Boundedness, monotonicity and asymptotic behavior of solutions of Caputo fractional differential equations

Tao Zhu

School of Mathematics and Physics, Nanjing Institute of Technology,
Nanjing, 211167, PR China
Email: zhutaoyzu@sina.cn

Abstract. By the Schauder fixed point theorem, we first investigate the boundedness and monotonicity of solutions of Caputo fractional differential equations. We also study the asymptotic behavior of solutions of Caputo fractional differential equations under some different conditions. We prove that the solutions of the Caputo fractional differential equations converge asymptotically to a constant as $t \to +\infty$. Finally, several examples are given to illustrate our main results.
Keywords: Boundedness; Monotonicity; Asymptotic behavior; Caputo fractional differential equations.
MSC2020: 34D05; 26A33; 34A08.

1. Introduction

The investigation of the properties of solutions for fractional differential equations has recently received a lot of attention. For example, Medved [19] proved that all solutions of fractional differential equation with the Caputo derivative are asymptotic to $a + bt$ as $t \to +\infty$. Diethelm et al. [9] gave characteristics regarding the asymptotic behavior of solutions to some classes of linear multi-order fractional differential equation systems. Grace and Zafer [10] studied the asymptotic behavior of nonoscillatory solutions of fractional differential equations. Ahmad et al. [3] studied the asymptotic behavior of solutions for a general class of fractional integro-differential equations. Ponce [21] obtained properties on the behavior of mild solutions to abstract fractional Cauchy problems for the Caputo and Riemann-Liouville fractional derivatives. Cong et al. [6] presented some distinct asymptotic properties of solutions to Caputo fractional differential equations. Kassim and Tatar [13] studied the asymptotic behavior of solutions of fractional differential equations with Hadamard fractional derivatives. Wang et al. [23] studied the long-term behavior of time fractional neutral functional differential equations.

Zhu [27] proved that all solutions of Riemann-Liouville fractional differential equation are attractive. Ky [16] devoted to discuss the asymptotic behavior of solutions to various classes of two-term fractional differential equations. Zhu [28] presented that Riemann-Liouville fractional differential equation has at least one globally attractive solution and $x(t) = x_0 t^{\beta-1} + o(t^{\beta-\gamma_1})$ as $t \to +\infty$.

Choi and Koo [5] proved that the solutions of the following Caputo fractional differential equation

$$D_C^\beta x(t) = \lambda x(t), \qquad \beta \in (0,1), \quad t \geq t_0$$

are nondecreasing in t, where $\lambda \geq 0$. Recently, Goodrich and Lizama [11] considered the following fractional differential equation

$$\begin{cases} D_C^\beta x(t) = f(t, x(t)), & \beta \in (1,2), \quad t \in [0, +\infty) \\ x(0) = 0, \quad x'(0) \geq 0. \end{cases} \tag{1.1}$$

They proved that all solutions of the equation (1.1) are nondecreasing whenever $f(t,x) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. Goodrich and Lizama [12] considered the existence of positive and nonincreasing solutions for the following boundary value problem

$$\begin{cases} D_C^\beta x(t) + \lambda f(t, x(t)) = 0, & \beta \in (1,2), \quad t \in (0,1) \\ x'(0) = 0, \quad x(1) = T. \end{cases} \tag{1.2}$$

where λ and T are nonnegative constants. Zhu [29] investigated the following Riemann-Liouville fractional differential equation

$$\begin{cases} D^\beta x(t) = l(t)\phi(x(t)) + k(t), & \beta \in (0,1), \quad t \in (0, +\infty) \\ \lim_{t \to 0^+} t^{1-\beta} x(t) = x_0. \end{cases} \tag{1.3}$$

Zhu presented that the equation (1.3) has at least one decreasing solution in $C_{1-\beta}^+(0, +\infty)$, Zhu also studied the asymptotic behavior of solution of the equation (1.3) under some different conditions.

In this paper, we first study the boundedness and monotonicity of solution of the following Caputo fractional differential equation

$$\begin{cases} D_C^\beta x(t) = f(t, x(t)), & \beta \in (0,1), \quad t \in (0, +\infty) \\ x(0) = x_0. \end{cases} \tag{1.4}$$

By the Schauder fixed point theorem, we prove that the equation (1.4) has at least one nonnegative, bounded and nondecreasing continuous solution on $[0, +\infty)$. Using the boundedness of solution, we obtain the asymptotic behavior of solution of the equation (1.4). We also investigate the asymptotic behavior of solution of the following Caputo fractional differential equation

$$\begin{cases} D_C^\beta x(t) = l(t)\varphi(x(t)) + k(t), & \beta \in (0,1), \quad t \in (0, +\infty) \\ x(0) = x_0. \end{cases} \tag{1.5}$$

We prove that the equation (1.5) has at least one nondecreasing solution $x \in C_0^+[0, +\infty)$ and converges asymptotically to a constant as $t \to +\infty$. To the best of our knowledge, there have been few papers to study the monotonicity and asymptotic behavior of the Caputo fractional differential equations. The results and the methods of the proof in this paper seem to be new.

The outline of this paper is as follows. In Section 2, we introduce some notations, definitions and theorems needed in our proofs. Some new results about the fractional integral functions, and the equivalence between the Caputo fractional differential equation and the Volterra fractional integral equation are also proved in this section. In Section 3, we give some sufficient conditions on f for the boundedness and monotonicity of solution of fractional differential equation (1.4). In Section 4, we investigate the asymptotic behavior of solution of Caputo fractional differential equation (1.5). We proved that the nondecreasing solution of the equation (1.5) converges asymptotically to a constant as $t \to +\infty$. We also give the the asymptotic behavior of solution of the equation (1.4) by using the boundedness of solution. In the last Section, we will give some examples to illustrate our results.

2. PRELIMINARIES

In this section, we first introduce some notations and definitions which will be needed later. Some new results of fractional integral functions are also discussed in this section. At last we give the equivalence between the Caputo fractional differential equation and the Volterra integral equation when f is continuous on $(0, +\infty) \times \mathbb{R}$.

We denote by $C^+[0, 1] = \{x : x \in C[0, 1]$ and $x(t) \geq 0$ for all $t \in [0, 1]\}$ and $C^+[0, +\infty) = \{x : x \in C[0, +\infty)$ and $x(t) \geq 0$ for all $t \in [0, +\infty)\}$. Let $C^1(0, +\infty) = \{x : x' \in C(0, +\infty)$ for all $t \in (0, +\infty)\}$. Let $C^+_*[0, +\infty) = \{x : x \in C^+[0, +\infty)$ and $x(t)$ is a nondecreasing function on $[0, +\infty)\}$. Let $C^+_0[0, +\infty) = \{x : x \in C^+[0, +\infty)$ and $\sup_{0 \leq t < +\infty} x(t)$ is finite $\}$. It is known that $C^+_0[0, +\infty)$ is a Banach space with the norm $\|x\| = \sup_{0 \leq t < +\infty} x(t)$. A family $U \subset C^+[0, +\infty)$ is relatively compact if and only if for each $T > 0$ the restriction to $[0, T]$ of all functions from U forms an equicontinuous and uniformly bounded set. A sequence x_n is convergent to x in $C^+[0, +\infty)$ if and only if x_n is uniformly convergent to x on compact subsets of $[0, +\infty)$. $L^p_{loc}[0, +\infty)$ $(p \geq 1)$ is the space of all real valued functions f for which $|f|^p$ is Lebesgue integrable over every bounded subinterval of $[0, +\infty)$. For $p, q > 0$, the Beta function is defined by

$$\mathbf{B}(p, q) = \int_0^1 (1 - s)^{p-1} s^{q-1} ds.$$

It is well known that $\mathbf{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, where Γ is the Gamma function and

$$\Gamma(p) = \int_0^{+\infty} s^{p-1} \exp(-s) ds.$$

From the Beta function, we can get

$$\int_0^t (t - s)^{p-1} s^{q-1} ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} t^{p+q-1}, \qquad t > 0.$$

For $\beta \in (0,1)$, the Riemann-Liouville fractional integral of order β is defined by

$$I^{\beta}\rho(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{\rho(s)}{(t-s)^{1-\beta}} ds,$$

where $\rho \in L^1[0,T]$. The Riemann-Liouville fractional derivative $D^{\beta}\rho$ is defined by

$$D^{\beta}\rho(t) = \frac{d}{dt} I^{1-\beta}\rho(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{\rho(s)}{(t-s)^{\beta}} ds, \text{ a.e. } t \in [0,T].$$

where $I^{1-\beta}\rho$ is an absolutely continuous (AC) function on $[0,T]$. If $\rho(0)$ exists and $I^{1-\beta}\rho \in AC[0,T]$, then the Caputo fractional derivative $D_C^{\beta}\rho$ is defined by

$$D_C^{\beta}\rho(t) = D^{\beta}(\rho(t) - \rho(0)).$$

From Theorem 2.1 in [15], If $\rho \in AC[0,T]$, then we can get

$$D_C^{\beta}\rho(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\rho'(s)}{(t-s)^{\beta}} ds, \text{ a.e. } t \in [0,T].$$

For more details about fractional calculus, we refer the reader to the texts [8, 15, 20, 22].

Now, we present some recent results of the following fractional integral function

$$y(t) = \int_0^t (t-s)^{\beta-1}\rho(s) ds.$$

In [7], if $\rho \in C_{\alpha}[0,+\infty)$ with $0 \le \alpha < \beta < 1$, then $y(t)$ is continuous on $[0,+\infty)$ and $y(0) = 0$. In [2], Agarwal et al. proved that $y(t)$ is continuous on $[0,1]$ when $\rho \in L^p[0,1]$, where $p > 1/\beta$. Becker et al. [4] proved that $y(t)$ is continuous on $(0,1]$ when $\rho \in C(0,1] \cap L^1[0,1]$. Zhu [28] proved that $y(t)$ is continuous on $(0,1]$ and $t^{1-\beta}y(t)$ is continuous on $[0,1]$ when $\int_0^1 s^{p(1-\beta)}|\rho(s)|^p ds$ exists, where $p > 1/\beta$.

Lemma 2.1. *[29] Let $0 < \beta < 1$, and suppose that $\rho \in C(0,+\infty) \cap L^1_{loc}[0,+\infty)$. Then*

$$\lim_{t \to +\infty} \int_0^t (t-s)^{\beta-1}\rho(s) ds = 0$$

when $\lim_{t \to +\infty} t^{\beta}\rho(t) = 0$.

Lemma 2.2. *[29] Let $0 < \beta < 1$, and suppose that $\rho \in C(0,+\infty) \cap L^1_{loc}[0,+\infty)$. Then*

$$\lim_{t \to +\infty} \int_0^t (t-s)^{\beta-1}\rho(s) ds = \frac{a\pi}{\sin(\beta\pi)}$$

when $\lim_{t \to +\infty} t^{\beta}\rho(t) = a$.

Lemma 2.3. *Let $0 < \beta < 1$. Suppose that $\rho \in L^p_{loc}[0,+\infty)$, where $p > 1/\beta$. Then*

$$\left| \int_0^t (t-s)^{\beta-1}\rho(s) ds \right| \le \Gamma^{1/q}\left(\frac{p\beta-1}{p-1}\right) \Gamma^{1/q}\left(\frac{p-p\beta}{p-1}\right) \left(\int_0^t s^{p\beta-1}|\rho(s)|^p ds\right)^{1/p},$$

where $q = \frac{p}{p-1}$.

Proof. Using the Hölder inequality, we have

$$\begin{aligned}\left|\int_0^t (t-s)^{\beta-1}\rho(s)ds\right| &\le \int_0^t (t-s)^{\beta-1}s^{1/p-\beta}s^{\beta-1/p}|\rho(s)|ds \\ &\le \left(\int_0^t (t-s)^{q(\beta-1)}s^{q-q\beta-1}ds\right)^{1/q}\left(\int_0^t s^{p\beta-1}|\rho(s)|^p ds\right)^{1/p} \\ &\le \Gamma^{1/q}\left(\frac{p\beta-1}{p-1}\right)\Gamma^{1/q}\left(\frac{p-p\beta}{p-1}\right)\left(\int_0^t s^{p\beta-1}|\rho(s)|^p ds\right)^{1/p}.\end{aligned} \tag{2.1}$$

□

Lemma 2.4. *Let $0<\beta<1$. Suppose that $\rho\in L^p_{loc}[0,+\infty)$, where $p>1/\beta$, and there exists a nonnegative constant K such that $|t^\beta\rho(t)|\le K$ for $t\in[1,+\infty)$. Then*

$$y(t)=\int_0^t (t-s)^{\beta-1}\rho(s)ds$$

is a continuous and bounded function on $[0,+\infty)$.

Proof. Since $\rho\in L^p_{loc}[0,+\infty)$, using Lemma 2.2 in [2], then we get that $y(t)$ is a continuous function on $[0,+\infty)$. We only need to prove that $|y(t)|$ is finite for $t\in[1,+\infty)$. For $t\ge1$, we have

$$y(t)=\int_0^1 (t-s)^{\beta-1}\rho(s)ds+\int_1^t (t-s)^{\beta-1}\rho(s)ds. \tag{2.2}$$

Using Lemma 2.3, we get

$$\begin{aligned}\left|\int_0^1 (t-s)^{\beta-1}\rho(s)ds\right| &\le \int_0^1 (1-s)^{\beta-1}|\rho(s)|ds \\ &\le \Gamma^{1/q}\left(\frac{p\beta-1}{p-1}\right)\Gamma^{1/q}\left(\frac{p-p\beta}{p-1}\right)\left(\int_0^1 s^{p\beta-1}|\rho(s)|^p ds\right)^{1/p}.\end{aligned} \tag{2.3}$$

For $t\ge1$, we have

$$\begin{aligned}\left|\int_1^t (t-s)^{\beta-1}\rho(s)ds\right| &\le \int_1^t (t-s)^{\beta-1}Ks^{-\beta}ds \\ &\le K\int_0^t (t-s)^{\beta-1}s^{-\beta}ds \\ &= K\Gamma(\beta)\Gamma(1-\beta) \\ &= \frac{K\pi}{\sin(\beta\pi)}.\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we get that $|y(t)|$ is finite for $t\in[1,+\infty)$. Therefore, $y(t)$ is a bounded function on $[0,+\infty)$. □

Lemma 2.5. *Let $0<\beta<\gamma<1$. Suppose that $\rho\in L^p_{loc}[0,+\infty)$, where $p>1/\beta$, and there exists a nonnegative constant K such that $|t^\gamma\rho(t)|\le K$ for $t\in[1,+\infty)$. Then*

$$y(t)=\int_0^t (t-s)^{\beta-1}\rho(s)ds$$

is a continuous and bounded function on $[0,+\infty)$ and $\lim_{t\to+\infty}y(t)=0$.

Proof. For $t \in [1, +\infty)$, we know that

$$|t^\beta \rho(t)| = |t^{\beta-\gamma} t^\gamma \rho(t)| \leq K t^{\beta-\gamma} \leq K. \tag{2.5}$$

From Lemma 2.4, we get that $y(t)$ is a continuous and bounded function on $[0, +\infty)$. Since $\beta < \gamma$, from (2.5), we get $\lim_{t\to+\infty} t^\beta \rho(t) = 0$. Using Lemma 2.1, we have $\lim_{t\to+\infty} y(t) = 0$. $\square$

Lemma 2.6. *Let $t_0 \geq 0$ and $0 < \beta < 1$. If $\rho \in C(t_0, +\infty) \cap L^1_{loc}[t_0, +\infty)$. Then*

$$h(t) = \int_{t_0}^t (t-s)^{\beta-1} \rho(s) ds$$

is continuous on $(t_0, +\infty)$.

Proof. Let $s = u + t_0$. Then we get

$$h(t) = \int_{t_0}^t (t-s)^{\beta-1} \rho(s) ds = \int_0^{t-t_0} (t - t_0 - u)^{\beta-1} \rho(u + t_0) du.$$

Let $\rho_0(t) = \rho(t + t_0)$ for $t > 0$, we have

$$h(t) = \int_0^{t-t_0} (t - t_0 - u)^{\beta-1} \rho_0(u) du.$$

Since $\rho \in C(t_0, +\infty) \cap L^1_{loc}[t_0, +\infty)$, then $\rho_0 \in C(0, +\infty) \cap L^1_{loc}[0, +\infty)$, and using Lemma 4.6 in [4], we get that

$$y(t) = \int_0^t (t-u)^{\beta-1} \rho_0(u) du$$

is continuous on $(0, +\infty)$. Therefore, we get that $h(t) = y(t - t_0)$ is continuous on $(t_0, +\infty)$. $\square$

Lemma 2.7. *[24] Let $0 \leq t_0 < t$ and $0 < \beta, \gamma < 1$. Then for $t > t_0$ we have*

$$\int_{t_0}^t (t-s)^{-\beta} (s - t_0)^{-\gamma} ds = (t - t_0)^{1-\beta-\gamma} \mathbf{B}(1-\beta, 1-\gamma).$$

Lemma 2.8. *Let $t_0 \geq 0$ and $0 < \beta < 1$. Suppose that $\rho \in C(t_0, +\infty) \cap L^1_{loc}[t_0, +\infty)$. If $(t - t_0)^\beta \rho(t)$ is a nonincreasing (nondecreasing) function on $(t_0, +\infty)$, then*

$$h(t) = \int_{t_0}^t (t-s)^{\beta-1} \rho(s) ds$$

is a nonincreasing (nondecreasing) function on $(t_0, +\infty)$.

Proof. Using Lemma 2.6, we get that $h(t)$ is a continuous function on $(t_0, +\infty)$. Let $t_0 < t_1 < t_2 < +\infty$. Since $(t - t_0)^\beta \rho(t)$ is a nonincreasing function on $(t_0, +\infty)$, and using Lemma 2.7, then we have

$$
\begin{aligned}
&h(t_2) - h(t_1)\\
&= \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} \rho(s) ds\\
&\quad + \int_{t_0}^{t_1} (t_2 - s)^{\beta-1} \rho(s) ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} \rho(s) ds\\
&= \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} (s - t_0)^{-\beta} (s - t_0)^{\beta} \rho(s) ds\\
&\quad + \int_{t_0}^{t_1} \left[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right] (s - t_0)^{-\beta} (s - t_0)^{\beta} \rho(s) ds\\
&\le \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} (s - t_0)^{-\beta} (t_1 - t_0)^{\beta} \rho(t_1) ds\\
&\quad + \int_{t_0}^{t_1} \left[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right] (s - t_0)^{-\beta} (t_1 - t_0)^{\beta} \rho(t_1) ds\\
&= (t_1 - t_0)^{\beta} \rho(t_1) \left[\int_{t_0}^{t_2} (t_2 - s)^{\beta-1} (s - t_0)^{-\beta} ds - \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} (s - t_0)^{-\beta} ds\right]\\
&= (t_1 - t_0)^{\beta} \rho(t_1) \left[\mathbf{B}(\beta, 1-\beta) - \mathbf{B}(\beta, 1-\beta)\right]\\
&= 0.
\end{aligned} \tag{2.6}
$$

Then we get that $h(t)$ is a nonincreasing function on $(t_0, +\infty)$.

Using the same procedure as in the proof of (2.6), we can also obtain that $h(t)$ is a nondecreasing function on $(t_0, +\infty)$ when $(t - t_0)^{\beta} \rho(t)$ is a nondecreasing function on $(t_0, +\infty)$. □

Remark 2.9. In [29], if $\rho \in C(0, +\infty) \cap L^1_{loc}[0, +\infty)$, then $y(t)$ is a nonincreasing (nondecreasing) function on $(0, +\infty)$ when $t^{\beta} \rho(t)$ is a nonincreasing (nondecreasing) function on $(0, +\infty)$. In Lemma 2.8, we can immediately obtain the result in [29] when $t_0 = 0$. The proof in this paper is significantly different from the proof of Lemma 2.6 in [29].

Now, we will give two examples to illustrate the applicability of the Lemma 2.8. For example, let

$$h(t) = \int_1^t \frac{(t - s)^{-1/2}}{\sqrt{s^2 - 1}} ds,$$

we know that $\rho(t) = \frac{1}{\sqrt{t^2 - 1}}$ is a nonincreasing function on $(1, +\infty)$, and $\sqrt{t - 1}\rho(t) = \frac{1}{\sqrt{t+1}}$ is a nonincreasing function on $(1, +\infty)$. Using Lemma 2.8, we get that $h(t)$ is a nonincreasing function on $(1, +\infty)$, and

$$h(t) \le \int_1^t (t - s)^{-1/2} (s - 1)^{-1/2} ds = \mathbf{B}(1/2, 1/2) = \pi, \qquad t \in (1, +\infty).$$

Let

$$h(t) = \int_1^t (t - s)^{-1/2} \sqrt{\frac{s}{s - 1}} ds,$$

we know that $\rho(t) = \sqrt{\frac{t}{t-1}}$ is a nonincreasing function on $(1, +\infty)$, and $\sqrt{t - 1}\rho(t) = \sqrt{t}$ is a nondecreasing function on $(1, +\infty)$. By Lemma 2.8, we obtain that $h(t)$ is a nondecreasing function on $(1,+\infty)$, and

$$h(t) \geq \int_1^t (t-s)^{-1/2} ds = 2\sqrt{t-1} \to +\infty$$

as $t \to +\infty$.

In [4, Lemma 4.8], Becker et al. obtained the following result of fractional integral operator.

Lemma 2.10. *[4] Let $\rho \in C(0,T] \cap L^1[0,T]$. Let $\beta, \gamma > 0$ with $\beta + \gamma \geq 1$. Then*

$$I^{\beta+\gamma}\rho(t) = \frac{1}{\Gamma(\beta+\gamma)} \int_0^t \frac{\rho(s)}{(t-s)^{1-\beta-\gamma}} ds$$

is continuous on $[0,T]$. Moreover,

$$I^{\beta+\gamma}\rho(t) = I^\beta I^\gamma \rho(t)$$

at each $t \in [0,T]$.

In [25, Proposition 3.2], If $0 < \alpha \leq \beta < 1$ (or if $\alpha = 0$ and $0 < \beta < 1$) and $u' \in C_\alpha$ then $I^\beta u \in C^1[0,T]$ if and only if $u(0) = 0$. However, I^β does not map $C^1[0,T]$ into $C^1[0,T]$ in general. Using Lemma 2.10, we can obtain the following result.

Lemma 2.11. *(1). If $\rho \in I^\beta W$, where $W = C(0,+\infty) \cap L^1_{loc}[0,+\infty)$, then $I^{1-\beta}\rho \in C^1(0,+\infty)$. (2). If $\rho \in C[0,+\infty)$ and $\rho' \in C(0,+\infty) \cap L^1_{loc}[0,+\infty)$, then $I^{1-\beta}\rho \in C^1(0,+\infty)$.*

Proof. (1). Let $\rho = I^\beta v$, where $v \in W$. Then using Lemma 2.10, we have

$$I^{1-\beta}\rho = I^{1-\beta} I^\beta v = Iv.$$

Since $v \in W$, then we have $I^{1-\beta}\rho \in C^1(0,+\infty)$ and $(I^{1-\beta}\rho)' = v$.

(2). Since $\rho \in C[0,+\infty)$ and $\rho' \in C(0,+\infty) \cap L^1_{loc}[0,+\infty)$, then using Lemma 2.10, we get

$$I^{1-\beta}\rho = I^{1-\beta}(I\rho' + \rho(0)) = I^{1-\beta} I\rho' + \frac{\rho(0)t^{1-\beta}}{\Gamma(2-\beta)} = II^{1-\beta}\rho' + \frac{\rho(0)t^{1-\beta}}{\Gamma(2-\beta)}.$$

Then $I^{1-\beta}\rho \in C^1(0,+\infty)$ and $(I^{1-\beta}\rho)' = I^{1-\beta}\rho' + \frac{\rho(0)t^{-\beta}}{\Gamma(1-\beta)}$.

□

In [24, Theorem 4.6], Webb gave an equivalence for a Caputo fractional differential equation and an integral equation when the nonlinearity is of the form $t^{-\gamma} f$. The case when there is no singular term $t^{-\gamma}$ is essentially well known, for example, see Diethelm [8, Lemma 6.2] and Kilbas and Marzan [14, Theorem 1]. Recently, Lan and Webb [18] obtained the following result when $f \in L^p$ for some $p > 1/\beta$. This result is very similar to Theorem 4.2 in [17].

Theorem 2.12. *[18, Lemma 4] Let $0 < \beta < 1$ and suppose that $f \in L^p$ for some $p > 1/\beta$. Then the following assertions are equivalent.*

(1) *$u \in C[0,T]$ satisfies*

$$u(t) = u_0 + I^\beta f(t) \tag{2.7}$$

for each $t \in [0,T]$.

(2) *$u \in C[0,T]$, and $I^{1-\beta}u \in AC$, and u is a solution of the following Caputo fractional initial value problem*

$$\begin{cases} D_C^\beta u(t) = f(t), \\ u(0) = u_0 \end{cases} \tag{2.8}$$

for a.e. $t \in (0,T]$.

We now give the equivalence between the Caputo fractional differential equation and the Volterra integral equation when f is continuous on $(0,+\infty) \times \mathbb{R}$.

Theorem 2.13. *Let $0 < \beta < 1$ and $p > 1/\beta$, let f be continuous on $(0,+\infty) \times \mathbb{R}$, and $g \in L^p_{Loc}[0,+\infty)$ for every $x \in C[0,+\infty)$, where $g(t) = f(t,x(t))$. If $x \in C[0,T]$ and $I^{1-\beta}x \in C^1(0,+\infty)$, and x satisfies the following Caputo fractional initial value problem*

$$\begin{cases} D_C^\beta x(t) = f(t,x(t)), & t \in (0,+\infty), \\ x(0) = x_0. \end{cases} \tag{2.9}$$

Then x satisfies the following Volterra integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds, \qquad t \in [0,+\infty). \tag{2.10}$$

If $x \in C[0,T]$ satisfies the equation (2.10), then x satisfies the equation (2.9).

Proof. If $x \in C[0,T]$ and $I^{1-\beta}x \in C^1(0,+\infty)$, suppose that x satisfies the equation (2.9). For $t \in (0,+\infty)$, we get

$$\begin{aligned} D_C^\beta x(t) &= D^\beta (x(t) - x(0)) \\ &= DI^{1-\beta}(x(t) - x_0) \\ &= f(t,x(t)). \end{aligned} \tag{2.11}$$

An integration of (2.11) yields

$$I^{1-\beta}(x(t) - x_0) = If(t,x(t)).$$

Using Lemma 2.10, we get

$$I(x(t) - x_0) = I^{1+\beta} f(t,x(t)) = II^\beta f(t,x(t)). \tag{2.12}$$

Differentiating (2.12), we obtain

$$x(t) - x_0 = I^\beta f(t,x(t)), \qquad t \in (0,+\infty). \tag{2.13}$$

Since $g \in L^p_{Loc}[0,+\infty)$, using Lemma 2.1 in [2], then we get

$$\lim_{t\to 0^+} I^\beta f(t,x(t)) = \lim_{t\to 0^+} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds = 0.$$

Together with the equality (2.13), then we deduce that

$$x(t) = x_0 + I^\beta f(t,x(t)), \qquad t \in [0,+\infty).$$

If $x \in C[0,T]$ satisfies the equation (2.10), since $g \in L^p_{loc}[0,+\infty)$, then we get

$$\lim_{t\to 0^+} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds = 0.$$

From equality (2.10), then we get $x(0) = x_0$. For $t \in (0, +\infty)$, using Lemma 2.10 and Lemma 2.11, we have

$$
\begin{aligned}
D_C^\beta x(t) &= D^\beta (x(t) - x(0)) \\
&= DI^{1-\beta}(x(t) - x_0) \\
&= DI^{1-\beta} I^\beta (f(t, x(t))) \\
&= DI(f(t, x(t))) \\
&= f(t, x(t)).
\end{aligned} \tag{2.14}
$$

Hence, x satisfies the equation (2.9). □

In [26, Theorem 2.4], using the Hölder inequality, Zhu obtained the following fractional integral inequality.

Theorem 2.14. *Let $0 < T \le +\infty$ and $0 < \alpha < \beta < 1$, $a(t)$ and $l(t)$ are continuous, nonnegative functions on $[0, T)$. Let $x(t)$ be a continuous, nonnegative function on $[0, T)$ with*

$$x(t) \le a(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s) x(s) ds.$$

Then

$$x(t) \le \left(A(t) + \int_0^t L(s) A(s) \exp\left(\int_s^t L(\tau) d\tau \right) ds \right)^\alpha \qquad t \in [0, T). \tag{2.15}$$

If $a(t)$ is a nondecreasing function on $[0, T)$, then the inequality (2.15) is reduced to

$$x(t) \le \left(A(t) \exp\left(\int_0^t L(s) ds \right) \right)^\alpha,$$

where $A(t) = 2^{1/\alpha - 1} a^{1/\alpha}(t)$ and $L(t) = \frac{2^{1/\alpha-1}}{\Gamma^{1/\alpha}(\beta)} \left(\Gamma(\frac{\beta-\alpha}{1-\alpha}) \Gamma(\frac{1-\beta}{1-\alpha}) \right)^{1/\alpha - 1} t^{\beta/\alpha - 1} l^{1/\alpha}(t)$.

Let $\alpha = 1/p$ in Theorem 2.14 and using the same proof of Theorem 2.14, we can obtain the following result.

Theorem 2.15. *Let $0 < \beta < 1$, let $a \in C^+[0, +\infty)$ and $l \in C^+(0, +\infty) \cap L^p_{loc}[0, +\infty)$, where $p > 1/\beta$. Let $x(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$x(t) \le a(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s) x(s) ds.$$

Then

$$x(t) \le \left(A(t) + \int_0^t L(s) A(s) \exp\left(\int_s^t L(\tau) d\tau \right) ds \right)^{1/p} \qquad t \in [0, +\infty). \tag{2.16}$$

If $a(t)$ is a nondecreasing function on $[0, +\infty)$, then the inequality (2.16) is reduced to

$$x(t) \le \left(A(t) \exp\left(\int_0^t L(s) ds \right) \right)^{1/p},$$

where $A(t) = 2^{p-1} a^p(t)$ and $L(t) = \frac{2^{p-1}}{\Gamma^p(\beta)} \left(\Gamma(\frac{p\beta-1}{p-1}) \Gamma(\frac{p-p\beta}{p-1}) \right)^{p-1} t^{p\beta-1} l^p(t)$.

Finally, the following fixed point theorems will be needed in the proof of the main results of this paper.

Theorem 2.16. *[1] If C is a nonempty, closed, convex and bounded subset of a Banach space E, and $F : C \to C$ is a continuous and completely continuous map. Then F has a fixed point in C.*

Theorem 2.17. *[1] Let E be a Banach space, C a closed, convex subset of E and $0 \in C$. Let $F : C \to C$ be a continuous and completely continuous map. Define*

$$E(F) = \{x \in C : x = \lambda F(x) \text{ for some } \lambda \in (0,1)\}. \tag{2.17}$$

Then either $E(F)$ is unbounded or F has a fixed point.

3. Boundedness and monotonicity of fractional differential equations

In this section, we study the boundedness and monotonicity of solution of the fractional differential equation (1.4). We first show that the fractional differential equation (1.4) has at least one nonnegative, nondecreasing and continuous solution on $[0, +\infty)$.

Theorem 3.1. *Let $x_0 \geq 0$ and $p > 1/\beta$. Let $f : (0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be continuous, if $g(t) = t^\beta f(t, x(t))$ is a nondecreasing function on $(0, +\infty)$ when $x(t)$ is a nonnegative, nondecreasing continuous function on $[0, +\infty)$. Suppose there are two nonnegative functions $l, k \in C(0, +\infty) \cap L^p_{loc}[0, +\infty)$ such that*

$$f(t, u) \leq l(t)u + k(t) \tag{3.1}$$

for all $t \in (0, +\infty)$ and $u \in [0, +\infty)$. Then the fractional differential equation (1.4) has at least one nonnegative, nondecreasing and continuous solution on $[0, +\infty)$.

Proof. For any $T > 0$, we first show that the fractional differential equation (1.4) has at least one solution $x \in C^+_*[0, T]$. Let us define the operator F by the formula

$$(Fx)(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds. \tag{3.2}$$

Step 1. We first prove $Fx \in C^+_*[0, T]$ when $x \in C^+_*[0, T]$. Since $x_0 \geq 0$ and $f(t, x(t))$ is a nonnegative function, then we get that Fx is a nonnegative function on $[0, T]$. Since $x(t)$ and $g(t)$ are nondecreasing functions, using Lemma 2.8, we get that Fx is a nondecreasing function. Since $l, k \in C(0, +\infty) \cap L^p_{Loc}[0, +\infty)$, using (3.1), then we get that Fx is a continuous function on $[0, T]$. Hence, $Fx \in C^+_*[0, T]$ when $x \in C^+_*[0, T]$.

Step 2. Now, we prove that the operator F is a compact operator. Let $U \in C^+_*[0, T]$ and $\|x\| \leq R$ for any $x \in U$, where $R > 0$. We only need to prove that $F(U)$ is uniformly bounded and equicontinuous on $[0, T]$. From (3.1), we get $f(t, x(t)) \leq Rl(t) + k(t)$ for every $x \in U$. Using the same procedure as in Lemma 2.1 in [2], we get that $F(U)$ is equicontinuous on $[0, T]$. Let

$x \in U$. Using Lemma 2.3, we have

$$
\begin{aligned}
(Fx)(t) &\le x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[l(s)x(s)+k(s)]ds \\
&\le x_0 + \frac{R}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k(s) ds \\
&\le x_0 + \frac{MR}{\Gamma(\beta)} \left(\int_0^t s^{p\beta-1} l^p(s) ds \right)^{1/p} + \frac{M}{\Gamma(\beta)} \left(\int_0^t s^{p\beta-1} k^p(s) ds \right)^{1/p} \\
&\le x_0 + \frac{MR}{\Gamma(\beta)} \left(\int_0^T s^{p\beta-1} l^p(s) ds \right)^{1/p} + \frac{M}{\Gamma(\beta)} \left(\int_0^T s^{p\beta-1} k^p(s) ds \right)^{1/p},
\end{aligned} \tag{3.3}
$$

where $M = \Gamma^{1/q}\left(\frac{p\beta-1}{p-1}\right)\Gamma^{1/q}\left(\frac{p-p\beta}{p-1}\right)$. This proves that the set $F(U)$ is bounded. By Ascoli-Arzela theorem, we know that FU is relatively compact. Therefore, the operator F is a compact operator.

Step 3. We show that F is continuous, that is $x_n \to x$ implies $Fx_n \to F$x. Since $x_n \to x$, let $\|x_n\|, \|x\| \le R$, then we have

$$
(t-s)^{\beta-1} |f(s, x_n(s)) - f(s, x(s))| \le 2(t-s)^{\beta-1}(Rl(s)+k(s)). \tag{3.4}
$$

Since $l, k \in L^p[0,T]$, from Lemma 2.3, we know that $\int_0^t (t-s)^{\beta-1}(Rl(s)+k(s))ds$ exists for all $t \in (0,T]$. For every $s \in (0,t)$, we know

$$
(t-s)^{\beta-1} f(s, x_n(s)) \to (t-s)^{\beta-1} f(s, x(s)) \tag{3.5}
$$

as $n \to +\infty$. From (3.4) and (3.5), using the Lebesgue dominated convergence theorem, then we have

$$
|(Fx_n)(t) - (Fx)(t)| = \frac{1}{\Gamma(\beta)} \left| \int_0^t (t-s)^{\beta-1}[f(s, x_n(s)) - f(s, x(s))]ds \right| \to 0
$$

as $n \to +\infty$. Therefore, $(Fx_n)(t) \to (Fx)(t)$ pointwise on $[0,T]$ as $n \to +\infty$. With the fact that F is a compact operator, then we get that $\|Fx_n - Fx\| \to 0$ as $n \to +\infty$, which implies the map F is continuous.

Step 4. If $x \in C_*^+[0,T]$ is any solution of

$$
x(t) = \lambda \left(x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \right), \tag{3.6}
$$

where $0 < \lambda < 1$, then we have

$$
\begin{aligned}
x(t) &\le x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)x(s) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k(s) ds \\
&\le x_0 + \frac{M}{\Gamma(\beta)} \left(\int_0^T s^{p\beta-1} k^p(s) ds \right)^{1/p} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)x(s) ds.
\end{aligned} \tag{3.7}
$$

Using Theorem 2.15, we get

$$
x(t) \le 2 \left(x_0 + \frac{M}{\Gamma(\beta)} \left(\int_0^T s^{p\beta-1} k^p(s) ds \right)^{1/p} \right) \exp\left(\int_0^T \frac{L(s)}{p} ds \right), \qquad t \in [0,T],
$$

where $L(t) = \frac{2^{p-1}}{\Gamma^p(\beta)}(\Gamma(\frac{p\beta-1}{p-1})\Gamma(\frac{p-p\beta}{p-1}))^{p-1} t^{p\beta-1} l^p(t)$ and $p > 1/\beta$. Therefore, $x(t)$ is a bounded function on $[0,T]$.

Finally, applying fixed point theorem 2.17, we get that there exists a function $x \in C_*^+[0,T]$ such that $Fx = x$. From Theorem 2.13, we know that the function x is also a solution of the fractional differential equation (1.4).

Since T can be chosen arbitrarily large, then we know that the fractional differential equation (1.4) has at least one solution in $C_*^+[0,+\infty)$. Thus, we complete the proof. □

Now, we begin to study the boundedness of solutions of Caputo fractional differential equation (1.4). We first discuss the boundedness of the equation (1.4) when $f(t,u) \le l(t)u^\mu + k(t)$, where $0 \le \mu < 1$.

Theorem 3.2. *Let $x_0 \ge 0$, $0 \le \mu < 1$ and $p > 1/\beta$. Let $f : (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ be continuous. Suppose there are two nonnegative functions $l, k \in C(0,+\infty) \cap L^p_{loc}[0,+\infty)$, and there exists a nonnegative constant K such that $t^\beta l(t) \le K$ and $t^\beta k(t) \le K$ for $t \in [1,+\infty)$. If f satisfies the following condition*

$$f(t,u) \le l(t)u^\mu + k(t) \tag{3.8}$$

for all $t \in (0,+\infty)$ and $u \in [0,+\infty)$. Then the fractional differential equation (1.4) has at least one nonnegative bounded solution on $[0,+\infty)$.

Proof. Let us define the operator F by the formula

$$(Fx)(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds, \tag{3.9}$$

where $x \in C_0^+[0,+\infty)$. Since $t^\beta l(t), t^\beta k(t) \le K$ for $t \in [1,+\infty)$, using Lemma 2.4, then there exist two nonnegative constants M_1, M_2 such that $M_1 = \sup_{0\le t<+\infty} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)ds$ and $M_2 = \sup_{0\le t<+\infty} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k(s)ds$. Let $R > 0$ sufficiently large such that

$$x_0 + M_1 R^\mu + M_2 \le R, \tag{3.10}$$

where $0 \le \mu < 1$. We define the subset U_R of $C_0^+[0,+\infty)$ as follows

$$U_R = \left\{ x : x \in C_0^+[0,+\infty) \text{ and } \|x\| = \sup_{0\le t<+\infty} x(t) \le R \right\}. \tag{3.11}$$

The set U_R is nonempty, convex, closed and bounded in $C_0^+[0,+\infty)$.

We can easily obtain that Fx is a nonnegative continuous function on $[0,+\infty)$ when $x \in C_0^+[0,+\infty)$. If $x \in U_R$, then we have

$$\begin{aligned} (Fx)(t) &\le x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [l(s)x^\mu(s) + k(s)]ds \\ &\le x_0 + M_1 R^\mu + M_2 \\ &\le R. \end{aligned} \tag{3.12}$$

Therefore, $Fx \in U_R$ when $x \in U_R$.

For any $T > 0$, using the same procedure as in the proof of Theorem 3.1, we know that $F : C^+[0,T] \to C^+[0,T]$ is a a continuous and completely continuous map. Therefore, F :

30 Apr 2024 06:12:24 PDT
231209-ZhuTao Version 4 - Submitted to J. Integr. Eq. Appl.

$C_0^+[0,+\infty) \to C_0^+[0,+\infty)$ is a continuous and completely continuous map. Using Theorem 2.16 and Theorem 2.13, we get that the fractional differential equation (1.4) has at least one solution in $C_0^+[0,+\infty)$. $\square$

Next, we will give the boundedness of Caputo fractional differential equation (1.4) for the case $\mu = 1$.

Theorem 3.3. *Let $x_0 \geq 0$, $0 < \beta < \gamma < 1$ and $p > 1/\beta$. Let $f : (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ be continuous. Suppose there are two nonnegative functions $l, k \in C(0,+\infty) \cap L^p_{loc}[0,+\infty)$, and there exists a nonnegative constant K such that $t^\gamma l(t) \leq K$ and $t^\beta k(t) \leq K$ for $t \in [1,+\infty)$. If f satisfies the following condition*

$$f(t,u) \leq l(t)u + k(t) \tag{3.13}$$

for all $t \in (0,+\infty)$ and $u \in [0,+\infty)$. Then the fractional differential equation (1.4) has at least one nonnegative bounded solution on $[0,+\infty)$.

Proof. If $x \in C_0^+[0,+\infty)$ and let $R = \sup_{0 \leq t < +\infty} x(t)$, then we have

$$\begin{aligned}(Fx)(t) &\leq x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[l(s)x(s) + k(s)]ds \\ &\leq x_0 + \frac{R}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}l(s)ds + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}k(s)ds \\ &\leq x_0 + M_1R + M_2.\end{aligned} \tag{3.14}$$

where M_1, M_2 are as defined in Theorem 3.2. Then we get $Fx \in C_0^+[0,+\infty)$ when $x \in C_0^+[0,+\infty)$. Similar to the proof of the Theorem 3.2, we can get that $F : C_0^+[0,+\infty) \to C_0^+[0,+\infty)$ is a continuous and completely continuous map.

If $x \in C_0^+[0,+\infty)$ is any solution of

$$x(t) = \lambda\left(x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,x(s))ds\right), \tag{3.15}$$

where $0 < \lambda < 1$, then we have

$$\begin{aligned}x(t) &\leq x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,x(s))ds \\ &\leq x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}l(s)x(s)ds + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}k(s)ds \\ &\leq x_0 + M_2 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}l(s)x(s)ds.\end{aligned} \tag{3.16}$$

Using Theorem 2.15, then we have

$$x(t) \leq 2(x_0 + M_2)\exp\left(\int_0^t \frac{L(s)}{p}ds\right),$$

where $L(t) = \frac{2^{p-1}}{\Gamma^p(\beta)}\left(\Gamma(\frac{p\beta-1}{p-1})\Gamma(\frac{p-p\beta}{p-1})\right)^{p-1} t^{p\beta-1}l^p(t)$ and $p > 1/\beta$. Since $0 < \beta < \gamma < 1$ and $t^\gamma l(t) \leq K$ for $t \in [1,+\infty)$, then we get that $t^{p\beta-1}l^p(t) \leq K^p t^{p\beta-p\gamma-1}$ for $t \in [1,+\infty)$, and the integral $\int_1^{+\infty} s^{p\beta-1}l^p(s)ds$ is convergent. Consequently, we obtain that $\int_0^{+\infty} s^{p\beta-1}l^p(s)ds$ is convergent. Therefore, we get that $\int_0^{+\infty} L(s)ds$ is finite, and

$$x(t) \le 2(x_0 + M_2)\exp\left(\int_0^{+\infty} \frac{L(s)}{p}ds\right), \qquad t \in [0, +\infty).$$

This proves that any solution of the equation (3.15) is a bounded function on $[0, +\infty)$.

Using Theorem 2.17 and Theorem 2.13, we get that the fractional differential equation (1.4) has at least one solution in $C_0^+[0, +\infty)$. □

From Theorem 3.1 and Theorem 3.2, we can immediately obtain the following result.

Theorem 3.4. *Let $x_0 \ge 0$, $0 \le \mu < 1$ and $p > 1/\beta$. Let $f : (0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be continuous, if $g(t) = t^\beta f(t, x(t))$ is a nondecreasing function on $(0, +\infty)$ when $x(t)$ is a nonnegative, nondecreasing continuous function on $[0, +\infty)$. Suppose there are two nonnegative functions $l, k \in C(0, +\infty) \cap L^p_{loc}[0, +\infty)$, and there exists a nonnegative constant K such that $t^\beta l(t) \le K$ and $t^\beta k(t) \le K$ for $t \in [1, +\infty)$. If f satisfies the following condition*

$$f(t, u) \le l(t)u^\mu + k(t) \tag{3.17}$$

for all $t \in (0, +\infty)$ and $u \in [0, +\infty)$. Then the fractional differential equation (1.4) has at least one nonnegative, bounded and nondecreasing continuous solution on $[0, +\infty)$.

4. ASYMPTOTIC BEHAVIOR OF FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, we first study the asymptotic behavior of the fractional differential equation (1.5).

Theorem 4.1. *Let $x_0 \ge 0$, $0 \le \mu < 1$ and $p > 1/\beta$. If nonnegative functions $l, k \in C(0, +\infty) \cap L^p_{loc}[0, +\infty)$ and $t^\beta l(t), t^\beta k(t)$ are two nondecreasing functions on $(0, +\infty)$, and suppose that $\lim_{t\to+\infty} t^\beta l(t) = a$ and $\lim_{t\to+\infty} t^\beta k(t) = b$. Let nonnegative and nondecreasing function $\varphi \in C[0, +\infty)$ and $\varphi(x) \le Mx^\mu$ for $x \in [0, +\infty)$, where M is a nonnegative constant. Then the fractional differential equation (1.5) has at least one nondecreasing solution $x \in C_0^+[0, +\infty)$ and $\lim_{t\to+\infty} x(t) = x^*$, where x^* is the minimal nonnegative solution that satisfies the following equation*

$$x = x_0 + \frac{\pi}{\Gamma(\beta)\sin(\beta\pi)}(a\varphi(x) + b). \tag{4.1}$$

Proof. Let

$$U^* = \left\{x : x \in C_*^+[0, +\infty) \text{ and } \|x\| = \sup_{0 \le t < +\infty} x(t) \le x^*\right\}. \tag{4.2}$$

We can easily get that Fx is a nonnegative nondecreasing continuous function on $[0, +\infty)$ when $x \in C_*^+[0, +\infty)$. If $x \in U^*$, since φ is a nondecreasing function on $[0, +\infty)$ and $t^\beta l(t), t^\beta k(t)$

are two nondecreasing functions on $(0,+\infty)$, then we have

$$
\begin{aligned}
(Fx)(t) &\leq x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[l(s)\varphi(x^*)+k(s)]ds \\
&\leq x_0 + \frac{\varphi(x^*)}{\Gamma(\beta)}\lim_{t\to+\infty}\int_0^t (t-s)^{\beta-1}l(s)ds + \frac{1}{\Gamma(\beta)}\lim_{t\to+\infty}\int_0^t (t-s)^{\beta-1}k(s)ds \\
&= x_0 + \frac{a\pi\varphi(x^*)}{\Gamma(\beta)\sin(\beta\pi)} + \frac{b\pi}{\Gamma(\beta)\sin(\beta\pi)} \\
&= x^*.
\end{aligned}
\tag{4.3}
$$

Therefore, $Fx \in U^*$ when $x \in U^*$.

The rest of the proof is similar to that of Theorem 3.2. We know that the fractional differential equation (1.5) has at least one solution $x \in U^*$ that satisfies the following fractional integral equation

$$
x(t) = x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[l(s)\varphi(x(s))+k(s)]ds, \qquad t \in [0,+\infty). \tag{4.4}
$$

Since $x(t)$ is a nonnegative, nondecreasing and bounded function on $[0,+\infty)$, we know that $\lim_{t\to+\infty} x(t)$ exists. Using Lemma 2.2, we get

$$
\begin{aligned}
\lim_{t\to+\infty} x(t) &= \lim_{t\to+\infty}\left[x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[l(s)\varphi(x(s))+k(s)]ds\right] \\
&= x_0 + \frac{1}{\Gamma(\beta)}\left[\lim_{t\to+\infty}\int_0^t (t-s)^{\beta-1}l(s)\varphi(x(s))ds + \lim_{t\to+\infty}\int_0^t (t-s)^{\beta-1}k(s)ds\right] \\
&= x_0 + \frac{1}{\Gamma(\beta)}\left[\frac{a\pi}{\sin(\beta\pi)}\phi\left(\lim_{t\to+\infty} x(t)\right) + \frac{b\pi}{\sin(\beta\pi)}\right].
\end{aligned}
\tag{4.5}
$$

From (4.1) and $\lim_{t\to+\infty} x(t) \leq x^*$, then we get $\lim_{t\to+\infty} x(t) = x^*$. Thus, we complete the proof. □

Remark 4.2. In Theorem 4.1, we know that the equation (4.1) has at least one nonnegative solution. In fact, let

$$
g(x) = x - x_0 - \frac{\pi}{\Gamma(\beta)\sin(\beta\pi)}(a\varphi(x)+b). \tag{4.6}
$$

We have

$$
g(0) = -x_0 - \frac{\pi}{\Gamma(\beta)\sin(\beta\pi)}(a\varphi(0)+b) \leq 0 \tag{4.7}
$$

and

$$
g(x) \geq x - x_0 - \frac{\pi}{\Gamma(\beta)\sin(\beta\pi)}(aMx^\mu+b). \tag{4.8}
$$

From (4.8), we get $\lim_{x\to+\infty} g(x) = +\infty$ for $0 \leq \mu < 1$. Therefore, there has at least one nonnegative solution x such that $g(x) = 0$, that is, x is a nonnegative solution of the equation (4.1).

If the equation (4.1) has a unique nonnegative solution x^*, from Theorem (4.1), then we can prove that the fractional differential equation (1.5) has at least one nondecreasing solution and $\lim_{t\to+\infty} x(t) = x^*$.

Now, we begin to study the asymptotic behavior of the fractional differential equation (1.4).

Theorem 4.3. *Let $x_0 \geq 0$, $0 < \beta < \gamma < 1$ and $p > 1/\beta$. Let $f : (0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be continuous. Let l, k and $k_1 \in C(0, +\infty) \cap L^p_{loc}[0, +\infty)$ be nonnegative functions. Suppose there exists a nonnegative constant K such that $t^\gamma l(t) \leq K$ for $t \in [1, +\infty)$, and there is a nonnegative constant b such that $\lim_{t\to+\infty} t^\beta k(t) = b$ and $\lim_{t\to+\infty} t^\beta k_1(t) = b$. If f satisfies the following condition*

$$k_1(t) \leq f(t, u) \leq l(t)u + k(t) \tag{4.9}$$

for all $t \in (0, +\infty)$ and $u \in [0, +\infty)$. Then the fractional differential equation (1.4) has at least one bounded solution $x \in C_0^+[0, +\infty)$ and $\lim_{t\to+\infty} x(t) = x_0 + \frac{b\pi}{\Gamma(\beta)\sin(\beta\pi)}$.

Proof. From Theorem 3.3, we know that the fractional differential equation (1.4) has at least one bounded solution $x \in C_0^+[0, +\infty)$ that satisfies the following fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds, \qquad t \in [0, +\infty). \tag{4.10}$$

Let $R = \sup_{0\leq t<+\infty} x(t)$, then we get

$$x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k_1(s) ds \leq x(t) \leq x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [Rl(s) + k(s)] ds. \tag{4.11}$$

Since $\lim_{t\to+\infty} t^\beta l(t) = 0$, $\lim_{t\to+\infty} t^\beta k(t) = b$ and $\lim_{t\to+\infty} t^\beta k_1(t) = b$, and using Lemma 2.2, we get

$$\lim_{t\to+\infty} \left[x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k_1(s) ds \right] = x_0 + \frac{b\pi}{\Gamma(\beta)\sin(\beta\pi)} \tag{4.12}$$

and

$$\lim_{t\to+\infty} \left[x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [Rl(s) + k(s)] ds \right] = x_0 + \frac{b\pi}{\Gamma(\beta)\sin(\beta\pi)}. \tag{4.13}$$

From (4.12) and (4.13), we get $\lim_{t\to+\infty} x(t) = x_0 + \frac{b\pi}{\Gamma(\beta)\sin(\beta\pi)}$. Thus, we complete the proof. □

Theorem 4.4. *Let $x_0 \geq 0$, $0 < \beta < \gamma < 1$ and $p > 1/\beta$. Let $f : (0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be continuous. Suppose there are two nonnegative functions $l, k \in C(0, +\infty) \cap L^p_{loc}[0, +\infty)$, and there exists a nonnegative constant K such that $t^\gamma l(t) \leq K$ and $t^\gamma k(t) \leq K$ for $t \in [1, +\infty)$. If f satisfies the following condition*

$$f(t, u) \leq l(t)u + k(t) \tag{4.14}$$

for all $t \in (0, +\infty)$ and $u \in [0, +\infty)$. Then the fractional differential equation (1.4) has at least one nonnegative bounded solution on $[0, +\infty)$ and $\lim_{t\to+\infty} x(t) = x_0$.

Proof. From Theorem 3.3, we know that the fractional differential equation (1.4) has at least one nonnegative bounded solution $x \in C_0^+[0, +\infty)$ that satisfies the following fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds, \qquad t \in [0, +\infty). \tag{4.15}$$

Let $R = \sup_{0 \le t < +\infty} x(t)$, then we get

$$x_0 \le x(t) \le x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [Rl(s) + k(s)] ds. \tag{4.16}$$

Since $\lim_{t\to+\infty} t^\beta l(t) = \lim_{t\to+\infty} t^\beta k(t) = 0$, using Lemma 2.1, we get

$$\lim_{t\to+\infty} \left[x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [Rl(s) + k(s)] ds \right] = x_0. \tag{4.17}$$

From (4.16) and (4.17), we get $\lim_{t\to+\infty} x(t) = x_0$. Thus, we complete the proof. □

5. Examples

Example 5.1. *Consider the following Caputo fractional differential equation*

$$\begin{cases} D_C^{1/2} x(t) = \frac{x(t)}{1+\sqrt{t}}, & t \in (0, +\infty), \\ x(0) = 1. \end{cases} \tag{5.1}$$

From Theorem 3.1 and using the same method in the proof of the Theorem 3.2 in [26], we get that the fractional differential equation (5.1) has a unique nonnegative nondecreasing continuous solution $x(t)$ on $[0, +\infty)$, and $x(t)$ satisfies the following Volterra fractional integral equation

$$x(t) = 1 + \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} \frac{x(s)}{1+\sqrt{s}} ds. \tag{5.2}$$

Suppose that $x(t)$ is a bounded function on $[0, +\infty)$. Since $x(t)$ is a nondecreasing continuous solution on $[0, +\infty)$, then $\lim_{t\to+\infty} x(t)$ exists. Hence, using Lemma 2.2, we obtain

$$\begin{aligned} \lim_{t\to+\infty} x(t) &= \lim_{t\to+\infty} \left[1 + \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} \frac{x(s)}{1+\sqrt{s}} ds \right] \\ &= 1 + \sqrt{\pi} \lim_{t\to+\infty} x(t) \end{aligned} \tag{5.3}$$

and $\lim_{t\to+\infty} x(t) = \frac{1}{1-\sqrt{\pi}}$. This contradicts the fact that $\lim_{t\to+\infty} x(t) \ge 0$. Hence, the fractional differential equation (5.1) has a unique nondecreasing unbounded solution on $[0, +\infty)$.

Example 5.2. *Consider the following Caputo fractional differential equation*

$$\begin{cases} D_C^{1/2} x(t) = \frac{\sqrt{x(t)}}{1+\sqrt{t}}, & t \in (0, +\infty), \\ x(0) = 1. \end{cases} \tag{5.4}$$

Using Theorem 4.1, we get that the fractional differential equation (5.4) has at least one nondecreasing solution $x \in C_0^+[0, +\infty)$ and $\lim_{t\to+\infty} x(t) = (\frac{\sqrt{\pi}+\sqrt{\pi+4}}{2})^2$.

Example 5.3. *Consider the following Caputo fractional differential equation*

$$\begin{cases} D_C^{1/2} x(t) = \frac{\sqrt{x(t)+t}}{1+t}, & t \in (0, +\infty), \\ x(0) = 1. \end{cases} \tag{5.5}$$

We know

$$\frac{\sqrt{t}}{1+t} \le \frac{\sqrt{x+t}}{1+t} \le \frac{\sqrt{x}+\sqrt{t}}{1+t} \le \frac{x+\sqrt{t}+1}{1+t}$$

for all $t \in (0,+\infty)$ and $x \in [0,+\infty)$. Using Theorem 4.3, then we obtain that the fractional differential equation (5.5) has at least one bounded solution $x \in C_0^+[0,+\infty)$ and $\lim_{t\to+\infty} x(t) = 1+\sqrt{\pi}$.

Example 5.4. *Consider the following Caputo fractional differential equation*

$$\begin{cases} D_C^{1/2} x(t) = \frac{\ln(1+tx(t))}{1+t^2}, & t \in (0,+\infty), \\ x(0) = 1. \end{cases} \tag{5.6}$$

Since

$$\frac{\ln(1+tx)}{1+t^2} \le \frac{tx}{1+t^2}$$

for all $t \in [0,+\infty)$ and $x \in [0,+\infty)$, and by Theorem 4.4, then we obtain that the fractional differential equation (5.6) has at least one bounded solution $x \in C_0^+[0,+\infty)$ and $\lim_{t\to+\infty} x(t) = 1$.

Acknowledgement

I thank the referee for pointing out some useful references and for valuable comments which led to improvements of the paper.

References

[1] Agarwal, R.P., Meehan, M., O'Regan, D.: Fixed point theory and applications. Cambridge University Press, Cambridge (2001)

[2] Agarwal, R.P., O'Regan, D., Staněk, S.: Positive solutions for mixed problems of singular fractional differential equations. Math. Nachr. 285, 27-41 (2012)

[3] Ahmad, A.M., Furati, K.M., Tatar, N.E.: Asymptotic behavior of solutions for a class of fractional integro-differential equations. Mediterr. J. Math. 15, 188 (2018)

[4] Becker, L.C., Burton, T.A., Purnaras, I.K.: Complementary equations: a fractional differential equation and a Volterra integral equation. Electron. J. Qual. Theory. Differ. Equ. 12, 1-24 (2015)

[5] Choi, S., Koo, N.: The monotonic property and stability of solutions of fractional differential equations. Nonlinear. Anal. 74, 6530-6536 (2011)

[6] Cong, N.D., Tuan, H.T., Trinh, H.: On asymptotic properties of solutions to fractional differential equations. J. Math. Anal. Appl. 484(2), 123759 (2020)

[7] Delbosco, D., Rodino, L.: Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl. 204, 609-625 (1996)

[8] Diethelm, K.: The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type. Lecture Notes in Mathematics, Vol.2004, Springer-Verlag, Berlin (2010)

[9] Diethelm, K., Siegmund, S., Tuan, H.T.: Asymptotic behavior of solutions of linear multi-order fractional differential systems. Fract. Calc. Appl. Anal. 20, 1165-1195 (2017)

[10] Grace, S.R., Zafer, A.: On the asymptotic behavior of nonoscillatory solutions of certain fractional differential equations. Eur. Phys. J. Spec. Top. 226, 3657-3665 (2017)

[11] Goodrich, C.S., Lizama, C.: Positivity, monotonicity, and convexity for convolution operators, Discrete Contin. Dyn. Syst. Series A. 40, 4961-4983 (2020)

[12] Goodrich, C.S., Lizama, C.: Existence and monotonicity of nonlocal boundary value problems: the one-dimensional case, Proc. Roy. Soc. Edinburgh Sect. A. 152, 1-27 (2022)

[13] Kassim, M.D., Tatar, N.E.: Asymptotic behavior of solutions of fractional differential equations with Hadamard fractional derivatives. Fract. Calc. Appl. Anal. 24(2), 483-508 (2021)

[14] Kilbas, A.A., Marzan, S.A.: Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiabl functions. Diff. Equ. 41(1), 84-89 (2005)

[15] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. In: North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam (2006)

[16] Ky, D.G., Thinh, L.V., Tuan, H.T.: Existence, uniqueness and asymptotic behavior of solutions to two-term fractional differential equations. Commun. Nonlinear. Sci. Numer. Simul. 115, 106751 (2022)
[17] Lan, K.Q.: Linear first order Riemann-Liouville fractional differential and perturbed Abel's integral equations. J. Differ. Equ. 306, 28-59 (2022)
[18] Lan, K.Q., Webb, J.R.L.: A new Bihari inequality and initial value problems of first order fractional differential equations. Fract. Calc. Appl. Anal. 26(3), 962-988 (2023)
[19] Medved, M.: On the asymptotic behavior of solutions of nonlinear differential equations of integer and also of non-integer order. Electron. J. Qual. Theory. Differ. Equ. Proc. 9th Coll. QTDE. 10, 1-9 (2012)
[20] Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications. Academic Press, San Diego (1999)
[21] Ponce, R.: Asymptotic behavior of mild solutions to fractional Cauchy problems in Banach spaces. Appl. Math. Lett. 105, 106322 (2020)
[22] Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Yverdon (1993)
[23] Wang, D., Xiao, A., Sun, S.: Asymptotic behavior of solutions to time fractional neutral functional differential equations. J. Comput. Appl. Math. 382, 113086 (2021)
[24] Webb, J.R.L.: Weakly singular Gronwall inequalities and applications to fractional differential equations. J. Math. Anal. Appl. 471, 692-711 (2019)
[25] Webb, J.R.L.: Initial value problems for Caputo fractional equations with singular nonlinearities. Electron. J. Differ. Equ. 117, 1-32 (2019)
[26] Zhu, T.: New Henry-Grownall integral inequalities and their applications to fractional differential equations. Bull. Braz. Math. Soc. 49(3), 647-657 (2018)
[27] Zhu, T.: Attractivity of solutions of Riemann-Liouville fractional differential equations. Electron. J. Qual. Theory. Differ. Equ. 52, 1-12 (2022)
[28] Zhu, T.: Global attractivity for fractional differential equations of Riemann-Liouville type. Fract. Calc. Appl. Anal. 26(5), 2264-2280 (2023)
[29] Zhu, T.: Monotonicity and asymptotic behavior of solutions for Riemann-Liouville fractional differential equations. https://doi.org/10.48550/arXiv.2308.15036