# Pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequential solutions to some semilinear difference equation in Banach spaces 

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#### Abstract

The main purpose of this paper is to investigate some existence results for pseudo $S$ asymptotically $(\omega, c)$-periodic sequential solutions to a semilinear difference equation of convolution type and a semilinear Weyl-like fractional difference equation in Banach spaces. For this purpose, we first give the definition of the pseudo $S$-asymptotically ( $\omega, c$ )periodic sequence and prove the completeness, convolution and superposition theorems for such a sequence in abstract spaces. We show some existence and uniqueness of pseudo $S$ asymptotically $(\omega, c)$-periodic sequential solutions under some different Lipschitz type conditions of the nonlinear force term with its second variable. We also consider the existence of pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequential solutions under a non-Lipschitz growth condition.


Keywords: Pseudo $S$-asymptotically $(\omega, c)$-periodic sequences; Weyl-like fractional difference equations; difference equation of convolution type; existence and uniqueness.

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## 1 Introduction

In this paper, we mainly consider some existence results for pseudo $S$-asymptotically ( $\omega, c$ )periodic sequential solutions to the following semilinear difference of convolution type

$$
\begin{equation*}
u(n+1)=A \sum_{k=-\infty}^{n} a(n-k) u(k+1)+\sum_{k=-\infty}^{n} b(n-k) p(k, u(k)), n \in \mathbb{Z}, \tag{1.1}
\end{equation*}
$$

where $a(k), b(k)$ are two suitable scalar-valued sequences to make Eq. (ㄸ.D) sense, the operator $A$ is closed and linear on a Banach space $X, p: \mathbb{Z} \times X \rightarrow X$ is a function satisfying some additional growth conditions which will be specified later, and $\mathbb{Z}$ denotes the set of all integers.

[^0]Let $a(k)=b(k)=\frac{\Gamma(k+\alpha)}{\Gamma(\alpha) \Gamma(n+1)}$ in Eq. (■.]) with Gamma function $\Gamma$ and arbitrarily nonnegative integer $k$, we will have the following semilinear fractional difference equation

$$
\begin{equation*}
\Delta^{\alpha} u(k)=A u(k+1)+f(k, u(k)), 0<\alpha \leq 1, k \in \mathbb{Z} \tag{1.2}
\end{equation*}
$$

where $\Delta^{\alpha}$ is understood in Weyl-like fractional difference operator, which was primarily introduced in [ [1, Definition 2.3]. To ensure the solvability of Eq. ([.2), we here assume that $1 \in \rho(A)$ (the resolvent set of $A$ ) and $\left\|(I-A)^{-1}\right\| \leq 1, I$ is the identity operator (see [ $\left.\mathbb{I}, \mathbb{Z}\right]$ ).

Here we first recall some known results for the existence of asymptotically sequential solutions to Eq. ([.لा) or ([2]). Keyantuo, Lizama, Rueda and Warma [I8] initially established the existence and uniqueness of weighted pseudo $S$-asymptotically $\omega$-periodic and weighted pseudo almost automorphic mild solutions to Eq. ([.لत) by introducing a discrete resolvent family (see Definition [2.2). Chang and Lü [II] investigated some existence results for weighted pseudo $S$ asymptotically $\omega$-antiperiodic mild solutions to Eq. ([.. $)$. Abadias and Lizama [I] presented the existence and uniqueness of almost automorphic solutions to Eq. ([.2) by introducing an $\alpha$-resolvent sequence. Alvarez and Lizama [3] considered the existence and uniqueness of weighted pseudo almost automorphic solutions and $S$-asymptotically $\omega$-periodic solutions to Eq. ([.2). Xia and Wang [27] established some sufficient conditions for the existence, uniqueness of weighted pseudo $S$-asymptotically $\omega$-periodic mild solutions to Eq. ( $\mathbb{L} 2)$ ). Cao, Samet and Zhou [9] showed some existence results for asymptotically almost periodic mild solutions to Eq. ([.2)). we notice that asymptotically sequential solutions to Eq. (ITI) or (LTV) in aforementioned works are considered under the framework of bounded sequences $\left(l^{\infty}(\mathbb{Z}, X)\right)$.

On the other hand, for the following second linear ordinary differential equation

$$
\begin{equation*}
x^{\prime \prime}(t)+q(t) x(t)=0, q(t+\omega)=q(t), \omega \neq 0, \tag{1.3}
\end{equation*}
$$

It is known from the Floquet theorem that Eq. ( $\llbracket .3)$ admits at least one constant $c \neq 0$ and one nontrivial solution $x(t)$ such that $x(t+\omega)=c x(t)$. The representation $x(t+\omega)=c x(t)$ includes some well-known functions such as the periodic function $(c=1)$, antiperiodic function ( $c=-1$, see [23, [24]), Bloch periodic function ( $c=e^{i k \omega}$, see [ [I], [.]]) and some unbounded functions $(|c| \neq 1)$, and $x(t)$ is called an $(\omega, c)$-periodic function. The vector-valued $(\omega, c)$-periodic function in abstract spaces was presented by Alvarez, Gómez and Pinto in [4]. Some generalized ( $\omega, c$ )periodic functions in abstract spaces were also introduced via different ergodic forms. For instance, $(\omega, c)$-asymptotically periodic functions and $(\omega, c)$-pseudo periodic functions in abstract spaces with their applications were studied by Alvarez, Castillo and Pinto in [5, 6$]$. The pseudo $S$-asymptotically ( $\omega, c$ )-periodic function in abstract spaces with applications was considered by Chang and Zhao in [ [ 2 ]. For more results on $(\omega, c)$-periodic functions with extensions and applications, we refer to [16, [20, 21, [22] and references cited therein. The ( $\omega, c$ )-periodic sequence in abstract spaces was presented by Alvarez, Díaz and Lizam in [7] as the discrete counterpart of the $(\omega, c)$-periodic function in [ 4$]$. The existence and uniqueness of $(\omega, c)$-periodic sequential solutions to Eq. ([.2) was established by Alvarez, Díaz and Lizam in [8], and such asymptotically sequential solutions to Eq. ([.2) may be unbounded. Since the influence of small perturbations on ( $\omega, c$ )-periodic sequences always exists, we naturally need to consider the generalized $(\omega, c)$ periodic sequences via some ergodic forms.

The first part of this paper is to present the definition and basic properties of the pseudo $S$-asymptotically $(\omega, c)$-periodic sequence in abstract spaces, which can be seen as the discrete
counterpart of the pseudo $S$-asymptotically ( $\omega, c$ )-periodic function in [ [12] and an extension of the ( $\omega, c$ )-periodic sequence in [ $[\boxed{]}]$. Noting that an $(\omega, c)$-periodic sequence may be unbounded $(|c| \neq 1)$, we can not deal with the pseudo $S$-asymptotically $(\omega, c)$-periodic sequence just under the framework of bounded sequences $\left(l^{\infty}(\mathbb{Z}, X)\right)$. We give the notion of the pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequence by constructing a suitable space $l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ for given $c \in \mathbb{C} \backslash 0, \omega \in \mathbb{Z}^{+}$(see Section 3). We prove the convolution theorem and the completeness of the space of such sequences with a suitable norm in $l_{\omega, c}^{\infty}(\mathbb{Z}, X)$. Some applicable superposition theorems are also shown for the Nemytskii's superposition operator of a pseudo $S$-asymptotically $(\omega, c)$-periodic sequence.

The second part of this paper is to discuss some existence results for pseudo $S$-asymptotically $(\omega, c)$-periodic sequential solutions to Eq. ( $\mathbb{L}$ ) and Eq. ( $\mathbb{L 2}$ ). Here we mainly apply the notion of the pseudo $S$-asymptotically $(\omega, c)$-periodic sequence and its properties given in the first part to Eq. ([.ل(). We prove some existence and uniqueness of pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequential mild solutions to Eq. ([.]) under some different globally Lipschitz type conditions. We also show some existence results for pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequential mild solutions to Eq. ( $\mathbb{\square}$ ) with a local Lipschitz or a non-Lipschitz growth condition respectively.

The remainder of the paper is outlined as follows: Section 2 is Preliminaries which include some basic notions, lemmas and notations which will be used throughout this paper. Section 3 is mainly focused the pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequence and its properties. Section 4 is mainly concerned with some existence results for pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequential solutions.

## 2 Preliminaries

Let notations $\mathbb{R}, \mathbb{Z}^{+}\left(\mathbb{R}^{+}\right), \mathbb{Z}_{+}\left(\mathbb{R}_{+}\right)$and $\mathbb{C}$ denote the set of all real numbers, positive integers (real numbers), nonnegative integers (real numbers) and complex numbers respectively. Let $X$ be a Banach space and $l^{\infty}(\mathbb{Z}, X)$ denotes the Banach space consists of all bounded sequences $u: \mathbb{Z} \rightarrow X$ with sup norm $\|u\|_{\infty}=\sup _{k \in \mathbb{Z}}\|u(k)\|$. The space $F(\mathbb{Z}, X)$ represents the set formed by all sequences $u: \mathbb{Z} \rightarrow X$. We denote by $\mathcal{B}(X)$ the space of all bounded linear operators from $X$ to $X$. For a given $v: \mathbb{Z}\left(\mathbb{Z}_{+}\right) \rightarrow \mathbb{C}$ and $1 \leq q<\infty$, if $\sum_{k \in \mathbb{Z}}^{\infty}|v(k)|^{q}<\infty\left(\sum_{k=0}^{\infty}|v(k)|^{q}<\infty\right)$, then $v$ is called a $q$-th summable sequence; particularly $v$ is called a summable sequence if $q=1$. For other unmentioned notations and definitions, we refer to [2, [4], [18].

Definition 2.1 [ $\mathbb{Z}]$ For given $c \in \mathbb{C} \backslash\{0\}, \omega \in \mathbb{Z}^{+}$, a sequence $f \in F(\mathbb{Z}, X)$ is said to be $(\omega, c)$-periodic if $f(k+\omega)=c f(k)$ for all $k \in \mathbb{Z} . \omega$ is called the $c$-period of $f$.

We denote by $P_{\omega, c}(\mathbb{Z}, X)$ the set of all $(\omega, c)$-periodic sequences from $\mathbb{Z}$ to $X$. When $c=1$ ( $\omega$-periodic case) we write $P_{\omega}(\mathbb{Z}, X)$ instead of $P_{\omega, 1}(\mathbb{Z}, X)$. Using the principal branch of the complex Logarithm, $c^{\frac{k}{\omega}}$ is defined as $c^{\frac{k}{\omega}}:=\exp \left(\frac{k}{\omega} \log (c)\right)=c^{\wedge}(k)$ and we will use notation $|c|(k)=\left|c^{\wedge}(k)\right|=|c|^{\frac{k}{\omega}}$ (see [ $[4]$ ).

Definition 2.2 [ $[8$, Definition 3.2] Let $A$ be a closed linear operator with its domain $D(A)$ defined on a Banach space $X$. Assume that $a, b$ are two scalar-valued sequences. An operator-
valued sequence $\{S(k)\}_{k \in \mathbb{Z}_{+}} \subset \mathcal{B}(X)$ is called a discrete resolvent family generated by $A$ if it verifies the following conditions:
(1) $S(k)(X) \subset D(A)$, and $S(k) A x=A S(k) x$ for all $x \in D(A)$ and $k \in \mathbb{Z}_{+}$;
(2) For $x \in D(A)$ and $k \in \mathbb{Z}_{+}, S(k) x=b(k) x+A \sum_{i=0}^{k} a(k-i) S(k) x$.

For more results on properties and applications of the discrete resolvent family $\{S(k)\}_{k \in \mathbb{Z}_{+}}$, we


We recall a compactness criterion established in [Z] and well-known fixed point theorems. Let $h: \mathbb{Z} \rightarrow \mathbb{R}^{+}$be a function satisfying that $h(n) \geq 1$ for all $n \in \mathbb{Z}$ and $h(n) \rightarrow \infty$ as $|n| \rightarrow \infty$. We consider the space

$$
C_{h}(\mathbb{Z}, X)=\left\{u: \mathbb{Z} \rightarrow X \left\lvert\, \lim _{|n| \rightarrow \infty} \frac{\|u(n)\|}{h(n)}=0\right.\right\},
$$

which is a Banach space endowed with the norm $\|u\|_{h}=\sup _{n \in \mathbb{Z}} \frac{\|u(n)\|}{h(n)}$, see [Z] for details.
Lemma 2.1 [ $\mathbb{Z}]$ A subset $\mathbb{K} \subseteq C_{h}(\mathbb{Z}, X)$ is a relatively compact set in $C_{h}(\mathbb{Z}, X)$ if it verifies the following conditions:
(i) The set $Q_{n}(\mathbb{K})=\left\{\frac{u(n)}{h(n)}: u \in \mathbb{K}\right\}$ is relatively compact in $X$ for all $n \in \mathbb{Z}$.
(ii) $\mathbb{K}$ is weighted equiconvergent at $\pm \infty$, that is for any $\epsilon>0$, there exists a constant $N>0$ such that $\|u(n)\| \leq \epsilon h(n)$ for each $|n| \geq N$ and all $u \in \mathbb{K}$.

Lemma 2.2 [ 17 , Banach fixed point theorem] Let $(Z, d)$ be a complete metric space and $\Upsilon$ : $Z \rightarrow Z$. If there exist constants $\bar{L} \in[0,1)$ and $n_{0} \in \mathbb{N}$ such that

$$
d\left(\Upsilon^{n_{0}}(x), \Upsilon^{n_{0}}(y)\right) \leq \bar{L} d(x, y), \text { for all } x, y \in Z,
$$

then $\Upsilon$ has a unique fixed point in $Z$, where $\Upsilon^{2} x=\Upsilon(\Upsilon x), \Upsilon^{3} x=\Upsilon\left(\Upsilon^{2} x\right), \cdots, \Upsilon^{n_{0}} x=$ $\Upsilon\left(\Upsilon^{n_{0}-1} x\right), \cdots$.

Lemma 2.3 [ 17 , Schaefer fixed point theorem] Let $Y$ be a normed linear space with $\mathbb{O} \subset Y$ convex and $0 \in \mathbb{O}$. Assume that $\Upsilon: \mathbb{O} \rightarrow \mathbb{O}$ is a completely continuous operator. If the set

$$
\epsilon(\Upsilon)=\{x \in \mathbb{O}: x=\lambda \Upsilon x \text { for some } \lambda \in(0,1)\}
$$

is bounded, then $\Upsilon$ has at least one fixed point in $\mathbb{O}$.

## 3 Pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequences

In this section, we introduce the notion of the pseudo $S$-asymptotically $(\omega, c)$-periodic sequence and establish some of its fundamental properties.
For given $c \in \mathbb{C} \backslash\{0\}, \omega \in \mathbb{Z}^{+}$, we define the following set

$$
l_{\omega, c}^{\infty}(\mathbb{Z}, X)=\left\{f \in F(\mathbb{Z}, X): \sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) f(n)\right\|\right\}<+\infty
$$

For the set $l_{\omega, c}^{\infty}(Z, X)$, we have the following property.
Proposition 3.1 Let $f \in F(\mathbb{Z}, X)$. Then $f \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ if and only if $f(n)=c^{\wedge}(n) u(n)$, $u \in l^{\infty}(\mathbb{Z}, X)$.

Proof: It is clear that $f \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ if $f(n)=c^{\wedge}(n) u(n)$ with $u \in l^{\infty}(\mathbb{Z}, X)$. For the inverse statement, let $f \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ and if we write $u(n)=c^{\wedge}(-n) f(n)$, then we have

$$
\|u\|_{\infty}=\sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) f(n)\right\|<+\infty
$$

So $u \in l^{\infty}(\mathbb{Z}, X)$ and $f(n)=c^{\wedge}(n) u(n)$.
Theorem 3.1 The set $l_{w, c}^{\infty}(\mathbb{Z}, X)$ is a Banach space with the norm

$$
\|f\|_{\omega, c}^{\infty}=\sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) f(n)\right\|
$$

Proof: It is clear that $f_{1}+f_{2} \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ and $k f \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ for each $f_{1}, f_{2} \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ and any $k \in \mathbb{C}$. Thus $l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ forms a vector space. It is also easy to check that $\|\cdot\|_{\omega, c}^{\infty}$ defines a norm in $l_{\omega, c}^{\infty}(\mathbb{Z}, X)$. Let $\left\{f_{p}\right\}_{p \in \mathbb{Z}_{+}} \subseteq l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ be a Cauchy sequence. By Proposition [.]l, we can rewrite $f_{p}(n)=c^{\wedge}(n) u_{p}(n)$ with $u_{p} \in l^{\infty}(\mathbb{Z}, X)$. The relationship $\left\|u_{p}-u_{q}\right\|_{\infty}=\left\|f_{p}-f_{q}\right\|_{\omega, c}^{\infty}$ implies that $\left\{u_{p}\right\}_{p \in \mathbb{Z}_{+}}$is also a Cauchy sequence in the Banach space $l^{\infty}(\mathbb{Z}, X)$, and thus there exists a sequence $u \in l^{\infty}(\mathbb{Z}, X)$ such that $\lim _{p \rightarrow \infty}\left\|u_{p}(n)-u(n)\right\|_{\infty}=0$. Consequently,

$$
\lim _{p \rightarrow \infty} \sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) c^{\wedge}(n) u_{p}(n)-c^{\wedge}(-n) c^{\wedge}(n) u(n)\right\|=0,
$$

that is, $f_{p}(n) \rightarrow f(n):=c^{\wedge}(n) u(n)$ with $\|\cdot\|_{\omega, c}^{\infty}$ in $l_{\omega, c}^{\infty}(\mathbb{Z}, X)$.
Definition 3.1 A sequence $f \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ is said to be $S$-asymptotically $(\omega, c)$-periodic if

$$
\lim _{|n| \rightarrow \infty}\left\|c^{\wedge}(-n)[f(n+\omega)-c f(n)]\right\|=0, \forall n \in \mathbb{Z}
$$

The collection of such sequences will be denoted by $S A P_{\omega, c}(\mathbb{Z}, X)$.
Lemma 3.1 Let $f_{1}, f_{2}, f \in S A P_{\omega, c}(\mathbb{Z}, X)$. Then the following results hold:
(1) $f_{1}+f_{2} \in S A P_{\omega, c}(\mathbb{Z}, X)$, and $k f \in S A P_{\omega, c}(\mathbb{Z}, X)$ for any $k \in \mathbb{C}$.
(2) The sequence $f_{a}(n):=f(n+a) \in S A P_{\omega, c}(\mathbb{Z}, X)$ for each $a \in \mathbb{Z}$.

Proof: (1) From the Definition [3.D, for any $\varepsilon>0$, there exists a constant $T_{\varepsilon}>0$ such that

$$
\left\|c^{\wedge}(-n)[f(n+\omega)-c f(n)]\right\| \leq \frac{\varepsilon}{|k|},\left\|c^{\wedge}(-n)\left[f_{i}(n+\omega)-c f_{i}(n)\right]\right\| \leq \frac{\varepsilon}{2}, i=1,2
$$

for each $|n|>T_{\varepsilon}$. Hence,

$$
\left\|c^{\wedge}(-n)[k f(n+\omega)-c k f(n)]\right\| \leq|k|\left\|c^{\wedge}(-n)[f(n+\omega)-c f(n)]\right\| \leq \varepsilon,
$$

and

$$
\begin{aligned}
& \left\|c^{\wedge}(-n)\left[f_{1}(n+\omega)+f_{2}(n+\omega)-c\left(f_{1}(n)+f_{2}(n)\right)\right]\right\| \\
\leq & \left\|c^{\wedge}(-n)\left[f_{1}(n+\omega)-c\left(f_{1}(n)\right)\right]\right\|+c^{\wedge}(-n)\left[f_{2}(n+\omega)-c\left(f_{2}(n)\right)\right] \| \\
\leq & \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .
\end{aligned}
$$

The above arguments imply $f_{1}+f_{2}, k f \in S A P_{\omega, c}(\mathbb{Z}, X)$.
(2) By the Definition [3.D, for any $\varepsilon>0$, there exists a constant $T_{\varepsilon}>0$ such that

$$
\left\|c^{\wedge}(-n)[f(n+\omega)-c f(n)]\right\| \leq \frac{\varepsilon}{\left|c^{\wedge}(a)\right|}
$$

for $|n|>T_{\varepsilon}$. Thus we can deduce that

$$
\begin{aligned}
& \left\|c^{\wedge}(-n)[f(n+a+\omega)-c f(n+a)]\right\| \\
= & \left\|c^{\wedge}(a) c^{\wedge}(-n-a)[f(n+a+\omega)-c f(n+a)]\right\| \\
= & \left|c^{\wedge}(a)\right|\left\|c^{\wedge}(-n-a)[f(n+a+\omega)-c f(n+a)]\right\| \leq \varepsilon
\end{aligned}
$$

for $|n|>T=\max \left\{T_{\varepsilon}-a, T_{\varepsilon}+a\right\}$, which implies that $f_{a} \in S A P_{\omega, c}(\mathbb{Z}, X)$.
Theorem 3.2 The space $\left(S A P_{\omega, c}(\mathbb{Z}, X),\|\cdot\|_{\omega, c}^{\infty}\right)$ is a Banach space.
Proof: Lemma [3.D implies that $S A P_{\omega, c}(\mathbb{Z}, X)$ is a vector space. Let $\left\{f_{p}\right\}_{p \in \mathbb{Z}_{+}} \subseteq S A P_{\omega, c}(\mathbb{Z}, X)$ converge to $f$ as $p \rightarrow \infty$. Then for any $\varepsilon>0$, we can choose suitable constants $N>0$ and $T_{\varepsilon}>0$ such that

$$
\begin{aligned}
& \left\|f_{p}-f\right\|_{\omega, c}^{\infty}=\sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n)\left[f_{p}(n)-f(n)\right]\right\| \leq \frac{\varepsilon}{3|c|} \\
& \left\|c^{\wedge}(-n)\left[f_{p}(n+\omega)-c f_{p}(n)\right]\right\| \leq \frac{\varepsilon}{3}
\end{aligned}
$$

for $p>N$ and $|n|>T_{\varepsilon}$. Thus

$$
\begin{aligned}
& \left\|c^{\wedge}(-n)[f(n+\omega)-c f(n)]\right\| \\
= & \left\|c^{\wedge}(-n)\left[f(n+\omega)-f_{p}(n+\omega)+f_{p}(n+\omega)-c f_{p}(n)+c f_{p}(n)-c f(n)\right]\right\| \\
\leq & |c|\left\|c^{\wedge}(-n-\omega)\left[f(n+\omega)-f_{p}(n+\omega)\right]\right\|+\| c^{\wedge}(-n)\left[f\left(n_{p}+\omega\right)-c f_{p}(n)\right] \\
& +|c|\left\|c^{\wedge}(-n)\left[f_{p}(n)-f(n)\right]\right\| \\
\leq & \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

which implies that the $S A P_{\omega, c}(\mathbb{Z}, X)$ is a closed sub-space of $l_{\omega, c}^{\infty}(\mathbb{Z}, X)$. Thus it is a Banach space equipped with $\|\cdot\|_{\omega, c}^{\infty}$.

Definition 3.2 A sequence $f \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ is called pseudo $S$-asymptotically $(\omega, c)$-periodic if it satisfies

$$
\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\|=0, k \in \mathbb{Z} .
$$

The collection of such functions will be denoted by $\operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$.
Lemma 3.2 Let $f_{1}, f_{2}, f \in \operatorname{DSA} P_{\omega, c}(\mathbb{Z}, X)$. Then the following results hold:
(1) $f_{1}+f_{2} \in D P S A P_{\omega, c}(\mathbb{Z}, X)$, and $l f \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ for any $l \in \mathbb{C}$.
(2) The sequence $f_{a}(n):=f(n+a) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ for each $a \in \mathbb{Z}$.

Proof: (1)By Definition [3.2, we have for any $l \in \mathbb{C}$ and $i=1,2$

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\|=0 \\
& \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f_{i}(k+\omega)-c f_{i}(k)\right]\right\|=0
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f_{1}(k+\omega)+f_{2}(k+\omega)-c\left(f_{1}(k)+f_{2}(k)\right)\right]\right\| \\
\leq & \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f_{1}(k+\omega)-c f_{1}(k)\right]\right\| \\
& +\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f_{2}(k+\omega)-c f_{2}(k)\right]\right\| \\
= & 0 \\
& \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n} \| c^{\wedge}(-k)[l f(k+\omega)-c l f(k)] \\
\leq & \lim _{n \rightarrow \infty}|l| \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\|=0 .
\end{aligned}
$$

It follows from above arguments that $f_{1}+f_{2}, l f \in \operatorname{DSSAP} P_{\omega, c}(\mathbb{Z}, X)$.
(2) By the Definition 3.2, we have

$$
\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\|=0, k \in \mathbb{Z}
$$

Thus for each $a \in \mathbb{Z}$

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+a+\omega)-c f(k+a)]\right\| \\
= & \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n+a}^{n+a}\left\|c^{\wedge}(-k+a)[f(k+\omega)-c f(k)]\right\| \\
\leq & \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n+|a|}^{n+|a|}\left\|c^{\wedge}(-k) c^{\wedge}(a)[f(k+\omega)-c f(k)]\right\| \\
\leq & \lim _{n \rightarrow \infty} \frac{\left|c^{\wedge}(a)\right|(n+|a|+1)}{n+\frac{1}{2}} \frac{1}{2(n+|a|+1)} \sum_{k=-n+|a|}^{n+|a|}\left\|c^{\wedge}(-k) c^{\wedge}(a)[f(k+\omega)-c f(k)]\right\| \\
= & 0
\end{aligned}
$$

which implies that $f_{a} \in D P S A P_{\omega, c}(Z, X)$.
Theorem 3.3 The space $\left(D P S A P_{\omega, c}(\mathbb{Z}, X),\|\cdot\|_{\omega, c}^{\infty}\right)$ is a Banach space.
Proof: It follows from Lemma $\mathbb{K 2}$ that $\operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ is a vector space. Let $\left\{f_{p}\right\}_{p \in \mathbb{Z}_{+}} \subseteq$ $\operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$ converge to $f$ as $n \rightarrow \infty$. Then for any $\varepsilon>0$, we can choose suitable constants $N>0$ and $n_{\varepsilon}$ such that

$$
\begin{aligned}
& \left\|f_{p}-f\right\|_{\omega, c}^{\infty}=\sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n)\left[f_{p}(n)-f(n)\right]\right\| \leq \frac{\varepsilon}{3|c|}, \\
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f_{p}(k+\omega)-c f_{p}(k)\right]\right\| \leq \frac{\varepsilon}{3}
\end{aligned}
$$

for $p>N$ and $n>n_{\varepsilon}$. Thus

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \\
= & \frac{1}{2 n+1} \sum_{k=-n}^{n} \| c^{\wedge}(-k)\left[f(k+\omega)-f_{p}(k+\omega)+f_{p}(k+\omega)\right. \\
& \left.-c f_{p}(k)+c f_{p}(k+\omega)\right]-c f(k) \| \\
\leq & \frac{1}{2 n+1} \sum_{k=-n}^{n}|c|\left\|c^{\wedge}(-k-\omega)\left[f(k+\omega)-f_{p}(k+\omega)\right]\right\| \\
& +\frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f_{p}(k+\omega)-c f_{p}(k)\right]\right\| \\
& +\frac{1}{2 n+1} \sum_{k=-n}^{n}|c|\left\|c^{\wedge}(-k)\left[f_{p}(k)-f(k)\right]\right\|
\end{aligned}
$$

$$
\begin{aligned}
& \leq|c|\left\|f_{p}-f\right\|_{\omega, c}^{\infty}+\frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f_{p}(k+\omega)-c f_{p}(k+\omega)\right]\right\| \\
& \quad+|c|\left\|f_{p}-f\right\|_{\omega, c}^{\infty} \\
& \leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

which implies that the $\operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$ is a closed sub-space of $l_{\omega, c}^{\infty}(\mathbb{Z}, X)$. So it is a Banach space equipped with $\|\cdot\|_{\omega, c}^{\infty}$.

Lemma 3.3 Let $f \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$. Then the following assertions are equivalent:
(a) $\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\|=0$.
(b) For each $\varepsilon>0, \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k \in M_{n, \varepsilon}(f)}=0$, where

$$
M_{n, \varepsilon}(f)=\left\{k \in[-n, n] \cap \mathbb{Z}:\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \geq \varepsilon\right\} .
$$

Proof: Since

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \\
= & \frac{1}{2 n+1} \sum_{[-n, n] \backslash k \in M_{n, \varepsilon}(f)}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \\
+ & \frac{1}{2 n+1} \sum_{k \in M_{n, \varepsilon}(f)}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \\
\geq & \frac{1}{2 n+1} \sum_{k \in M_{n, \varepsilon}(f)}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \\
\geq & \frac{\varepsilon}{2 n+1} \sum_{k \in M_{n, \varepsilon}(f)} \geq 0
\end{aligned}
$$

we can verify the assertion(b) if the assertion (a) is true. On the other hand,

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \\
= & \frac{1}{2 n+1} \sum_{k \in[-n, n] \backslash M_{n, \varepsilon}(f)}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\| \\
+ & \frac{1}{2 n+1} \sum_{k \in M_{n, \varepsilon}(f)}\left\|c^{\wedge}(-k)[f(k+\omega)-c f(k)]\right\|
\end{aligned}
$$

$$
\leq\left(1-\frac{1}{2 n+1} \sum_{k \in M_{n, \varepsilon}(f)}\right) \varepsilon+\frac{1}{2 n+1} 2|c|\|f\|_{\omega, c}^{\infty} \sum_{k \in M_{n, \varepsilon}(f)},
$$

thus we can confirm the assertion (a) by the truth of the assertion (b).
We have the following convolution property.
Theorem 3.4 Let $\{S(n)\}_{n \in \mathbb{Z}_{+}} \subseteq \mathcal{B}(X)$. Assume further that $\left\{c^{\wedge}(-n) S(n)\right\}_{n \in \mathbb{Z}_{+}}$is summalbe. If $f \in D P S A P_{\omega, c}(\mathbb{Z}, X)$, then

$$
u(n):=\sum_{k=-\infty}^{n} S(n-k) f(k) \in \operatorname{DPSAP}_{\omega, c}(Z, X)
$$

Proof: Since $\left\{c^{\wedge}(-n) S(n)\right\}_{n \in \mathbb{Z}_{+}}$is summable and $f \in D P S A P_{\omega, c}(\mathbb{Z}, X)$, we have

$$
\sum_{k=0}^{\infty}\left\|c^{\wedge}(-k) S(k)\right\|<\infty, f(n)=c^{\wedge}(n) p(n), p(n) \in l^{\infty}(\mathbb{Z}, X)
$$

Hence $u(n)=\sum_{k=-\infty}^{n} S(n-k) f(k)=c^{\wedge}(n) \sum_{k=0}^{\infty} c^{\wedge}(-k) S(k) p(n-k)$ and

$$
\begin{aligned}
& \left\|\sum_{k=0}^{\infty} c^{\wedge}(-k) S(k) p(n-k)\right\| \leq \sum_{k=0}^{\infty}\left\|c^{\wedge}(-k) S(k) p(n-k)\right\| \\
\leq & \|p\|^{\infty} \sum_{k=0}^{\infty}\left\|c^{\wedge}(-k) S(k)\right\|<\infty
\end{aligned}
$$

which hints that $\sum_{k=0}^{\infty} c^{\wedge}(-k) S(k) p(n-k) \in l^{\infty}(\mathbb{Z}, X), u(n) \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$. It also follows from the Fubini theorem that

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[u(k+\omega)-c u(k)]\right\| \\
= & \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[\sum_{m=-\infty}^{k+\omega} S(k+\omega-m) f(m)-c \sum_{m=-\infty}^{k} S(k-m) f(m)\right]\right\| \\
= & \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[\sum_{m=-\infty}^{k} S(k-m) f(m+\omega)-c \sum_{m=-\infty}^{k} S(k-m) f(m)\right]\right\| \\
= & \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k) \sum_{m=0}^{\infty} S(m)[f(k-m+\omega)-c f(k-m)]\right\| \\
\leq & \frac{1}{2 n+1} \sum_{k=-n}^{n} \sum_{m=0}^{\infty}\left\|c^{\wedge}(-k) S(m)[f(k-m+\omega)-c f(k-m)]\right\|
\end{aligned}
$$

$$
=\sum_{m=0}^{\infty}\left\|c^{\wedge}(-m) S(m)\right\|\left[\frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k+m)[f(k-m+\omega)-c f(k-m)]\right\| .\right.
$$

By $f \in D P S A P_{\omega, c}(\mathbb{Z}, X)$, Lemma $\operatorname{B.2}(2)$ and the dominated convergence theorem, we have

$$
\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[u(k+\omega)-c u(k)]\right\|=0 .
$$

Thus $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.
For given $\phi \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ and $f \in F(\mathbb{Z} \times X, X)$, we define the Nemytskii's superposition operator by $N_{f}(\phi)(\cdot):=f(\cdot, \phi(\cdot))$. We present the following superposition theorems for $N_{f}(\phi)$.

Theorem 3.5 Let $f \in F(\mathbb{Z} \times X, X)$ satisfy the following conditions :
(A1) (a) For any bounded subset $Q \subseteq X, \sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) f\left(n, c^{\wedge}(n) x\right)\right\|<\infty$ uniformly for $x \in Q$.
(b) $\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[f(k+\omega, c x)-c f(k, x)]\right\|=0$ uniformly for $x \in X$.
(A2) There exists a constant $L>0$ such that for all $x, y \in X$ and $n \in \mathbb{Z}$,

$$
\|f(n, x)-f(n, y)\| \leq L\|x-y\|
$$

Then for each $\phi \in D P S A P_{\omega, c}(\mathbb{Z}, X), N_{f}(\phi) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.
Proof: For each $\phi \in D P S A P_{\omega, c}(\mathbb{Z}, X)$, we have

$$
\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[\phi(k+\omega)-c \phi(k)]\right\|=0, k \in \mathbb{Z}
$$

On the other hand,

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[N_{f}(\phi)(k+\omega)-c N_{f}(\phi)(k)\right]\right\| \\
= & \frac{1}{2 n+1} \sum_{k=-n}^{n} \| c^{\wedge}(-k)\left[f(k+\omega, \phi(k+\omega))-c f\left(k, \frac{1}{c} \phi(k+\omega)\right)\right. \\
+ & \left.c f\left(k, \frac{1}{c} \phi(k+\omega)\right)-c f(k, \phi(k))\right] \| \\
\leq & \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f(k+\omega, \phi(k+\omega))-c f\left(k, \frac{1}{c} \phi(k+\omega)\right)\right]\right\| \\
+ & \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[c f\left(k, \frac{1}{c} \phi(k+\omega)\right)-c f(k, \phi(k))\right]\right\|
\end{aligned}
$$

$$
=I_{1}+I_{2} .
$$

By (A1)(b), we have $I_{1} \rightarrow 0$ as $n \rightarrow \infty$. For $I_{2}$, we have

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[c f\left(k, \frac{1}{c} \phi(k+\omega)\right)-c f(k, \phi(k))\right]\right\| \\
\leq & \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left|c^{\wedge}(-k+\omega)\right|\left\|\left[f\left(k, \frac{1}{c} \phi(k+\omega)\right)-f(k, \phi(k))\right]\right\| \\
\leq & \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left|c^{\wedge}(-k+\omega)\right| L\left\|\frac{1}{c} \phi(k+\omega)-\phi(k)\right\| \\
= & L \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[\phi(k+\omega)-c \phi(k)]\right\|=0 .
\end{aligned}
$$

So $\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[N_{f}(\phi)(k+\omega)-c N_{f}(\phi)(k)\right]\right\|=0$, i.e. $N_{f}(\phi) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.
Theorem 3.6 Let $f \in F(\mathbb{Z} \times X, X)$ satisfy (A1) and the following condition:
(A3) $f_{n}(z):=c^{\wedge}(-n) f\left(n, c^{\wedge}(n) z\right)$ is uniformly continuous for $z$ in any bounded subset of $X$ uniformly in $n \in \mathbb{Z}$; that is, for any $\varepsilon>0$ and any bounded subset $Q \subseteq X$, there exists $\delta$ such that $x, y \in Q$ and $\|x-y\|<\delta$ imply that

$$
\left\|f_{n}(x)-f_{n}(y)\right\|=\left\|c^{\wedge}(-n)\left[f\left(n, c^{\wedge}(n) x\right)-f\left(n, c^{\wedge}(n) y\right)\right]\right\| \leq \varepsilon, n \in \mathbb{Z}
$$

Then for each $\phi \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X), N_{f}(\phi) \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$.
Proof: From (A1)(a), we have $f(\cdot, \phi(\cdot)) \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ and there exists a constant $M>0$ such that $\sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) f\left(n, c^{\wedge}(n) x\right)\right\| \leq M$ for all $x \in Q$. Meanwhile,

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[N_{f}(\phi)(k+\omega)-c N_{f}(\phi)(k)\right]\right\| \\
\leq & \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f(k+\omega, \phi(k+\omega))-c f\left(k, \frac{1}{c} \phi(k+\omega)\right)\right]\right\| \\
+ & \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[c f\left(k, \frac{1}{c} \phi(k+\omega)\right)-c f(k, \phi(k))\right]\right\| \\
= & I_{1}+I_{2} .
\end{aligned}
$$

It is obvious that $\lim _{n \rightarrow \infty} I_{1}=0$. For $I_{2}$, we choose bounded subset $Q=\left\{c^{\wedge}(-k) \phi(k): k \in \mathbb{Z}\right\}$. If $\| c^{\wedge}(-n)[\phi(k+\omega)-c \phi(k)]<|c| \delta$, then

$$
\left\|c^{\wedge}(-k-\omega) \phi(k+\omega)-c^{\wedge}(-k) \phi(k)\right\|=\left\|\frac{1}{c} c^{\wedge}(-k)[\phi(k+\omega)-c \phi(k)]\right\|<\delta .
$$

By (A3), we have

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[c f\left(k, \frac{1}{c} \phi(k+\omega)\right)-c f(k, \phi(k))\right]\right\| \\
= & |c| \frac{1}{2 n+1} \sum_{k=-n}^{n} \| c^{\wedge}(-k)\left[f\left(k, c^{\wedge}(k) c^{\wedge}(-k-\omega) \phi(k+\omega)\right)\right. \\
& \left.-f\left(k, c^{\wedge}(k) c^{\wedge}(-k) \phi(k)\right)\right] \| \\
= & |c| \frac{1}{2 n+1} \sum_{k \in[-n, n] \backslash M_{n,|c| \delta}(\phi)} \| c^{\wedge}(-k)\left[f\left(k, c^{\wedge}(k) c^{\wedge}(-k-\omega) \phi(k+\omega)\right)\right. \\
& \left.-f\left(k, c^{\wedge}(k) c^{\wedge}(-k) \phi(k)\right)\right] \| \\
& +|c| \frac{1}{2 n+1} \sum_{M_{n,|c| \delta}(\phi)} \| c^{\wedge}(-k)\left[f\left(k, c^{\wedge}(k) c^{\wedge}(-k-\omega) \phi(k+\omega)\right)\right. \\
& \left.-f\left(k, c^{\wedge}(k) c^{\wedge}(-k) \phi(k)\right)\right] \| \\
\leq & \left(1-\frac{1}{2 n+1} \sum_{k \in M_{n,|c| \delta}(\phi)}\right)|c| \varepsilon+2 M|c| \frac{1}{2 n+1} \sum_{k \in M_{n,|c| \delta}(\phi)} .
\end{aligned}
$$

By $\phi \in D P S A P_{\omega, c}(\mathbb{Z}, X)$, Lemma [3.3] and the arbitrariness of $\varepsilon$, we have $\lim _{n \rightarrow \infty} I_{2}=0$. Hence, $\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[N_{f}(\phi)(k+\omega)-c N_{f}(\phi)(k)\right]\right\|=0$, i.e. $N_{f}(\phi) \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$. We recall the following set.

$$
\mathfrak{g}(n)=\left\{v: \mathbb{Z} \rightarrow \mathbb{R}_{+} \text {and } \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n} v(k)<\infty\right\}
$$

Theorem 3.7 Let $f \in F(\mathbb{Z} \times X, X)$ satisfy (A1) and the following condition:
(A4) There exists a sequence $\mathfrak{L}(\cdot) \in \mathfrak{g}(n)$ such that for any $\varepsilon>0$ and any bounded subset $Q \subseteq X$, there is a constant $\delta>0$ satisfying

$$
\left\|c^{\wedge}(-n)\left[f\left(n, c^{\wedge}(n) x\right)-f\left(n, c^{\wedge}(n) y\right)\right]\right\| \leq \mathfrak{L}(n) \varepsilon
$$

for all $x, y \in Q$ with $\|x-y\|<\delta$ and $n \in \mathbb{Z}$.
Then for each $\phi \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X), N_{f}(\phi) \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$.
Proof: From (A1)(a), we have $f(\cdot, \phi(\cdot)) \in l_{\omega, c}^{\infty}(\mathbb{Z}, X)$ and there exists a constant $M>0$ such that $\sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) f\left(n, c^{\wedge}(n) x\right)\right\| \leq M$ for all $x \in Q$.
On the other hand,

$$
\frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[N_{f}(\phi)(k+\omega)-c N_{f}(\phi)(k)\right]\right\|
$$

$$
\begin{aligned}
& \leq \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[f(k+\omega, \phi(k+\omega))-c f\left(k, \frac{1}{c} \phi(k+\omega)\right)\right]\right\| \\
& +\frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[c f\left(k, \frac{1}{c} \phi(k+\omega)\right)-c f(k, \phi(k))\right]\right\| \\
& =I_{1}+I_{2} .
\end{aligned}
$$

It is clearly that $\lim _{n \rightarrow \infty} I_{1}=0$. For $I_{2}$, we choose bounded subset $Q=\left\{c^{\wedge}(-k) \phi(k): k \in \mathbb{Z}\right\}$. If $\| c^{\wedge}(-n)[\phi(k+\omega)-c \phi(k)]<|c| \delta$, then

$$
\left\|c^{\wedge}(-k-\omega) \phi(k+\omega)-c^{\wedge}(-k) \phi(k)\right\|=\left\|\frac{1}{c} c^{\wedge}(-k)[\phi(k+\omega)-c \phi(k)]\right\|<\delta
$$

By (A4), we have

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[c f\left(k, \frac{1}{c} \phi(k+\omega)\right)-c f(k, \phi(k))\right]\right\| \\
= & |c| \frac{1}{2 n+1} \sum_{k=-n}^{n} \| c^{\wedge}(-k)\left[f\left(k, c^{\wedge}(k) c^{\wedge}(-k-\omega) \phi(k+\omega)\right)\right. \\
& \left.-f\left(k, c^{\wedge}(k) c^{\wedge}(-k) \phi(k)\right)\right] \| \\
= & |c| \frac{1}{2 n+1} \sum_{k \in[-n, n] \backslash M_{n,|c| \delta}(\phi)} \| c^{\wedge}(-k)\left[f\left(k, c^{\wedge}(k) c^{\wedge}(-k-\omega) \phi(k+\omega)\right)\right. \\
& \left.-f\left(k, c^{\wedge}(k) c^{\wedge}(-k) \phi(k)\right)\right] \| \\
& +|c| \frac{1}{2 n+1} \sum_{M_{n,|c| \delta}(\phi)} \| c^{\wedge}(-k)\left[f\left(k, c^{\wedge}(k) c^{\wedge}(-k-\omega) \phi(k+\omega)\right)\right. \\
& \left.-f\left(k, c^{\wedge}(k) c^{\wedge}(-k) \phi(k)\right)\right] \| \\
\leq & |c| \varepsilon \frac{1}{2 n+1} \sum_{k=-n}^{n} \mathfrak{L}(k)+2 M|c| \frac{1}{2 n+1} \sum_{k \in M_{n,|c| \delta}(\phi)} .
\end{aligned}
$$

From Lemma [3.3, $\mathfrak{L}(\cdot) \in \mathfrak{g}(n)$ and the arbitrariness of $\varepsilon$, we have $\lim _{n \rightarrow \infty} I_{2}=0$, and thus $\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)\left[N_{f}(\phi)(k+\omega)-c N_{f}(\phi)(k)\right]\right\|=0$, i.e. $\quad N_{f}(\phi) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.

Corollary 3.1 Let $f \in F(\mathbb{Z} \times X, X)$ satisfy (A1) and the following condition:
(C1) For any bounded subset $Q \subseteq X$, there exists a sequence $\mathfrak{L}(\cdot) \in \mathfrak{g}(n)$ such that

$$
\left\|c^{\wedge}(-n)\left[f\left(n, c^{\wedge}(n) x\right)-f\left(n, c^{\wedge}(n) y\right)\right]\right\| \leq \mathfrak{L}(n)\|x-y\|
$$

for all $x, y \in Q$ and $n \in \mathbb{Z}$.
Then for each $\phi \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X), N_{f}(\phi) \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$.

Proof: It is easy to check that (A4) in Theorem 3.7 holds if (C1) is satisfied. So the conclusion is a direct consequence of Theorem [3.7.

Corollary 3.2 Let $f \in F(\mathbb{Z} \times X, X)$ satisfy (A1) and the following condition:
(C2) For any bounded subset $Q \subseteq X$, there exists a function $\mathfrak{L}(n) \in l^{p}\left(\mathbb{Z}, R_{+}\right)(1 \leq p<\infty)$ such that

$$
\left\|c^{\wedge}(-n)\left[f\left(n, c^{\wedge}(n) x\right)-f\left(n, c^{\wedge}(n) y\right)\right]\right\| \leq \mathfrak{L}(n)\|x-y\|
$$

for all $x, y \in Q$ and $n \in \mathbb{Z}$.
Then for each $\phi \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X), N_{f}(\phi) \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$.
Proof: Since

$$
\begin{aligned}
& \frac{1}{2 n+1} \sum_{k=-n}^{n} \mathfrak{L}(k)=\frac{1}{2 n+1} \sum_{k=-n}^{n} 1 \times \mathfrak{L}(k) \\
\leq & \frac{1}{2 n+1}\left[\sum_{k=-n}^{n} \mathfrak{L}(k)^{p}\right]^{\frac{1}{p}}\left[\sum_{k=-n}^{n} 1^{1-p}\right]^{\frac{1}{1-p}} \\
= & \frac{1}{2 n+1}\left[\sum_{k=-n}^{n} \mathfrak{L}(k)^{p}\right]^{\frac{1}{p}}[2 n+1]^{\frac{1}{1-p}} \\
= & \frac{1}{(2 n+1)^{\frac{1}{p}}}\left[\sum_{k=-n}^{n} \mathfrak{L}(k)^{p}\right]^{\frac{1}{p}} \\
\leq & {\left[\sum_{k=-n}^{n} \mathfrak{L}(k)^{p}\right]^{\frac{1}{p}}, }
\end{aligned}
$$

we have that $\mathfrak{L}(\cdot) \in \mathfrak{g}(n)$ if $\mathfrak{L} \in l^{p}\left(\mathbb{Z}, R_{+}\right)(1 \leq p<\infty)$. So (C1) in Corollary $\left.{ }^{2}\right]$ holds by (C2) and the assertion is true.

## 4 Existence results

In this section,we show some existence results for pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequential mild solutions to Eq. ([.]).

Definition 4.1 Let $A$ generate a discrete resolvent family $\{S(k)\}_{k \in Z_{+}} \subseteq \mathcal{B}(X)$ and $p: \mathbb{Z} \times X \rightarrow$ $X$. A sequence $u: \mathbb{Z} \rightarrow X$ is called a mild solution to Eq. (■.ل]) if $k \rightarrow S(n-k) p(k, u(k))$ is summalbe on $\mathbb{Z}$ for each $n \in \mathbb{Z}$ and $u$ verifies

$$
u(n+1)=\sum_{k=-\infty}^{n} S(n-k) p(k, u(k)), n \in \mathbb{Z}
$$

In what follows, we always assume that $c \in \mathbb{C} \backslash 0, \omega \in \mathbb{Z}^{+}$and $Q$ is bounded subset of $X$. Let us list the following hypotheses:
(DA) Let the operator $A$ generate a discrete resolvent family $\{S(k)\}_{k \in Z_{+}} \subseteq \mathcal{B}(X)$ and $c^{\wedge}(-n) S(n)$ is summable.
(D1) (1) For any bounded subset $Q \subseteq X, \sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) p\left(n, c^{\wedge}(n) x\right)\right\|<\infty$ uniformly for $x \in Q$.
(2) $\lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|c^{\wedge}(-k)[p(k+\omega, c x)-c p(k, x)]\right\|=0$ uniformly for $x \in X$.

Lemma 4.1 Let conditions (DA) and (D1) hold. Assume further that $p: \mathbb{Z} \times X \rightarrow X$ satisfies the following condition:
(D2) There exists a function $L_{p}: \mathbb{Z}_{+} \rightarrow \mathbb{R}_{+}$such that for each $r \geq 0$ and all $u, v$ in any bounded subset $Q$ of $X$ with $\|u\| \leq r,\|v\| \leq r$,

$$
\left\|c^{\wedge}(-k)\left[p\left(k, c^{\wedge}(k) u\right)-p\left(k, c^{\wedge}(k) v\right)\right]\right\| \leq L_{p}(r)\|u-v\|, k \in \mathbb{Z} .
$$

Then for each $u \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X), P(n):=\sum_{k=-\infty}^{n} S(n-k) p(k, u(k)) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.
Proof: For each given $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X),\left\{c^{\wedge}(-k) u(k)\right\}$ is bounded and there exists a bounded subset $Q$ of $X$ such that $c^{\wedge}(-k) u(k) \in Q$ for all $k \in \mathbb{Z}$. It follows from the condition (D2) that $c^{\wedge}(-k) p\left(k, c^{\wedge}(k) u\right)$ is uniformly continuous on the bounded subset $Q$ uniformly for $k \in \mathbb{Z}$. Thus we have from Theorem [3.5 together with conditions (D1) and (D2) that $p(\cdot, u(\cdot)) \in$ $\operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ if $u \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$. It further follows from the condition (DA) and Theorem 3.4 that $P(n) \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ for each $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.

Lemma 4.2 Assume that conditions (DA) and (D1) hold. Let $p: \mathbb{Z} \times X \rightarrow X$ be a function that satisfies the following condition:
(D3) There exists a summable function $L_{p}(\cdot): \mathbb{Z} \rightarrow \mathbb{R}_{+}$such that

$$
\left\|c^{\wedge}(-k)\left[p\left(k, c^{\wedge}(k) u\right)-p\left(k, c^{\wedge}(k) v\right)\right]\right\| \leq L_{p}(k)\|u-v\|, \forall u, v \in Q, k \in \mathbb{Z}
$$

Then $P(n):=\sum_{k=-\infty}^{n} S(n-k) p(k, u(k)) \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ whenever $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$.
Proof: It follows from Corollary 5.2 together with conditions (D1) and (D3) that $p(\cdot, u(\cdot)) \in$ $D P S A P_{\omega, c}(\mathbb{Z}, X)$ for each $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$. Thus we have from Theorem $[3.4$ with the condition (DA) that $P(n) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ whenever $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.

Lemma 4.3 Let $p: \mathbb{Z} \times X \rightarrow X$ be a function that satisfies assumption (D1). Assume further that the following conditions hold:
(D4) There exists a $q$-th summable $(1 \leq q<\infty)$ function $\mathscr{L}_{p}(\cdot): \mathbb{Z} \rightarrow \mathbb{R}_{+}$such that

$$
\left\|c^{\wedge}(-k)\left[p\left(k, c^{\wedge}(k) u\right)-p\left(k, c^{\wedge}(k) v\right)\right]\right\| \leq \mathscr{L}_{p}(k)\|u-v\|, \forall u, v \in Q, k \in \mathbb{Z} .
$$

(D5) There exist constants $\delta>0, \mathcal{M}>0$ such that $\left\|c^{\wedge}(-k) S(k)\right\| \leq \mathcal{M} e^{-\delta k}$ for all $k \in \mathbb{Z}_{+}$.
Then $P(n):=\sum_{k=-\infty}^{n} S(n-k) p(k, u(k)) \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ whenever $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$.
Proof: The condition (D5) implies that the condition (DA) holds. It follows from Corollary $\left[3.2\right.$ together with conditions (D1) and (D4) that $p(\cdot, u(\cdot)) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ for each $u \in$ $D P S A P_{\omega, c}(\mathbb{Z}, X)$. Thus $P(n) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ whenever $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ through Theorem [3.4.

Lemma 4.4 Let conditions (DA) and (D1) hold. Assume that $p \in F(\mathbb{Z} \times X, X)$ satisfies the following condition:
(D6) $p_{n}(z):=c^{\wedge}(-n) p\left(n, c^{\wedge}(n) z\right)$ is uniformly continuous for $z$ in any bounded subset of $X$ uniformly in $n \in \mathbb{Z}$.
Then $P(n):=\sum_{k=-\infty}^{n} S(n-k) p(k, u(k)) \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ whenever $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$.
Proof: The proof can be conducted directly through Theorems 3.4 and 3.6 together with conditions (DA), (D1) and (D6).

The following result is concerned with a local Lipschitz growth condition on the function $c^{\wedge}(-k) p\left(k, c^{\wedge}(k) \cdot\right)$ for all $k \in \mathbb{Z}$.

Theorem 4.1 Suppose that conditions (DA), (D1) and (D2) are satisfied. Assume further that the following condition hold:
(D7) $\sup _{r>0}\left[|c| r-r L_{p}(r)\|S\|_{c}\right]>\|S\|_{c} \sup _{k \in \mathbb{Z}}\left\|c^{\wedge}(-k) p(k, 0)\right\|$, where $\|S\|_{c}=\sum_{k=0}^{\infty}\left\|c^{\wedge}(-k) S(k)\right\|$.
Then Eq.([.لD) has a mild solution $u \in \operatorname{DSSAP} P_{\omega, c}(\mathbb{Z}, X)$, which is unique on an arbitrarily closed ball $\mathbb{B}_{r}$ of the space $D P S A P_{\omega, c}(\mathbb{Z}, X)$ with its center at 0 and radius $r$ satisfying the condition (D7).

Proof: Let the operator $\mathcal{P}: D P S A P_{\omega, c}(\mathbb{Z}, X) \rightarrow D P S A P_{\omega, c}(\mathbb{Z}, X)$ be defined by

$$
\begin{equation*}
(\mathcal{P} u)(n)=\sum_{k=-\infty}^{n-1} S(n-1-k) p(k, u(k)) . \tag{4.1}
\end{equation*}
$$

We obtain from Lemma $\mathbb{L} .1$ that $\mathcal{P}$ is well-defined for each $u \in \operatorname{DPSA} P_{\omega, c}(\mathbb{Z}, X)$. We also deduce from the condition (D7) that there exists a constant $r>0$ such that

$$
\begin{equation*}
|c| r-r L_{p}(r)\|S\|_{c}>\|S\|_{c} \sup _{k \in \mathbb{Z}}\left\|c^{\wedge}(-k) p(k, 0)\right\| . \tag{4.2}
\end{equation*}
$$

Let $\mathbb{B}=\left\{u \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X):\|u\|_{\omega, c}^{\infty} \leq r\right\}$, which is a closed subset of the Banach space $\operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$. We show that $\mathcal{P}(\mathbb{B}) \subseteq \mathbb{B}$. For each $u \in \mathbb{B}$ and all $n \in \mathbb{Z}$, we have

$$
\left\|c^{\wedge}(-n)(\mathcal{P} u)(n)\right\|
$$

$$
\begin{aligned}
\leq & \left|c^{\wedge}(-n)\right| \sum_{k=-\infty}^{n-1}\left\|c^{\wedge}(k) S(n-1-k)\right\|\left\|c^{\wedge}(-k) p\left(k, c^{\wedge}(k) c^{\wedge}(-k) u(k)\right)\right\| \\
\leq & \frac{1}{|c|} \sum_{k=-\infty}^{n-1}\left\|c^{\wedge}(k-n+1) S(n-1-k)\right\|\left\|c^{\wedge}(-k)\left[p\left(k, c^{\wedge}(k) c^{\wedge}(-k) u(k)\right)-p(k, 0)\right]\right\| \\
& +\frac{1}{|c|} \sum_{k=-\infty}^{n-1}\left\|c^{\wedge}(k-n+1) S(n-1-k)\right\|\left\|c^{\wedge}(-k) p(k, 0)\right\| \\
\leq & \frac{1}{|c|}\left[\sup _{k \in \mathbb{Z}}\left\|c^{\wedge}(-k) p(k, 0)\right\|+r L_{p}(r)\right]\|S\|_{c}
\end{aligned}
$$

from which with ( $\mathbb{4 . 2 1 )}$ ) we have $\|\mathcal{P} u\|_{\omega, c}^{\infty} \leq r$ and thus $\mathcal{P}(\mathbb{B}) \subseteq \mathbb{B}$.
Next it is shown for the contraction of $\mathcal{P}$ on the set $\mathbb{B}$. From (4.2), we have $|c| r-r L_{p}(r)\|S\|_{c}>$ 0 , i.e. $\frac{1}{|c|} L_{p}(r)\|S\|_{c}<1$. Thus for each $u, v \in \mathbb{B}$ and all $n \in \mathbb{Z}$, we have

$$
\begin{aligned}
\left\|c^{\wedge}(-n)(\mathcal{P} u)(n)-(\mathcal{P} v)(n)\right\| & =\left\|\sum_{k=-\infty}^{n-1} S(n-1-k)(p(k, u(k))-p(k, v(k)))\right\| \\
& \leq \sum_{k=-\infty}^{n-1}\|S(n-1-k)\|\|p(k, u(k))-p(k, v(k))\| \\
& \leq \frac{1}{|c|} L_{p}(r)\|S\|_{c}\|u-v\|_{\omega, c}^{\infty},
\end{aligned}
$$

which implies $\|\mathcal{P} u-\mathcal{P} v\|_{\omega, c}^{\infty} \leq \frac{1}{|c|} L_{p}(r)\|S\|_{c}\|u-v\|_{\omega, c}^{\infty}$. Thus $\mathcal{P}$ is a contraction on $\mathbb{B}$ and admits a unique fixed point $u \in \mathbb{B}$, which is also a mild solution $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ to Eq. ([.]).

Next we establish some existence and uniqueness of pseudo $S$-asymptotically ( $\omega, c$ )-periodic sequential mild solutions to Eq. ( $\mathbb{L}$ ) under global Lipschitz growth conditions on the function $c(-k) p\left(k, c^{\wedge}(k) \cdot\right)$ for all $k \in \mathbb{Z}$. The first result is concerned with a Lipschitz constant coefficient $L>0$.

Corollary 4.1 Assume that conditions (DA), (D1) and (D2) hold. If $L_{p}(\cdot) \equiv L>0$ with $\frac{1}{|c|} L\|S\|_{c}<1$, then Eq. (【.-D) has a unique mild solution $u \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$.

Proof: Since $0<\frac{1}{|c|} L\|S\|_{c}<1$ and $\|S\|_{c} \sup _{k \in \mathbb{Z}}\left\|c^{\wedge}(-k) p(k, 0)\right\|<\infty$, there exists a constant $r *>0$ such that $r *>\frac{\|S\|_{c}|c| \sup _{k \in \mathbb{Z}}\left\|c^{\wedge}(-k) p(k, 0)\right\|}{1-\frac{1}{|c|} L\|S\|_{c}}$. Thus for all $r>r *$, we have $|c| r-$ $r L\|S\|_{c}>\|S\|_{c} \sup _{k \in \mathbb{Z}}\left\|c^{\wedge}(-k) p(k, 0)\right\|$, i.e. ( (K.2) is satisfied for all $r>r *$. It is inferred from the proof of Theorem 1.0 that Eq. ([.]) admits a unique solution $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$.

The second result is involved in the Lipschitz growth condition on $c^{\wedge}(-k) p\left(k, c^{\wedge}(k) \cdot\right)$ for all $k \in \mathbb{Z}$ with a summable coefficient $\mathcal{L}_{p}: \mathbb{Z} \rightarrow \mathbb{R}_{+}$.

Theorem 4.2 Let conditions (DA), (D1) and (D3) hold. Then Eq.(L. ${ }^{(\mathbb{L}) \text { ) admits a unique mild }}$ solution $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$.

Proof: Let the operator $\mathcal{P}$ be defined by ( $\mathbb{L} .1$ ). For each $u \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$, we have from Lemma $\mathbb{4 . 2}$ that $(\mathcal{P} u)(n) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$. Thus $\mathcal{P}: D P S A P_{\omega, c}(\mathbb{Z}, X) \rightarrow D P S A P_{\omega, c}(\mathbb{Z}, X)$ is well-defined. It is deduced by the condition (DA) that $\sup _{n \in \mathbb{Z}}\left\|c^{\wedge}(-n) S(n)\right\| \leq M$ for a suitable constant $M>0$. For each $u, v \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$ and all $n \in \mathbb{Z}$, we have

$$
\begin{aligned}
& \left\|c^{\wedge}(-n)[(\mathcal{P} u)(n)-(\mathcal{P} v)(n)]\right\| \\
\leq & \left|c^{\wedge}(-n)\right| \sum_{k=-\infty}^{n-1}\|S(n-1-k)\|\|p(k, u(k))-p(k, v(k))\| \\
= & \frac{1}{|c|} \sum_{k=-\infty}^{n-1}\left\|c^{\wedge}(k+1-n) S(n-1-k)\right\|\left\|c^{\wedge}(-k)[p(k, u(k))-p(k, v(k))]\right\| \\
\leq & \frac{M}{|c|}\left(\sum_{k=-\infty}^{n-1} \mathcal{L}_{p}(k)\right)\|u-v\|_{\omega, c}^{\infty} .
\end{aligned}
$$

Generally, by [[3], Lemma 3.2.] and mathematical induction, we have

$$
\begin{aligned}
& \left\|c^{\wedge}(-n)\left[\left(\mathcal{P}^{m} u\right)(n)-\left(\mathcal{P}^{m} v\right)(n)\right]\right\| \\
\leq & \frac{1}{|c|} \sum_{k=-\infty}^{n-1}\left\|c^{\wedge}(k+1-n) S(n-1-k)\right\| \| c^{\wedge}(-k)\left[p\left(k,\left(\mathcal{P}^{m-1} u\right)(k)\right)\right. \\
& \left.-p\left(k,\left(\mathcal{P}^{m-1} v\right)(k)\right)\right] \| \\
\leq & \frac{(M)^{m}}{|c|^{m}(m-1)!}\left[\sum_{k=-\infty}^{n-1} \mathcal{L}_{p}(k)\left(\sum_{j=-\infty}^{k-1} \mathcal{L}_{p}(j)\right)^{m-1}\right]\|u-v\|_{\omega, c}^{\infty} \\
\leq & \frac{(M)^{m}}{|c|^{m} m!}\left(\sum_{k=-\infty}^{n-1} \mathcal{L}_{p}(k)\right)^{m}\|u-v\|_{\omega, c}^{\infty} \\
\leq & \frac{\left(\frac{M}{|c|}\left\|\mathcal{L}_{p}\right\|_{1}\right)^{m}}{m!}\|u-v\|_{\omega, c}^{\infty}
\end{aligned}
$$

which implies that $\left\|\mathcal{P}^{m} u-\mathcal{P}^{m} v\right\|_{\omega, c}^{\infty} \leq \frac{\left(\frac{M}{|c|}\left\|\mathcal{L}_{p}\right\|_{1}\right)^{m}}{m!}\|u-v\|_{\omega, c}^{\infty}$. Since $\frac{\left(\frac{M}{|c|}\left\|\mathcal{L}_{p}\right\|_{1}\right)^{m}}{m!}<1$ for a sufficiently large $m \in \mathbb{Z}^{+}$, we conclude from the Banach fixed point theorem (see Lemma [2.2) that $\mathcal{P}$ has a unique fixed point $u \in \operatorname{DSS} P_{\omega, c}(\mathbb{Z}, X)$ which is a mild solution to Eq. ( (L.]).

The third result is related to the Lipschitz growth condition on $c^{\wedge}(-k) p\left(k, c^{\wedge}(k) \cdot\right)$ for all $k \in \mathbb{Z}$ with a $q$-th summable coefficient $\mathscr{L}_{p}: \mathbb{Z} \rightarrow \mathbb{R}_{+}$.

Theorem 4.3 Suppose that conditions (D1), (D4) and (D5) are satisfied. Then Eq. (ח. ${ }^{(\mathbb{D}) \text { ) has }}$ a unique mild solution $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ provided that $\left\|\mathscr{L}_{p}\right\|_{q} \leq \frac{|c|}{\mathcal{M}}\left(\frac{e^{\tilde{q} \delta}}{e^{\tilde{d} \delta}-1}\right)^{-1 / \tilde{q}}$, where $\tilde{q}=\frac{q}{q-1},\left\|\mathscr{L}_{p}\right\|_{q}:=\left(\sum_{j=-\infty}^{\infty}(\mathscr{L}(j))^{q}\right)^{1 / q}, 1<q<\infty$.

Proof: Similarly, consider the operator $\mathcal{P}$ defined by ( 4.0 ). It follows from Lemma 4.3 that $(\mathcal{P} u)(n) \in D P S A P_{\omega, c}(\mathbb{Z}, X)$ for each $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$. For all $n \in \mathbb{Z}$ and each $u, v \in$ $\operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$, we have

$$
\begin{aligned}
& \|c(-n)[(\mathcal{P} u)(n)-(\mathcal{P} v)(n)]\| \\
\leq & \frac{1}{|c|} \sum_{k=-\infty}^{n-1}\|c(-n+1+k) S(n-1-k)\|\|c(-k)[p(k, u(k))-p(k, v(k))]\| \\
\leq & \frac{1}{|c|} \sum_{k=-\infty}^{n-1}\|c(-n+k+1) S(n-1-k)\| \mathscr{L}_{p}(k)\|u-v\|_{\omega, c}^{\infty} \\
= & \frac{1}{|c|} \sum_{k=0}^{\infty}\|c(-k) S(k)\| \mathscr{L}_{p}(n-1-k)\|u-v\|_{\omega, c}^{\infty} \\
\leq & \frac{\mathcal{M}}{|c|}\left\|\mathscr{L}_{p}\right\|_{q}\left(\sum_{m=0}^{\infty} e^{-m \tilde{q} \delta}\right)^{1 / \tilde{q}}\|u-v\|_{\omega, c}^{\infty} \\
\leq & \frac{\mathcal{M}}{|c|}\left\|\mathscr{L}_{p}\right\|_{q}\left(\frac{e^{\tilde{q} \delta}}{e^{\tilde{q} \delta}-1}\right)^{1 / \tilde{q}}\|u-v\|_{\omega, c}^{\infty} .
\end{aligned}
$$

Therefore,

$$
\|\mathcal{P} u-\mathcal{P} v\|_{\omega, c}^{\infty} \leq \frac{\mathcal{M}}{|c|}\left\|\mathscr{L}_{p}\right\|_{q}\left(\frac{e^{\tilde{q} \delta}}{e^{\tilde{q} \delta}-1}\right)^{1 / \tilde{q}}\|u-v\|_{\omega, c}^{\infty}
$$

which implies that $\mathcal{P}$ is a contraction from $\left\|\mathscr{L}_{p}\right\|_{q} \leq \frac{|c|}{\mathcal{M}}\left(\frac{e^{\tilde{q} \delta}}{e^{\tilde{q} \delta}-1}\right)^{-1 / \tilde{q}}$. Thus Eq. ([.]) has a unique mild solution $u \in \operatorname{DSSA} P_{\omega, c}(\mathbb{Z}, X)$.

Finally, we investigate the existence of pseudo $S$-asymptotically ( $\omega, c$ )-periodic mild solutions to Eq. ([.]l) with non-Lipschitz growth condition on the function $c(-k) p\left(k, c^{\wedge}(k) \cdot\right), k \in \mathbb{Z}$.

Theorem 4.4 Suppose that conditions (DA), (D1) and (D6) are satisfied. Assume further that the following conditions hold:
(D8) There exists a nondecreasing function $W_{p}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$such that $\|p(k, u)\| \leq W_{p}(\|u\|)$ for all $k \in \mathbb{Z}$ and $u \in X$.
(D9) For each $\xi>0, \lim _{|n| \rightarrow \infty} \frac{1}{h(n)} \sum_{k=-\infty}^{n-1}\|S(n-k-1)\| W_{p}(\xi h(k))=0$.
(D10) For each $\epsilon>0$, there exists $\delta>0$ such that for every $u, v \in C_{h}(\mathbb{Z}, X),\|u-v\|_{h} \leq \delta$ implies that $\sum_{k=-\infty}^{n}\|S(n-k)\|\|p(k, u(k))-p(k, v(k))\| \leq \epsilon$ for all $n \in \mathbb{Z}$.
(D11) $\liminf _{r \rightarrow \infty} \frac{\beta(r)}{r}<1$, where $\beta(r)=\sup _{n \in \mathbb{Z}}\left[\frac{1}{h(n+1)} \sum_{k=-\infty}^{n}\|S(n-k)\| W_{p}(r h(k))\right]$.
(D12) For all $a, b \in \mathbb{Z}$ with $a \leq b$ and each $r>0$, the set $\{p(k, u): k \in[a, b],\|u\| \leq r\}$ is relatively compact in $X$.
(D13) For each $\epsilon>0$, there exists $\delta>0$ such that for every $u, v \in C_{h}(\mathbb{Z}, X),\|u-v\|_{h} \leq \delta$ implies that $\left\|c^{\wedge}(-k)[p(k, u(k))-p(k, v(k))]\right\| \leq \epsilon$ for all $k \in \mathbb{Z}$.

Then Eq. ([.ل]) admits at least one mild solution $u \in D P A S P_{\omega, c}(\mathbb{Z}, X)$.
Proof: Let $B_{r}(Y)$ be a closed ball with center at 0 and radius $r$ in the space $Y$. Define the operator $\mathcal{P}$ on $C_{h}(\mathbb{Z}, X)$ in the form ([प.]). For each $u \in C_{h}(\mathbb{Z}, X)$, it follows from the condition (D8) that

$$
\|(\mathcal{P} u)(n)\| \leq \sum_{k=-\infty}^{n-1}\|S(k)\| W_{p}(\|u(k)\|) \leq \sum_{k=-\infty}^{n-1}\|S(k)\| W_{p}\left(\|u\|_{h} h(k)\right)
$$

Thus we have $\lim _{|n| \rightarrow \infty} \frac{\|(\mathcal{P} u)(n)\|}{h(n)}=0$ by the condition (D9), which shows that $\mathcal{P}: C_{h}(\mathbb{Z}, X) \rightarrow$ $C_{h}(\mathbb{Z}, X)$ is well-defined. We shall prove that $\mathcal{P}$ admits a fixed point $u \in D P S A P_{\omega, c}(\mathbb{Z}, X)$. For the sake of convenience, we divide the main proof into the following steps.

Step 1.We show that $\mathcal{P}$ is completely continuous. Firstly, we show that $\mathcal{P}$ is continuous. For any $\epsilon>0$, let $\delta>0$ be chosen in the condition (D10). If $u, v \in C_{h}(\mathbb{Z}, X)$ with $\|u-v\|_{h} \leq \delta$, then we have

$$
\|(\mathcal{P} u)(n)-(\mathcal{P} v)(n)\| \leq \sum_{k=-\infty}^{n-1}\|S(n-k-1)\|\|p(k, u(k))-p(k, v(k))\| \leq \epsilon
$$

Since $h(n) \geq 1$, we obtain that $\frac{\|(\mathcal{P} u)(n)-(\mathcal{P} v)(n)\|}{h(n)} \leq \epsilon$ for all $n \in \mathbb{Z}$. Thus $\|\mathcal{P} u-\mathcal{P} v\|_{h} \leq \epsilon$, which proves the claim.

Secondly, let $V=\mathcal{P}\left(B_{r}\left(C_{h}(\mathbb{Z}, X)\right)\right)$ be defined by $v=\mathcal{P} u$ for $u \in B_{r}\left(C_{h}(\mathbb{Z}, X)\right)$. We show that $Q_{n}(V):=\left\{\frac{v(n)}{h(n)}: v \in V\right\}$ is relatively compact in $X$ for each $n \in \mathbb{Z}$. From the condition (D9), for any $\epsilon>0$ we can choose $a \in \mathbb{Z}^{+}$such that $\frac{1}{h(n)} \sum_{k=a}^{\infty}\|S(k)\| W_{p}(r h(n-k-1)) \leq \epsilon$. For each $u \in B_{r}\left(C_{h}(\mathbb{Z}, X)\right)$, we have

$$
\begin{aligned}
\frac{v(n)}{h(n)}= & \frac{1}{h(n)} \sum_{k=0}^{a-1} S(k) p(n-k-1, u(n-k-1)) \\
& +\frac{1}{h(n)} \sum_{k=a}^{\infty} S(k) p(n-k-1, u(n-k-1)) \\
= & \frac{a}{h(n)}\left[\frac{1}{a} \sum_{k=0}^{a-1} S(k) p(n-k-1, u(n-k-1))\right]
\end{aligned}
$$

$$
+\frac{1}{h(n)} \sum_{k=a}^{\infty} S(k) p(n-k-1, u(n-k-1)) .
$$

By $\frac{1}{h(n)}\left\|\sum_{k=a}^{\infty} S(k) p(n-k-1, u(n-k-1))\right\| \leq \frac{1}{h(n)} \sum_{k=a}^{\infty}\|S(k)\| W_{p}(r h(n-k-1)) \leq \epsilon$, we obtain that $\frac{v(n)}{h(n)} \in \frac{a}{h(n)} \overline{\operatorname{conv}(\mathcal{K})}+B_{\epsilon}(X)$, where $\overline{\operatorname{conv}(\mathcal{K})}$ denotes the convex hull of $\mathcal{K}$ and

$$
\mathcal{K}=\bigcup_{k=0}^{a-1}\{S(k) p(\xi, u): \xi \in[n-a, n-1] \bigcap \mathbb{Z},\|u\| \leq r \max h(\xi)\} .
$$

Since $Q_{n}(V) \subseteq \frac{a}{h(n)} \overline{\operatorname{conv}(\mathcal{K})}+B_{\epsilon}(X)$ and $\mathcal{K}$ is relatively compact from the condition (D12), we infer that $Q_{n}(V)$ is relatively compact in $X$ for all $n \in \mathbb{Z}$. On the other hand, it follows from conditions (D8)-(D9) that

$$
\frac{\|v(n)\|}{h(n)} \leq \frac{1}{h(n)} \sum_{k=-\infty}^{n-1}\|S(n-k-1)\| W_{p}(r h(k)) \rightarrow 0,|n| \rightarrow \infty
$$

and this convergence is independent of $u \in B_{r}\left(C_{h}(\mathbb{Z}, X)\right)$. Hence by Lemma [.工, $V$ is relatively compact in $C_{h}(\mathbb{Z}, X)$.

Step 2. We show that the set $\mathcal{Q}:=\left\{u^{\lambda}: u^{\lambda}=\lambda \mathcal{P}\left(u^{\lambda}\right), \lambda \in(0,1)\right\}$ is bounded. Let $u^{\lambda} \in C_{h}(\mathbb{Z}, X)$ be a solution to the equation $u^{\lambda}=\lambda \mathcal{P}\left(u^{\lambda}\right)$ for some $\lambda \in(0,1)$. Then we have the estimate

$$
\left\|u^{\lambda}(n)\right\| \leq \sum_{k=-\infty}^{n-1}\|S(n-k-1)\| W_{p}\left(\left\|u^{\lambda}\right\|_{h} h(k)\right) \leq h(n) \beta\left(\left\|u^{\lambda}\right\|_{h}\right) .
$$

Thus we have

$$
\frac{\left\|u^{\lambda}\right\|_{h}}{\beta\left(\left\|u^{\lambda}\right\|_{h}\right)} \leq 1
$$

and together with the condition (D11), we can prove the assertion.
Step 3. We show that $\mathcal{P}$ has a fixed point $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$. It is known from Lemma 4.4 that $\mathcal{P}\left(D P S A P_{\omega, c}(\mathbb{Z}, X)\right) \subseteq \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$. Consequently, we can consider the operator $\mathcal{P}:{\overline{\operatorname{DPSAP}}{ }_{\omega, c}(\mathbb{Z}, X)}^{h} \rightarrow{\overline{\operatorname{DPSAP}}{ }_{\omega, c}(\mathbb{Z}, X)}^{h}$ (the closure in $C_{h}(\mathbb{Z}, X)$ ) and $\mathcal{P}$ is completely continuous from Steps 1-2. Since $\mathcal{Q}$ is bounded, we can obtain that $\mathcal{P}$ has a fixed point $u \in{\overline{\operatorname{DPSAP}}{ }_{\omega, c}(\mathbb{Z}, X)}^{h}$ by the Schaefer fixed point theorem (see Lemma [2.3). Let $\left\{u_{n}\right\} \subseteq D P S A P_{\omega, c}(\mathbb{Z}, X)$ be such that $\left\|u_{n}-u\right\|_{h} \rightarrow 0$ as $n \rightarrow \infty$. Then we have

$$
\begin{aligned}
\left\|\mathcal{P} u_{n}-u\right\|_{\omega, c}^{\infty} & =\left\|\mathcal{P} u_{n}-\mathcal{P} u\right\|_{\omega, c}^{\infty} \\
& \leq \sup _{n \in \mathbb{Z}}|c(-n)|\left(\sum_{k=-\infty}^{n-1}\|S(n-k-1)\|\left\|p\left(k, u_{n}(k)\right)-p(k, u(k))\right\|\right)
\end{aligned}
$$

$$
\leq \sup _{n \in \mathbb{Z}} \frac{1}{|c|}\left(\sum_{k=-\infty}^{n-1}\|c(-n+k+1) S(n-k-1)\|\left\|c(-k)\left[p\left(k, u_{n}(k)\right)-p(k, u(k))\right]\right\|\right)
$$

which implies by the condition (D13) that $\left\|\mathcal{P} u_{n}-u\right\|_{\omega, c}^{\infty} \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $\mathbb{Z}$. Since $u_{n} \in \operatorname{DPSAP} P_{\omega, c}(\mathbb{Z}, X)$, we have $u \in \operatorname{DSSAP} P_{\omega, c}(\mathbb{Z}, X)$, i.e. Eq. ([.لD) admits a pseudo $S$-asymptotically $(\omega, c)$-periodic sequential mild solution.

At the end of this paper, we give some examples to illustrate the validity of our main results. For more interesting examples and applications corresponding to Eq. ([.ل(), we can refer to $[\boxed{1}, 3, \boxed{3}, \boxed{]}]$ and references therein.

Example 4.1 Let $X:=L^{2}[0,1]$. Assume that $a(n)=\frac{1}{a}\left(1-\frac{1}{(1+a)^{n+1}}\right)$ and $b(n)=1-\frac{n-1}{2^{n+2}}$. We consider the following differential-difference equation

$$
\begin{equation*}
u(n+1, x)=\sum_{k=-\infty}^{n} a(n-k) \frac{\partial^{2}}{\partial x^{2}} u(k+1, x)+\sum_{k=-\infty}^{n} b(n-k) p(k, u(k, x)), n \in \mathbb{Z}, x \in[0,1] . \tag{4.3}
\end{equation*}
$$

Define the operator $A:=\frac{\partial^{2}}{\partial x^{2}} u$ on $X$ with domain $D(A):=\left\{u \in L^{2}[0, \pi]: u^{\prime \prime} \in L^{2}[0,1] ; u(0)=\right.$ $u(1)=0\}$. Thus Eq. (4.3) can be converted into Eq. (【.]) with $u(k):=u(k, \cdot)$. Moreover, the operator $A$ generates a bounded analytic $C_{0}$-semigroup on $X$ (see [15., Example 4.8]). It also follows from [18, Theorem 3.9] that the operator $A$ with above choices of $a, b$ generates a summable discrete resolvent family $\{S(k)\}_{k \in \mathbb{Z}_{+}}$satisfying $\|S(k)\| \leq M, M>0$ for all $k \in \mathbb{Z}$.

Take $p(k, u)=\eta \frac{\sin (u+1)}{e^{|2 k|}}(\eta>0)$ and $c=e$. Thus we have for all $\omega \in \mathbb{Z}_{+}$

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\|p(k+\omega, u)-c p(k,-u)\| e^{-\frac{k}{\omega}} \\
= & \lim _{n \rightarrow \infty} \frac{1}{2 n+1} \sum_{k=-n}^{n}\left\|\eta \frac{\sin (u+1)}{e^{|2 k|}}-c \eta \frac{\sin (-u+1)}{e^{|2 k|}}\right\| e^{-\frac{k}{\omega}} \\
\leq & \lim _{n \rightarrow \infty} \frac{\eta}{2 n+1} \sum_{k=-n}^{n}\left(\frac{1}{e^{|k|}} \frac{e^{-\frac{k}{\omega}}}{e^{|k|}}+\frac{c e^{-\frac{k}{\omega}}}{e^{|2 k|}}\right) \\
\leq & \lim _{n \rightarrow \infty} \frac{\eta}{2 n+1}\left[\left(\sum_{k=0}^{n} \frac{e^{-\frac{k}{\omega}}}{e^{2 k}}+\frac{c e^{-\frac{k}{\omega}}}{e^{2 k}}\right)+\left(\sum_{k=0}^{n} \frac{e^{\frac{k}{\omega}}}{e^{2 k}}+\frac{c e^{\frac{k}{\omega}}}{e^{2 k}}\right)\right]=0
\end{aligned}
$$

and $\sup _{n \in \mathbb{Z}} c(-n) p\left(n, c^{\wedge}(n) x\right)=\sup _{k \in \mathbb{Z}} c(-n) \eta \frac{\sin \left(c^{\wedge}(n) x+1\right)}{e^{|2 n|}}<\infty$.
It shows that the condition (D1) holds. On the other hand, we have

$$
\begin{aligned}
\left\|c(-k)\left[p\left(k, c^{\wedge}(k) u\right)-p\left(k, c^{\wedge}(k) v\right)\right]\right\|_{X}^{2} & =\left\|c(-k)\left[\eta \frac{\sin \left(c^{\wedge}(k) u+1\right)}{e^{|2 k|}}-\eta \frac{\sin \left(c^{\wedge}(k) v+1\right)}{e^{|2 k|}}\right]\right\|_{X}^{2} \\
& \leq|c(-k)|^{2} \frac{\eta^{2}}{e^{4 k \mid}\left\|\sin \left(c^{\wedge}(k) u+1\right)-\sin \left(c^{\wedge}(k) v+1\right)\right\|_{X}^{2}} \\
& \leq|[c(-k)]|^{2} \eta^{2}\left\|c^{\wedge}(k) u-c^{\wedge}(k) v\right\|_{X}^{2}
\end{aligned}
$$

$$
\leq \eta^{2}\|u-v\|_{X}^{2}
$$

for each $u, v \in X, k \in \mathbb{Z}$. Hence for $\eta$ small enough, Eq. (4.3) admits a unique solution $u \in D P A S P_{\omega, c}(\mathbb{Z}, X)$ via Corollary 4.d.

Let $a(k)=b(k)=\frac{\Gamma(k+\alpha)}{\Gamma(\alpha) \Gamma(n+1)}, A:=\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda<0$. Then we can have the following scalar-valued difference equation

$$
\begin{equation*}
\Delta^{\alpha} u(n)=\lambda u(n+1)+f(n), 0<\alpha \leq 1, n \in \mathbb{Z} \tag{4.4}
\end{equation*}
$$

where $\Delta^{\alpha}$ is fractional difference in Weyl-like sense, $f \in l_{\omega, c}^{\infty}(\mathbb{Z}, \mathbb{C})$. It is known from [ $\mathbb{1}$, Theorem 3.5] that $\lambda$ generates a summable discrete resolvent family $\left\{S_{\alpha}(n)\right\}_{n \in \mathbb{Z}_{+}}$given by

$$
S_{\alpha}(n)=\frac{(-1)^{n}}{n!}\left(\left(s^{\alpha}-\lambda\right)^{-1}\right)_{\left.\right|_{s=1}}^{(n)} .
$$

Thus the solution of Eq. (4.4) can be formulated in the form $u(n+1)=\sum_{k=-\infty}^{n} S_{\alpha}(n-k) f(k)$. It follows from Theorem [3.4 that $u \in \operatorname{DPSAP}_{\omega, c}(\mathbb{Z}, X)$ if $f \in D P S A P_{\omega, c}(\mathbb{Z}, X)$. Here we give numerical simulations for solutions of Eq. (4.4) whenever $\alpha=1 / 2, \lambda=-1 / 10$ and let $\omega=4$. In Figure 四, we have $f_{1}(k)=e^{\frac{k}{8}} \cos (\pi k / 2)+e^{-|k|^{4}}, k \in \mathbb{Z}$ for $c=e^{\frac{1}{2}}$, whereas in Figure [】, we have $f_{2}(k)=e^{\frac{k}{16}} \cos (\pi k / 2)+e^{-|k|^{3}}, k \in \mathbb{Z}$ for $c=e^{\frac{1}{4}}$. The notation denotes values of $u$ at $n$.


Figure 1: Solution $u(n)$ for the case $f_{1}$ on the interval $[-16,16]$.


Figure 2: Solution $u(n)$ for the case $f_{2}$ on the interval $[-16,16]$.

## Statements and Declarations

## Competing interests

The authors declare no competing interest.
Authors' contributions
All authors contribute equally to this manuscript.
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