two coverings are not complementary. The curve C also admits a pair of generic 3-to-1 coverings to an elliptic curve which is not in the isomorphism class of E_1. For example, we can take

$$\phi_2\colon C \to E_2, \quad (x,y) \mapsto \left(\frac{-2x^3+4x^2+5x+2}{x(2x^2+4x+3)}, \frac{2(2x+1)(2x^2+1)y}{x^2(2x^2+4x+3)^2}\right),$$

$$\tilde{\phi}_2\colon C \to E_2, \quad (x,y) \mapsto \left(\frac{2x^3+5x^2+4x-2}{3x^2+4x+2}, \frac{2(x+2)(x^2+2)y}{(3x^2+4x+2)^2}\right),$$

where $E_2\colon y^2 = x^3 - x^2 + x + 3$ and $j(E_2) = 64/9$. We note that $\tilde{\phi}_2 = \phi_2 \circ \eta$ and that ϕ_1 and ϕ_2 are complementary, as are $\tilde{\phi}_1$ and $\tilde{\phi}_2$. The covering ϕ_2 ramifies at the two points with $x = -3/2$, whereas the covering $\tilde{\phi}_2$ ramifies at the two points with $x = -2/3$. If $\mathrm{char}(K) = 0$ then the ramification occurs above points of infinite order. The involution η induces complementary coverings of degree two from C to the same two elliptic curves. The corresponding maps are

$$\varphi_1 : C \to E_1, \quad (x,y) \mapsto \left(\frac{81x}{(x-1)^2}, \frac{81y}{(x-1)^3} \right),$$

$$\varphi_2 : C \to E_2, \quad (x,y) \mapsto \left(-\frac{x^2+x+1}{(x+1)^2}, \frac{y}{(x+1)^3} \right).$$

The curves E_1 and E_2 are 5-isogenous and, up to composition with $[-1]$, the restriction of the isogeny to the 3-torsion yields the isomorphism $E_1[3] \xrightarrow{\sim} E_2[3]$ from Lemma 2.3. This is elaborated in §5, where this example is revisited (see Example 5.11). Note that this example demonstrates that the converse of Lemma 2.5 is false, in the sense that coverings that exchange $P(x)$ and $Q(x)$ need not be complementary. This is just one such example from a one-dimensional family that is discussed in §3.3.2.

Example 3.7. Let C be the genus-2 curve defined by the affine model

$$y^2 = (x+2)(x^2+x+1)(8x^3+9x^2+12x+4).$$

Then the morphisms

$$\phi_1(x,y) = \left(-\frac{3x^2}{(x+2)(x^2+x+1)}, \frac{(x^3-3x-4)y}{(x+2)^2(x^2+x+1)^2} \right),$$

$$\phi_2(x,y) = \left(\frac{(x+2)^3}{8x^3+9x^2+12x+4}, \frac{(x+2)(13x^2+4x+4)y}{(8x^3+9x^2+12x+4)^2} \right),$$

are complementary 3-to-1 coverings of elliptic curves that are respectively defined by the affine plane models $E_1 : y^2 = x^3+6x^2+21x+8$ and $E_2 : y^2 = x^3-3x^2+3x$. We have $j(E_1) = 432$ and $j(E_2) = 0$.

Example 3.8. Let C be the genus-2 curve defined by the affine model

$$y^2 = (x-2)(x+1)(2x-1)(16x^3+9x^2-12x+4).$$

Then the morphisms

$$\phi_1(x,y) = \left(-\frac{9x^2}{(x-2)(x+1)(2x-1)}, \frac{(2x^3+3x-4)y}{(x-2)^2(x+1)^2(2x-1)^2} \right),$$

$$\phi_2(x,y) = \left(\frac{2(x-2)^3}{16x^3+9x^2-12x+4}, \frac{2(x-2)(5x+2)(7x-2)y}{(16x^3+9x^2-12x+4)^2} \right),$$

are complementary 3-to-1 coverings of elliptic curves that are respectively defined by the affine plane models $E_1 : y^2 = x^3+6x^2+9x+8$ and $E_2 : y^2 = x^3+9x^2+18x$. We have $j(E_1) = -216$ and $j(E_2) = 1728$.

Example 3.9. Let $K = K(q)$ with $q^2 = -15$ and consider the following polynomials in $K[x]$:

$$\begin{aligned}
P(x) &= 4x^3 + (9q-15)x^2 - (24q+24)x + 256,\\
Q(x) &= 64x^3 + (18q-126)x^2 - (192q+192)x + 1024,\\
R(x) &= x^3 + (6q+6)x - 128,\\
S(x) &= (6q-19)x^3 - (54q+138)x^2 + (24q+168)x + 96q - 352.
\end{aligned}$$

Let C be the genus-2 curve defined by the affine model $y^2 = P(x)Q(x)$. Then the morphisms

$$\phi_1(x,y) = \left(\frac{(18-30q)x^2}{P(x)}, \frac{8R(x)y}{P(x)^2}\right), \quad \phi_2(x,y) = \left(\frac{16(x-2+2q)^3}{Q(x)}, \frac{32S(x)y}{Q(x)^2}\right),$$

are complementary degree-3 coverings of elliptic curves

$$\begin{aligned}
E_1 &: y^2 = x^3 + 12x^2 + (3q+57)x + 64,\\
E_2 &: y^2 = x^3 + (3q-9)x^2 - (15q+9)x,
\end{aligned}$$

respectively. An isomorphism $E_1 \xrightarrow{\sim} E_2$ is given by $(x,y) \mapsto (x+7-q, y)$ and the j-invariant of the two curves is $j(E_1) = j(E_2) = \frac{1}{2}(297+81q)$.

3.3. ***Complementary coverings of isogenous curves.*** The curves C in Examples 3.2–3.6 have an additional automorphism of order two that induces a complementary degree-3 covering. In this section we describe such curves in full generality. We also describe the curves C with complementary degree-3 coverings to two elliptic curves that are twists of each other. Plane models of the corresponding elliptic curves and formulas for the isogenies between them are omitted here, but can be found in [14]. Curves of genus two with additional involutions and a $(3,3)$-split Jacobian have already appeared in [30], but the analysis therein is erroneous.[1]

3.3.1. *Families of complementary coverings of twists.* It is worth asking for which curves C we have $j(E_1) = j(E_2)$. If we equate the two j-invariants in (3.4) we obtain equations in the weighted projective space $\mathbf{P}(1,2,3)$. Let us first suppose that $\mathrm{char}(K) \neq 3$. Then the said equations define a union of two curves of genus zero, namely:

$$\mathcal{X}_1 : 4a^3b^3c - 108a^3c^3 - a^2b^5 + 108a^2b^2c^2 - 54ab^4c + 8b^6 + 27b^3c^2 = 0, \tag{3.12}$$

$$\begin{aligned}
\mathcal{Y}_1 : 16a^6b^6 &- 864a^6b^3c^2 + 11664a^6c^4 - 324a^5b^5c + 8748a^5b^2c^3 - 81a^4b^7\\
&+ 14580a^4b^4c^2 - 157464a^4bc^4 - 864a^3b^6c - 215784a^3b^3c^3 + 78732a^3c^5\\
&+ 324a^2b^8 + 30618a^2b^5c^2 + 2125764a^2b^2c^4 - 5832ab^7c - 314928ab^4c^3\\
&- 6377292abc^5 + 37908b^6c^2 + 255879b^3c^4 + 8503056c^6 = 0.
\end{aligned} \tag{3.13}$$

The two curves have five distinct (geometric) points of intersection. Two intersection points, namely $[1:0:0]$ and $[3:3:1]$, are singularities on both curves and do not define genus-2 curves. Another intersection point is $[6:12:10]$, which defines the isomorphism class from Example 3.3

[1] The j-invariants of the elliptic curves covered 2-to-1 are incorrect, it is wrongly claimed that there are only finitely many cases defined over $\mathbf{Q}$ (up to isomorphism), and the isomorphism class of $y^2 = 4x^5 + 7x^3 + 3x$ is omitted.

and is singular on the second curve. Let $\omega \in \overline{K}$ denote a primitive third root of unity. Then the remaining two points of intersection of $\mathcal{X}_1$ and $\mathcal{Y}_1$ are $[3:6+6\omega^{\pm 1}:4+6\omega^{\pm 1}]$. These two points define isomorphism classes of C for which $j(E_1)=j(E_2)=1728$ (see also Example 5.10).

Both $\mathcal{X}_1$ and $\mathcal{Y}_1$ are birational to $\mathbf{P}^1$, after extending K if necessary in the case of the latter. One example of a birational map $\mathbf{P}^1 \dashrightarrow \mathcal{X}_1$ is

$$t \mapsto [3(t+1)(t+3):48(t+1)t^2:64(3t+1)t^3]. \tag{3.14}$$

Accordingly, let $t \in K$ be such that $t(3t+1)(t^2-6t-3) \neq 0$ and let $q \in \overline{K}$ be such that $q^2 = t^2-6t-3$. Let C be the hyperelliptic curve defined by the affine plane model $y^2 = P(x)Q(x)$, where

$$P(x) = 4tx^3 + 3(t+1)(t+3)x^2 + 12t(t+1)x + 4t(3t+1),$$
$$Q(x) = 4(3t+1)x^3 + 9(t+1)^2x^2 + 6(t+1)(3t+1)x + (3t+1)^2.$$

This is a curve defined by (3.2), such that $[a:b:c]$ is the image of t under the map (3.14). The curve C admits a degree-3 covering $\phi\colon C \to E$, where E is the elliptic curve defined by

$$E\colon y^2 = x^3 + 3t(t+1)x^2 - \frac{3t^2(t+1)(2t^2+9t+3)}{t^2-6t-3}x + t^3(3t+1),$$

with j-invariant

$$j(E) = \frac{27(t-3)^3(t+1)^3}{t^3}. \tag{3.15}$$

The curve C admits an automorphism η of order two, defined over $K(q)$, such that $\phi \circ \eta\colon C \to E$ is complementary to ϕ. One such automorphism is given by

$$\eta(x,y) = \left(-\frac{(3t+1)((t+1)x+3t+1)}{4(2t+1)x+(3t+1)(t+1)}, \frac{(3t+1)^3q^3y}{(4(2t+1)x+(3t+1)(t+1))^3} \right).$$

A K-rational complementary covering $\tilde{\phi}\colon C \to \tilde{E}$, where $\tilde{E}$ is a twist of E, can be obtained by composing with $(x,y) \mapsto (x,qy)$. By Corollary 2.7, the curve C also admits complementary degree-2 coverings to a pair of elliptic curves, say E_1 and E_2, all defined over $K(q)$. Furthermore, the elliptic curves E_1 and E_2 are 3-isogenous to E over $K(q)$. Setting $s = (3+q+t)/(3-q-t)$, so that $t = -(s^2-s+1)/(s+1)$ and $q = (s^2+2s-2)/(s+1)$, we have

$$j(E_1) = -\frac{27s^3(9s^3+8)^3}{s^3+1}, \quad j(E_2) = -\frac{3(s-2)^3(s^3-78s^2+84s-80)^3}{(s+1)^9(s^2-s+1)}.$$

The j-invariant of E becomes

$$j(E) = -\frac{27s^3(s^3-8)^3}{(s^3+1)^3}.$$

If $3(s^2+2s-2)$ is a square in $K(s)$ then E is isomorphic over $K(s)$ to the projective plane curve defined by the equation $x^3+y^3+z^3+3sxyz=0$, while C is isomorphic over $K(s)$ to the hyperelliptic curve defined by the affine plane model

$$\begin{aligned} 3(3s^2-4s+2)y^2 = {} & (s+1)^4x^6 - 3(6s^4-56s^3+84s^2-72s+25)x^4 \\ & + 9(9s^4-12s^3+6s^2+4s-5)x^2 - 9. \end{aligned} \tag{3.16}$$

The relevance of this fact will become apparent in §5, particularly Theorem 5.6. Note that we have $t^2-6t-3\neq 0$ by assumption, which implies $s^2+2s-2\neq 0$. For $s^2+2s-2=0$, i.e. for $s=-1\pm\sqrt{3}$, equation (3.16) defines a curve of genus two that is in the isomorphism class of the curve considered in Example 3.5, which admits two complementary coverings of degree three that both have special ramification.

For $\mathcal{Y}_1$ there is a birational map $\mathbf{P}^1 \dashrightarrow \mathcal{Y}_1$ over $K(\omega)$, for example $\beta(t)=[p_1(t):p_2(t):p_3(t)]$, where

$$\begin{aligned} p_1(t) &= -3\omega(t^2-\omega t-3\omega^2)(t^2-6t-3),\\ p_2(t) &= 48t^2(t^2+3\omega t-3\omega^2)(t^2-6t-3),\\ p_3(t) &= 64t^3(t^2+3t-3)(t^2-6t-3)^2. \end{aligned} \tag{3.17}$$

For all $t\in K$ such that $t(t^2+3t-3)(t^2-6t-3)(t^2+(\omega^2-1)t+3\omega^2)\neq 0$, a genus-2 curve C defined by (3.2) with $[a:b:c]=[p_1(t):p_2(t):p_3(t)]$ admits complementary degree-3 coverings to a pair of isomorphic elliptic curves E_1 and E_2, all defined over $K(\omega)$, such that

$$j(E_1)=j(E_2)=\frac{27(t-3)^3(t+1)^3}{t^3}. \tag{3.18}$$

Computing the absolute invariants of C, we find that they are K-rational so if $\omega\notin K$ then C might or might not admit a model over K (see [11, 25]). Note that (3.18) matches (3.15) and that it is invariant under $t\mapsto -3/t$. Moreover, $\beta(-3/t)$ is obtained from $\beta(t)$ by replacing ω by ω^2, so any two curves that are respectively defined by $\beta(t)$ and $\beta(-3/t)$ are twists of each other.

If $i\in\overline{K}$ is such that $i^2=-1$ then we have a birational map $\mathbf{P}^1 \dashrightarrow \mathcal{Y}_1$ over $K(i)$, for example

$$t\mapsto[-6t((2+3i)t^2-6+9i):6it(t^2+4it+3)(t^2+6t-3)^2:4t^2(3t^2+2t-9)(t^2+6t-3)^3].$$

The corresponding genus-2 curve admits complementary degree-3 coverings of elliptic curves E_1 and E_2 whose j-invariant is $j(E_1)=j(E_2)=1728(t^2-6t-3)^3(t^2+6t-3)^{-3}$. Similarly, replacing t by $-3/t$ corresponds to replacing i by $-i$.

Curves defined by the points of $\mathcal{Y}_1$ do not have complementary coverings that are obtained from one another by composing with an involution, except for the three points of intersection with $\mathcal{X}_1$. Indeed, using Lemma 1.1, one can show that for $[a:b:c]\in\mathcal{Y}_1(\overline{K})\setminus\mathcal{X}_1(\overline{K})$ there are no fractional linear transformations that send the roots of $P(x)$ to the roots of $Q(x)$ and vice versa, with the exception of two isomorphism classes, namely those of the curves[2]

$$\begin{aligned} C_1\colon y^2 &= x^6+3x^4-6x^2-8,\\ C_2\colon y^2 &= 8x^6-2040x^5-2244x^4-5840x^3-4230x^2-4014x-837. \end{aligned}$$

It is easily verified that the involutions do not induce a complementary covering for both of these exceptional cases.

Now let us suppose that $\mathrm{char}(K)=3$. Equating the j-invariants in (3.4), and assuming $b\neq 0$ because the curve C is not of genus two if $b=0$, we obtain equations that define the following

[2]If $\mathrm{char}(K)=19$ then the second isomorphism class is that of $C_2\colon y^2=x^5-x^3+4x$.

two curves of genus zero:

$$\mathcal{X}_1\colon a^3c - a^2b^2 - b^3 = 0,$$
$$\mathcal{Y}_1\colon a = 0.$$

A birational map $\mathbf{P}^1 \dashrightarrow \mathcal{X}_1$ is given by $t \mapsto [t : t : t+1]$. We find that for every $t \in K$ such that $t(t+1) \neq 0$, the model (3.2) with $(a,b,c) = (t,t,t+1)$ defines a genus-2 curve that admits a non-hyperelliptic involution given by

$$(x,y) \mapsto \left(\frac{(t+1)x}{(x-t-1)}, \frac{(t+1)^3 y}{(x-t-1)^3}\right).$$

By applying a suitable isomorphism to this model, we obtain a twist of the genus-2 curve

$$C\colon y^2 = x^6 + t(t+1)x^4 + t^3(t+1)x^2 + 2t^2.$$

The curve C admits a degree-3 map to an elliptic curve E whose modular invariant is $j(E) = t^3$. A complementary degree-3 map can be obtained by pre-composing a given one with an involution of C, given by $(x,y) \mapsto (-x, \pm y)$. The curve C also admits a pair of complementary covering maps of degree two to elliptic curves E_1 and E_2 whose j-invariants are $j(E_1) = t$ and $j(E_2) = t^9$. There exist a separable 3-isogeny $E \to E_1$ and an inseparable 3-isogeny $E \to E_2$.

The family defined by $\mathcal{Y}_1$ behaves differently in characteristic 3. One obvious map $\mathbf{P}^1 \dashrightarrow \mathcal{Y}_1$ is given by $t \mapsto [0 : t : 1]$. Thus for every $t \in K \setminus \{0\}$ we have a genus-2 curve that is defined by the affine plane model

$$C\colon y^2 = x^6 + t^2x^5 + (t^3-1)x^3 + 1 \tag{3.19}$$

and admits a pair of complementary coverings of degree three to the supersingular elliptic curves $E_1\colon y^2 = x^3 + tx + 1$ and $E_2\colon y^2 = x^3 - tx + 1$, whose j-invariant is $j(E_1) = j(E_2) = 0$. The curve C does not have additional involutions.

Remark. For a generic pair of elliptic curves E_1 and E_2 over $\overline{K}$, there are twelve pairwise distinct isomorphism classes of C such that $\mathrm{Jac}(C)$ is $(3,3)$-isogenous to $E_1 \times E_2$. The exceptions are elliptic curves that are 2-isogenous or have a large automorphism group. If the two elliptic curves are geometrically isomorphic then there are generically nine pairwise distinct isomorphism classes of such C, with the same exceptions. This is explained in §§4–5 of the paper (see Corollary 5.7). The analysis above gives insight into the reasons behind the three "missing" isomorphism classes – six of the twelve curves come as three pairs of twists.

3.3.2. *Families of complementary coverings of isogenous curves.* In this subsection we describe two additional families of curves of genus two that have a $(3,3)$-split Jacobian and additional involutions. Using Lemma 1.1, we can impose on the variables a,b,c the condition of existence of a fractional linear transformation $\mu(x) \in \overline{K}(x)$ of order two that permutes the roots of $P(x)Q(x)$. Since $\mu(x)$ is an involution, we have $\mu(x) = (ux+v)/(wx-u)$ for some $u,v,w \in \overline{K}$ such that $u^2 + vw \neq 0$. The condition that $\mu(x)$ permutes the roots of $P(x)Q(x)$ is equivalent to $P(x)Q(x)$ dividing $\mathrm{Res}_y((wy-u)x - (uy+v), P(y)Q(y))$. Imposing $u^2+vw \neq 0$ and $\mathrm{Disc}_x(P(x)Q(x)) \neq 0$ and eliminating the variables u,v,w from the resulting equations, we obtain an equation in a,b,c

that defines a union of three curves of genus zero, with an exception that is explained further below. These three curves include $\mathcal{X}_1$, defined by (3.12), and

$$\mathcal{X}_5\colon 4a^3c^3+a^2b^5-28a^2b^2c^2-18ab^4c+468abc^3+85b^3c^2-2160c^4=0, \tag{3.20}$$

$$\begin{aligned}\mathcal{X}_8\colon\ & 128a^6c^6+32a^5b^8c-1280a^5b^5c^3+11168a^5b^2c^5-16a^4b^{10}+320a^4b^7c^2\\ & -10864a^4b^4c^4+374040a^4bc^6+54a^3b^9c+24624a^3b^6c^3-781106a^3b^3c^5\\ & +2092500a^3c^7+81a^2b^{11}-16535a^2b^8c^2+443087a^2b^5c^4-1503225a^2b^2c^6\\ & +2250ab^{10}c-69300ab^7c^3+274410ab^4c^5-1215000abc^7-324b^{12}\\ & +16929b^9c^2-333187b^6c^4+3459375b^3c^6-11390625c^8=0.\end{aligned} \tag{3.21}$$

A birational map $\mathbf{P}^1 \dashrightarrow \mathcal{X}_5$ is given by

$$t\mapsto[-(t-4)(t^2+1):2(t+2)(t^2+1):2(t^2+1)^2]. \tag{3.22}$$

After applying a suitable isomorphism to the model (3.2) with $[a:b:c]$ defined by (3.22), we obtain the following. If $t\in K$ is such that $(2t-11)(t^2+1)\neq 0$ then the genus-2 curve given by

$$C\colon y^2=x^6-(4t^2-12t-5)x^4+(8t^2+72t-13)x^2-(2t-11)^2 \tag{3.23}$$

admits complementary covering maps of degree three to a pair of elliptic curves E_1 and E_2 whose modular invariants are

$$j(E_1)=\frac{64(t^2-6t+4)^3}{2t-11},\quad j(E_2)=\frac{64(t^2+114t+124)^3}{(2t-11)^5}.$$

Moreover, there exists a separable 5-isogeny $E_1\to E_2$ and the involution $(x,y)\mapsto(-x,y)$ of C induces complementary covering maps of degree two from C to the same pair of elliptic curves. For $t=1$ we obtain the curve $y^2=x^6+13x^4+67x^2-81$, which is a twist of the curve from Example 3.6.

The exception mentioned above occurs if $\operatorname{char}(K)=5$. In that case (3.20) defines a union of two curves, one of which admits the parametrization given by (3.22), while the other is defined by $a=0$, which can be parametrized by $t\mapsto[0:t:1]$. After applying a suitable isomorphism to the corresponding model of C, we obtain the following family of curves. Let $t\in K$ be such that $t^3+3\neq 0$. Then the hyperelliptic curve given by

$$C\colon y^2=x^6+2t^2x^4+t(3t^3+1)x^2-(t^3+3)^2 \tag{3.24}$$

admits complementary covering maps, of degree two and of degree three, to elliptic curves E_1 and E_2 whose modular invariants are given by

$$j(E_1)=\frac{3t^{15}}{t^{15}+3},\quad j(E_2)=\frac{3t^3}{t^3+3}.$$

If $t\neq 0$ then there exists a separable 5-isogeny $E_1\to E_2$. If $t=0$ then E_1 and E_2 are isomorphic to the elliptic curve $y^2=x^3+1$, which is supersingular. The isomorphism class of the corresponding curve $C\colon y^2=x^6+1$ is omitted by the family defined by (3.23) if $\operatorname{char}(K)=5$. All other curves defined by (3.24) are isomorphic to a curve defined by (3.23) in characteristic 5. Indeed, if $t\neq 0$ then (3.24) is isomorphic to the element of (3.23) defined by the parameter $2(t^3+2)/(t^3+3)$.

A birational map $\mathbf{P}^1 \to \mathcal{X}_8$ is given by

$$t \mapsto [-32t^4 - 8t^3 + 20t^2 + 4t + 1 : 4t(4t^2 + 7t + 1) : 8t^2(4t^2 + 3t + 1)].$$

After applying a suitable isomorphism to the corresponding model defined by (3.2), we obtain the following family of curves. Let $t \in K$ be such that $t(t^2-1)(4t^2+3t+1) \neq 0$. Then

$$C\colon y^2 = ((t+1)^2x^2 - t^2) \cdot (4tx^2 - 1) \cdot ((t-1)x^2 - t - 1)$$

is a curve of genus two that admits complementary covering maps of degree three to 8-isogenous elliptic curves E_1 and E_2 whose modular invariants are given by

$$j(E_1) = \frac{4(t^4 + 60t^3 + 134t^2 + 60t + 1)^3}{t(t-1)^8(t+1)^2}, \quad j(E_2) = \frac{16(16t^4 - 16t^2 + 1)^3}{t^2(t-1)(t+1)}.$$

The involution $(x, y) \mapsto (-x, y)$ of C induces complementary covering maps $C \to \tilde{E}_1$ and $C \to \tilde{E}_2$ of degree two, where $\tilde{E}_1$ and $\tilde{E}_2$ are elliptic curves whose modular invariants are

$$j(\tilde{E}_1) = \frac{16(t^4 + 14t^2 + 1)^3}{t^2(t-1)^4(t+1)^4}, \quad j(\tilde{E}_2) = \frac{256(t^4 - t^2 + 1)^3}{t^4(t-1)^2(t+1)^2}.$$

Furthermore, there exist 2-isogenies $E_1 \to \tilde{E}_1$, $E_2 \to \tilde{E}_2$, and $\tilde{E}_1 \to \tilde{E}_2$.

4. Gluing two elliptic curves along the n-torsion

In the previous section we started with a maximal covering $C \to E_1$ of degree n and constructed a complementary curve E_2. In the remaining sections we adopt a different approach. We start with two elliptic curves E_1, E_2 and aim to construct a curve of genus two whose Jacobian is (n,n)-isogenous to $E_1 \times E_2$ via an isogeny whose kernel is prescribed. This approach can be found in [16]. We begin by recalling useful definitions and results.

Let A be an abelian variety over K and let $\lambda\colon A \to A^\vee$ be a polarization. Suppose that $m \in \mathbf{N}$ is coprime to $\operatorname{char}(K)$ and such that $\operatorname{Ker}(\lambda) \subset A[m]$. Let

$$e_m\colon A[m](\overline{K}) \times A^\vee[m](\overline{K}) \to \mu_m$$

denote the Weil pairing. Then we can associate to λ a skew-symmetric pairing

$$e_\lambda\colon \operatorname{Ker}(\lambda) \times \operatorname{Ker}(\lambda) \to \mu_m$$

that is defined for every pair (P,Q) of geometric points as $e_\lambda(P,Q) = e_m(P, \lambda(R))$, where R is such that $[m]R = Q$. This does not depend on R or m (see [26, §16]).

Lemma 4.1. *Let $\varphi\colon A \to B$ be an isogeny whose degree is coprime to $\operatorname{char}(K)$ and let $\lambda\colon A \to A^\vee$ be a polarization induced by a line bundle $\mathscr{L}$. Then the following are equivalent:*

1. *There exists a line bundle $\mathscr{M}$ on B such that $\mathscr{L} = \varphi^*(\mathscr{M})$, inducing a polarization $\lambda'\colon B \to B^\vee$,*
2. *$\operatorname{Ker}(\varphi) \subset \operatorname{Ker}(\lambda)$ and e_λ is trivial on $\operatorname{Ker}(\varphi) \times \operatorname{Ker}(\varphi)$.*

Proof. See Proposition 16.8 in [26] or Theorem 2 and its Corollary in [27, §23]. □

Corollary 4.2. *Let $\phi_1 \colon C \to E_1$ be a maximal covering of an elliptic curve by a curve of genus two, such that $\deg(\phi_1) = n$ is coprime to $\mathrm{char}(K)$. Let E_2 be a complementary elliptic curve and let $\alpha \colon E_1[n] \xrightarrow{\sim} E_2[n]$ be the induced canonical isomorphism, as in Lemma 2.3. Then α inverts the Weil pairing, i.e.*

$$e_n(P,Q) = e_n(\alpha(P),\alpha(Q))^{-1} \tag{4.1}$$

for all $P,Q \in E_1[n](\overline{K})$.

Lemma 4.1 provides a criterion for deciding when a polarization descends through an isogeny. In view of the lemma and its corollary, our starting data are two elliptic curves E_1, E_2 and an isomorphism $\alpha \colon E_1[n] \xrightarrow{\sim} E_2[n]$ that is *anti-symplectic* with respect to the Weil pairing, which is to say that α satisfies (4.1) for all $P,Q \in E_1[n](\overline{K})$, where n is coprime to $\mathrm{char}(K)$.

We assume that $E_1 \times E_2$ is equipped with the usual principal polarization, given by the divisor

$$\Theta = E_1 \times \{O_2\} + \{O_1\} \times E_2.$$

Let $\Gamma_\alpha \subset (E_1 \times E_2)[n]$ be the graph of α and let $\varphi \colon E_1 \times E_2 \to (E_1 \times E_2)/\Gamma_\alpha$ be the canonical map, which is clearly an isogeny. We denote the quotient $(E_1 \times E_2)/\Gamma_\alpha$ by J.

Lemma 4.3. *The isogeny $\varphi \colon E_1 \times E_2 \to J$ induces a principal polarization on J, defined by an effective divisor $C \in \mathrm{Div}(J \otimes \overline{K})$ such that $\varphi^*(C)$ is linearly equivalent to $n\Theta$. If n is odd then there exists a unique such C that is fixed by $[-1]$.*

Proof. See [16, pp. 156–157]. □

It is a well known result of Weil (see [32, Satz 2] or [17, pp. 86–87]) that any principally polarized abelian surface over $\overline{K}$ is either a Jacobian or a product of two elliptic curves (with the usual polarizations). Therefore the question of whether or not J is a Jacobian reduces to the question of whether or not the divisor C is irreducible. The following two lemmas will prove useful.

Lemma 4.4. *The divisor C is irreducible if and only if the divisor $D = \varphi^*(C)$ is irreducible. If C is reducible then E_1, E_2, and the two irreducible components of C are isogenous.*

Proof. See Propositions 1.3 and 1.4 in [16]. □

Lemma 4.5. *If $\alpha \colon E_1[n] \xrightarrow{\sim} E_2[n]$ is the restriction of an isogeny $f \colon E_1 \to E_2$ of degree $n-1$ then C is reducible.*

Proof. This is a special case of the general reducibility criterion for C, given in [21, §2]. □

5. Gluing two elliptic curves along the 3-torsion

In this section we deal with the case $n = 3$, given two elliptic curves from the Hesse pencil. From now on, unless specified otherwise, we assume that the field K is of characteristic $\mathrm{char}(K) \notin \{2,3\}$ and that it contains a primitive third root of unity that we denote by ω.

5.1. *Prerequisites.* The one-dimensional family of curves given by

$$E_a\colon x^3+y^3+z^3+3axyz=0, \tag{5.1}$$

is called the *Hesse pencil*. Every $a\in K$ defines an elliptic curve E_a, except if $a^3=-1$. Throughout this section, we assume that $O=[-1:1:0]$ is the identity element so that the inversion morphism is given by $[x:y:z]\mapsto[y:x:z]$, the addition morphism is given by

$$([x_1:y_1:z_1],\,[x_2:y_2:z_2])\mapsto[y_1^2x_2z_2-y_2^2x_1z_1\ :\ x_1^2y_2z_2-x_2^2y_1z_1\ :\ z_1^2x_2y_2-z_2^2x_1y_1],$$

and the duplication morphism is given by $[x:y:z]\mapsto[y(x^3-z^3)\ :\ x(z^3-y^3)\ :\ z(y^3-x^3)]$. We denote the set of elliptic curves in the Hesse pencil by $\mathcal{H}$. The j-invariant of E_a is

$$j(E_a)=-\frac{27a^3(a^3-8)^3}{(a^3+1)^3} \tag{5.2}$$

and $j\colon\mathcal{H}\to\mathbf{A}^1$ is 12-to-1, except above $j=0$ and $j=1728$. In fact, the elements of the set

$$\mathcal{S}(a)=\left\{a,\,a\omega,\,a\omega^2,\,\frac{2-a}{1+a},\,\frac{2-a}{1+a}\omega,\,\frac{2-a}{1+a}\omega^2,\,\frac{2-a\omega}{1+a\omega},\right.$$
$$\left.\frac{2-a\omega}{1+a\omega}\omega,\,\frac{2-a\omega}{1+a\omega}\omega^2,\,\frac{2-a\omega^2}{1+a\omega^2},\,\frac{2-a\omega^2}{1+a\omega^2}\omega,\,\frac{2-a\omega^2}{1+a\omega^2}\omega^2\right\} \tag{5.3}$$

define isomorphic elliptic curves; the isomorphisms $E_a\xrightarrow{\sim}E_{a\omega}$ and $E_a\xrightarrow{\sim}E_{(2-a)/(1+a)}$ are respectively given by

$$[x:y:z]\mapsto[\omega x:\omega y:z],$$
$$[x:y:z]\mapsto[\omega x+\omega^2y+z:\omega^2x+\omega y+z:x+y+z].$$

The 3-torsion subgroup of every elliptic curve in $\mathcal{H}$ is fully K-rational and given by $xyz=0$. Therefore the same nine points in $\mathbf{P}^2$ are the 3-torsion points of every element of $\mathcal{H}$ and each of the nine points can be given by homogeneous coordinates that are a permutation of $\{0,1,-\omega^k\}$, where $k\in\{0,1,2\}$. In fact, the Hesse pencil is exactly the family of all cubics passing through these nine points.

There is a known converse in the form of the following lemma. We include a short direct proof because we have not seen it in the literature.

Lemma 5.1. *Every elliptic curve over K with fully K-rational 3-torsion is isomorphic to an element of the Hesse pencil.*

Proof. Let E be an elliptic curve such that $\#E[3](K)=9$ and suppose that it is given by the Weierstraß equation

$$F(x,y,z)=-y^2z+x^3+axz^2+bz^3=0,$$

where $a,b\in K$ and $4a^3+27b^2\neq0$. The Hessian of $F(x,y,z)$ is given, up to multiplication by a non-zero constant, by the polynomial $H(x,y,z)=3xy^2+3ax^2z+9bxz^2-a^2z^3$. The intersection of E and the curve defined by $H(x,y,z)=0$ consists of the nine inflection points of E, which are all K-rational by assumption. Computing the intersection yields the division polynomial

$$3x^4+6ax^2+12bx-a^2, \tag{5.4}$$

which must split completely over K. Suppose that the (necessarily pairwise distinct) roots of (5.4) are $t_1,t_2,t_3,t_4\in K$. Expanding $3(x-t_1)(x-t_2)(x-t_3)(x-t_4)$ and equating with (5.4) gives

$$t_4=-t_1-t_2-t_3, \qquad -2a=t_1^2+t_1t_2+t_2^2+t_1t_3+t_2t_3+t_3^2,$$
$$4b=(t_1+t_2)(t_1+t_3)(t_2+t_3), \qquad a^2=3t_1t_2t_3(t_1+t_2+t_3).$$

Eliminating a and b, and renaming the roots if necessary, we conclude that the point (t_1,t_2,t_3) satisfies

$$t_1^2+\omega t_2^2+\omega^2t_3^2-2\omega^2t_1t_2-2\omega t_1t_3-2t_2t_3=0.$$

Let $s_1,s_2,s_3,s_4\in K$ be such that the points of order three on E are given by $[t_i:\pm s_i:1]$. Replacing s_i by $-s_i$ for various $i\in\{1,2,3\}$ if necessary, we may assume $s_1s_2+\omega s_1s_3=\omega^2s_2s_3$. Now let

$$t=\frac{(1+2\omega)t_1+(1+3\omega)t_2-(1+3\omega^2)t_3}{t_3-t_2}, \quad u=\frac{(1+2\omega)(t_1-t_3)}{s_1+\omega s_3}.$$

We have $t^3\neq -1$ and

$$u^2=-\frac{12}{t_1+\omega t_2+\omega^2t_3}, \quad au^4=-3t(t^3-8), \quad bu^6=-2(t^6+20t^3-8).$$

Finally, the elliptic curve defined by the Weierstraß equation

$$-y^2z+x^3-3t(t^3-8)xz^2-2(t^6+20t^3-8)z^3=0$$

is isomorphic to the element of $\mathcal{H}$ defined by $x^3+y^3+z^3+3txyz=0$ via the isomorphism

$$[x:y:z]\mapsto[3tx-(1+2\omega)y+3(t^3+4)z\ :\ 3tx+(1+2\omega)y+3(t^3+4)z\ :\ 6(x-3t^2z)]. \qquad \square$$

Let $S=[-1:0:1]$ and $T=[-\omega:1:0]$. These two points generate $E[3]$ for every elliptic curve $E\in\mathcal{H}$. For every $E\in\mathcal{H}$ we fix the group isomorphism $\rho\colon E[3]\xrightarrow{\sim}(\mathbf{Z}/3\mathbf{Z})^2$ that is defined by $\rho(S)=(1,0)$ and $\rho(T)=(0,1)$. The Weil pairing on $E[3]$ is completely determined by the value $e_3(S,T)$ and one can easily find that $e_3(S,T)=\omega$. For example, using the construction in [31, Ch. III §8], one finds that $e_3(S,T)=g(P+T)/g(P)$, where

$$g=\frac{x^2z+y^2x+z^2y}{xyz}\in K(E)$$

and $P\in E(\overline{K})\setminus E[3]$ is a point such that $g(P)$ and $g(P+T)$ are both non-zero. It follows that the Weil pairing on $E[3]$ is given by

$$e_3(P,Q)=\omega^{\det(\rho(P),\rho(Q))}$$

and we interpret it as the determinant

$$\det\colon(\mathbf{Z}/3\mathbf{Z})^2\times(\mathbf{Z}/3\mathbf{Z})^2\to\mathbf{Z}/3\mathbf{Z}.$$

Since $\mathrm{Aut}((\mathbf{Z}/3\mathbf{Z})^2) \cong \mathrm{GL}_2(\mathbf{Z}/3\mathbf{Z})$ is a group of order 48, every anti-symplectic isomorphism $E_1[3] \xrightarrow{\sim} E_2[3]$ corresponds to one of the 24 elements of the coset $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\mathrm{SL}_2(\mathbf{Z}/3\mathbf{Z})$. However, since each isomorphism can be composed with $[-1]$, we are left with twelve distinct cases, meaning that there are generically twelve distinct isomorphism classes of principally polarized abelian surfaces that can be obtained by gluing two elliptic curves along the 3-torsion.

Before we deal with the general case, we will consider an important example in which gluing two elliptic curves along the 3-torsion does not result in a Jacobian.

Example 5.2. Let $t \in K$ be such that $t(t^3-1)(8t^3+1) \neq 0$, let $a = -(1+2t^3)/(3t^2)$, and let $b = (1-4t^3)/(3t)$. Then the elliptic curves

$$E_1\colon x^3+y^3+z^3+3axyz = 0,$$
$$E_2\colon x^3+y^3+z^3+3bxyz = 0,$$

each have a K-rational point of order two. Indeed, the point $T_1 = [t:t:1]$ lies on E_1 and the point $T_2 = [1:1:-2t]$ lies on E_2. The map $f\colon E_1 \to E_2$, defined by

$$[x:y:z] \mapsto [f_1(x,y,z) : f_2(x,y,z) : f_3(x,y,z)],$$

where

$$f_1(x,y,z) = x(-2t^2y^2 - t^2xy + t^2x^2 - yz + 2t^3xz + tz^2),$$
$$f_2(x,y,z) = y(-2t^2x^2 - t^2xy + t^2y^2 - xz + 2t^3yz + tz^2),$$
$$f_3(x,y,z) = tz(x+y+tz)(x+y-2tz),$$

is an isogeny whose kernel is the group of order two generated by T_1. The kernel of the dual isogeny $f^\vee\colon E_2 \to E_1$ is generated by T_2. Restricting f to the 3-torsion, we obtain the isomorphism $\alpha\colon E_1[3] \xrightarrow{\sim} E_2[3]$ that corresponds to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbf{Z}/3\mathbf{Z})$. By Lemma 4.5, $(E_1 \times E_2)/\Gamma_\alpha$ is not a Jacobian (with the induced polarization). In fact, it can be shown that it is isomorphic to $E_1 \times E_2$ as a principally polarized abelian surface, as follows. Let $\varphi \in \mathrm{End}(E_1 \times E_2)$ be the endomorphism $(P,Q) \mapsto ([3]P, f(P)-Q)$. It is readily seen that $\mathrm{Ker}(\varphi) = \Gamma_\alpha$. Let D_1 be the image of $E_1 \times \{O_2\}$ under $(P,Q) \mapsto (P, f(P)) + (T_1,T_2)$, let D_2 be the image of $\{O_1\} \times E_2$ under $(P,Q) \mapsto (-f^\vee(Q), Q) + (T_1,T_2)$, let F_1 be the image of $E_1 \times \{O_2\}$ under the translation $(P,Q) \mapsto (P,Q) + (T_1,T_2)$, and let F_2 be the image of D_2 under the involution $(P,Q) \mapsto (-P,Q)$. Then the divisor $D_1 + D_2$ is linearly equivalent to 3Θ and it is the pull-back $\varphi^*(F_1+F_2)$. This is verified by direct computation (see also Theorem 5.6 and the discussion preceding it). Since $D_1 \cong E_1 \cong F_1$ and $D_2 \cong E_2 \cong F_2$ as elliptic curves, it follows that $(E_1 \times E_2)/\Gamma_\alpha \cong E_1 \times E_2$ as principally polarized abelian surfaces, by the Torelli theorem. We note that a and b satisfy

$$3a^2b^2 + a^3 + b^3 - 3ab + 2 = 0, \tag{5.5}$$

which is an equation describing a singular affine plane curve of genus zero.

We will now consider the isomorphism $E_1[3] \xrightarrow{\sim} E_2[3]$ from Example 5.2 in full generality. From now on, we fix $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. All of the constructions that follow are completely analogous and yield identical results if one fixes $-\alpha$ instead. Let E_1 and E_2 be two elliptic curves in $\mathcal{H}$, corresponding to parameters a and b, respectively. Let A and G respectively denote the images of $E_1 \times E_2$ and Γ_α in $\mathbf{P}^8$ under the Segre embedding

$$\sigma\colon ([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]) \mapsto [x_1x_2 : x_1y_2 : x_1z_2 : y_1x_2 : y_1y_2 : y_1z_2 : z_1x_2 : z_1y_2 : z_1z_2].$$

The identity element of A is $O_A = [1 : -1 : 0 : -1 : 1 : 0 : 0 : 0 : 0]$ and the inversion morphism $[-1]_A$ is given by

$$[X_1 : X_2 : \cdots : X_9] \mapsto [X_5 : X_4 : X_6 : X_2 : X_1 : X_3 : X_8 : X_7 : X_9]. \tag{5.6}$$

Let $\Theta = \sigma(E_1 \times \{O_2\}) + \sigma(\{O_1\} \times E_2)$ and let D denote the effective divisor on A that is linearly equivalent to 3Θ, invariant under $[-1]_A$, and invariant under the translations by the points of G. Let $\varphi\colon A \to J$ denote the isogeny with kernel G and let $C = \varphi(D)$. The divisor D can be determined explicitly and it is this fact that will ultimately allow us to determine how C depends on the curves E_1 and E_2.

5.2. *The computations.* In this subsection we will go over the steps that lead to the Igusa–Clebsch invariants, as well as an affine plane model, of a genus-2 curve whose Jacobian is isomorphic to $(E_1 \times E_2)/\Gamma_\alpha$ as a principally polarized abelian surface, if such a curve exists.

The first step is computing the ideal $I = I(A)$ that defines A as a variety in $\mathbf{P}^8$. This is a relatively simple ideal computation and is omitted here.

Lemma 5.3. *Let $\mathcal{W}$ denote the union of the set of geometric points of order two on $\sigma(E_1 \times \{O_2\})$ and the set of geometric points of order two on $\sigma(\{O_1\} \times E_2)$. Then any hyperplane section on A that is invariant under $[-1]_A$ contains either $\mathcal{W}$ or its complement in $A[2](\overline{K})$.*

Proof. The two eigenspaces of (5.6) are respectively generated by the sets

$$\begin{aligned} S_1 &= \{X_1 + X_5, X_2 + X_4, X_3 + X_6, X_7 + X_8, X_9\},\\ S_2 &= \{X_1 - X_5, X_2 - X_4, X_3 - X_6, X_7 - X_8\}. \end{aligned} \tag{5.7}$$

By adding the corresponding linear forms from (5.7) to I, we find that $A[2](\overline{K})$ consists of six points that are in the zero locus of the ideal generated by S_1 (i.e. the points of $\mathcal{W}$) and ten points that are in the zero locus of the ideal generated by S_2 (the remaining points). Since every linear form that is an eigenvector for $[-1]_A$ is a linear combination of the elements of exactly one of these two sets, the claim follows. □

Corollary 5.4. *The quotient $J = A/G$ is not a Jacobian if and only if $D(\overline{K})$ contains exactly one point from the set $A[2](\overline{K}) \setminus \mathcal{W}$.*

Proof. By the previously mentioned theorem of Weil, the divisor C is either a curve of genus two or a union of two elliptic curves that meet in a 2-torsion point. Since $[-1]_J$ induces an involution ι on the irreducible components of C, we conclude that $C(\overline{K})$ contains exactly six points fixed by ι if and only if it is irreducible and that it contains exactly seven points fixed by ι if and only if it is reducible. Since $\deg(\varphi)$ is odd, the restriction of φ to the 2-torsion is an isomorphism and there is exactly one geometric point of $(E_1 \times E_2)[2]$ above each point of $C(\overline{K})$ that is fixed by ι. Therefore $D(\overline{K})$ contains at most seven 2-torsion points. By Lemma 5.3, $D(\overline{K})$ contains at least the order-2 points of $\sigma(E_1 \times \{O_2\})$ and $\sigma(\{O_1\} \times E_2)$ and the claim follows. □

A fact crucial to our approach is that the translations by the points of $A[3]$ are linear. In fact, they can be extended to automorphisms of $\mathbf{P}^8$, which is a consequence of the fact that A is embedded into $\mathbf{P}^8$ via an embedding corresponding to $L(3\Theta)$. This can also be shown directly, using the addition formulas. In particular, the group of translations by the points of G is generated by the following two automorphisms of $\mathbf{P}^8$:

$$[X_1 : X_2 : \cdots : X_9] \mapsto [X_5 : X_6 : X_4 : X_8 : X_9 : X_7 : X_2 : X_3 : X_1],$$
$$[X_1 : X_2 : \cdots : X_9] \mapsto [X_1 : \omega X_2 : \omega^2 X_3 : \omega^2 X_4 : X_5 : \omega X_6 : \omega X_7 : \omega^2 X_8 : X_9].$$

From this we immediately determine that the nine effective divisors that are invariant under the action of G and linearly equivalent to 3Θ are the hyperplane sections defined by the following nine linear forms:

$$\begin{aligned}
L_1 &= X_1 + X_5 + X_9, & L_6 &= \omega^2 X_3 + \omega X_4 + X_8,\\
L_2 &= \omega X_1 + \omega^2 X_5 + X_9, & L_7 &= X_2 + X_6 + X_7,\\
L_3 &= \omega^2 X_1 + \omega X_5 + X_9, & L_8 &= \omega X_2 + \omega^2 X_6 + X_7,\\
L_4 &= X_3 + X_4 + X_8, & L_9 &= \omega^2 X_2 + \omega X_6 + X_7.\\
L_5 &= \omega X_3 + \omega^2 X_4 + X_8, & &
\end{aligned}$$

We note that the divisor D, that is invariant under $[-1]_A$, is defined by $L_1 = 0$ and does not contain O_A. Now we can compute the scheme that is the intersection of D and the nine points of $A[2](\overline{K})$ that are not 2-torsion points on $\sigma(E_1 \times \{O_2\})$ or $\sigma(\{O_1\} \times E_2)$ and apply Corollary 5.4. Dehomogenizing by setting $X_9 = 1$, taking the corresponding ideal in the ring $K[X_1, \ldots, X_8, a, b]$, and eliminating the X_i gives

$$3a^2b^2 + a^3 + b^3 - 3ab + 2 = 0. \tag{5.8}$$

Note that this matches (5.5). This yields the following result, which is a special case of Theorem 3 in [21] (cf. Proposition 2.2. and Corollary 2.3 in [16]).

Proposition 5.5. *The principally polarized abelian surface $J = A/G$ is a product of two elliptic curves if and only if (5.8) holds, i.e. if and only if $\alpha\colon E_1[3] \xrightarrow{\sim} E_2[3]$ is the restriction of a 2-isogeny.*

Proof. If $a, b \in K \setminus \{-1, -\omega, -\omega^2\}$ are such that (5.8) holds then we are precisely in the situation described in Example 5.2, with $t = (1-ab)/(a^2+b)$, and we have $J \cong E_1 \times E_2$. Suppose that the curves E_1 and E_2 are 2-isogenous over K and that the isogeny $E_1 \to E_2$ restricts to $\alpha\colon E_1[3] \xrightarrow{\sim} E_2[3]$. The kernel of the isogeny is generated by a K-rational point of order two. As $[-1]$ is defined on the elements of $\mathcal{H}$ by $[x : y : z] \mapsto [y : x : z]$, such a point is of the form $[t : t : 1]$, where t is a root of $p(x) = 2x^3 + 3ax^2 + 1 \in K[x]$. Since $p(t) = 0$, we have $p(x) = (x-t)(2x^2 + (3a+2t)x + 2t^2 + 3at)$, whence $a = -(2t^3+1)/(3t^2)$. Since α is the restriction of the 2-isogeny, we must have $b = (1-4t^3)/(3t)$, as in Example 5.2. This is because all other possible values of b are obtained by post-composing the 2-isogeny from the example with an isomorphism to another element of $\mathcal{H}$, which results in a 2-isogeny that does not restrict to α. Thus we have shown that (5.8) holds if and only if α is the restriction of a 2-isogeny. By the argument in Example 5.2, we have that (5.8) implies that C is reducible, so that J is not a Jacobian. On the other hand, Corollary 5.4 and the discussion that follows it show that if C is reducible then C and D both contain seven points of order two, which implies (5.8). □

Equation (5.8) can be thought of as the analogue of $\Phi_2(j(E_1), j(E_2)) = 0$ that is specific to our choice of α. Here Φ_2 denotes the classical modular polynomial

$$\Phi_2(X,Y) = X^3 + Y^3 - X^2Y^2 + 1488(X^2Y + XY^2) - 162000(X^2+Y^2) \\ + 40773375XY + 8748000000(X+Y) - 157464000000000.$$

The abelian surface J can be found explicitly, in principle, as the quotient of the variety A under the group action of G, where G acts by point translation (see Lecture 10 in [19] for example). Since φ^* is injective and $\dim_K(L(nC)) = n^2$ for every $n \in \mathbf{N}$, we have that the subspace $L(nD)^G = \varphi^*(L(nC))$ of G-invariants of $L(nD)$ is of dimension n^2 for every $n \in \mathbf{N}$. In particular, since $\mathscr{L}(3C)$ is very ample, by finding nine linearly independent G-invariant elements of $L(3D)$, we can obtain φ as a map to $\mathbf{P}^8$. We can take the nine G-invariant forms L_i^3 for this purpose. Unsurprisingly, explicitly computing $\varphi(A)$ is not feasible.

Let us assume that $3a^2b^2 + a^3 + b^3 - 3ab + 2 \neq 0$ so that C is irreducible. It follows that the global sections of $\mathscr{L}(2C)$ define the canonical map $\kappa \colon J \to J/[-1]$, where $J/[-1]$ is a Kummer surface in $\mathbf{P}^3$ (see Proposition 4.23 in [17]). Therefore the four-dimensional G-invariant subspace $L(2D)^G \subset L(2D)$ defines the composition $\psi = \kappa \circ \varphi$. We have that $\psi(D)$ is a conic in $\mathbf{P}^3$ and that the image under ψ of the 2-torsion points that lie on D consists of six pairwise distinct (geometric) points on $\psi(D)$ that are the branch locus of the canonical 2-to-1 map $C \to \psi(D)$. By finding a K-rational point on the conic $\psi(D)$, we obtain an isomorphism $\psi(D) \xrightarrow{\sim} \mathbf{P}^1$. The image in $\mathbf{P}^1$ of the six branch points gives us a sextic that defines a plane model of a hyperelliptic curve that is in the isomorphism class of C. We can directly compute the absolute invariants from this model. We may take the following four G-invariant forms to define ψ:

$$X_2X_4 + X_3X_7 + X_6X_8, \quad X_2X_3 + X_4X_6 + X_7X_8, \\ X_2X_8 + X_3X_6 + X_4X_7, \quad X_1^2 + X_5^2 + X_9^2. \tag{5.9}$$

An alternative approach is to compute the curve $C = \varphi(D)$ directly, compute the canonical divisor K_C, and then find the image in $\mathbf{P}^1$ of the six points of $J[2](\overline{K})$ that lie on C, under the canonical map defined by $L(K_C)$. However, this method is significantly less efficient.

We make an important observation. The absolute invariants of C, as functions of the parameters a and b, will have certain symmetries. For example, the abelian surface $E_1 \times E_2$ is isomorphic to $E_2 \times E_1$ and the isomorphism, which is just a permutation of the homogeneous coordinates, leaves G intact so that $(E_1 \times E_2)/\Gamma_\alpha$ and $(E_2 \times E_1)/\Gamma_\alpha$ give the same absolute invariants. Similarly, the same invariants are obtained if one starts with a pair $(E_1, E_2) \in \mathcal{H}^2$ defined by parameters $(a\omega, b\omega^2)$ or $(a\omega^2, b\omega)$.

Remark. Recall that each isomorphism class of elliptic curves has twelve representatives in $\mathcal{H}$, with the exception of the isomorphism classes defined by j-invariants 0 and 1728. It is a natural question to ask which of the corresponding 144 isomorphic pairs $(E_1, E_2) \in \mathcal{H}^2$ result in the same isomorphism class of the principally polarized abelian surface $(E_1 \times E_2)/\Gamma_\alpha$. It turns out that the pairs are partitioned into twelve sets of twelve pairs such that in each set all pairs give the same isomorphism class. After taking automorphisms into account, there are twelve choices for $\alpha\colon E_1[3] \xrightarrow{\sim} E_2[3]$. All twelve choices can be reduced to the case we are considering by replacing E_1 and E_2 by different elements of $\mathcal{H}$ in their respective isomorphism classes. In fact, given two curves $E_1, E_2 \in \mathcal{H}$, all isomorphism classes of $(E_1 \times E_2)/\Gamma_\alpha$ that can be obtained by varying α can also be obtained by fixing α and E_1 and varying E_2 in its isomorphism class (or fixing α and E_2 and varying E_1 in its isomorphism class). This is made concrete in the Appendix. Our choice of α is motivated by the simplicity of the equation (5.8) and the equations defining the divisor D, both avoiding unnecessary primitive third roots of unity.

To obtain the absolute invariants of C as functions of (a,b), the first thing we do is make several degree estimates. For example, we can take a and b to be two large integers of comparable height, such as two large consecutive primes. We can also take a to be a large integer and $b \in \{0,1\}$. This gives us estimates for the degrees of particular monomials that appear in the invariants. Next we notice that the discriminant J_{10}, that appears in the denominators, is going to be zero for choices of (a,b) that either do not define a pair of elliptic curves or do not define a quotient J that is a Jacobian. By factoring the invariants obtained for various choices of $a,b \in \mathbf{Z}$ and combining this information with the degree estimates, we conclude that, up to multiplication by the fifth power of a non-zero constant, J_{10} equals

$$9(a^3+1)(b^3+1)(3a^2b^2+a^3+b^3-3ab+2)^{12}. \tag{5.10}$$

To obtain the numerators, we use interpolation. We compute the absolute invariants of C for many choices of (a,b) and multiply them in each case by (5.10). We conclude from the aforementioned symmetries that the numerators are also linear combinations of monomials $a^m b^n$ such that $m \equiv n \pmod 3$. This significantly reduces the number of non-zero coefficients and makes the computation reasonably fast. Using the bounds we obtained on the degrees and the coefficients, we interpolate over finite fields $\mathbf{F}_p$ for a suitable set of primes p and then lift the results using the Chinese remainder theorem. The Igusa–Clebsch invariants are then obtained from the Igusa invariants using the formulas in [25].

It takes a bit more effort to determine a plane model for C. Directly computing $C = \varphi(D)$, a curve in $\mathbf{P}^8$, in terms of variables a and b seems infeasible in general (but can be done for concrete values of a and b). However, one can find a suitable automorphism of $\mathbf{P}^8$ and a projection to $\mathbf{P}^3$, such that the composition $\tilde{\varphi}\colon \mathbf{P}^8 \to \mathbf{P}^3$ is defined over the prime field of characteristic $\mathrm{char}(K)$ and does not depend on a and b, and such that $\tilde{C} = \tilde{\varphi}(D)$ is the intersection of a cubic and a quadric. More concretely, there is a model of C given by

$$\tilde{C}\colon \begin{cases} F(x_1,x_2,x_3) = 0 \\ G(x_1,x_2,x_3) = x_4^2, \end{cases} \tag{5.11}$$

where $\deg(F) = 3$ and $\deg(G) = 2$ and x_1, x_2, x_3, x_4 are homogeneous coordinates in $\mathbf{P}^3$. The hyperelliptic involution ι on $\tilde{C}$ is defined by $x_4 \mapsto -x_4$. This model is simple enough to be determined in terms of a and b by the same method as the invariants. A plane model of C can then be obtained from the model in $\mathbf{P}^3$ relatively easily, as follows. Let $Y = x_4$, let Z be a linear form in x_1, x_2, x_3 that defines the hyperplane section divisor on $\tilde{C}$ that is of the form $3P + 3\iota(P)$

for some $P \in \tilde{C}(\overline{K})$, and let X be a linear form in x_1, x_2, x_3 that defines a hyperplane section divisor on $\tilde{C}$ that is of the form $2P + 2\iota(P) + Q + \iota(Q)$ for some $Q \in \tilde{C}(\overline{K}) \setminus \{P, \iota(P)\}$. Coefficients of Z can be determined (up to multiplication by a non-zero scalar) by substituting $Z = 0$ into the first equation of (5.11) and equating the discriminants of the resulting cubic and its derivative with zero, thus ensuring that the cubic has a triple root. To find a suitable linear form X, we can compute the equations defining the divisor $P + \iota(P)$, which amounts to computing the reduced subscheme of the intersection of $Z = 0$ and $\tilde{C}$, and take for X a linear combination of the two equations that are linear. If we set $x = X/Z$ and $y = Y/Z$ then the image of $\tilde{C}$ under the map $[x_1 : x_2 : x_3 : x_4] \mapsto (x, y)$ is an affine plane curve of the form $y^2 g(x)^2 = f(x)$ with $\deg(f) = 6$ and $\deg(g) \leqslant 1$. If $\deg(g) = 0$, we have a Weierstraß equation for C. Otherwise, a Weierstraß equation can be obtained by computing the image of the said curve under $(x, y) \mapsto (g(x), g(x)y)$.

The image of the composition $E_1 \times E_2 \xrightarrow{\varphi} J \xrightarrow{\kappa} J/[-1]$ defined by the quadratic forms in (5.9) can also be computed using similar methods. This yields the Kummer surface of J as a quartic in $\mathbf{P}^3$. We omit the equation here, but it can be found in [14].

If $3a^2b^2 + a^3 + b^3 - 3ab + 2 = 0$, so that D is reducible, we can compute $\varphi(D)$ directly, working over the field $K(t)$ and setting $a = -(1+2t^3)/(3t^2)$ and $b = (1-4t^3)/(3t)$, as in Example 5.2. We find that $\varphi(D)$ is a union of two elliptic curves in $\mathbf{P}^8$ that are respectively isomorphic to E_1 and E_2 and meet transversally at a point of order two. This implies that the principally polarized abelian surfaces $E_1 \times E_2$ and $(E_1 \times E_2)/\Gamma_\alpha$ are isomorphic, as stated already in Example 5.2.

We summarize our results in the following theorem.

Theorem 5.6. *Let K be a field of characteristic $\mathrm{char}(K) \notin \{2,3\}$ and let $\omega \in \overline{K}$ be a primitive third root of unity, not necessarily in K. Let $a, b \in K$ be such that $(a^3+1)(b^3+1) \neq 0$ and let E_a and E_b be the elliptic curves given by the models*

$$E_a \colon x^3 + y^3 + z^3 + 3axyz = 0, \quad E_b \colon x^3 + y^3 + z^3 + 3bxyz = 0,$$

with the identity element $O = [-1 : 1 : 0]$. Let $\mathcal{A}(a,b)$ denote the principally polarized abelian surface $E_a \times E_b$, with the polarization defined by the divisor $\Theta = E_a \times \{O\} + \{O\} \times E_b$. Let $\alpha \colon E_a[3] \xrightarrow{\sim} E_b[3]$ be the isomorphism defined by

$$[-1 : 0 : 1] \mapsto [-1 : 0 : 1], \quad [-\omega : 1 : 0] \mapsto [-\omega^2 : 1 : 0],$$

let Γ_α denote its graph, and let $\mathcal{J}(a,b)$ denote the principally polarized abelian surface $\mathcal{A}(a,b)/\Gamma_\alpha$, with the induced polarization. If $3a^2b^2 + a^3 + b^3 - 3ab + 2 = 0$ then $\mathcal{J}(a,b)$ is isomorphic to $\mathcal{A}(a,b)$ and E_a and E_b are 2-isogenous. Otherwise, $\mathcal{J}(a,b)$ is isomorphic to the Jacobian of a curve C of genus two whose Igusa–Clebsch invariants are as follows:

$$\begin{aligned} I_2 = 72(&9a^6b^6 - 30(a^7b^4 + a^4b^7) - 88a^5b^5 + a^8b^2 + a^2b^8 + 54(a^6b^3 + a^3b^6) + 65a^4b^4 \\ &- 32(a^7b + ab^7) - 104(a^5b^2 + a^2b^5) + 40(a^6 + b^6) + 44a^3b^3 + 100(a^4b + ab^4) \\ &- 68a^2b^2 + 16(a^3 + b^3) + 112ab - 20), \end{aligned}$$

$$\begin{aligned} I_4 = 36(&3a^2b^2 + a^3 + b^3 - 3ab + 2)^4(9a^4b^4 + 240a^3b^3 + 8(a^4b + ab^4) + 240a^2b^2 \\ &+ 160(a^3 + b^3) + 256ab + 320), \end{aligned}$$

$$\begin{aligned}
I_6 = {} & 72(3a^2b^2+a^3+b^3-3ab+2)^4(729a^{10}b^{10}-3402(a^{11}b^8+a^8b^{11}) \\
& +30456a^9b^9+81(a^{12}b^6+a^6b^{12})-70794(a^{10}b^7+a^7b^{10})-201555a^8b^8 \\
& -2160(a^{11}b^5+a^5b^{11})+60(a^{12}b^3+a^3b^{12})+106560(a^9b^6+a^6b^9) \\
& -148932a^7b^7-121608(a^{10}b^4+a^4b^{10})+480(a^{11}b^2+a^2b^{11}) \\
& -358740(a^8b^5+a^5b^8)-8(a^{12}+b^{12})+156928(a^9b^3+a^3b^9) \\
& +336444a^6b^6-50160(a^{10}b+ab^{10})+81072(a^7b^4+a^4b^7) \\
& -462096a^5b^5-167112(a^8b^2+a^2b^8)+84224(a^9+b^9) \\
& +455568(a^6b^3+a^3b^6)+761040a^4b^4+181152(a^7b+ab^7) \\
& -93600(a^5b^2+a^2b^5)+219552(a^6+b^6)+383424a^3b^3 \\
& +564480(a^4b+ab^4)+88512a^2b^2+74624(a^3+b^3)+314112ab-55040),
\end{aligned}$$

$$I_{10} = 36864(a^3+1)(b^3+1)(3a^2b^2+a^3+b^3-3ab+2)^{12}.$$

Moreover, the K-isomorphism class of C is determined by the following:

(1) If $a^3+b^3+3ab-1 \neq 0$ then an affine plane model of C is given by

$$-3dy^2 = (dc_1x^3+c_2(a,b)x^2+dc_3(a,b)x+d^2)(d^2x^3+dc_3(b,a)x^2+c_2(b,a)x+dc_1),$$

where

$$\begin{aligned}
d &= 3a^2b^2+a^3+b^3-3ab+2, & c_2(a,b) &= 9(1-ab)(a^2+b)(2b^4-a^3b+3ab^2+3a^2+b), \\
c_1 &= -3a^2b^2-4a^3-4b^3-6ab+1, & c_3(a,b) &= 3(-b^4+2a^3b-3a^2-2b).
\end{aligned}$$

(2) If $a=b=1/2$ then C has a model given by $y^2=(3x^3+6x^2+3x+4)(3x^3-6x^2+3x-4)$ and if $\{a,b\}=\{\omega/2,\omega^2/2\}$ then C has a model given by $y^2=x^6+6x^4-39x^2+48$. These two curves are isomorphic over $K(\omega)$, but not over K if $\omega \notin K$.

(3) If $a+b=1$ and $a,b \neq 1/2$ then C has a model $y^2=A(x)B(x)$, where

$$\begin{aligned}
A(x) &= (a^2-a+1)^2x^3-6(a-1)(a^3+1)x^2+3(a-2)(4a^2-a-2)x-6(a^2+2a-2), \\
B(x) &= (a^2-a+1)^2x^3-6a(a-2)(a^2-a+1)x^2-3(a+1)(4a^2-7a+1)x-6(a^2-4a+1).
\end{aligned}$$

(4) If $a\omega+b\omega^2=1$ and $(a,b)\neq(\omega^2/2,\omega/2)$ then C has a model $y^2=A(x)B(x)$, where

$$\begin{aligned}
A(x) &= x^3-6\omega(a^2-\omega)x^2+3\omega(4a^3-9\omega^2a^2+4)x-6\omega(a^2+2a\omega^2-2\omega)(a+1)(a+\omega), \\
B(x) &= x^3-6a(a\omega-2)x^2-3\omega(4a^3-3a^2\omega^2-6a\omega+1)x-6w(a+1)(a+w)(a^2-4a\omega^2+\omega).
\end{aligned}$$

Replacing ω by ω^2 gives a model of C if, instead, $a\omega^2+b\omega=1$ and $(a,b)\neq(\omega/2,\omega^2/2)$. □

Theorem 5.6 allows us to freely choose elliptic curves E_1 and E_2 and determine the curves C of genus two whose Jacobian can be obtained by gluing E_1 and E_2 along the 3-torsion, and do so without computing any Gröbner bases, making assumptions about the ramification behaviour of the corresponding covering maps $C \to E_i$, or making assumptions regarding existence of isogenies between E_1 and E_2. The downside is that we do not necessarily obtain an affine plane model of C over the smallest possible field. However, if the field K is perfect then it is possible to obtain such a model from the Igusa invariants, or from the Igusa–Clebsch invariants if $\operatorname{char}(K) \neq 5$ (see [11, 25]). If K is a number field then the work of Bruin, Sijsling, and Zotine [10] allows one to verify numerically over $\mathbf{C}$ that a curve C as in Theorem 5.6 indeed has a $(3,3)$-split Jacobian. The theorem also makes it possible to determine the equation satisfied by the Igusa–Clebsch invariants of curves of genus two with a $(3,3)$-split Jacobian (see [14]).

The following corollary to the theorem gives additional context to the families of curves discussed in §3.3.1.

Corollary 5.7. *There are generically nine unique isomorphism classes of Jacobians, as principally polarized abelian surfaces, that can be obtained by gluing along the 3-torsion two elliptic curves that are geometrically isomorphic.*

Proof. With notation as in Theorem 5.6, suppose that $j(E_a) = j(E_b)$. Then b is an element of the set $\mathcal{S}(a)$, defined by (5.3). Computing for all $b \in \mathcal{S}(a)$ the Igusa–Clebsch invariants of the curve C whose Jacobian is obtained by gluing E_a and E_b, we find that there are nine unique isomorphism classes of such C. In fact, we have the following isomorphisms of principally polarized abelian surfaces:

$$\mathcal{J}(a, \tfrac{2-a}{1+a}\omega) \cong \mathcal{J}(a, \tfrac{2-a\omega}{1+a\omega}), \quad \mathcal{J}(a, \tfrac{2-a}{1+a}\omega^2) \cong \mathcal{J}(a, \tfrac{2-a\omega^2}{1+a\omega^2}), \quad \mathcal{J}(a, \tfrac{2-a\omega}{1+a\omega}\omega^2) \cong \mathcal{J}(a, \tfrac{2-a\omega^2}{1+a\omega^2}\omega). \qquad \square$$

Remark. Setting $a = b = s$ in Theorem 5.6 results in a curve C that is isomorphic to (3.16). We also note that the three pairs of isomorphic principally polarized abelian surfaces in Corollary 5.7 are Jacobians of curves defined by $\mathcal{Y}_1$ in §3.3.1.

We conclude this section by revisiting some of the examples from §3.

Example 5.8. Let $a = b = 0$ so that $E_1 = E_2 = E$, where E is the curve $x^3 + y^3 + z^3 = 0$. Note that $j(E) = 0$. By Theorem 5.6 we have that $(E \times E)/\Gamma_\alpha \cong \operatorname{Jac}(C)$, where C is a genus-2 curve with Igusa–Clebsch invariants $[-90 : 720 : -15480 : 144]$. These are easily verified to be the invariants of the curve $C\colon y^2 = (x^3+1)(4x^3+1)$ from Example 3.2.

Example 5.9. By (5.2) there are exactly four elliptic curves $E \in \mathcal{H}$ with $j(E) = 0$. Computing the Igusa–Clebsch invariants using Theorem 5.6, we conclude that, up to $\overline{K}$-isomorphism, there are precisely two distinct ways of gluing two such curves along the 3-torsion, unless $\operatorname{char}(K) = 5$. The curves defined by $a = 0$ and $b = 2$ yield $[12006 : 2250000 : 10139625000 : 316406250000]$, which are the invariants of $C\colon y^2 = (x^3 + 6x^2 + 12x + 10)(10x^3 + 36x^2 + 60x + 25)$ from Example 3.3.

Example 5.10. Let $a = -1 + \sqrt{3}$. If we set $b = a$ then these parameters define $E_1 = E_2 = E \in \mathcal{H}$ with $j(E) = 1728$ and Theorem 5.6 implies that $(E \times E)/\Gamma_\alpha$ is principally polarized by a curve of genus two whose Igusa–Clebsch invariants are $[774 : 9648 : 2763360 : 27648]$. These are easily verified to be the invariants of the curve $C\colon y^2 = x(x^2+1)(4x^2+3)$ from Example 3.5. If we instead set $b = a\omega^{\pm 1}$, we obtain the remaining two isomorphism classes of curves C of genus two whose Jacobian is $(3,3)$-isogenous to $E_1 \times E_2$ with $j(E_1) = j(E_2) = 1728$, unless $\operatorname{char}(K) = 7$, in which case there is only one additional such isomorphism class. The corresponding degree-3 coverings are both generic and the Igusa–Clebsch invariants of C are

$$[762 + 822\omega^{\pm 1} : -20064 + 3060\omega^{\pm 1} : -5121464 - 3754864\omega^{\pm 1} : -2456032 + 1243024\omega^{\pm 1}].$$

Example 5.11. Let C be the genus-2 curve from Example 3.6, defined by

$$y^2 = x(2x^2+4x+3)(3x^2+4x+2).$$

Then C has Igusa–Clebsch invariants $[86 : 13456 : 471968 : 6718464]$. As we have seen, $\mathrm{Jac}(C)$ is $(3,3)$-isogenous to some $E_1 \times E_2$. We can use Theorem 5.6 to determine the j-invariants of the elliptic curves E_1 and E_2, as follows. Consider the ideal of $K[\lambda, a, b, j_1, j_2]$ generated by the equations arising from the following:

(1) the Igusa–Clebsch invariants of C equal the expressions from Theorem 5.6, up to multiplication by appropriate powers of $\lambda \neq 0$;
(2) the j-invariant $j_1 = j(E_1)$ equals $j(E_a)$, given by the expression (5.2);
(3) the j-invariant $j_2 = j(E_2)$ equals $j(E_b)$, given by the corresponding expression in b.

Eliminating the variables a, b, λ from the ideal leaves two equations, namely

$$j_2^2 + \frac{873302912}{59049} j_2 - \frac{55918260224}{531441} = 0, \quad j_1 + j_2 + \frac{873302912}{59049} = 0,$$

whence we conclude that $\{j(E_1), j(E_2)\} = \{64/9, -873722816/59049\}$. After choosing suitable curves $E_1, E_2 \in \mathcal{H}$ that are twists of the elliptic curves from Example 3.6, verifying that the 5-isogeny $E_1 \to E_2$ induces the isomorphism $\alpha\colon E_1[3] \xrightarrow{\sim} E_2[3]$ is straightforward.

A. Appendix

In §5 we fixed generators $S = [-1 : 0 : 1]$ and $T = [-\omega : 1 : 0]$ of the 3-torsion subgroup for all curves $E_a \in \mathcal{H}$ and we analyzed the gluing of a pair of elliptic curves $E_a, E_b \in \mathcal{H}$ along the 3-torsion via the isomorphism $\alpha\colon E_a[3] \xrightarrow{\sim} E_b[3]$ defined by $\alpha(S) = S$ and $\alpha(T) = -T$, identified with the element $\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbf{Z}/3\mathbf{Z})$. If $\tilde\alpha\colon E_a[3] \xrightarrow{\sim} E_b[3]$ is some other anti-symplectic isomorphism then gluing the curves E_a and E_b via $\tilde\alpha$ can be accomplished by composing with an elliptic curve isomorphism $\eta\colon E_b \xrightarrow{\sim} E_{\tilde b}$ such that $\eta \circ \tilde\alpha\colon E_a[3] \xrightarrow{\sim} E_{\tilde b}[3]$ equals α or $[-1] \circ \alpha$. The following table indicates which parameter $\tilde b \in \mathcal{S}(b)$ corresponds to which isomorphism $\tilde\alpha\colon E_a[3] \xrightarrow{\sim} E_b[3]$, such that $(E_a \times E_b)/\Gamma_{\tilde\alpha} \cong (E_a \times E_{\tilde b})/\Gamma_\alpha$ as principally polarized abelian surfaces:

$\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&1\\0&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&0\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&1\\1&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&0\\1&2\end{smallmatrix}\right]$
b	$b\omega$	$b\omega^2$	$\frac{2-b}{1+b}$	$\frac{2-b}{1+b}\omega$	$\frac{2-b}{1+b}\omega^2$	$\frac{2-b\omega}{1+b\omega}$	$\frac{2-b\omega}{1+b\omega}\omega$	$\frac{2-b\omega}{1+b\omega}\omega^2$	$\frac{2-b\omega^2}{1+b\omega^2}$	$\frac{2-b\omega^2}{1+b\omega^2}\omega$	$\frac{2-b\omega^2}{1+b\omega^2}\omega^2$

The corresponding isomorphisms $E_b \xrightarrow{\sim} E_{\tilde b}$ can be deduced easily from the two isomorphisms given in §5.1. Every abelian surface obtained by gluing along the 3-torsion two elements of $\mathcal{H}$ that are respectively isomorphic to E_a and E_b is isomorphic to one of the twelve principally polarized abelian surfaces $(E_a \times E_{\tilde b})/\Gamma_\alpha$. In particular, we have isomorphisms

$$(E_a \times E_b)/\Gamma_\alpha \cong (E_{a\omega} \times E_{b\omega^2})/\Gamma_\alpha \cong (E_{(2-a)/(1+a)} \times E_{(2-b)/(1+b)})/\Gamma_\alpha. \quad \text{(A.1)}$$

Isomorphisms between $(E_a \times E_b)/\Gamma_\alpha$ and the remaining nine abelian surfaces can be easily obtained by composition. We make (A.1) explicit in the generic case. Suppose that $a, b \in K$ are such that $(3a^2b^2 + a^3 + b^3 - 3ab + 2)(a^3 + b^3 + 3ab - 1) \neq 0$ and let $C(a,b)$ denote the hyperelliptic curve defined in statement *(1)* of Theorem 5.6. Then an isomorphism $C(a,b) \xrightarrow{\sim} C(a\omega, b\omega^2)$ is given by $(x,y) \mapsto (\omega x, y)$, while an isomorphism $C(a,b) \xrightarrow{\sim} C((2-a)/(1+a), (2-b)/(1+b))$ is given by

$$(x,y) \mapsto \left(-\frac{p(a,b)x+d}{dx+p(b,a)}, \frac{qy}{(dx+p(b,a))^3} \right),$$

where

$$\begin{aligned} p(a,b) &= 3a^3b - a^3 - b^3 + 3ab^2 - 3a^2 - 3ab - 3b + 1, \\ q &= 729(1+2\omega)(ab-1)^3(a^2+b^2-ab+a+b+1)^3(a+1)^5(b+1)^5, \\ d &= 3a^2b^2 + a^3 + b^3 - 3ab + 2. \end{aligned}$$

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