

BOUNDEDNESS OF HARDY TYPE OPERATORS ON Q TYPE SPACES ASSOCIATED WITH WEIGHTS

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ABSTRACT. In this paper, we investigate the boundedness of weight Hardy operators U_ψ and the corresponding Cesàro type average operators V_ψ on Q type spaces $\mathcal{Q}_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$. Moreover, we provide explicit norms for both U_ψ and V_ψ on $\mathcal{Q}_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$ under the assumption of the integrability of ψ .

1. Introduction

Let $f \in L^1(\mathbb{R})$. The classical Hardy operator is defined as

$$(1) \quad Uf(x) = \frac{1}{x} \int_0^x f(y) dy, \quad x \neq 0.$$

The adjoint operator of U is the classical Cesàro average operator:

$$Vf(x) := \begin{cases} \int_x^\infty f(y) \frac{dy}{y}, & x > 0; \\ -\int_{-\infty}^x f(y) \frac{dy}{y}, & x < 0. \end{cases}$$

In addition, $U + V$ becomes the Calderón maximal operator:

$$(U + V)f(x) = \frac{1}{x} \int_0^x f(y) dy + \int_x^\infty \frac{f(y)}{y} dy, \quad x > 0,$$

see Bennett, Devore and Sharpley [4]. It is obvious that the operator U can be dominated by the Hardy-Littlewood maximal function: $|Uf| \leq M(f)$, and the famous Hardy integral inequality holds: for $1 < p < \infty$,

$$\|Uf\|_{L^p(\mathbb{R})} \leq \frac{p}{p-1} \|f\|_{L^p(\mathbb{R})},$$

where the constant $p/(p-1)$ is the best possible, see Hardy, Littlewood and Pólya [16]. In many branches of analysis such as approximation theory, differential equations, the theory of function spaces, etc., the Hardy integral inequality and its variants have played an important role. Compared with the Hardy-Littlewood maximal function, the study of the Hardy operator and its generalizations may not be as delicate as that of maximal operators, but still requires the use of certain beautiful and

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elegant ideas. In the last decades, the boundedness of the Hardy operator and related topics have attracted the attention of many mathematicians. We refer the reader to Andersen and Muckenhoupt [1], Edmunds and Evans [10], Giang and Móricz [14], Golubov [15], Long and Wang [18], Móricz [19] and the references therein for further information.

In [5], Carton-Lebrun and Fosset introduced a class of integral operators, called weighted Hardy type operators, as a generalization of U . Given a nonnegative function $\psi : [0, 1] \rightarrow [0, \infty)$. The weighted Hardy type operator U_ψ is defined as

$$U_\psi f(x) := \int_0^1 f(tx) \psi(t) dt, \quad x \in \mathbb{R}^n.$$

Accordingly, as the adjoint operator of U_ψ , the weighted Cesáro average operator of V_ψ is defined by

$$V_\psi f(x) := \int_0^1 f\left(\frac{x}{t}\right) t^{-n} \psi(t) dt, \quad x \in \mathbb{R}^n.$$

The weighted Hardy type operator U_ψ and the weighted Cesáro average operator V_ψ are adjoint mutually:

$$\int_{\mathbb{R}^n} g(x) U_\psi f(x) dx = \int_{\mathbb{R}^n} f(x) V_\psi g(x) dx,$$

where $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ with $1 < p, q < \infty$ and $1/p + 1/q = 1$. In [5], Carton-Lebrun and Fosset proved that if $t^{1-n}\psi(t)$ is bounded on $[0, 1]$ then U_ψ is bounded on $BMO(\mathbb{R}^n)$. In [25], Xiao determined the corresponding operator norms of U_ψ and V_ψ , respectively, which sharpens and extends the main result of [5]. For further progress on this topic, we refer to Chu, Fu and Wu [7], Fu, Liu and Lu [12], Fu and Lu [13], Tang and Zhai [21], Tran [22] and the references therein.

The main purpose of this paper is to investigate the boundedness of Hardy type operators on a class of Q type spaces related with weights. Initially, the Q type spaces of analytic functions, denoted by $Q_p(\mathbb{D})$, were introduced as the extensions of the holomorphic BMO type space $BMOA(\mathbb{D})$ on the unit disk \mathbb{D} , see Aulaskari, Xiao and Zhao [2]. In 2001, Essén, Janson, Peng and Xiao introduced the space $Q_\alpha(\mathbb{R}^n)$ in [11] as a generalization of $Q_p(\mathbb{D})$ in high-dimensional Euclidean settings. From the view of the theory of function spaces, $Q_\alpha(\mathbb{R}^n)$ can be seen as a class of differential function spaces between the Sobolev spaces $W^{2,s}(\mathbb{R}^n)$ and the bounded mean oscillation space $BMO(\mathbb{R}^n)$. In recent decades, Q type spaces and their generalizations have been extensively studied. As a class of differential function spaces, Q type spaces have been extensively applied to the study of harmonic analysis, differential equations and potential theory, etc. For more information, we refer to Chen, Li and Lou [6], Dafni and Xiao [9], Wu and Xie [24], Xiao [27], Li and Zhai [17] and the references therein.

Let $\mathcal{K} : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing function. The Q type spaces related with the weight function $\mathcal{K}(\cdot)$ is defined as follows.

Definition 1.1. Let $1 \leq p, q < \infty$, $\lambda \geq 0$, and $n \geq 1$. The space $Q_{\mathcal{K}, \lambda}^{p, q}(\mathbb{R}^n)$ is defined as the set of all measurable functions satisfying

$$\|f\|_{Q_{\mathcal{K}, \lambda}^{p, q}(\mathbb{R}^n)}^q := \sup_{I \subset \mathbb{R}^n} (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^p dx \right)^{q/p} \mathcal{K}\left(\frac{|y|}{\ell(I)}\right) \frac{dy}{|y|^{qn}} < \infty,$$

where the symbol \sup_I denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n .

The space $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ covers many classical Q type spaces. Below we list several examples.

Example 1.1. ([6, Definition 1.1]) Let $1 \leq q < \infty$, $n \geq 1$. The space $Q_{\mathcal{H}}^q(\mathbb{R}^n)$ is defined as the set of all measurable functions satisfying

$$\|f\|_{Q_{\mathcal{H}}^q(\mathbb{R}^n)}^q := \sup_{I \subset \mathbb{R}^n} \int_I \int_I \frac{|f(x) - f(y)|^q}{|x - y|^{qn}} \mathcal{H}\left(\frac{|x - y|}{\ell(I)}\right) dx dy < \infty,$$

where the symbol \sup_I denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n .

Example 1.2. ([8, Definition 1.1]) Let $1 \leq q < \infty$, $\lambda \geq 0$, and $n \geq 1$. The space $Q_{\mathcal{H},\lambda}^q(\mathbb{R}^n)$ is defined as the set of all measurable functions satisfying

$$\|f\|_{Q_{\mathcal{H},\lambda}^q(\mathbb{R}^n)}^q := \sup_{I \subset \mathbb{R}^n} (\ell(I))^{-\lambda n} \int_I \int_I \frac{|f(x) - f(y)|^q}{|x - y|^{qn}} \mathcal{H}\left(\frac{|x - y|}{\ell(I)}\right) dx dy < \infty,$$

where the symbol \sup_I denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n .

Example 1.3. ([28, Definition 2.6]) Let $1 \leq q \leq p < \infty$ and $0 \leq \alpha < \min\{1, n/q\}$. The Besov- Q space $Q_{p,q}^\alpha(\mathbb{R}^n)$ is defined to be the set of all functions satisfying

$$\|f\|_{Q_{p,q}^\alpha(\mathbb{R}^n)}^q := \sup_{I \subset \mathbb{R}^n} (\ell(I))^{(q-p)n/p} \int_I \int_I \frac{|f(x) - f(y)|^q}{|x - y|^{n+q\alpha}} dx dy < \infty,$$

where the symbol \sup_I denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n . If $\lambda = q\alpha/n - q - q/p + 2$ and $\mathcal{H}(t) = t^{qn-n-q\alpha}$, then $Q_{\mathcal{H},\lambda}^{q,q}(\mathbb{R}^n) = Q_{p,q}^\alpha(\mathbb{R}^n)$. Specially, the spaces $Q_{\mathcal{H},\lambda}^{2,2}(\mathbb{R}^n)$ have been introduced by Bao and Wulan in [3]. In particular, with special values for p , q and α , we can obtain that $Q_{n/\alpha,2}^\alpha(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n)$, where $Q_\alpha(\mathbb{R}^n)$ was introduced by Essén, Janson, Peng, and Xiao in [11]. If $\mathcal{H}(t) = t^{n-2\alpha}$, then $Q_{\mathcal{H},0}^{2,2}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n)$. Additionally, we can also obtain that $Q_{n/(\alpha+\beta-1),2}^{\alpha-\beta+1}(\mathbb{R}^n) = Q_\alpha^\beta(\mathbb{R}^n)$, where $Q_\alpha^\beta(\mathbb{R}^n)$ was introduced by Li and Zhai in [17]. If $\lambda = (4 - 4\beta)/n$ and $\mathcal{H}(t) = t^{n-2(\alpha-\beta+1)}$, then $Q_{\mathcal{H},\lambda}^{2,2}(\mathbb{R}^n) = Q_\alpha^\beta(\mathbb{R}^n)$.

Inspired by the works of [25] and [21], it leads to two purposes of this paper naturally. The first one is to classify the condition of ψ such that the operators U_ψ and V_ψ are bounded on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. The second is to determine the corresponding operator norms.

In Section 2, we investigate various properties of $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. For instance, the spaces $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ are invariant under affine transformations. In addition, we provide the proof of the completeness of $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ and the non-trivial conditions. At last, the inclusion relations between $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ and several classical spaces are discussed.

In Section 3, as the main results of this paper, we prove that if the function ψ satisfies

$$\int_0^1 t^{n+\lambda n/q-n/p-n/q} \psi(t) dt < \infty,$$

the weighted Hardy operator U_ψ is bounded on the space $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. Also, let

$$\int_0^1 t^{n/p+n/q-2n-\lambda n/q} \psi(t) dt < \infty.$$

The adjoint operator V_ψ is bounded on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. Moreover, we obtain explicit norms for U_ψ and V_ψ on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ under specific conditions, see Theorems 3.1 and 3.2. We point out that these boundedness results generalize the related results in [25, 21], and are new for the special cases $Q_{\mathcal{H}}(\mathbb{R}^n)$ (cf. [3]) and $Q_{\mathcal{H},\lambda}^q(\mathbb{R}^n)$ (cf. [6]). Section 4 is devoted to several special examples of weight functions $\mathcal{H}(\cdot)$ satisfying the conditions listed in Section 3, including logarithmic functions and sine functions.

Some notations:

- (i) Let \mathbb{R}^n be the n -dimensional Euclidean space with the Euclidean norm $|x|$ and the Lebesgue measure dx . The symbol I denotes a cube in \mathbb{R}^n with the edges parallel to the coordinate axes. The sidelength of I is $\ell(I)$ and the volume is denoted by $|I|$. Denote by $tI, t > 0$, the cube with the same center as I and sidelength $t\ell(I)$.
- (ii) $U \approx V$ shows that there is a constant $c > 0$ such that $c^{-1}V \leq U \leq cV$. If $U \leq cV$, then we write $U \lesssim V$. Similarly, we write $V \gtrsim U$ if $V \geq cU$.
- (iii) In this paper, we assume that the weight function $\mathcal{H}(\cdot) : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing. Throughout the rest of this paper, we assume that $\mathcal{H}(t) \approx \mathcal{H}(2t)$, $t \in [0, \infty)$.

2. Some basic properties of $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$

The invariance of $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ under affine transformations is as follows.

Proposition 2.1. *Let $1 \leq p, q < \infty$ and $\lambda \geq 0$.*

- (i) $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ is rotation invariant;
- (ii) $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ is translation invariant;
- (iii) If $f_t(x) = t^{(1/p+1/q-1-\lambda/q)n} f(tx)$, $t > 0$, then $\|f_t\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)} = \|f\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)}$.

Proof. (i) From Definition 1.1, it is easy to see that if we replace the cube I with center x_1 and the side length $\ell(I)$ by the ball $B(x_1, \sqrt{n}\ell(I))$, the space newly obtained is the same as $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and any orthogonal matrix \mathcal{A} of order n , we have $|x| = |x\mathcal{A}|$. Set $\omega(x) = x\mathcal{A}$. For any ball $B \subset \mathbb{R}^n$ with radius $r(B)$, we obtain $\|f \circ \omega\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)} = \|f\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)}$ since

$$\begin{aligned} & \sup_{B \subset \mathbb{R}^n} (r(B))^{-\lambda n} \int_{|y| < 2r(B)} \left(\int_B |f(x\mathcal{A} + y\mathcal{A}) - f(x\mathcal{A})|^p dx \right)^{q/p} \mathcal{H}\left(\frac{|y\mathcal{A}|}{r(B)}\right) \frac{dy}{|y\mathcal{A}|^{qn}} \\ &= \sup_{B \subset \mathbb{R}^n} (r(B))^{-\lambda n} \int_{\{\xi | \xi = y\mathcal{A}, |y| < 2r(B)\}} \left(\int_{\{\eta | \eta = x\mathcal{A}, x \in B\}} |f(\eta + \xi) - f(\eta)|^p d\eta \right)^{q/p} \mathcal{H}\left(\frac{|\xi|}{r(B)}\right) \frac{d\xi}{|\xi|^{qn}} \\ &= \sup_{B \subset \mathbb{R}^n} (r(B))^{-\lambda n} \int_{|\xi| < 2r(B)} \left(\int_B |f(\eta + \xi) - f(\eta)|^p d\eta \right)^{q/p} \mathcal{H}\left(\frac{|\xi|}{r(B)}\right) \frac{d\xi}{|\xi|^{qn}}. \end{aligned}$$

(ii) For any $f \in Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$, we have

$$\begin{aligned} & \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y+2x_0) - f(x+x_0)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y+x_0|}{\ell(I)} \right) \frac{dy}{|y+x_0|^{qn}} \\ &= \int_{B(x_0, \sqrt{n}\ell(I))} \left(\int_{I+x_0} |f(\eta+\xi) - f(\eta)|^p d\eta \right)^{q/p} \mathcal{H} \left(\frac{|\xi|}{\ell(I)} \right) \frac{d\xi}{|\xi|^{qn}}, \end{aligned}$$

hence $\|f(\cdot+x_0)\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)} = \|f(\cdot)\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)}$.

(iii) Assume that I is a cube in \mathbb{R}^n and let $f \in Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. By a change of variables, we can get

$$\begin{aligned} & (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f_t(x+y) - f_t(x)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\ &= (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I t^{(1+p/q-p-\lambda p/q)n} |f(tx+ty) - f(tx)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\ &= (\ell(tI))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(tx+ty) - f(tx)|^p dt dx \right)^{q/p} \mathcal{H} \left(\frac{|ty|}{\ell(tI)} \right) \frac{d(ty)}{|ty|^{qn}} \\ &= (\ell(tI))^{-\lambda n} \int_{|\xi| < \sqrt{n}\ell(tI)} \left(\int_{tI} |f(\eta+\xi) - f(\eta)|^p d\eta \right)^{q/p} \mathcal{H} \left(\frac{|\xi|}{\ell(tI)} \right) \frac{d\xi}{|\xi|^{qn}}, \end{aligned}$$

which means that $\|f_t\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)}^q = \|f\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)}^q$. □

Theorem 2.2. Let $1 \leq p, q < \infty$, $a > 0$, and $\lambda \geq \frac{aq}{(p+a)p}$. Then

$$Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n) \supseteq Q_{\mathcal{H},\lambda-\frac{aq}{(p+a)p}}^{p+a,q}(\mathbb{R}^n).$$

Proof. If $f \in Q_{\mathcal{H},\lambda-\frac{aq}{(p+a)p}}^{p+a,q}(\mathbb{R}^n)$, then for any cube I in \mathbb{R}^n , by applying Hölder's inequality, we have

$$\begin{aligned} & (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\ & \leq (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\left(\int_I |f(x+y) - f(x)|^{p+a} dx \right)^{p/(p+a)} \left(\int_I 1 dx \right)^{a/(p+a)} \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\ & = (\ell(I))^{\frac{aqn}{(p+a)p}} (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^{p+a} dx \right)^{q/(p+a)} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\ & \leq \|f\|_{Q_{\mathcal{H},\lambda-\frac{aq}{(p+a)p}}^{p+a,q}(\mathbb{R}^n)}^q, \end{aligned}$$

which gives $f \in Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. So $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n) \supseteq Q_{\mathcal{H},\lambda-\frac{aq}{(p+a)p}}^{p+a,q}(\mathbb{R}^n)$. □

Let I be a cube centered at x_I with side length $\ell(I)$. Denote by tI , $t > 0$, the cube with the center x_I and side length $t\ell(I)$. Because of the translation invariance of $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ and the assumption that

$\mathcal{H}(t) \approx \mathcal{H}(2t)$, the following equivalent definition of $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ can be obtained immediately via change of variables.

Proposition 2.3. *Let $1 \leq p, q < \infty$ and $\lambda \geq 0$. Then $f \in Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ if and only if*

$$(2) \quad \sup_{I \subset \mathbb{R}^n} (\ell(I))^{-\lambda n} \int_I \left(\int_I \frac{|f(x) - f(y)|^p}{|x - y|^{pn}} \mathcal{H}^{p/q} \left(\frac{|x - y|}{\ell(I)} \right) dx \right)^{q/p} dy < \infty,$$

where the symbol \sup_I denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n .

Next, we introduce the space $\mathcal{L}_{q,\gamma}(\mathbb{R}^n)$.

Definition 2.1. *Let $1 \leq q < \infty$, and $0 \leq \gamma < 1 + q/n$. The Campanato space $\mathcal{L}_{q,\gamma}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f satisfying*

$$\|f\|_{\mathcal{L}_{q,\gamma}(\mathbb{R}^n)}^q := \sup_{I \subset \mathbb{R}^n} (\ell(I))^{-\gamma n} \int_I |f(x) - f_I|^q dx < \infty,$$

where the symbol \sup_I denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n , $f_I = (\ell(I))^{-n} \int_I f(x) dx$.

Following the procedure of [8, Proposition 2.10], we can get the following inclusion relationship between $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ and $\mathcal{L}_{p,\gamma}(\mathbb{R}^n)$.

Theorem 2.4. *Let $1 \leq p, q < \infty$ and $0 \leq \lambda < 1 + q/p + q/n - q$, $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n) \subseteq \mathcal{L}_{p,p(1+\lambda/q-1/q)}(\mathbb{R}^n)$.*

Remark 2.1. *As a corollary of Theorem 2.4, we can see that for $1 \leq p, q < \infty$ and $\lambda > 1 + q/p + q/n - q$, the elements in $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ are constants. Hence in the rest of this paper, we always assume that $0 \leq \lambda < 1 + q/p + q/n - q$.*

Notice that $\|f\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)} = 0$ if and only if f is constant almost everywhere. The following theorem shows that $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ is completed.

Theorem 2.5. *Let $1 \leq p, q < \infty$ and $0 \leq \lambda < 1 + q/p + q/n - q$. $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ is a Banach space.*

Proof. Let $\{f_m\}$ be a Cauchy sequence in $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$. From Theorem 2.4, we know that $\{f_m\}$ is also a Cauchy sequence in $\mathcal{L}_{p,p(1+\lambda/q-1/q)}(\mathbb{R}^n)$. Since $\mathcal{L}_{p,p(1+\lambda/q-1/q)}(\mathbb{R}^n)$ is a Banach space, $\{f_m\}$ converges to a function f according to the norm of $\mathcal{L}_{p,p(1+\lambda/q-1/q)}(\mathbb{R}^n)$. For any cube I , we have

$$(\ell(I))^{-(1+\lambda/q-1/q)pn} \int_I |(f_m(x) - f_{mI}) - (f(x) - f_I)|^p dx \leq \|f_m - f\|_{\mathcal{L}_{p,p(1+\lambda/q-1/q)}(\mathbb{R}^n)}^p.$$

Then, we have

$$\lim_{m \rightarrow \infty} \int_I |(f_m(x) - f_{mI}) - (f(x) - f_I)|^p dx = 0.$$

Therefore, there exists a subsequence $\{f_{m_k}\}_{k \in \mathbb{Z}_+}$ such that

$$\lim_{k \rightarrow \infty} f_{m_k}(x) - f_{m_k I} = f(x) - f_I$$

a.e. $x \in I$, thereby

$$\begin{aligned} (f_{m_l}(y) - f(y)) - (f_{m_l}(x) - f(x)) &= \lim_{l \rightarrow \infty} \left\{ f_{m_k}(x) - f_{m_{k_l}} - (f(x) - f_l) \right. \\ &\quad \left. - (f_{m_k}(y) - f_{m_{k_l}} - (f(y) - f_l)) \right. \\ &\quad \left. + (f_{m_l}(y) - f(y)) - (f_{m_l}(x) - f(x)) \right\} \\ &= \lim_{l \rightarrow \infty} \left\{ (f_{m_k}(x) - f_{m_l}(x)) - (f_{m_k}(y) - f_{m_l}(y)) \right\}, \end{aligned}$$

a.e. $x, y \in I$, using Fatou's lemma, we obtain

$$\begin{aligned} &(\ell(I))^{-\lambda n} \int_I \left(\int_I \frac{|f_{m_l}(x) - f(x) - (f_{m_l}(y) - f(y))|^p}{|x - y|^{pn}} \mathcal{H}^{p/q} \left(\frac{|x - y|}{\ell(I)} \right) dx \right)^{q/p} dy \\ &= (\ell(I))^{-\lambda n} \int_I \left(\int_I \lim_{l \rightarrow \infty} \frac{|f_{m_k}(x) - f_{m_l}(x) - (f_{m_k}(y) - f_{m_l}(y))|^p}{|x - y|^{pn}} \mathcal{H}^{p/q} \left(\frac{|x - y|}{\ell(I)} \right) dx \right)^{q/p} dy \\ &\leq \lim_{l \rightarrow \infty} (\ell(I))^{-\lambda n} \int_I \left(\int_I \frac{|(f_{m_k}(x) - f_{m_l}(x)) - (f_{m_k}(y) - f_{m_l}(y))|^p}{|x - y|^{pn}} \mathcal{H}^{p/q} \left(\frac{|x - y|}{\ell(I)} \right) dx \right)^{q/p} dy \\ &\leq \lim_{l \rightarrow \infty} \|f_{m_k} - f_{m_l}\|_{Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)}^q. \end{aligned}$$

Thus,

$$\|f_{m_l} - f\|_{Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)} \leq \lim_{l \rightarrow \infty} \|f_{m_k} - f_{m_l}\|_{Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)},$$

which implies that $\{f_{m_l}\}$ converges to f by the norm of $Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)$. Since

$$\|f_m - f\|_{Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)} \leq \|f_{m_l} - f\|_{Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)} + \|f_{m_l} - f_m\|_{Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)},$$

$\{f_m\}$ converges to f by the norm $\|\cdot\|_{Q_{\mathcal{H}, \lambda}^{p, q}(\mathbb{R}^n)}$, which completes the proof of Theorem 2.5. \square

Let f be a differential function on \mathbb{R}^n . Denote by ∇f the gradient of f , i.e.,

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Definition 2.2. Let $1 < p < \infty$, $0 \leq \gamma < 2 + p/n - p$ and $f \in C^1(\mathbb{R}^n)$. The space $CIS_\gamma^p(\mathbb{R}^n)$ is defined as the set of all measurable functions satisfying

$$\|f\|_{CIS_\gamma^p(\mathbb{R}^n)}^p := \sup_{I \subseteq \mathbb{R}^n} (\ell(I))^{n-pn-\gamma n+p} \int_I |\nabla f(x)|^p dx < \infty,$$

where the symbol \sup_I denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n .

Remark 2.2. Specially, for $p = 2$ and $\gamma = 2$, $CIS_0^2(\mathbb{R}^n)$ comes back to the conformally invariant Sobolev type space $CIS(\mathbb{R}^n)$ introduced by Xiao in [26].

Proposition 2.6. Let $n/(n-1) < p < 2n/(n-1)$ and $0 \leq \gamma < 2 + p/n - p$. The space $CIS_\gamma^p(\mathbb{R}^n)$ is non-trivial.

Proof. Define

$$f_0(x) := (1 + |x|^2)^{(\gamma-2+p)n/(2p)}.$$

Below we prove that $f_0(x) \in CIS_\gamma^p(\mathbb{R}^n)$. A direct computation gives, for $i = 1, 2, \dots, n$,

$$\partial_{x_i} f_0(x) = n(\gamma/p - 2/p + 1)(1 + |x|^2)^{(\gamma-2+p)n/(2p)-1} x_i,$$

which implies that

$$|\nabla f_0(x)| \lesssim (1 + |x|^2)^{(\gamma-2+p)n/2-p/2}.$$

Then

$$(\ell(I))^{n-pn-\gamma n+p} \int_I |\nabla f_0(x)|^p dx \lesssim (\ell(I))^{n-pn-\gamma n+p} \int_I (1 + |x|^2)^{(\gamma-2+p)n/2-p/2} dx.$$

Denote by x_0 the center of I . We divide the rest of the proof into two cases.

Case 1: $|x_0| \leq 2\ell(I)$. For $x \in I$, $|x| \leq |x - x_0| + |x_0| < 3\ell(I)$. When $p > n/(n-1)$ and $\gamma \geq 0$, we can get $\gamma n - n + pn - p > 0$ and

$$\begin{aligned} & (\ell(I))^{n-pn-\gamma n+p} \int_I |\nabla f_0(x)|^p dx \\ & \lesssim (\ell(I))^{n-pn-\gamma n+p} \int_0^{3\ell(I)} (1 + |x|^2)^{(\gamma-2+p)n/2-p/2} |x|^{n-1} d|x| \\ & \lesssim (\ell(I))^{n-pn-\gamma n+p} \int_0^{3\ell(I)} (1 + |x|)^{(\gamma-2+p)n-p} |x|^{n-1} d|x| \\ & \lesssim (\ell(I))^{n-pn-\gamma n+p} \int_0^{3\ell(I)} |x|^{\gamma n - n + pn - p - 1} d|x| \lesssim 1. \end{aligned}$$

Case 2: $|x_0| > 2\ell(I)$. For this case, if $x \in I$, then $|x| \geq |x_0| - |x - x_0| > \ell(I)$. Since $\gamma < 2 + p/n - p$, i.e., $p - (\gamma - 2 + p)n > 0$, we obtain

$$\begin{aligned} & (\ell(I))^{n-pn-\gamma n+p} \int_I |\nabla f_0(x)|^p dx \\ & \lesssim (\ell(I))^{n-pn-\gamma n+p} \int_I \frac{dx}{(1 + |x|)^{p-(\gamma-2+p)n}} \\ & \lesssim (\ell(I))^{n-pn-\gamma n+p} \frac{1}{(1 + \ell(I))^{p-(\gamma-2+p)n}} \int_I 1 dx \lesssim 1. \end{aligned}$$

So we can see that

$$\sup_{I \subseteq \mathbb{R}^n} (\ell(I))^{n-pn-\gamma n+p} \int_I |\nabla f_0(x)|^p dx < \infty,$$

which completes the proof. \square

Theorem 2.7. Let $1 \leq q < \infty$, $n/(n-1) < p < 2n/(n-1)$ and $\max\{0, 1 - q/p\} \leq \lambda < 1 + q/p + q/n - q$. The space $Q_{\mathcal{K}, \lambda}^{p, q}(\mathbb{R}^n)$ is non-trivial if and only if

$$(3) \quad \int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{(q-1)(n-1)}} dt < \infty.$$

Proof. Necessity. We apply the idea of [11, Theorem 2.3]. Firstly, we assume that

$$\int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{(q-1)(n-1)}} dt = \infty.$$

Let $f \in Q_{\mathcal{K}, \lambda}^{p, q}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ be nonconstant and real. Then there exists a point $x_0 = (x_1^0, x_2^0, \dots, x_n^0)$ such that $\nabla f(x_0) \neq 0$. According to the Householder reflector [23, page 71], there exists an orthogonal matrix $\mathcal{A} = (a_{ij})$, $i, j = 1, 2, \dots, n$, such that

$$\nabla f(x_0) \mathcal{A} = \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) a_{i1}, \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) a_{i2}, \dots, \frac{\partial f}{\partial x_i}(x_0) a_{in} \right) = (|\nabla f(x_0)|, 0, \dots, 0).$$

Let $h(x) = f(x \mathcal{A}^\top)$, where \mathcal{A}^\top is the transpose matrix of \mathcal{A} and $\det(\mathcal{A}^\top) \neq 0$. By Proposition 2.1, we can get $h \in Q_{\mathcal{K}, \lambda}^{p, q}(\mathbb{R}^n)$. There exists a point $y_0 = (y_1^0, y_2^0, \dots, y_n^0)$ such that $y_0 \mathcal{A}^\top = x_0$, i.e., $x_j^0 = \sum_{i=1}^n y_j^0 a_{ji}$. In addition, it holds

$$\begin{cases} \frac{\partial h}{\partial y_1}(y_0) = \frac{\partial f}{\partial y_1}(y_0 \mathcal{A}^\top) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(y_0 \mathcal{A}^\top) a_{j1} = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x_0) a_{j1} = |\nabla f(x_0)|; \\ \frac{\partial h}{\partial y_i}(y_0) = 0 \quad i = 2, 3, \dots, n. \end{cases}$$

Therefore, $\nabla h(y_0) = (\nabla f(x_0), 0, \dots, 0)$. Note that $h \in CIS_Y^p(\mathbb{R}^n)$. There exists $\delta > 0$ and a cube I centered at y_0 such that if $x \in I$, $\frac{\partial h(x)}{\partial x_1} > 2\delta$ and $\frac{\partial h(x)}{\partial x_i} < \delta$, $i = 2, 3, \dots, n$. Then

$$\begin{aligned} & (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |h(x+y) - h(x)|^p dx \right)^{q/p} \mathcal{K}\left(\frac{|y|}{\ell(I)}\right) \frac{dy}{|y|^{qn}} \\ & \geq (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \delta^p dx \right)^{q/p} \mathcal{K}\left(\frac{|y|}{\ell(I)}\right) \frac{dy}{|y|^{qn-q}} \\ & \gtrsim (\ell(I))^{-\lambda n + qn/p} \int_{|y| < \sqrt{n}\ell(I)} \mathcal{K}\left(\frac{|y|}{\ell(I)}\right) \frac{dy}{|y|^{qn-q}} \\ & \gtrsim (\ell(I))^{-\lambda n + qn/p} \int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{(q-1)(n-1)}} dt = \infty, \end{aligned}$$

which indicates that $h \notin Q_{\mathcal{K}, \lambda}^{p, q}(\mathbb{R}^n)$ and this is a contradiction. The above results show that $Q_{\mathcal{K}, \lambda}^{p, q}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ is trivial if

$$\int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{(q-1)(n-1)}} dt = \infty.$$

For $f \in Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, denote by $f * g$ the convolution:

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

It follows from Minkowski's inequality and Proposition 2.1 (ii) that $f * g \in Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$ with

$$\|f * g\|_{Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)} \leq \|f\|_{Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)} \int_{\mathbb{R}^n} g(y)dy.$$

Especially, if g is a smooth function with compact support, then $f * g \in Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ is a constant a.e. on \mathbb{R}^n . By [11, Theorem 2.3], there exists a sequence $\{g_n > 0\}$ with

$$\int_{\mathbb{R}^n} g_n(x)dx = 1$$

and the support of g_n shrinking to 0 such that $f * g_n \rightarrow f$ as $n \rightarrow \infty$ a.e. on \mathbb{R}^n . It follows that f is a constant a.e. on \mathbb{R}^n .

Sufficiency. We follow the idea of [26, Theorem 4.1]. Assume that $f \in CIS_{p\lambda/q+1-p/q}^p(\mathbb{R}^n)$ and \mathcal{K} satisfies (3). Given a cube I . We get

$$\begin{aligned} & \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^p dx \right)^{q/p} \mathcal{K}\left(\frac{|y|}{\ell(I)}\right) \frac{dy}{|y|^{qn}} \\ &= \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \left| \int_0^1 \nabla f(x+ty) \cdot y dt \right|^p dx \right)^{q/p} \mathcal{K}\left(\frac{|y|}{\ell(I)}\right) \frac{dy}{|y|^{qn}} \\ &\leq \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \left(\int_0^1 |\nabla f(x+ty)| dt \right)^p dx \right)^{q/p} \mathcal{K}\left(\frac{|y|}{\ell(I)}\right) \frac{dy}{|y|^{qn-q}}. \end{aligned}$$

Minkowski's inequality gives

$$\begin{aligned}
 & \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\
 & \leq \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \left(\int_0^1 |\nabla f(x+ty)|^p dt \right)^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn-q}} \right)^{1/q} \\
 & \leq \int_0^1 \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |\nabla f(x+ty)|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn-q}} \right)^{1/q} dt \\
 & \leq \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_{3\sqrt{n}I} |\nabla f(v)|^p dv \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn-q}} \right)^{1/q} \\
 & \lesssim \|f\|_{CIS_{p\lambda/q+1-p/q}^p(\mathbb{R}^n)} \left((\ell(I))^{qn-q-n} \int_{|y| < \sqrt{n}\ell(I)} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn-q}} \right)^{1/q} \\
 & \lesssim \|f\|_{CIS_{p\lambda/q+1-p/q}^p(\mathbb{R}^n)} \left(\int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{(q-1)(n-1)}} dt \right)^{1/q} < \infty,
 \end{aligned}$$

which indicates that $CIS_{p\lambda/q+1-p/q}^p(\mathbb{R}^n) \subseteq Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$. Obviously, $Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$ is non-trivial. \square

Moreover, there are also inclusion relationships between $Q_{\mathcal{K},\lambda}^{q,q}(\mathbb{R}^n)$ and Besov spaces.

Definition 2.3. Let $0 < \alpha < 1$ and $1 \leq p, q < \infty$. The space $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ is defined as the set of all measurable functions satisfying

$$\|f\|_{\dot{B}_p^{\alpha,q}(\mathbb{R}^n)}^q := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx \right)^{q/p} \frac{dy}{|y|^{n+q\alpha}} < \infty.$$

Theorem 2.8. If $1 \leq q \leq p$, $\lambda = q\alpha/n - q - q/p + 2$ and $\mathcal{K}(t) \lesssim t^{qn-n-q\alpha}$. Then $\dot{B}_p^{\alpha,q}(\mathbb{R}^n) \subseteq Q_{\mathcal{K},\lambda}^{q,q}(\mathbb{R}^n)$.

Proof. Suppose $f \in \dot{B}_p^{\alpha,q}(\mathbb{R}^n)$. By Hölder's inequality, we obtain that for any cube I in \mathbb{R}^n ,

$$\begin{aligned}
 & (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^q dx \right)^{q/q} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\
 & \leq (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^p dx \right)^{q/p} (\ell(I))^{(p-q)n/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\
 & \lesssim (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^p dx \right)^{q/p} (\ell(I))^{(p-q)n/p} \left(\frac{|y|}{\ell(I)} \right)^{qn-n-q\alpha} \frac{dy}{|y|^{qn}} \\
 & = (\ell(I))^{-\lambda n + n - qn/p - qn + n + q\alpha} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(x+y) - f(x)|^p dx \right)^{q/p} \frac{dy}{|y|^{n+q\alpha}} \\
 & \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx \right)^{q/p} \frac{dy}{|y|^{n+q\alpha}} \\
 & = \|f\|_{\dot{B}_p^{\alpha,q}(\mathbb{R}^n)}^q < \infty,
 \end{aligned}$$

which implies $f \in Q_{\mathcal{K},\lambda}^{q,q}(\mathbb{R}^n)$, i.e., $\dot{B}_p^{\alpha,q}(\mathbb{R}^n) \subseteq Q_{\mathcal{K},\lambda}^{q,q}(\mathbb{R}^n)$. \square

3. Boundedness of weighted Hardy operator on $Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$

In this section, we will discuss the boundedness of U_ψ and its adjoint operator V_ψ on $Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$. We assume that the weight function $\mathcal{K}(\cdot)$ satisfies the following condition:

$$(4) \quad \int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{qn-n+1}} dt < \infty.$$

Theorem 3.1. Let $\psi : [0, 1] \rightarrow [0, \infty]$ be a function and $1 \leq p, q < \infty$, $0 \leq \lambda < 1 + q/p + q/n - q$, and $n \geq 1$. Then $U_\psi : Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n) \rightarrow Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$ exists as a bounded operator if

$$(5) \quad \int_0^1 t^{n+\lambda n/q - n/p - n/q} \psi(t) dt < \infty.$$

Moreover, when (5) holds and $1 < q + \lambda < 1 + q/p$, $\mathcal{K}(\cdot)$ satisfies (4), the norm of U_ψ on $Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n) \rightarrow Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)} = \int_0^1 t^{n+\lambda n/q - n/p - n/q} \psi(t) dt.$$

Proof. We assume that (5) holds. If $f \in Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$, by Theorem 2.5, $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ is a Banach space. Then for any cube $I \subset \mathbb{R}^n$, by use of Minkowski's inequality, we have

$$\begin{aligned} & \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |U_\psi f(x+y) - U_\psi f(x)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &= \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \left| \int_0^1 (f(tx+ty) - f(tx)) \psi(t) dt \right|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &\leq \int_0^1 \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(tx+ty) - f(tx)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \psi(t) dt. \end{aligned}$$

Changing the variables: $u = tx$ and $v = ty$, we get

$$\begin{aligned} & \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f(tx+ty) - f(tx)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &= \left((\ell(tI))^{-\lambda n} \int_{|v| < \sqrt{n}\ell(tI)} \left(\int_{tI} |f(u+v) - f(u)|^p du \right)^{q/p} \mathcal{H} \left(\frac{|v|}{\ell(tI)} \right) \frac{dv}{|v|^{qn}} \right)^{1/q} t^{n(1-1/p-1/q+\lambda/q)}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |U_\psi f(x+y) - U_\psi f(x)|^p dx \right)^{q/p} \mathcal{H} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &\leq \|f\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)} \int_0^1 t^{n-n/p-n/q+\lambda n/q} \psi(t) dt, \end{aligned}$$

which implies the boundedness of U_ψ on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$.

Naturally, if U_ψ is bounded on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$, then we can choose the function

$$f_1(x) = \begin{cases} -|x|^{n-n/p-n/q+\lambda n/q}, & x \in \mathbb{R}_l^n; \\ +|x|^{n-n/p-n/q+\lambda n/q}, & x \in \mathbb{R}_r^n. \end{cases}$$

where \mathbb{R}_l^n and \mathbb{R}_r^n denote the left and right halves of \mathbb{R}^n , separated by the hyperplane $x_1 = 0$ (x_1 is the first coordinate of $x \in \mathbb{R}^n$).

We compute the norm of $f_1(x)$ on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ as follows. For any cube I in \mathbb{R}^n , if $1 < q + \lambda < 1 + q/p$, $\int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{qn-n+1}} dt < \infty$, we have

$$\begin{aligned} & (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |f_1(x+y) - f_1(x)|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\ & \lesssim (\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_{3I} |f_1(x)|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \\ & = (\ell(I))^{-\lambda n} \left(\int_{3I \cap \{|x| \leq \ell(3I)\}} |x|^{pn-n+(\lambda-1)pn/q} dx + \int_{3I \cap \{|x| > \ell(3I)\}} |x|^{pn-n+(\lambda-1)pn/q} dx \right)^{q/p} \\ & \quad \times \left(\int_{|y| < \sqrt{n}\ell(I)} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right) \\ & \lesssim (\ell(I))^{-\lambda n} \left(\int_0^{\ell(3I)} |x|^{pn-1+(\lambda-1)pn/q} dx + \int_{3I} (\ell(I))^{pn-n+(\lambda-1)pn/q} dx \right)^{q/p} \int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{qn-n+1}} dt \\ & \lesssim \int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{qn-n+1}} dt < \infty, \end{aligned}$$

which implies that $0 \neq \|f_1\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)}^q < \infty$, i.e. $f_1 \in Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$.

Noting that

$$U_\psi f_1(x) = f_1(x) \int_0^1 t^{n-n/p-n/q+\lambda n/q} \psi(t) dt,$$

we have

$$\|U_\psi\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)} = \int_0^1 t^{n-n/p-n/q+\lambda n/q} \psi(t) dt.$$

Therefore, the proof of the Theorem 3.1 is complete. \square

For the boundedness of V_ψ on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$, similarly, we have

Theorem 3.2. Let $\psi: [0, 1] \rightarrow [0, \infty)$ be a function, $0 \leq \lambda < 1 + q/p + q/n - q$, $n \geq 1$, and $1 \leq p, q < \infty$. Then $V_\psi: Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n) \rightarrow Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ exists as a bounded operator if

$$(6) \quad \int_0^1 t^{n/p+n/q-2n-\lambda n/q} \psi(t) dt < \infty.$$

Moreover, when (6) holds and $1 < q + \lambda < 1 + q/p$, $\mathcal{K}(\cdot)$ satisfies (4), the norm of V_ψ on $Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)$ is given by

$$\|V_\psi\|_{Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n) \rightarrow Q_{\mathcal{H},\lambda}^{p,q}(\mathbb{R}^n)} = \int_0^1 t^{n/p+n/q-2n-\lambda n/q} \psi(t) dt.$$

Proof. Suppose (6) holds. If $f \in Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$, then for any cube I in \mathbb{R}^n , applying Minkowski's inequality, we have

$$\begin{aligned} & \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |V_\psi f(x+y) - V_\psi f(x)|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &= \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \left| \int_0^1 \left(f\left(\frac{x+y}{t}\right) - f\left(\frac{x}{t}\right) \right) t^{-n} \psi(t) dt \right|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &\leq \int_0^1 \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \left| f\left(\frac{x+y}{t}\right) - f\left(\frac{x}{t}\right) \right|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} t^{-n} \psi(t) dt. \end{aligned}$$

Letting $u = \frac{x}{t}$ and $v = \frac{y}{t}$, we get

$$\begin{aligned} & \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I \left| f\left(\frac{x+y}{t}\right) - f\left(\frac{x}{t}\right) \right|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &= \left(\left(\ell\left(\frac{I}{t}\right) \right)^{-\lambda n} \int_{|v| < \sqrt{n}\ell\left(\frac{I}{t}\right)} \left(\int_I |f(u+v) - f(u)|^p du \right)^{q/p} \mathcal{K} \left(\frac{|v|}{\ell\left(\frac{I}{t}\right)} \right) \frac{dv}{|v|^{qn}} \right)^{1/q} t^{n/p+n/q-2n-\lambda n/q}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left((\ell(I))^{-\lambda n} \int_{|y| < \sqrt{n}\ell(I)} \left(\int_I |V_\psi f(x+y) - V_\psi f(x)|^p dx \right)^{q/p} \mathcal{K} \left(\frac{|y|}{\ell(I)} \right) \frac{dy}{|y|^{qn}} \right)^{1/q} \\ &\leq \|f\|_{Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)} \int_0^1 t^{n/p+n/q-2n-\lambda n/q} \psi(t) dt < \infty, \end{aligned}$$

which implies the boundedness of V_ψ on $Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)$.

For the second half of the theorem, By a way similar to Theorem 3.1, take the function

$$f_1(x) = \begin{cases} -|x|^{n-n/p-n/q+\lambda n/q}, & x \in \mathbb{R}_l^n; \\ +|x|^{n-n/p-n/q+\lambda n/q}, & x \in \mathbb{R}_r^n. \end{cases}$$

Noting that $1 < q + \lambda < 1 + q/p$ and $\int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{qn-n+1}} dt < \infty$, we can get

$$0 \neq \|f_1\|_{Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)}^q \lesssim \int_0^{\sqrt{n}} \frac{\mathcal{K}(t)}{t^{qn-n+1}} dt < \infty.$$

And we have

$$V_\psi f_1 = f_1(x) \int_0^1 t^{n/p+n/q-2n-\lambda n/q} \psi(t) dt.$$

In that case

$$\|V_\psi\|_{Q_{\mathcal{K},\lambda}^{p,q}(\mathbb{R}^n)} = \int_0^1 t^{n/p+n/q-2n-\lambda n/q} \psi(t) dt.$$

Therefore, the proof of the Theorem 3.2 is complete. \square

Corollary 3.3. Let $\psi : [0, 1] \rightarrow [0, \infty]$ be a function, $n \geq 1$, and $1 \leq q < \infty$.

(i) $U_\psi : Q_{\mathcal{K}}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}}^q(\mathbb{R}^n)$ exists as a bounded operator if

$$(7) \quad \int_0^1 t^{(q-2)n/q} \psi(t) dt < \infty.$$

Moreover, when (7) holds and $1 < q < 2$, $\mathcal{K}(\cdot)$ satisfies (4), the norm of U_ψ on $Q_{\mathcal{K}}^q(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{Q_{\mathcal{K}}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}}^q(\mathbb{R}^n)} = \int_0^1 t^{(q-2)n/q} \psi(t) dt.$$

(ii) $V_\psi : Q_{\mathcal{K}}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}}^q(\mathbb{R}^n)$ exists as a bounded operator if

$$(8) \quad \int_0^1 t^{(2-2q)n/q} \psi(t) dt < \infty.$$

Moreover, when (8) holds and $1 < q < 2$, $\mathcal{K}(\cdot)$ satisfies (4), the norm of V_ψ on $Q_{\mathcal{K}}^q(\mathbb{R}^n)$ is given by

$$\|V_\psi\|_{Q_{\mathcal{K}}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}}^q(\mathbb{R}^n)} = \int_0^1 t^{(2-2q)n/q} \psi(t) dt.$$

Corollary 3.4. Let $\psi : [0, 1] \rightarrow [0, \infty]$ be a function, $0 \leq \lambda < 2 + q/n - q$, $n \geq 1$, and $1 \leq q < \infty$.

(i) $U_\psi : Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n)$ exists as a bounded operator if

$$(9) \quad \int_0^1 t^{(q-2+\lambda)n/q} \psi(t) dt < \infty.$$

Moreover, when (9) holds and $1 < q + \lambda < 2$, $\mathcal{K}(\cdot)$ satisfies (4), the norm of U_ψ on $Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n)} = \int_0^1 t^{(q-2+\lambda)n/q} \psi(t) dt.$$

(ii) $V_\psi : Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n)$ exists as a bounded operator if

$$(10) \quad \int_0^1 t^{(2-2q-\lambda)n/q} \psi(t) dt < \infty.$$

Moreover, when (10) holds and $1 < q + \lambda < 2$, $\mathcal{K}(\cdot)$ satisfies (4), the norm of V_ψ on $Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n)$ is given by

$$\|V_\psi\|_{Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n) \rightarrow Q_{\mathcal{K}, \lambda}^q(\mathbb{R}^n)} = \int_0^1 t^{(2-2q-\lambda)n/q} \psi(t) dt.$$

4. Applications

In Section 3, the integrability of $\mathcal{K}(\cdot)$ plays an important role. Below we will provide several examples of the weight functions \mathcal{K} . The first one is a generalization of the example introduced by Cui, Li and Lou [8].

Example 4.1. Let $\beta > 0$, $1 \leq q < \infty$ and $qn - n < m$, we define $\mathcal{K}_1(\cdot)$ as

$$\mathcal{K}_1(t) := \begin{cases} \frac{t^m}{\left| \ln \left(\frac{t}{e\sqrt{n}} \right) \right|^\beta}, & 0 < t < \sqrt{n}; \\ t^m, & t \geq \sqrt{n}. \end{cases}$$

Then $\mathcal{K}_1(\cdot)$ satisfies (4).

Proof. By the definition of $\mathcal{K}_1(\cdot)$, when $0 < t < \sqrt{n}$, a direct computation gives

$$\frac{\mathcal{K}_1(t)}{t^{qn-n+1}} = \frac{1}{t^{qn-n-m+1} \left| \ln \left(\frac{t}{e\sqrt{n}} \right) \right|^\beta}.$$

So we have

$$\begin{aligned} \int_0^{\sqrt{n}} \frac{\mathcal{K}_1(t)}{t^{qn-n+1}} dt &= \lim_{u \rightarrow 0^+} \int_u^{\sqrt{n}} \frac{1}{t^{qn-n-m+1} \left| \ln \left(\frac{t}{e\sqrt{n}} \right) \right|^\beta} dt \\ &\leq \lim_{u \rightarrow 0^+} \int_u^{\sqrt{n}} \frac{1}{t^{qn-n-m+1}} dt \\ &\lesssim \lim_{u \rightarrow 0^+} t^{m-qn+n} \Big|_u^{\sqrt{n}} \lesssim 1. \end{aligned}$$

It is obvious that $\mathcal{K}_1(\cdot)$ is a non-decreasing function. □

Example 4.2. Let $1 \leq q < \infty$, $\beta > -1$, $m > 0$, and $qn - n - m < \min\{0, -\beta\}$. Define $\mathcal{K}_2(\cdot)$ as

$$\mathcal{K}_2(t) := \begin{cases} t^m \left(\ln \left(\frac{e\sqrt{n}}{t} \right) \right)^\beta, & 0 < t < \sqrt{n}; \\ t^m, & t \geq \sqrt{n}. \end{cases}$$

Then $\mathcal{K}_2(\cdot)$ satisfies (4).

Proof. We first prove that the integral is convergent. Let $x = \ln(e\sqrt{n}/t)$. Then

$$\begin{aligned} \int_0^{\sqrt{n}} \frac{\mathcal{K}_2(t)}{t^{qn-n+1}} dt &= \int_0^{\sqrt{n}} t^{m-qn+n-1} \left(\ln \left(\frac{e\sqrt{n}}{t} \right) \right)^\beta dt \\ &\lesssim \int_1^\infty e^{x(qn-n-m)} x^\beta dx \\ &\lesssim \int_{m-qn+n}^\infty e^{-y} y^\beta dy \\ &\leq \int_0^\infty e^{-y} y^{(\beta+1)-1} dy = \Gamma(\beta+1). \end{aligned}$$

1 Next, we prove the monotonicity of $\mathcal{K}_2(\cdot)$. In fact, we only need to verify that $e^{x(qn-n-m)}x^\beta$ is
 2 monotonically decreasing for $x > 1$. Since $qn - n - m + \beta < 0$, we have,

$$3 \left(e^{x(qn-n-m)}x^\beta \right)' = e^{x(qn-n-m)}x^{\beta-1}((qn-n-m)x + \beta) < 0.$$

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 6 Therefore, $\mathcal{K}_2(\cdot)$ is a non-decreasing function. □

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 8 Some other example is related to sine functions.

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 10 **Example 4.3.** Let $\beta > 0$, $m > 0$, $1 \leq q < \infty$ and $qn - n < m + \beta$. Define $\mathcal{K}_3(\cdot)$ as

$$11 \mathcal{K}_3(t) := \begin{cases} t^m \sin\left(\frac{\pi t}{2\sqrt{n}}\right)^\beta, & 0 < t < \sqrt{n}; \\ t^m, & t \geq \sqrt{n}. \end{cases}$$

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 16 Then $\mathcal{K}_3(\cdot)$ satisfies (4).

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 18 *Proof.* We use direct computation to prove the above example, when $0 < t < \sqrt{n}$, we have

$$19 \int_0^{\sqrt{n}} \frac{\mathcal{K}_3(t)}{t^{qn-n+1}} dt = \lim_{u \rightarrow 0^+} \int_u^{\sqrt{n}} t^{m-qn+n-1} \sin\left(\frac{\pi t}{2\sqrt{n}}\right)^\beta dt$$

$$20 \lesssim \lim_{u \rightarrow 0^+} \int_u^{\sqrt{n}} t^{m+\beta-qn+n-1} dt$$

$$21 \lesssim \lim_{u \rightarrow 0^+} t^{m+\beta-qn+n} \Big|_u^{\sqrt{n}} \lesssim 1.$$

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 26 And by the definition, we know that $\mathcal{K}_3(\cdot)$ is a non-decreasing function. □

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 29 **Example 4.4.** Let $m \geq \beta > 0$, $1 \leq q < \infty$, and $qn - n < m - \beta$. Define $\mathcal{K}_4(\cdot)$ as

$$30 \mathcal{K}_4(t) := \begin{cases} \frac{t^m}{\sin\left(\frac{\pi t}{2\sqrt{n}}\right)^\beta}, & 0 < t < \sqrt{n}; \\ t^m, & t \geq \sqrt{n}. \end{cases}$$

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 35 Then $\mathcal{K}_4(\cdot)$ satisfies (4).

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 38 *Proof.* We still need to prove the convergence of the integral first. Since $\sin(\pi t/(2\sqrt{n}))^\beta$ is a concave
 39 function when $0 < t < \sqrt{n}$, we can easily obtain $\sin(\pi t/(2\sqrt{n}))^\beta > t/\sqrt{n}$, so we have

$$40 \frac{1}{\sin(\pi t/(2\sqrt{n}))^\beta} > \frac{\sqrt{n}}{t}.$$

Then,

$$\begin{aligned} \int_0^{\sqrt{n}} \frac{\mathcal{K}_4(t)}{t^{qn-n+1}} dt &= \lim_{u \rightarrow 0^+} \int_u^{\sqrt{n}} \frac{t^{m-qn+n-1}}{\sin\left(\frac{\pi t}{2\sqrt{n}}\right)^\beta} dt \\ &\lesssim \lim_{u \rightarrow 0^+} \int_u^{\sqrt{n}} t^{m-qn+n-1-\beta} dt \\ &\lesssim \lim_{u \rightarrow 0^+} t^{m-\beta-qn+n}|_u^{\sqrt{n}} \lesssim 1. \end{aligned}$$

Let $x = \pi t / (2\sqrt{n})$, we only need to prove that $\frac{x^m}{(\sin x)^\beta}$ is an increasing function when $0 < x < \pi/2$. We obtain that

$$\left(\frac{x^m}{(\sin x)^\beta} \right)' = (m \sin x - \beta \cos x \cdot x) \frac{x^{m-1} (\sin x)^{\beta-1}}{(\sin x)^{2\beta}}.$$

And because $\beta \leq m$ and $\tan x > x$ when $0 < x < \pi/2$. Therefore, we have $\frac{x^m}{(\sin x)^\beta}$ is an increasing function, which implies $\mathcal{K}_4(\cdot)$ is a non-decreasing function. \square

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