Exponential stabilization of laminated beams with Gurtin-Pipkin thermal law: the case of equal speeds

Djellali Fayssal [1], Victor R. Cabanillas[2,3], and Adel M. Al-Mahdi*[4,5]

[1] *Laboratory of Mathematics, Dynamics and Modelization, Badji-Mokhtar Annaba University, Annaba 23000, Algeria*
[2] Universidad de Lima, Programa de Estudios Generales, Avenida Javier Prado Este 4600, Lima 15023, Perú
[3] Universidad Nacional Mayor de San Marcos, Facultad de Ciencias Matemáticas, Calle Germán Amézaga 375, Lima 15081, Perú
[4] Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
[5] *The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
[1] faycel.djellali@gmail.com
[2] VCABANIL@ulima.edu.pe
[3] almahdi@kfupm.edu.sa

* Corresponding author

Abstract

This paper considers a thermoelastic laminated beam system with thermal dissipation governed by the Gurtin-Pipkin law. The exponential stability problem of this system has been addressed by Djellali [1] and Liu and Zhao [2], who showed that the exponential stability of the system depends on complicated stability numbers involving the system wave propagation speeds and several other physical parameters of the system. In this paper, we improve the existing results in the literature by showing that the exponential decay of the energy associated with the system depends solely on the ratio between the system speeds.

Keywords: Laminated beams; Gurtin-Pipkin thermal law; stability number; exponential stability.
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1 Introduction

In a recent paper, Liu and Zhao [2] considered a laminated beam system with structural dissipation coupled with a parabolic equation representing the thermal effect given by the Gurtin-Pipkin law, considering that the temperature affects the effective rotation angle equation. More precisely, Liu and Zhao considered the following system

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3w - \psi\right)_{tt} - D\left(3w - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) + \delta\theta_x = 0, \\ I_\rho w_{tt} - Dw_{xx} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0, \\ k\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(x, t-s)\,ds + \delta\left(3w - \psi\right)_{tx} = 0, \end{cases} \tag{1.1}$$

where the functions φ, ψ, $3w-\psi$, θ and g represent the transverse displacement of the beam, the rotation angle, the effective rotation angle, the relative temperature and the memory kernel, respectively. In addition, w is proportional to the amount of slip along the interface. The laminated beam model (without considering thermal effects) was derived and proposed by Hansen and Spies (see [3], [4]). In equation $(1.1)_4$, the heat difference obeys the Gurtin-Pipkin law given by

$$q=-\int_0^\infty g(s)\theta_x\left(x,t-s\right)ds,$$

where q represents the heat flux depending on the history of the temperature gradient due to the kernel g. The function $g=g(s)$ is the relaxation kernel of the thermal conductivity, which is a bounded convex function on $\mathbb{R}^+$ with total mass

$$\int_0^\infty g(s)ds=1. \tag{1.2}$$

The positive constantsρ, G, I_ρ, D, γ and α denote the beam density, the shear stiffness, the mass moment of inertia, the flexural stiffness, the adhesive stiffness of the beams and the adhesive damping parameter, respectively. The constants β, k and δ are positive physical parameters from thermoelasticity theory.

Under some assumptions on the relaxation function g, Liu and Zhao [2] proved the well-posedness of the system (1.1) and studied the stability of the associated semigroup. They introduced the stability number χ_g defined by

$$\chi_g=\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\left(1-\frac{\beta}{g(0)}\frac{kG}{\rho}\right)-\frac{\beta}{g(0)}\frac{G\delta^2}{\rho I_\rho}, \tag{1.3}$$

and proved that the energy associated with the system (1.1) is exponentially stable if and only if $\chi_g=0$. For which they used the perturbed energy method and a characterization of exponential stability known as the Gearhart-Herbst-Prüss-Huang theorem.

Our aim in this work is to study system (1.1) with the following boundary conditions

$$\begin{cases}\varphi_x(0,t)=\psi(0,t)=w(0,t)=\theta_x(0,t)=0, & t>0,\\ \varphi_x(1,t)=\psi(1,t)=w(1,t)=\theta_x(1,t)=0, & t>0,\end{cases} \tag{1.4}$$

and initial data

$$\begin{cases}\varphi(x,0)=\varphi_0,\ \varphi_t(x,0)=\varphi_1,\ \psi(x,0)=\psi_0,\ \psi_t(x,0)=\psi_1, & x\in(0,1),\\ w(x,0)=w_0,\ w_t(x,0)=w_1,\ \theta(x,0)=\theta_0, & x\in(0,1).\end{cases} \tag{1.5}$$

Then, exploiting the energy method, we will obtain an exponential decay result of system (1.1)-(1.5) provided that the condition

$$\chi=\frac{G}{\rho}-\frac{D}{I_\rho}=0, \tag{1.6}$$

is satisfied.

Before starting the analysis and proof of our results, we briefly discuss recent achievements in the stability of beams subjected to different thermal laws.

Thermoelastic laminated beams under Fourier's law

Liu and Zhao [5] studied the following system of laminated beams with and without frictional structural damping under the thermal effect described by Fourier's law

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3w - \psi\right)_{tt} - D\left(3w - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) + \delta\theta_x = 0, \\ I_\rho w_{tt} - D w_{xx} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0, \\ k\theta_t - \alpha\theta_{xx} + 3\delta\left(3w - \psi\right)_{tx} = 0, \end{cases} \tag{1.7}$$

Using the perturbed energy method, the authors showed that when $\alpha \geq 0$, the system is the system is exponentially stable if and only if $\chi = \frac{G}{\rho} - \frac{D}{I_\rho} = 0$.

Still on thermoelastic laminated beam systems with thermal Fourier law, Apalara [6] considered a system similar to (1.7) with heat distribution acting on the interfacial slip. A model for the same type of thermoelastic beams with memory term working on the rotational equation was also studied by Liu and Zhao [7]. More recently, Djellali [8] studied a model in which the heat distribution acts on the transverse displacement and rotational equations, proving the exponential decay of the associated energy if and only if $\chi = 0$. Among other works in the context of thermoelastic and thermoviscoelastic laminated beams with heat flow governed by Fourier's law, we can cite Raposo et al. [9], Choucha et al. [10], Enyi and Mukiawa [11], Nonato et al. [12], Quispe et al. [13] and Cabanillas et al. [14].

Thermoelastic laminated beams under Cattaneo's law

Fourier's law predicts the instantaneous propagation of thermal signals (see [15]); for this reason, it has been the subject of some criticism in the scientific community due to the evidence (based on physical experiments) that thermal motion is a wave-like mechanism. This impasse lies in the physical paradox of the infinite propagation speed of the signals of the Fourier equation, which is a characteristic of parabolicity. For this reason, several "non-classical" theories have been proposed as an alternative to the Fourier heat propagation model. One of these theories is due to Maxwell and Cattaneo [16], whose heat propagation model introduces a small thermal relaxation parameter that transforms the heat transfer equation into a hyperbolic equation. The resulting propagation law is the Maxwell-Cattaneo law, popularly known as the *second sound*. Thus, by replacing Fourier's law with Cattaneo's law in (1.7), we obtain

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3w - \psi\right)_{tt} - D\left(3w - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) + \delta\theta_x = 0, \\ I_\rho w_{tt} - D w_{xx} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0, \\ k\theta_t + q_x + \delta\left(3w - \psi\right)_{tx} = 0, \\ \tau q_t + \alpha q + \theta_x = 0, \end{cases} \tag{1.8}$$

where θ represents the temperature difference and q is the heat flux. Note that the system (1.8) reduces to (1.7) when $\tau = 0$.

More recently, Djellali et al. [17] considered the system (1.8) and proved exponential stability under the condition $\chi = 0$, where χ is the stability number defined in (1.6), which depends only on the system wave propagation speeds, thus improving the stability conditions found by other authors, whose results depend on complicated stability numbers that not only depend on the wave speeds but also involve several other parameters of the system parameters. For example, Apalara [18] introduced the stability number

$$\chi_A = \left(1 - \frac{\tau\kappa G}{\rho}\right)\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right) - \frac{\tau\delta^2 G}{\rho I_\rho},$$

showing that the system (1.8) is exponentially stable when $\chi_A = 0$ and polynomially stable otherwise. The same result was proved by Feng [19] for a model similar to (1.8) but with coupling $\delta\varphi_{tx}$ instead of the coupling $\delta\left(3w - \psi\right)_{tx}$ in the fourth equation of (1.8) and with a stability number χ_F defined by

$$\chi_F = \tau\delta^2 D - (D\rho - GI_\rho)\left(\frac{\tau k D}{I_\rho} - 1\right).$$

The work of Feng [19] was complemented by Wang and Liu [20], who proved the lack of exponential stability when $\chi_F \neq 0$. Mukiawa et al. [21] studied a model similar to (1.8) to which they added an effective memory term on the angular displacement equation, proving stability results dependent on the relationship between the system wave speeds.

Thermoelastic laminated beams under Gurtin- Pipkin

The Gurtin and Pipkin theory [22] provides another alternative to describe the temperature evolution in the laminated beam system. The Gurtin-Pipkin law addresses the need to link heat flow with thermal memory. It is a general nonlinear theory in which thermal disturbances propagate at a finite speed.

The inclusion of the Gurtin-Pipkin thermal law as a dissipative effect in systems such as the Timoshenko, de Bresse, porous and Rao-Nakra systems has been the subject of several studies in recent years (see, for example, [23], [24], [25], [26], [27], [28], [29]).

In the context of laminated beams with thermal effect obeying the Gurtin-Pipkin law, as mentioned at the beginning of this section, Liu and Zhao [2] obtained the exponential stability of the system (1.1) depending on the stability number χ_g defined in (1.3). Subsequently, Djellali [1] showed that the system

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x + \sigma\theta_x = 0, \\ I_\rho\left(3w - \psi\right)_{tt} - D\left(3w - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) - \sigma\theta = 0, \\ I_\rho w_{tt} - Dw_{xx} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0, \\ k\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(x, t - s)\, ds + \sigma\varphi_{tx} + \sigma\left(3w - \psi\right)_t = 0, \end{cases}$$

is exponentially stable if and only if $\chi_D = 0$, where

$$\chi_D = \left(\frac{I_\rho}{D} - \frac{\rho}{G}\right)\left(\frac{kD}{I_\rho} - \frac{g(0)}{\beta}\right) + \frac{\sigma^2}{G}.$$

The case of infinite-length laminated beams with heat conduction described by the Gurtin-Pipkin law was addressed by Guesmia [30]. The author showed that the thermoelastic dissipation generated by the Gurtin-Pipkin law is strong enough to stabilize the system at least polynomially, even when only the second or third equation of the laminated system is controlled. For more results in thermoelasticity of type III, we refer to [31], [32], and

[33] and for thermoelasticity with Coleman-Gurtin's and Gurtin-Pipkin's we refer to the recent works in [34], [35], and [36].

The rest of the paper is structured as follows: In Section 2, we formulate the hypotheses about the relaxation function g that we will use throughout this manuscript and establish the functional framework on which we will work. In Section 3, following the ideas of Dafermos [37], we introduce an auxiliary variable that allows us to write our system as an abstract Cauchy problem. Then, we construct a semigroup of contractions $S(t)$ on a suitable Hilbert space $\mathcal{H}$, with which we establish the well-posedness of our system. Section 4 contains the central result. Using the energy method, we prove some lemmas and construct a Lyapunov functional that allows us to prove the exponential stability of the system under the hypothesis of equality of the wave propagation speeds of the system. In section 5, we study the lack of exponential stability. Finally, Section 6 presents our conclusions and discusses some open problems.

2 Assumptions and functional setting

Proceeding as in [24], let us denote by μ the function on $\mathbb{R}^+$ satisfying

$$\int_0^\infty \mu(s)ds = g(0), \tag{2.1}$$

whose relation with the relaxation function g is given by

$$\mu(s) = -g'(s),$$

the prime denoting the derivative with respect to s, and a requirement that g has a total mass 1 translates to

$$\int_0^\infty s\mu(s)ds = 1.$$

Assume that the following conditions are valid:

i) μ is an absolutely continuous non-negative and non-increasing function on $\mathbb{R}^+$ such that

$$\mu(0) = \lim_{s\to 0} \mu(s) \in (0,\infty).$$

ii) For almost every $s > 0$, there exists $\nu > 0$ such that

$$\mu'(s) + \nu\mu(s) \leq 0.$$

Hereafter, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the usual norm in the Lebesgue space $L^2(0,1)$. Moreover, we introduce the Hilbert subspace

$$L^2_*(0,1) = \left\{ u \in L^2(0,1);\ \int_0^1 u(x)dx = 0 \right\},$$

of the zero-mean functions, and its Hilbert subspaces

$$H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1) \quad \text{and} \quad H^2_*(0,1) = H^2(0,1) \cap L^2_*(0,1).$$

To deal with the memory term in equation $(1.1)_4$, we introduce the space

$$\mathcal{M} = L^2_\mu\left(\mathbb{R}^+; H^1_*(0,1)\right),$$

endowed with the inner product

$$\langle h, f\rangle_{\mathcal{M}} = \int_0^\infty \mu(s)\,\langle h_x(s), f_x(s)\rangle_{L^2}\,ds, \quad \text{for any } h, f \in \mathcal{M},$$

along with the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$

$$T\eta = -\eta',$$

with domain

$$\mathcal{D}(T) = \left\{\eta \in \mathcal{M};\ \eta' \in \mathcal{M},\ \lim_{s\to 0}\|\eta_x(s)\| = 0\right\},$$

the prime stands for weak derivative with respect to the internal variable $s \in \mathbb{R}^+$. The phase space of our problem will be

$$\mathcal{H} = H^1_*(0,1)\times L^2_*(0,1)\times H^1_0(0,1)\times L^2(0,1)\times H^1_0(0,1)\times L^2(0,1)\times L^2_*(0,1)\times \mathcal{M},$$

equipped with inner product

$$\begin{aligned}\left\langle U, \tilde{U}\right\rangle_{\mathcal{H}} =& \rho\,\langle \varphi_t, \tilde{\varphi}_t\rangle + I_\rho\left\langle 3w_t - \psi_t, 3\tilde{w}_t - \tilde{\psi}_t\right\rangle + 3I_\rho\,\langle w_t, \tilde{w}_t\rangle + k\left\langle \theta, \tilde{\theta}\right\rangle \\ &+ G\left\langle \psi - \varphi_x, \tilde{\psi} - \tilde{\varphi}_x\right\rangle + D\left\langle 3w_x - \psi_x, 3\tilde{w}_x - \tilde{\psi}_x\right\rangle + 3D\,\langle w_x, \tilde{w}_x\rangle \\ &+ 4\gamma\,\langle w, \tilde{w}\rangle + \frac{1}{\beta}\,\langle \eta, \tilde{\eta}\rangle_{\mathcal{M}},\end{aligned}$$

for $U = (\varphi, \varphi_t, 3w-\psi, 3w_t-\psi_t, w, w_t, \theta, \eta)$, $\tilde{U} = \left(\tilde{\varphi}, \tilde{\varphi}_t, 3\tilde{w}-\tilde{\psi}, 3\tilde{w}_t-\tilde{\psi}_t, \tilde{w}, \tilde{w}_t, \tilde{\theta}, \tilde{\eta}\right)$ belonging to $\mathcal{H}$, and the induced norm

$$\begin{aligned}\|U\|^2_{\mathcal{H}} =& \rho\|\varphi_t\|^2 + I_\rho\|3w_t-\psi_t\|^2 + 3I_\rho\|w_t\|^2 + k\|\theta\|^2 + G\|\psi-\varphi_x\|^2 \\ &+ D\|3w_x-\psi_x\|^2 + 3D\|w_x\|^2 + 4\gamma\|w\|^2 + \frac{1}{\beta}\|\eta\|^2_{\mathcal{M}}.\end{aligned} \tag{2.2}$$

The following observation plays a crucial role in this work.

Remark 2.1. *For every $\eta \in \mathcal{D}(T)$, the non-negative functional*

$$\Gamma[\eta] = -\int_0^\infty \mu'(s)\,\|\eta_x(s)\|^2\,ds,$$

is well defined, and the following identity holds [38]

$$-\Gamma[\eta] = 2\,\langle T\eta, \eta\rangle_{\mathcal{M}}. \tag{2.3}$$

Moreover, from hypothese (ii), we conclude that

$$\nu\,\|\eta\|^2_{\mathcal{M}} \le \Gamma[\eta]. \tag{2.4}$$

3 The contraction semigroup

In the same spirit of [37], we introduce for $s>0$ the new auxiliary variable $\eta=\eta^t\,(x,s):(0,1)\times[0,\infty)\times\mathbb{R}^+\longmapsto\mathbb{R}$, formally defined as

$$\eta^t(x,s)=\int_0^s\theta(x,t-r)\,dr, \tag{3.1}$$

which satisfies the following boundary conditions

$$\eta^t_x(0,s)=\;\eta^t_x(1,s)=\;0,$$

and the further "boundary condition"

$$\lim_{s\to0}\eta^t(x,s)=\;0.$$

Hence, η^t satisfies the equation

$$\eta^t_t=-\eta^t_s+\theta(t),$$

where η^t_t and η^t_s are the derivatives of $\eta(x,t,s)$ with respect to t and s respectively. Thus, our system reduces to

$$\begin{cases}\rho\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0,\\ I_\rho\left(3w-\psi\right)_{tt}-D\left(3w-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)+\delta\theta_x=0,\\ I_\rho w_{tt}-Dw_{xx}+G\left(\psi-\varphi_x\right)+\frac{4}{3}\gamma w+\frac{4}{3}\alpha w_t=0,\\ k\theta_t-\frac{1}{\beta}\int_0^\infty\mu(s)\eta_{xx}(s)\,ds+\delta\left(3w-\psi\right)_{tx}=0,\\ \eta_t=T\eta+\theta.\end{cases} \tag{3.2}$$

Now, denoting the effective rotation angle by $\xi=3w-\psi$ and introducing the state vector

$$U(t)=\left(\varphi(t),\phi(t),\xi(t),v(t),w(t),W(t),\theta(t),\eta^t\right)^T,$$

where $\phi=\varphi_t, v=\xi_t$ and $W=w_t$. Then, system (3.2) can be rewritten as

$$\begin{cases}U_t(t)=\mathcal{A}U(t), \qquad t>0,\\ U(0)=U_0=\left(\varphi_0,\varphi_1,\xi_0,\xi_1,w_0,w_1,\theta_0,\eta_0\right)^T,\end{cases} \tag{3.3}$$

the solution of (3.3) at time $t>0$ reads

$$U(t)=S(t)U_0=e^{t\mathcal{A}}U_0,$$

where $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is the linear operator defined as

$$\mathcal{A}U(t)=\begin{pmatrix}\phi\\ -\frac{G}{\rho}\left(3w-\xi-\varphi_x\right)_x\\ v\\ \frac{1}{I_\rho}\left[D\xi_{xx}+G\left(3w-\xi-\varphi_x\right)-\delta\theta_x\right]\\ W\\ \frac{1}{I_\rho}\left[Dw_{xx}-G\left(3w-\xi-\varphi_x\right)-\frac{4\gamma}{3}w-\frac{4\alpha}{3}W\right]\\ \frac{1}{k}\left[\frac{1}{\beta}\int_0^\infty\mu(s)\eta_{xx}(s)\,ds-\delta v_x\right]\\ T\eta+\theta\end{pmatrix}.$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ U \in \mathcal{H} \left| \begin{array}{c} \varphi_x \in H_0^1(0,1) \\ \phi \in H_*^1(0,1) \\ \xi \in H^2(0,1) \\ v \in H_0^1(0,1) \\ w \in H^2(0,1) \\ W \in H_0^1(0,1) \\ \theta_x \in H_0^1(0,1) \\ \theta \in H_*^1(0,1) \\ \eta \in \mathcal{D}(T) \\ \int_0^\infty \mu(s)\eta(s)ds \in H_*^2(0,1) \end{array} \right. \right\}.$$

Theorem 3.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup*

$$S(t) = e^{t\mathcal{A}} : \mathcal{H} \to \mathcal{H}.$$

Theorem 3.1 can be proved by means of the classical Lumer-Phillips theorem [39]. For applying the Lumer-Phillips Theorem to equations with memory in the context of past history, see, for example, [40].

Furthermore, for η we have the following explicit representation formula (see [38])

$$\eta^t(s) = \begin{cases} \int_0^s \theta(t-r)\, dr, & s \le t, \\ \eta_0(s-t) + \int_0^t \theta(t-r)\, dr, & s > t. \end{cases}$$

Remark 3.2. *As was observed in [41], the choice of zero-mean function spaces for φ and θ and their derivatives is consistent. Indeed, calling*

$$\Pi = \int_0^1 \varphi(x,t)dx, \quad \textit{and} \quad \Sigma(t) = \int_0^1 \theta(x,t)dx,$$

and integrating equations $(3.2)_1$ and $(3.2)_4$ on $(0,1)$, we obtain

$$\rho\,\ddot{\Pi}(t) = 0, \quad \textit{and} \quad k\,\dot{\Sigma}(t) = 0.$$

Hence, if $\Pi(0) = \dot{\Pi}(0) = \Sigma(0) = \dot{\Sigma}(0) = 0$, it follows that $\Pi(t) \equiv \Sigma(t) \equiv 0$.

4 Exponential stability

In this section, we prove an exponential decay result when the propagation speeds of the system waves (3.2) are equal, i.e., when condition (1.6) is satisfied. First, for any initial data $U_0 \in \mathcal{H}$, we define the energy functional $E(t)$ by (2.2) and we also have

$$E(t) = \|S(t)U_0\|^2_{\mathcal{H}} = \|U\|^2_{\mathcal{H}}\,. \tag{4.1}$$

The main result of this section is given by the following theorem.

Theorem 4.1. *The semigroup $S(t)$ is exponentially stable if $\chi = 0$.*

4.1 Some auxiliary functionals

In order to prove Theorem 4.1, we will define some functionals and prove some estimates for them.

Lemma 4.2. *The energy functional defined by* (4.1) *satisfies*

$$\begin{aligned}\frac{d}{dt}E(t) =&2\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} = \frac{2}{\beta}\langle T\eta,\, \eta\rangle_{\mathcal{M}} - 8\alpha \left\| w_t \right\|^2 \\ =&\frac{1}{\beta}\int_0^\infty \mu'(s) \left\| \eta_x(s) \right\|^2 ds - 8\alpha \left\| w_t \right\|^2 = -\frac{1}{\beta}\Gamma\left[\eta\right] - 8\alpha \left\| w_t \right\|^2 . \end{aligned} \tag{4.2}$$

Proof. Multiplying the first four equations in (3.2) by $\varphi_t, (3w_t - \psi_t), w_t$ and θ_t respectively, integrating on $(0,1)$, using the integration by parts formula, the boundary conditions and then adding the obtained results, we arrive at

$$\begin{aligned}&\frac{d}{dt}\Big[\rho \left\|\varphi_t\right\|^2 + G\left\|\psi - \varphi_x\right\|^2 + I_\rho \left\|3w_t - \psi_t\right\|^2 + D\left\|3w_x - \psi_x\right\|^2 \\ &\quad + 3I_\rho \left\|w_t\right\|^2 + 3D\left\|w_x\right\|^2 + 4\gamma \left\|w\right\|^2 + \rho_3 \left\|\theta\right\|^2\Big] \\ &= -8\alpha \left\|w_t\right\|^2 + \frac{2}{\beta}\int_0^1\int_0^\infty \mu(s)\theta(t)\eta_{xx}(s)dx\, ds.\end{aligned} \tag{4.3}$$

On the other hand, from the fifth equation in (3.2), we get

$$\frac{2}{\beta}\int_0^1\int_0^\infty \mu(s)\theta(t)\eta_{xx}(s)dx\, ds = -\frac{1}{\beta}\frac{d}{dt}\|\eta\|^2_{\mathcal{M}} + \frac{1}{\beta}\int_0^\infty \mu'(s)\left\|\eta_x(s)\right\|^2 ds. \tag{4.4}$$

Finally, combining (4.3) and (4.4) completes the proof. □

In the following lemmas, we will denote c_0, a positive generic constant whose value may not necessarily be the same from line to line.

Lemma 4.3. *Let*

$$\Theta(t) = \int_0^x \theta(y,t)dy,$$

and let the functional

$$I_1(t) = \frac{\rho_3 I_\rho}{\delta}\left\langle (3w-\psi)_t\,, \Theta(t)\right\rangle. \tag{4.5}$$

Then, for any $\varepsilon_1, \varepsilon_2 > 0$, the functional I_1 satisfies the estimate

$$\begin{aligned}\frac{d}{dt}I_1(t) \leq& -\frac{I_\rho}{2}\|3w_t - \psi_t\|^2 + \varepsilon_1\|3w_x - \psi_x\|^2 + \varepsilon_2\|\psi - \varphi_x\|^2 \\ &+ c_0\left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\|\theta\|^2 + c_0\Gamma\left[\eta\right].\end{aligned} \tag{4.6}$$

Proof. Direct computations, using equations $(3.2)_2$ and $(3.2)_4$, integrating by parts and bearing in mind that

$$\int_0^1 \theta(x,t)\, dx = 0, \tag{4.7}$$

we get

$$\frac{d}{dt}I_1(t) = -I_\rho\|3w_t-\psi_t\|^2+k\|\theta\|^2+\frac{I_\rho}{\delta\beta}\int_0^\infty \mu(s)\left\langle \eta_x(s),(3w-\psi)_t\right\rangle ds$$
$$-\frac{kD}{\delta}\left\langle (3w-\psi)_x\,,\theta\right\rangle+\frac{Gk}{\delta}\left\langle (\psi-\varphi_x)\,,\Theta(t)\right\rangle. \tag{4.8}$$

By Young's and Cauchy-Schwarz inequalities, for any $\varepsilon_1, \varepsilon_2 > 0$, we obtain

$$\begin{aligned}-\frac{kD}{\delta}\left\langle (3w-\psi)_x\,,\theta\right\rangle &\le c_0\|3w_x-\psi_x\|\|\theta\|\\ &\le \varepsilon_1\|3w_x-\psi_x\|^2+\frac{c_0}{\varepsilon_1}\|\theta\|^2,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{Gk}{\delta}\left\langle (\psi-\varphi_x)\,,\Theta(t)\right\rangle &\le c_0\|\psi-\varphi_x\|\|\Theta\|\\ &\le \varepsilon_2\|\psi-\varphi_x\|^2+\frac{c_0}{\varepsilon_2}\|\Theta\|^2\\ &\le \varepsilon_2\|\psi-\varphi_x\|^2+\frac{c_0}{\varepsilon_2}\|\theta\|^2.\end{aligned} \tag{4.10}$$

Applying the same inequalities and taking into account the estimate (2.4), we obtain

$$\begin{aligned}\frac{I_\rho}{\delta\beta}\int_0^\infty \mu(s)\left\langle \eta_x(s),(3w-\psi)_t\right\rangle ds &\le c_0\|3w_t-\psi_t\|\|\eta\|_{\mathcal{M}}\\ &\le \frac{I_\rho}{2}\|3w_t-\psi_t\|^2+c_0\Gamma\left[\eta\right],\end{aligned} \tag{4.11}$$

Thus, (4.6) follows by simple substitution of (4.9)-(4.11) into (4.8). □

Lemma 4.4. *Let*

$$\Phi(t)=\int_0^x \varphi(y,t)dy,$$

Then, the functional I_2 defined by

$$I_2(t) = -\rho\left\langle \Phi_t,\, w\right\rangle + I_\rho\left\langle w_t,\, w\right\rangle+\frac{2}{3}\alpha\|w\|^2, \tag{4.12}$$

satisfies, for any $\varepsilon_3 > 0$, the following estimate

$$\frac{d}{dt}I_2(t) \le -D\|w_x\|^2-\frac{4}{3}\gamma\|w\|^2+\varepsilon_3\|\varphi_t\|^2+c_0\left(1+\frac{1}{\varepsilon_3}\right)\|w_t\|^2. \tag{4.13}$$

Proof. A simple derivation of $I_2(t)$, use of $(3.2)_1$ and $(3.2)_3$, integrating by parts and bearing in mind that

$$\int_0^1 \varphi(x,t)\,dx = 0,$$

we obtain

$$\frac{d}{dt}I_2(t) = -D\|w_x\|^2-\frac{4}{3}\gamma\|w\|^2+I_\rho\|w_t\|^2-\rho\left\langle \Phi_t,\, w_t\right\rangle.$$

Use of Young and Cauchy-Schwarz inequalities, gives (4.13). □

Lemma 4.5. *The functional*

$$I_3(t) = I_\rho \langle \psi - \varphi_x, w_t \rangle - I_\rho \langle w_x, \varphi_t \rangle \ . \tag{4.14}$$

satisfies, for $\varepsilon_4 > 0$,

$$\begin{aligned} \frac{d}{dt} I_3(t) \leq & - \frac{G}{2} \|\psi - \varphi_x\|^2 + \varepsilon_4 \|3w_t - \psi_t\|^2 + c_0 \left(1 + \frac{1}{\varepsilon_4}\right) \|w_t\|^2 \\ & + c_0 \|w_x\|^2 + I_\rho \left(\frac{G}{\rho} - \frac{D}{I_\rho}\right) \langle (\psi - \varphi_x)_x , w_x \rangle . \end{aligned} \tag{4.15}$$

Proof. Differentiating $I_3(t)$, using the first and third equations in (3.2) and integrating by parts we have

$$\begin{aligned} \frac{d}{dt} I_3(t) = & I_\rho \langle \psi - \varphi_x, w_{tt} \rangle - I_\rho \langle w_x, \varphi_{tt} \rangle + I_\rho \langle \psi_t, w_t \rangle \\ = & - G \|\psi - \varphi_x\|^2 - \frac{4}{3} \gamma \langle \psi - \varphi_x, w \rangle - \frac{4}{3} \alpha \langle \psi - \varphi_x, w_t \rangle \\ & + I_\rho \langle \psi_t, w_t \rangle + I_\rho \left(\frac{G}{\rho} - \frac{D}{I_\rho}\right) \langle (\psi - \varphi_x)_x , w_x \rangle . \end{aligned}$$

Using the simple relation $\psi_t = 3w_t - (3w_t - \psi_t)$, we have

$$\begin{aligned} \frac{d}{dt} I_3(t) = & -G \|\psi - \varphi_x\|^2 - \frac{4}{3} \gamma \langle \psi - \varphi_x, w \rangle - \frac{4}{3} \alpha \langle \psi - \varphi_x, w_t \rangle + 3 I_\rho \|w_t\|^2 \\ & - I_\rho \langle (3w_t - \psi_t) , w_t \rangle + I_\rho \left(\frac{G}{\rho} - \frac{D}{I_\rho}\right) \langle (\psi - \varphi_x)_x , w_x \rangle . \end{aligned} \tag{4.16}$$

Thanks to the inequalities of Young and Poincaré, we obtain

$$-\frac{4}{3} \gamma \langle \psi - \varphi_x, w \rangle \leq \frac{G}{4} \|\psi - \varphi_x\|^2 + c_0 \|w_x\|^2, \tag{4.17}$$

and

$$-\frac{4}{3} \alpha \langle \psi - \varphi_x, w_t \rangle \leq \frac{G}{4} \|\psi - \varphi_x\|^2 + c_0 \|w_t\|^2. \tag{4.18}$$

Moreover, for any $\varepsilon_4 > 0$, we get

$$-I_\rho \langle (3w_t - \psi_t) , w_t \rangle \leq \varepsilon_4 \|3w_t - \psi_t\|^2 + \frac{c_0}{\varepsilon_4} \|w_t\|^2. \tag{4.19}$$

Substituting (4.17)-(4.19) into (4.16) yield (4.15). □

Lemma 4.6. *The functional*

$$I_4(t) = -\frac{k}{g(0)} \int_0^\infty \mu(s) \left\langle \theta(t), \eta^t(s) \right\rangle ds, \tag{4.20}$$

satisfies, for $\varepsilon_4 > 0$, *the estimate*

$$\frac{d}{dt} I_4(t) \leq -\frac{k}{2} \|\theta\|^2 + \varepsilon_4 \|3w_t - \psi_t\|^2 + c_0 \left(1 + \frac{1}{\varepsilon_4}\right) \Gamma[\eta] . \tag{4.21}$$

Proof. Direct computation, using equations $(3.2)_4$,$(3.2)_5$ and integrating by parts, we obtain

$$\frac{d}{dt}I_4(t) = -\frac{k}{g(0)}\int_0^\infty \mu(s)\langle T\eta(s),\theta(t)\rangle\,ds + \frac{1}{\beta g(0)}\left\|\int_0^\infty \mu(s)\eta_x(s)ds\right\|^2$$
$$-\frac{\delta}{g(0)}\int_0^\infty \mu(s)\langle \eta_x(s),(3w-\psi)_t\rangle\,ds\;-\;k\|\theta\|^2. \tag{4.22}$$

By virtue of Young's inequality and applying (2.4), we find

$$\begin{aligned}-\frac{k}{g(0)}\int_0^\infty \mu(s)\langle T\eta(s),\theta(t)\rangle\,ds &= -\frac{k}{g(0)}\int_0^\infty \mu'(s)\langle \eta(s),\theta(t)\rangle\,ds\\ &\le c_0\|\theta\|\sqrt{\Gamma[\eta]}\\ &\le \frac{k}{2}\|\theta\|^2+c_0\Gamma[\eta]\ ,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\frac{1}{\beta g(0)}\left\|\int_0^\infty \mu(s)\eta_x(s)ds\right\|^2 &\le\; c_0\|\eta\|^2_{\mathcal{M}}\\ &\le\; c_0\Gamma[\eta]\ ,\end{aligned} \tag{4.24}$$

$$\begin{aligned}-\frac{\delta}{g(0)}\int_0^\infty \mu(s)\langle \eta_x(s),(3w-\psi)_t\rangle\,ds &\le c_0\|3w_t-\psi_t\|\|\eta\|_{\mathcal{M}}\\ &\le \varepsilon_4\|3w_t-\psi_t\|^2+\frac{c_0}{\varepsilon_4}\Gamma[\eta]\ .\end{aligned} \tag{4.25}$$

Simple substitution of (4.23)-(4.25) into (4.22) gives (4.21). □

Lemma 4.7. *The functional $I_5(t)$ defined by*

$$I_5(t) = -\rho\langle\varphi_t(t),\varphi(t)\rangle\,, \tag{4.26}$$

satisfies, the following estimate

$$\frac{d}{dt}I_5(t) \le -\rho\|\varphi_t\|^2+\frac{D}{8}\|3w_x-\psi_x\|^2+c_0\|\psi-\varphi_x\|^2+c_0\|w_x\|^2. \tag{4.27}$$

Proof. Differentiating I_5, using the first equation in (3.2), integrating by parts and using the simple equality $\varphi_x = -(\psi-\varphi_x)-(3w-\psi)+3w$, we get

$$\frac{d}{dt}I_5(t) = -\rho\|\varphi_t\|^2+G\|\psi-\varphi_x\|^2+G\langle\psi-\varphi_x,3w-\psi\rangle-3G\langle\psi-\varphi_x,w\rangle\,.$$

Estimate (4.27) follows by exploiting Young's and Poincaré inequalities. □

Lemma 4.8. *The functional $I_6(t)$ defined by*

$$I_6(t) = I_\rho\langle 3w_t-\psi_t,\,3w-\psi\rangle\,, \tag{4.28}$$

satisfies, the estimate

$$\frac{d}{dt}I_6(t) \le -\frac{D}{2}\|3w_x-\psi_x\|^2+I_\rho\|3w_t-\psi_t\|^2+c_0\|\psi-\varphi_x\|^2+c_0\|\theta\|^2. \tag{4.29}$$

Proof. By exploiting $I_6(t)$, using the second equation in (3.2) and integrating by parts, we find

$$I_\rho\,(3w-\psi)_{tt} - D\,(3w-\psi)_{xx} - G\,(\psi-\varphi_x) + \delta\theta_x = 0$$

$$\begin{aligned}\frac{d}{dt}I_6(t) =& - D\|3w_x-\psi_x\|^2 + I_\rho\|3w_t-\psi_t\|^2 + G\,\langle 3w-\psi, \psi-\varphi_x\rangle \\ &+ \delta\,\langle 3w_x-\psi_x, \theta\rangle\,.\end{aligned}$$

Estimate (4.29) easily follows thanks to Young's and Poincaré inequalities. □

Proof of Theorem 4.1

We define the Lyapunov functional $\mathcal{F}(t)$ by

$$\mathcal{F}(t) = NE(t) + N_1I_1(t) + N_2I_2(t) + N_3I_3(t) + N_4I_4(t) + I_5(t) + I_6(t), \tag{4.30}$$

where $N, N_4\,(i=1:4)$ are positive constants to be chosen appropriately later.

Differentiating (4.30), applying the estimates (4.2), (4.6), (4.13), (4.15), (4.21), (4.27), (4.29), and adding the obtained results, we obtain

$$\begin{aligned}\mathcal{F}^{'}(t) \le& -\left(\frac{I_\rho}{2}N_1 - N_3\varepsilon_4 - N_4\varepsilon_4 - I_\rho\right)\|3w_t-\psi_t\|^2 - (DN_2 - c_0N_3 - c_0)\,\|w_x\|^2 \\ &- \left(\frac{G}{2}N_3 - N_1\varepsilon_2 - 2c_0\right)\|\psi-\varphi_x\|^2 - \left(\frac{D}{2} - \frac{D}{8} - N_1\varepsilon_1\right)\|3w_x-\psi_x\|^2 \\ &- (\rho - N_2\varepsilon_3)\,\|\varphi_t\|^2 - \left(\frac{k}{2}N_4 - N_1c_0\left(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right) - c_0\right)\|\theta\|^2 - \left(\frac{4}{3}\gamma N_2\right)\|w\|^2 \\ &- \left(8\alpha N - N_3c_0\left(1+\frac{1}{\varepsilon_4}\right) - c_0N_2\left(1+\frac{1}{\varepsilon_3}\right)\right)\|w_t\|^2 \\ &- \left(\frac{N}{\beta} - c_0N_1 - N_4\left(c_0\left(1+\frac{1}{\varepsilon_4}\right)\right)\right)\Gamma\,[\eta] + N_3\left(I_\rho\left(\frac{G}{\rho} - \frac{D}{I_\rho}\right)\langle(\psi-\varphi_x)_x\,, w_x\rangle\right)\end{aligned}$$

Then, taking

$$\varepsilon_1 = \frac{D}{8N_1}, \qquad \varepsilon_2 = \frac{GN_3}{4N_1}, \qquad \varepsilon_3 = \frac{\rho}{2N_2}, \qquad \varepsilon_4 = \frac{I_\rho N_1}{4\,(N_3+N_4)},$$

we arrive at

$$\begin{aligned}\mathcal{F}^{'}(t) \le& -\left[\frac{I_\rho}{4}N_1 - I_\rho\right]\|3w_t-\psi_t\|^2 - [DN_2 - c_0N_3 - c_0]\,\|w_x\|^2 \\ &- \left[\frac{G}{4}N_3 - c_0\right]\|\psi-\varphi_x\|^2 - \frac{D}{4}\|3w_x-\psi_x\|^2 - \frac{\rho}{2}\|\varphi_t\|^2 \\ &- \left[\frac{k}{2}N_4 - c_0N_1\left(N_1 + \frac{N_1}{N_3} + 1\right)\right]\|\theta\|^2 - \frac{4}{3}\gamma N_2\|w\|^2 \\ &- \left[8\alpha N - c_0N_2\,(N_2+1) - c_0N_3\left(1+\frac{N_3+N_4}{N_1}\right)\right]\|w_t\|^2 \\ &- \left[\frac{N}{\beta} - c_0N_1 - c_0N_4\left(1+\frac{N_3+N_4}{N_1}\right)\right]\Gamma\,[\eta]\,.\end{aligned}$$

We will carefully choose our parameters so that the bracketed terms are positive. First, we choose N_1 and N_3 large enough so that

$$\frac{I_\rho}{4}N_1 - I_\rho > 0, \qquad \text{and} \qquad \frac{G}{4}N_3 - c_0 > 0.$$

Then, we select N_2 and N_4 large enough so that

$$DN_2 - c_0N_3 - c_0 > 0, \qquad \text{and} \qquad \frac{k}{2}N_4 - c_0N_1\left(N_1 + \frac{N_1}{N_3} + 1\right) > 0.$$

Finally, we choose N very large enough so that

$$8\alpha N - c_0N_2\left(N_2+1\right) - c_0N_3\left(1 + \frac{N_3+N_4}{N_1}\right) > 0,$$

and

$$\frac{N}{\beta} - c_0N_1 - c_0N_4\left(1 + \frac{N_3+N_4}{N_1}\right) > 0,$$

and, further, $\mathcal{L} \sim E$.

Consequently, there exists a positive constant τ_1 such that

$$\mathcal{L}'(t) \leq -\tau_1 E(t), \quad \text{for all } t \geq 0.$$

Using the fact that $\mathcal{L} \sim E$, we infer that

$$\mathcal{L}'(t) \leq \; -\tau_2\,\mathcal{L}(t), \qquad \tau_2 > 0, \quad \text{for all } t \geq 0.$$

A simple integration of the last inequality over $(0, t)$, gives

$$\mathcal{L}(t) \leq \; \mathcal{L}(0)\, e^{-\tau_2\, t}, \quad \text{for all } t \geq 0.$$

Consequently, by virtue of $\mathcal{L} \sim E$, we conclude the exponential stability of the energy. The proof is now complete.

5 Lack of exponential stability

In the previous section, we proved that if $\chi = \frac{G}{\rho} - \frac{D}{I_\rho} = 0$, the semigroup $S\left(t\right)$ associated with the system (3.2) is exponentially stable. In this section, we will prove that if the condition $\chi = 0$ is not valid, the semigroup $S\left(t\right)$ is not exponentially stable. For this purpose, we will use the following characterization of the exponential stability due to Gearhart, Huang and Pruss (see [42], [43], [44]).

Theorem 5.1. *Let $\varrho\left(\mathcal{A}\right)$ be the resolvent set of the operator $\mathcal{A}$ and $S\left(t\right) = e^{t\mathcal{A}}$ be the C_0-semigroup of contractions generated by $\mathcal{A}$. Then $S\left(t\right)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(\mathcal{A}) \tag{5.1}$$

and

$$\limsup_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{5.2}$$

Now, we establish our result of the lack of exponential stability.

Theorem 5.2. *Let* $\gamma_g = 1 - \frac{\beta}{g(0)}\frac{kG}{\rho}$ *and let* χ_g *be the stability number defined in (1.3). If* $\chi \neq 0$ *and* $\gamma_g = 0$, *or if* $\chi\chi_g\gamma_g \neq 0$, *the semigroup* $S(t) = e^{t\mathcal{A}}$ *generated by* $\mathcal{A}$ *is not exponentially stable.*

Proof. We will prove that if $\chi \neq 0$, the associated semigroup $S(t)$ is not exponentially stable. More precisely, we will prove that if $\chi \neq 0$, the condition (5.2) is not satisfied. Equivalently, we will show that there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that

$$|\lambda_n| \to \infty \tag{5.3}$$

and

$$\lim_{n\to\infty} \|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

Equivalently, we must find a bounded sequence $(F_n) \subset \mathcal{H}$ such that

$$\lim_{n\to\infty} \|(i\lambda_n I - \mathcal{A})^{-1}F_n\|_{\mathcal{H}} = \infty.$$

For each $n \in \mathbb{N}$, let us denote

$$U_n = (i\lambda_n I - \mathcal{A})^{-1}F_n, \tag{5.4}$$

where

$$U_n = \left(\varphi_n, \phi_n, \xi_n, v_n, w_n, W_n, \theta_n, \eta_n^t\right) \quad \text{and} \quad F_n = \left(f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6, f_n^7\right).$$

With this notation, we will find a bounded sequence $(F_n) \subset \mathcal{H}$ such that the sequence $(U_n) \subset D(\mathcal{A})$ defined in (5.5) verifies

$$\lim_{n\to\infty} \|U_n\|_{\mathcal{H}} = \infty. \tag{5.5}$$

Note that the equation (5.4) can be written as

$$i\lambda_n U_n - \mathcal{A}U_n = F_n, \;\; n \in \mathbb{N}, \tag{5.6}$$

which, when decomposed into its components, gives rise to the following system of equations

$$i\lambda_n\varphi_n - \phi_n = f_n^1 \;\text{ in } H_*^1, \tag{5.7}$$

$$i\lambda_n\phi_n + \frac{G}{\rho}\left(3w_n - \xi_n - \varphi_{n,x}\right)_x = f_n^2 \;\text{ in } L_*^2, \tag{5.8}$$

$$i\lambda_n\xi_n - v_n = f_n^3 \;\text{ in } H_0^1, \tag{5.9}$$

$$i\lambda_n v_n - \frac{1}{I_\rho}\left[D\xi_{n,xx} + G\left(3w_n - \xi_n - \varphi_{n,x}\right) - \delta\theta_{n,x}\right] = f_n^4 \;\text{ in } L^2, \tag{5.10}$$

$$i\lambda_n w_n - W_n = f_n^5 \;\text{ in } H_0^1, \tag{5.11}$$

$$i\lambda_n W_n - \frac{1}{I_\rho}\left[Dw_{n,xx} - G\left(3w_n - \xi_n - \varphi_{n,x}\right) - \frac{4\gamma}{3}w_n - \frac{4\alpha}{3}W_n\right] = f_n^6 \;\text{ in } L^2, \tag{5.12}$$

$$i\lambda_n\theta_n - \frac{1}{k}\left[\frac{1}{\beta}\int_0^\infty \mu(s)\eta_{n,xx}(s)\,ds - \delta v_{n,x}\right] = f_n^7 \;\text{ in } L_{g_1}^2, \tag{5.13}$$

$$i\lambda_n\eta_n - T\eta_n - \theta_n = f_n^8 \ \text{ in } L^2_{g_2}, \tag{5.14}$$

From here, let us consider the sequence

$$F_n = \left(0, \frac{\cos(n\pi x)}{\rho}, 0, 0, 0, 0, 0, 0\right).$$

Then, from equations (5.7), (5.9) and (5.11) , we obtain $\phi_n = i\lambda_n\varphi_n, v_n = i\lambda_n\xi_n$ and $W_n = i\lambda_n w_n$. Replacing in the other equations, we obtain

$$\begin{aligned}
-\lambda_n^2\varphi_n + \frac{G}{\rho}\left(3w_n - \xi_n - \varphi_{n,x}\right)_x &= \frac{\cos(n\pi x)}{\rho} \quad \text{in} L^2_*, \\
-\lambda_n^2\xi_n - \frac{1}{I_\rho}\left[D\xi_{n,xx} + G\left(3w_n - \xi_n - \varphi_{n,x}\right) - \delta\theta_{n,x}\right] &= 0 \text{ in } L^2, \\
-\lambda_n^2 w_n - \frac{1}{I_\rho}\left[Dw_{n,xx} - G\left(3w_n - \xi_n - \varphi_{n,x}\right) - \frac{4\gamma}{3}w_n - \frac{4\alpha}{3}i\lambda_n w_n\right] &= 0 \ \text{ in } L^2, \\
i\lambda_n\theta_n - \frac{1}{k}\left[\frac{1}{\beta}\int_0^\infty \mu(s)\eta_{n,xx}(s)\,ds - \delta i\lambda_n\xi_{n,x}\right] &= 0 \text{ in } L^2_{g_1}, \\
i\lambda_n\eta_n - T\eta_n - \theta_n &= 0 \ \text{ in } L^2_{g_2},
\end{aligned}$$

According to the boundary conditions, we define

$$\begin{gathered}
\varphi_n(x) = A_n\cos(n\pi x), \quad \xi_n(x) = B_n\sin(n\pi x), \quad w_n(x) = C_n\sin(n\pi x), \\
\theta_n(x) = H_n\cos(n\pi x), \quad \eta_n(x) = R_n\cos(n\pi x),
\end{gathered} \tag{5.15}$$

where (A_n), (B_n), (C_n), (H_n) and (R_n) are real sequences to be fixed later. Replacing (5.15) in the last system of equations, we get

$$\left\{\begin{aligned}
\left(G(n\pi)^2 - \rho\lambda_n^2\right)A_n - Gn\pi B_n + 3Gn\pi C_n &= 1 \\
-n\pi G A_n + \left(D(n\pi)^2 - \lambda_n^2 I_\rho + G\right)B_n - 3GC_n - \delta n\pi H_n &= 0 \\
3Gn\pi A_n - 3GB_n + \left(3D(n\pi)^2 - 3\lambda_n^2 I_\rho + 9G + 4\gamma + 4\alpha i\lambda_n\right)C_n &= 0 \\
\delta\beta i\lambda_n n\pi B_n + i\lambda_n k\beta H_n + \textstyle\int_0^\infty \mu(s)(n\pi)^2 R_n(s)\ ds &= 0 \\
i\lambda_n R_n(s) + R_n'(s) - H_n &= 0.
\end{aligned}\right. \tag{5.16}$$

Integrating the last equation, we get

$$R_n(s) = \frac{H_n}{i\lambda_n}\left(1 - e^{-i\lambda_n s}\right). \tag{5.17}$$

Substituting (5.17) into the fourth equation of (5.16), we obtain

$$-\delta\beta\lambda_n^2 n\pi B_n - \lambda_n^2 k\beta H_n + (n\pi)^2 H_n\int_0^\infty \mu(s)\,ds - (n\pi)^2 H_n\int_0^\infty \mu(s)\left(e^{-i\lambda_n s}\right)\ ds = 0.$$

Then, applying the condition (2.1) on μ and denoting

$$\widehat{F}(\lambda_n) = \int_0^\infty \mu(s)\left(e^{-i\lambda_n s}\right)\ ds,$$

the system (5.16) takes the form

$$
\begin{cases}
\left(G\left(n\pi\right)^2-\rho\lambda_n^2\right)A_n-Gn\pi B_n+3Gn\pi C_n=1\\
-n\pi GA_n+\left(D\left(n\pi\right)^2-\lambda_n^2I_\rho+G\right)B_n-3GC_n-\delta n\pi H_n=0\\
3Gn\pi A_n-3GB_n+\left(3D\left(n\pi\right)^2-3\lambda_n^2I_\rho+9G+4\gamma+4\alpha i\lambda_n\right)C_n=0\\
-\delta\beta\lambda_n^2n\pi B_n+\left(-\lambda_n^2k\beta+\left(n\pi\right)^2g\left(0\right)-\left(n\pi\right)^2\widehat{F}\left(\lambda_n\right)\right)H_n=0,
\end{cases}
$$

If we define

$$
\begin{aligned}
&p_1=G\left(n\pi\right)^2-\rho\lambda_n^2, \qquad p_2=D\left(n\pi\right)^2-\lambda_n^2I_\rho+G\\
&p_3=3D\left(n\pi\right)^2-3\lambda_n^2I_\rho+9G+4\gamma+4\alpha i\lambda_n\\
&p_4=-\lambda_n^2k\beta+\left(n\pi\right)^2g\left(0\right)-\left(n\pi\right)^2\widehat{F}\left(\lambda_n\right),
\end{aligned}
$$

the previous system is written as

$$
\begin{cases}
p_1A_n-Gn\pi B_n+3Gn\pi C_n=1\\
-n\pi GA_n+p_2B_n-3GC_n-\delta n\pi H_n=0\\
3Gn\pi A_n-3GB_n+p_3C_n=0\\
-\delta\beta\lambda_n^2n\pi B_n+p_4H_n=0.
\end{cases}
\tag{5.18}
$$

Then, we choose the sequence (λ_n) defined as

$$
\lambda_n=\sqrt{\frac{G}{\rho}}\left(n\pi\right). \tag{5.19}
$$

Thus, we obtain

$$
p_1=0,\quad p_2=\left(n\pi\right)^2P_n,\quad p_3=\left(n\pi\right)^2Q_n,\quad p_4=\left(n\pi\right)^2S_n, \tag{5.20}
$$

where

$$
P_n=-I_\rho\chi+\frac{G}{\left(n\pi\right)^2}\ ,\quad Q_n=-3\chi I_\rho+\frac{9G+4\gamma}{\left(n\pi\right)^2}+\frac{4\alpha i}{n\pi}\sqrt{\frac{G}{\rho}}
$$

and

$$
S_n=g\left(0\right)\gamma_g-\widehat{F}\left(\lambda_n\right),\quad \text{with } \gamma_g=1-\frac{\beta}{g\left(0\right)}\frac{kG}{\rho}.
$$

By making $n\to\infty$, we obtain the following convergences

$$
P_n\to-I_\rho\chi\ ,\quad Q_n\to-3I_\rho\chi\ ,\quad S_n\to g\left(0\right)\gamma_g\ . \tag{5.21}
$$

The last convergence follows from the Riemann-Lebesgue lemma which guarantees that $\widehat{F}\left(\lambda_n\right)\to0$. Replacing (5.20) in the system (5.18), this becomes

$$
\begin{cases}
-Gn\pi B_n+3Gn\pi C_n=1\\
-n\pi GA_n+\left(n\pi\right)^2P_nB_n-3GC_n-\delta n\pi H_n=0\\
3Gn\pi A_n-3GB_n+\left(n\pi\right)^2Q_nC_n=0\\
-\delta\beta Gn\pi B_n+\rho S_nH_n=0.
\end{cases}
\tag{5.22}
$$

Using Cramer's rule, we obtain

$$
A_n=\frac{-\left(9G^2\rho S_n-\pi^4n^4\rho P_nQ_nS_n+\pi^4Gn^4\beta\delta^2Q_n\right)}{\pi^2G^2n^2\left(18G\rho S_n-9\pi^2n^2\rho P_nS_n-\pi^2n^2\rho Q_nS_n+9\pi^2Gn^2\beta\delta^2\right)}.
$$

Applying the convergences in (5.21), we get

$$\lim A_n = \frac{\pi^4 \left(-3I_\rho \chi\right) \left(\rho I_\rho g\left(0\right) \gamma_g \left(-\chi\right) - \beta G \delta^2\right)}{\pi^2 G^2 \left(12\pi^2 \rho I_\rho \chi g\left(0\right) \gamma_g + 9\pi^2 \beta G \delta^2\right)}$$
$$= \frac{I_\rho^2 \rho g\left(0\right) \chi \chi_g}{G^2 \left(4\rho I_\rho g\left(0\right) \chi_g + \beta G \delta^2\right)}\,, \tag{5.23}$$

where χ_g is the Liu–Zhao stability number defined in (1.3).

Thus, if $\chi \neq 0$ and $\gamma_g = 0$, we have

$$\lim_{n\to\infty} A_n = \frac{I_\rho \chi}{3G^2} \neq 0.$$

On the other hand, if $\chi \chi_g \gamma_g \neq 0$, we get

$$\lim_{n\to\infty} A_n \neq 0,$$

provided that the denominator of (5.23) is not zero, i.e., provided that

$$\chi_g \neq -\frac{\beta G \delta^2}{4g\left(0\right) \rho I_\rho}.$$

On the other hand, since the determinant Δ_n of the system (5.21) is of order 4, that is, $\Delta_n = O\left(n^4\right)$, we conclude that

$$B_n = \frac{-9\pi^3 G^2 n^3 \delta}{\Delta_n} \to 0 \qquad \text{and} \qquad C_n = \frac{-3\pi^3 G^2 n^3 \delta}{\Delta_n} \to 0.$$

Then $\|w_n\|$, $\|\xi_n\| \to 0$. Consequently

$$\begin{aligned}\|U_n\|_{\mathcal{H}} \geq\ & G\|3w_n - \xi_n - \varphi_{n,x}\| \geq G\left(\|\varphi_{n,x}\| - 3\|w_n\| - \|\xi_n\|\right)\\ &= G\left(\|A_n\left(n\pi\right)\cos\left(n\pi x\right)\| - 3\|w_n\| - \|\xi_n\|\right)\\ &= G\left(|A_n|\sqrt{\frac{n\pi}{2}} - 3\|w_n\| - \|\xi_n\|\right) \to \infty.\end{aligned}$$

This completes the proof. □

6 Conclusions and open problem

In this paper, we proved an exponential stability result for laminated beams with structural damping combined with heat conduction given by the Gurtin-Pipkin law under the single condition of equal wave speeds instead of complicated stability conditions involving the wave speeds and several other system parameters. A suitable definition and handling of the Lyapunov functionals have allowed us to simplify the exponential stability condition. The study of polynomial stability when $\chi \neq 0$ is an interesting open problem that we will address later.

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The authors declare no competing interests.

Author contributions

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